Article

Not peer-reviewed version

On Fuzzy γI-Continuity and γI-Irresoluteness via K-Fuzzy γI-Open Sets

Islam M. Taha * and Maha Mohammed Saeed

Posted Date: 12 March 2025

doi: 10.20944/preprints202503.0839.v1

Keywords: fuzzy ideals; fuzzy topology; k-fuzzy γI-open set; fuzzy γI-continuity; fuzzy γI-irresoluteness; fuzzy γI-openness; fuzzy γI-regular space; fuzzy γI-normal space

Article

On Fuzzy $\gamma\mathcal{I}$-Continuity and $\gamma\mathcal{I}$-Irresoluteness via k-Fuzzy $\gamma\mathcal{I}$-Open Sets

Islam M. Taha [1,*] and Maha M. Saeed [2]

[1] Department of Mathematics, Faculty of Science, Sohag University, Sohag, Egypt
[2] Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia
* Correspondence: itaha@ut.edu.sa

Abstract: In this article, we explored and investigated a novel class of fuzzy sets, called k-fuzzy $\gamma\mathcal{I}$-open (k-F$\gamma\mathcal{I}$-open) sets in fuzzy ideal topological spaces ($\mathcal{FITS}s$) based on Šostak′s sense. The class of k-F$\gamma\mathcal{I}$-open sets is contained in the class of k-fuzzy strong β-$\mathcal{I}$-open (k-FS$\beta\mathcal{I}$-open) sets and contains all k-fuzzy pre-$\mathcal{I}$-open (k-FP$\mathcal{I}$-open) sets and k-fuzzy semi-$\mathcal{I}$-open (k-FS$\mathcal{I}$-open) sets. We also introduced and studied the interior and closure operators with respect to the classes of k-F$\gamma\mathcal{I}$-open sets and k-F$\gamma\mathcal{I}$-closed sets. However, we defined and discussed novel types of fuzzy $\mathcal{I}$-separation axioms using k-F$\gamma\mathcal{I}$-closed sets, called k-F$\gamma\mathcal{I}$-regular spaces and k-F$\gamma\mathcal{I}$-normal spaces. Thereafter, we displayed and studied the notion of fuzzy $\gamma\mathcal{I}$-continuity (F$\gamma\mathcal{I}$-continuity) using k-F$\gamma\mathcal{I}$-open sets. Furthermore, we presented and characterized the notions of fuzzy weak $\gamma\mathcal{I}$-continuity (FW$\gamma\mathcal{I}$-continuity) and fuzzy almost $\gamma\mathcal{I}$-continuity (FA$\gamma\mathcal{I}$-continuity), which are weaker forms of F$\gamma\mathcal{I}$-continuity. Finally, we introduced and investigated some new fuzzy $\gamma\mathcal{I}$-mappings via k-F$\gamma\mathcal{I}$-open sets and k-F$\gamma\mathcal{I}$-closed sets, called F$\gamma\mathcal{I}$-open mappings, F$\gamma\mathcal{I}$-closed mappings, F$\gamma\mathcal{I}$-irresolute mappings, F$\gamma\mathcal{I}$-irresolute open mappings, and F$\gamma\mathcal{I}$-irresolute closed mappings.

Keywords: fuzzy ideals; fuzzy topology; k-fuzzy $\gamma\mathcal{I}$-open set; fuzzy $\gamma\mathcal{I}$-continuity; fuzzy $\gamma\mathcal{I}$-irresoluteness; fuzzy $\gamma\mathcal{I}$-openness; fuzzy $\gamma\mathcal{I}$-regular space; fuzzy $\gamma\mathcal{I}$-normal space

Mathematics Subject Classification: 54A05; 54A40; 54C05; 54C08; 54D15

---

1. Introduction

The concept of a fuzzy set of a nonempty set Z is a mapping $\rho : Z \to I$ (where $I = [0,1]$). This concept was first defined in 1965 by Zadeh [1]. The integration between fuzzy sets and some uncertainty approaches such as rough sets and soft sets has been investigated in [2–4]. The concept of a fuzzy topology was presented in 1968 by Chang [5]. Several authors have successfully generalized the theory of general topology to the fuzzy setting with crisp methods. According to Šostak [6], the notion of a fuzzy topology being a crisp subclass of the class of fuzzy sets and fuzziness in the notion of openness of a fuzzy set have not been considered, which seems to be a drawback in the process of fuzzification of a topological space. Therefore, Šostak [6] defined a novel definition of a fuzzy topology as the concept of openness of fuzzy sets. It is an extension of a fuzzy topology defined by Chang [5]. Thereafter, many researchers (Ramadan [7], Chattopadhyay et. al. [8], El Gayyar et. al. [9], Höhle and Šostak [10], Ramadan et. al. [11], Kim et. al. [12], Abbas [13,14], Kim and Abbas [15], Aygun and Abbas [16,17], Li and Shi [18,19], Shi and Li [20], Fang and Guo [21], El-Dardery et. al. [22], Kalaivani and Roopkumar [23], Solovyov [24], Minana and Šostak [25]) have redefined the same notion and investigated fuzzy topological spaces ($\mathcal{FTS}s$) being unaware of Šostak′s work.

The generalizations of fuzzy open sets plays an effective role in a fuzzy topology through their ability to improve on many results, or to open the door to explore and discuss several fuzzy topological notions such as fuzzy continuity [7,8], fuzzy connectedness [8], fuzzy compactness [8,9], fuzzy separation axioms [18], etc. Overall, the notions of k-fuzzy pre-open (k-FP-open) sets, k-fuzzy semi-open

 doi:10.20944/preprints202503.0839.v1

(k-FS-open) sets, k-fuzzy β-open (k-Fβ-open) sets, and k-fuzzy α-open (k-Fα-open) sets were presented and investigated by the authors of [12,14] in $\mathcal{FTS}s$ based on Šostak's sense [6]. Also, Kim et al. [12] defined and discussed some weaker forms of fuzzy continuity, called FS-continuity (resp. FP-continuity and Fα-continuity) between $\mathcal{FTS}s$ based on Šostak's sense. Abbas [14] explored and characterized the concepts of Fβ-continuous (resp. Fβ-irresolute) mappings between $\mathcal{FTS}s$ in the sense of Šostak. Also, Kim and Abbas [15] defined some new types of k-fuzzy compactness on $\mathcal{FTS}s$ in the sense of Šostak. Furthermore, the notions of k-fuzzy γ-open (k-Fγ-open) sets and k-fuzzy γ-closed (k-Fγ-closed) sets were defined and discussed by the authors of [26] on $\mathcal{FTS}s$ in the sense of Šostak [6].

A novel concept of fuzzy local function, called k-fuzzy local function was presented and investigated by Taha and Abbas [27] in an $\mathcal{FITS}$ $(Z, \zeta, \mathcal{I})$ based on Šostak's sense [6]. Moreover, the concepts of fuzzy lower (resp. upper) weakly and almost $\mathcal{I}$-continuous multifunctions were displayed and investigated by Taha and Abbas [27]. Also, Taha [28–30] introduced the notions of k-FS$\mathcal{I}$-open sets, k-FP$\mathcal{I}$-open sets, k-F$\alpha\mathcal{I}$-open sets, k-F$\beta\mathcal{I}$-open sets, k-FS$\beta\mathcal{I}$-open sets, k-F$\delta\mathcal{I}$-open sets, and k-GF$\mathcal{I}$-closed sets in an $\mathcal{FITS}$ $(Z, \zeta, \mathcal{I})$ based on Šostak's sense. Overall, Taha [29–31] presented the notions of fuzzy upper (resp. lower) generalized $\mathcal{I}$-continuous (resp. pre-$\mathcal{I}$-continuous, semi-$\mathcal{I}$-continuous, α-$\mathcal{I}$-continuous, δ-$\mathcal{I}$-continuous, and strong β-$\mathcal{I}$-continuous) multifunctions via fuzzy ideals [32].

The purpose of this study is as follows. Section 2 contains many basic results and notions that help in understanding the obtained results. In Section 3, we present and study a novel class of fuzzy sets, called k-F$\gamma\mathcal{I}$-open sets in $\mathcal{FITS}s$ based on Šostak's sense. This class is contained in the class of k-FS$\beta\mathcal{I}$-open sets and contains all k-F$\alpha\mathcal{I}$-open sets, k-FP$\mathcal{I}$-open sets, and k-FS$\mathcal{I}$-open sets. We also define and discuss the closure and interior operators with respect to the classes of k-F$\gamma\mathcal{I}$-open sets and k-F$\gamma\mathcal{I}$-closed sets. Furthermore, we introduce new types of fuzzy $\mathcal{I}$-separation axioms using k-F$\gamma\mathcal{I}$-closed sets, called k-F$\gamma\mathcal{I}$-regular spaces and k-F$\gamma\mathcal{I}$-normal spaces, and study some properties of them. In Section 4, we present and investigate the concept of F$\gamma\mathcal{I}$-continuous mappings using k-F$\gamma\mathcal{I}$-open sets. Also, we display and characterize the concepts of FA$\gamma\mathcal{I}$-continuous and FW$\gamma\mathcal{I}$-continuous mappings, which are weaker forms of F$\gamma\mathcal{I}$-continuous mappings. In Section 5, we explore and discuss some new F$\gamma\mathcal{I}$-mappings using k-F$\gamma\mathcal{I}$-open sets and k-F$\gamma\mathcal{I}$-closed sets, called F$\gamma\mathcal{I}$-open mappings, F$\gamma\mathcal{I}$-closed mappings, F$\gamma\mathcal{I}$-irresolute mappings, F$\gamma\mathcal{I}$-irresolute open mappings, and F$\gamma\mathcal{I}$-irresolute closed mappings. In the last section, we close this work with proposed future articles and conclusions.

2. Preliminaries

In this study, non-empty sets will be denoted by Z, Y, X, etc. On Z, I^Z is the class of all fuzzy sets. For any fuzzy set $\omega \in I^Z$, $\omega^c(z) = 1 - \omega(z)$, for each $z \in Z$. Also, for $s \in I$, $\underline{s}(z) = s$, for each $z \in Z$.

A fuzzy point z_s on Z is a fuzzy set, is defined as follows: $z_s(v) = s$ if $v = z$, and $z_s(v) = 0$ for any $v \in Z - \{z\}$. Moreover, we say that z_s belongs to $\omega \in I^Z$ ($z_s \in \omega$), if $s \leq \omega(z)$. On Z, $P_s(Z)$ is the class of all fuzzy points.

On Z, a fuzzy set $\nu \in I^Z$ is a quasi-coincident with $\rho \in I^Z$ ($\nu\ \mathcal{Q}\ \rho$), if there is $z \in Z$, with $\nu(z) + \rho(z) > 1$. Otherwise, ν is not a quasi-coincident with ρ ($\nu\ \overline{\mathcal{Q}}\ \rho$).

The difference between $\nu, \rho \in I^Z$ [27] is defined as follows:

$$\nu \overline{\wedge} \rho = \begin{cases} \underline{0}, & \text{if} \quad \nu \leq \rho, \\ \nu \wedge \rho^c, & \text{otherwise.} \end{cases}$$

Lemma 2.1. [33] Let $\omega, \rho \in I^Z$. Thus,

(1) $\omega\ \mathcal{Q}\ \rho$ iff there is $z_s \in \omega$ such that $z_s\ \mathcal{Q}\ \rho$,

(2) if $\omega\ \mathcal{Q}\ \rho$, then $\omega \wedge \rho \neq \underline{0}$,

(3) $\omega\ \overline{\mathcal{Q}}\ \rho$ iff $\omega \leq \rho^c$,

(4) $\omega \leq \rho$ iff $z_s \in \omega$ implies $z_s \in \rho$ iff $z_s\ \mathcal{Q}\ \omega$ implies $z_s\ \mathcal{Q}\ \rho$ iff $z_s\ \overline{\mathcal{Q}}\ \rho$ implies $z_s\ \overline{\mathcal{Q}}\ \omega$,

(5) $z_s\ \overline{\mathcal{Q}}\ \bigvee_{i \in \Gamma} \omega_i$ iff there is $i_\circ \in \Gamma$ such that $z_s\ \overline{\mathcal{Q}}\ \omega_{i_\circ}$.

 doi:10.20944/preprints202503.0839.v1

Definition 2.1. [6,7] A mapping $\zeta : I^Z \longrightarrow I$ is called a fuzzy topology on Z if it satisfies the following conditions:

(1) $\zeta(\underline{1}) = \zeta(\underline{0}) = 1$.
(2) $\zeta(\omega \wedge \rho) \geq \zeta(\omega) \wedge \zeta(\rho)$, for each $\omega, \rho \in I^Z$.
(3) $\zeta(\bigvee_{i\in\Gamma} \omega_i) \geq \bigwedge_{i\in\Gamma} \zeta(\omega_i)$, for each $\omega_i \in I^Z$.
Thus, (Z, ζ) is called a fuzzy topological space ($\mathcal{FTS}$) based on Šostak's sense.

Definition 2.2. [7,12] A fuzzy mapping $\mathbb{P} : (Z, \zeta) \longrightarrow (Y, \Im)$ is called

(1) fuzzy continuous if $\zeta(\mathbb{P}^{-1}(\rho)) \geq \Im(\rho)$, for every $\rho \in I^Y$;
(2) fuzzy open if $\Im(\mathbb{P}(\omega)) \geq \zeta(\omega)$, for every $\omega \in I^Z$;
(3) fuzzy closed if $\Im((\mathbb{P}(\omega))^c) \geq \zeta(\omega^c)$, for every $\omega \in I^Z$.

Definition 2.3. [8,11] In an $\mathcal{FTS}$ (Z, ζ), for each $\omega \in I^Z$ and $k \in I_\circ$ (where $I_\circ = (0, 1]$), we define fuzzy operators C_ζ and $I_\zeta : I^Z \times I_\circ \to I^Z$ as follows:

$$C_\zeta(\omega, k) = \bigwedge \{\nu \in I^Z : \omega \leq \nu,\ \zeta(\nu^c) \geq k\}.$$

$$I_\zeta(\omega, k) = \bigvee \{\nu \in I^Z : \nu \leq \omega,\ \zeta(\nu) \geq k\}.$$

Definition 2.4. [12,14,26] Let (Z, ζ) be an $\mathcal{FTS}$ and $k \in I_\circ$. A fuzzy set $\omega \in I^Z$ is called

(1) k-F-open if $\omega = I_\zeta(\omega, k)$;
(2) k-FP-open if $\omega \leq I_\zeta(C_\zeta(\omega, k), k)$;
(3) k-FS-open if $\omega \leq C_\zeta(I_\zeta(\omega, k), k)$;
(4) k-FR-open if $\omega = I_\zeta(C_\zeta(\omega, k), k)$;
(5) k-Fα-open if $\omega \leq I_\zeta(C_\zeta(I_\zeta(\omega, k), k), k)$;
(6) k-Fβ-open if $\omega \leq C_\zeta(I_\zeta(C_\zeta(\omega, k), k), k)$;
(7) k-Fγ-open if $\omega \leq C_\zeta(I_\zeta(\omega, k), k) \vee I_\zeta(C_\zeta(\omega, k), k)$.

Remark 2.1. [12,14,26] From the previous definitions, we have the following diagram.

k-**FP-open set**

↗ ↘

k-**F-open set** ⟶ *k*-**Fff-open set** ⟶ *k*-**Ffl-open set** ⟶ *k*-**Ffi-open set**

↘ ↗

k-**FS-open set**

Definition 2.5. [12,14,26] A fuzzy mapping $\mathbb{P} : (Z, \zeta) \longrightarrow (Y, \Im)$ is called FS-continuous (resp. FP-continuous, Fα-continuous, Fβ-continuous, and Fγ-continuous) if $\mathbb{P}^{-1}(\omega)$ is an k-FS-open (resp. k-FP-open, k-Fα-open, k-Fβ-open, and k-Fγ-open) set, for every $\omega \in I^Y$ with $\Im(\omega) \geq k$ and $k \in I_\circ$.

Definition 2.6. [26] In an $\mathcal{FTS}$ (Z, ζ), for each $\omega \in I^Z$ and $k \in I_\circ$, we define fuzzy operators γC_ζ and $\gamma I_\zeta : I^Z \times I_\circ \to I^Z$ as follows:

$$\gamma C_\zeta(\omega, k) = \bigwedge \{\nu \in I^Z : \omega \leq \nu,\ \nu \text{ is } k\text{-F}\gamma\text{-closed}\}.$$

$$\gamma I_\zeta(\omega, k) = \bigvee \{\nu \in I^Z : \nu \leq \omega,\ \nu \text{ is } k\text{-F}\gamma\text{-open}\}.$$

Definition 2.7. [32] A fuzzy ideal $\mathcal{I}$ on Z, is a map $\mathcal{I} : I^Z \longrightarrow I$ that satisfies the following:

(1) $\forall\, \omega, \nu \in I^Z$ and $\omega \leq \nu \ \Rightarrow\ \mathcal{I}(\nu) \leq \mathcal{I}(\omega)$.
(2) $\forall\, \omega, \nu \in I^Z \ \Rightarrow\ \mathcal{I}(\omega \vee \nu) \geq \mathcal{I}(\omega) \wedge \mathcal{I}(\nu)$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 12 March 2025 doi:10.20944/preprints202503.0839.v1

Moreover, $\mathcal{I}_0$ is the simplest fuzzy ideal on Z, and is defined as follows:

$$\mathcal{I}_0(\nu) = \begin{cases} 1, & \text{if} \quad \nu = \underline{0}, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.8. [27] Let $(Z, \zeta, \mathcal{I})$ be an $\mathcal{FITS}$, $k \in I_\circ$, and $\omega \in I^Z$. Then the k-fuzzy local function ω_k^* of ω is defined as follows:

$$\omega_k^* = \bigwedge\{\rho \in I^Z \ : \ \mathcal{I}(\omega \barwedge \rho) \geq k, \ \zeta(\rho^c) \geq k\}.$$

Remark 2.2. [27] If we take $\mathcal{I} = \mathcal{I}_0$, for each $\omega \in I^Z$ we have:

$$\omega_k^* = \bigwedge\{\rho \in I^Z \ : \ \omega \leq \rho, \ \zeta(\rho^c) \geq k\} = C_\zeta(\omega, k).$$

Definition 2.9. [27] Let $(Z, \zeta, \mathcal{I})$ be an $\mathcal{FITS}$, $k \in I_\circ$, and $\omega \in I^Z$. Then we define fuzzy operator C_ζ^* : $I^Z \times I_\circ \to I^Z$ as follows:

$$C_\zeta^*(\omega, k) = \omega \ \vee \ \omega_k^*.$$

Now if, $\mathcal{I} = \mathcal{I}_0$ then $C_\zeta^*(\omega, k) = \omega \vee \omega_k^* = \omega \vee C_\zeta(\omega, k) = C_\zeta(\omega, k)$ for each $\omega \in I^Z$.

Theorem 2.1. [27] Let $(Z, \zeta, \mathcal{I})$ be an $\mathcal{FITS}$, $k \in I_\circ$, and $\omega, \rho \in I^Z$. The operator $C_\zeta^* : I^Z \times I_\circ \to I^Z$ satisfies the following properties:
(1) $C_\zeta^*(\underline{0}, k) = \underline{0}$.
(2) $\omega \leq C_\zeta^*(\omega, k) \leq C_\zeta(\omega, k)$.
(3) If $\omega \leq \rho$, then $C_\zeta^*(\omega, k) \leq C_\zeta^*(\rho, k)$.
(4) $C_\zeta^*(\omega \vee \rho, k) = C_\zeta^*(\omega, k) \vee C_\zeta^*(\rho, k)$.
(5) $C_\zeta^*(\omega \wedge \rho, k) \leq C_\zeta^*(\omega, k) \wedge C_\zeta^*(\rho, k)$.
(6) $C_\zeta^*(C_\zeta^*(\omega, k), k) = C_\zeta^*(\omega, k)$.

Definition 2.10. [28,30] Let $(Z, \zeta, \mathcal{I})$ be an $\mathcal{FITS}$ and $k \in I_\circ$. A fuzzy set $\omega \in I^Z$ is called
(1) k-FS$\mathcal{I}$-open if $\omega \leq C_\zeta^*(I_\zeta(\omega, k), k)$;
(2) k-FP$\mathcal{I}$-open if $\omega \leq I_\zeta(C_\zeta^*(\omega, k), k)$;
(3) k-F$\alpha\mathcal{I}$-open if $\omega \leq I_\zeta(C_\zeta^*(I_\zeta(\omega, k), k), k)$;
(4) k-F$\beta\mathcal{I}$-open if $\omega \leq C_\zeta(I_\zeta(C_\zeta^*(\omega, k), k), k)$;
(5) k-FS$\beta\mathcal{I}$-open if $\omega \leq C_\zeta^*(I_\zeta(C_\zeta^*(\omega, k), k), k)$;
(6) k-FR$\mathcal{I}$-open if $\omega = I_\zeta(C_\zeta^*(\omega, k), k)$.

Definition 2.11. A fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Y, \Im)$ is called F$\alpha\mathcal{I}$-continuous (resp. FP$\mathcal{I}$-continuous, FS$\mathcal{I}$-continuous, and FS$\beta\mathcal{I}$-continuous) if $\mathbb{P}^{-1}(\omega)$ is an k-F$\alpha\mathcal{I}$-open (resp. k-FP$\mathcal{I}$-open, k-FS$\mathcal{I}$-open, and k-FS$\beta\mathcal{I}$-open) set, for each $\omega \in I^Y$ with $\Im(\omega) \geq k$ and $k \in I_\circ$.

Some basic notations and results that we need in the sequel are found in [7-9,27-31].

3. On k-Fuzzy $\gamma\mathcal{I}$-Open Sets

Definition 3.1. Let $(Z, \zeta, \mathcal{I})$ be an $\mathcal{FITS}$ and $k \in I_\circ$. A fuzzy set $\rho \in I^Z$ is called an k-F$\gamma\mathcal{I}$-open set if $\rho \leq C_\zeta^*(I_\zeta(\rho, k), k) \vee I_\zeta(C_\zeta^*(\rho, k), k)$.

Remark 3.1. The complement of k-F$\gamma\mathcal{I}$-open sets are k-F$\gamma\mathcal{I}$-closed sets.

Lemma 3.1. Every k-F$\gamma\mathcal{I}$-open set is k-Fγ-open [26].

Proof. The proof follows from Definitions 2.4, 3.1, and Theorem 2.1(2). □

Remark 3.2. If we take $\mathcal{I} = \mathcal{I}_0$, then k-F$\gamma\mathcal{I}$-open set and k-Fγ-open set [26] are equivalent.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 12 March 2025 doi:10.20944/preprints202503.0839.v1

Remark 3.3. The converse of Lemma 3.1 fails as Example 3.1 will show.

Example 3.1. Define $\zeta, \mathcal{I} : I^Z \longrightarrow I$ as follows:

$$\zeta(\rho) = \begin{cases} 1, & \text{if} \quad \rho \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if} \quad \rho = \underline{0.7}, \\ \frac{1}{3}, & \text{if} \quad \rho = \underline{0.3}, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{I}(\nu) = \begin{cases} 1, & \text{if} \quad \nu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \nu \leq \underline{0.6}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $\underline{0.6}$ is an $\frac{1}{3}$-Fγ-open set, but it is not $\frac{1}{3}$-F$\gamma\mathcal{I}$-open.

Proposition 3.1. In an $\mathcal{FITS}$ $(Z, \zeta, \mathcal{I})$, for each $\omega \in I^Z$ and $k \in I_\circ$. Then
(1) each k-FP$\mathcal{I}$-open set [28] is k-F$\gamma\mathcal{I}$-open;
(2) each k-F$\gamma\mathcal{I}$-open set is k-FS$\beta\mathcal{I}$-open [30];
(3) each k-FS$\mathcal{I}$-open set [28] is k-F$\gamma\mathcal{I}$-open.

Proof. (1) If ω is an k-FP$\mathcal{I}$-open set. Then

$$\begin{aligned} \omega &\leq I_\zeta(C^*_\zeta(\omega, k), k) \\ &\leq I_\zeta(C^*_\zeta(\omega, k), k) \vee I_\zeta(\omega, k) \\ &\leq I_\zeta(C^*_\zeta(\omega, k), k) \vee C^*_\zeta(I_\zeta(\omega, k), k). \end{aligned}$$

Thus, ω is k-F$\gamma\mathcal{I}$-open.
(2) If ω is an k-F$\gamma\mathcal{I}$-open set. Then

$$\begin{aligned} \omega &\leq C^*_\zeta(I_\zeta(\omega, k), k) \vee I_\zeta(C^*_\zeta(\omega, k), k) \\ &\leq C^*_\zeta(I_\zeta(C^*_\zeta(\omega, k), k), k) \vee I_\zeta(C^*_\zeta(\omega, k), k) \\ &\leq C^*_\zeta(I_\zeta(C^*_\zeta(\omega, k), k), k). \end{aligned}$$

Thus, ω is k-FS$\beta\mathcal{I}$-open.
(3) If ω is an k-FS$\mathcal{I}$-open set. Then

$$\begin{aligned} \omega &\leq C^*_\zeta(I_\zeta(\omega, k), k) \\ &\leq C^*_\zeta(I_\zeta(\omega, k), k) \vee I_\zeta(\omega, k) \\ &\leq C^*_\zeta(I_\zeta(\omega, k), k) \vee I_\zeta(C^*_\zeta(\omega, k), k). \end{aligned}$$

Thus, ω is k-F$\gamma\mathcal{I}$-open. □

Remark 3.4. From the previous discussions and definitions, we have the following diagram.

$$
\begin{array}{ccccc}
 & & k\text{-}\mathbf{FP}\mathcal{I}\text{-}\mathbf{open\ set} & & \\
 & \nearrow & \downarrow & & \\
k\text{-}\mathbf{Fff}\mathcal{I}\text{-}\mathbf{open\ set} & \longrightarrow & k\text{-}\mathbf{Ffl}\mathcal{I}\text{-}\mathbf{open\ set} & \longrightarrow & k\text{-}\mathbf{FSfi}\mathcal{I}\text{-}\mathbf{open\ set} \\
 & \searrow & \uparrow & & \\
 & & k\text{-}\mathbf{FS}\mathcal{I}\text{-}\mathbf{open\ set} & &
\end{array}
$$

Remark 3.5. The converse of the above diagram fails as Examples 3.2, 3.3, and 3.4 will show.

Example 3.2. Let $Z = \{z_1, z_2\}$ and define $\omega, \rho, \lambda \in I^Z$ as follows: $\omega = \{\frac{z_1}{0.4}, \frac{z_2}{0.3}\}$, $\rho = \{\frac{z_1}{0.5}, \frac{z_2}{0.4}\}$, $\lambda = \{\frac{z_1}{0.4}, \frac{z_2}{0.5}\}$. Define $\zeta, \mathcal{I} : I^Z \longrightarrow I$ as follows:

$$
\zeta(\nu) = \begin{cases} 1, & \text{if} \quad \nu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{4}, & \text{if} \quad \nu = \rho, \\ \frac{1}{2}, & \text{if} \quad \nu = \omega, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{I}(\mu) = \begin{cases} 1, & \text{if} \quad \mu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \mu < \underline{0.3}, \\ 0, & \text{otherwise}. \end{cases}
$$

Thus, λ is an $\frac{1}{4}$-F$\gamma\mathcal{I}$-open set, but it is not $\frac{1}{4}$-FP$\mathcal{I}$-open.

Example 3.3. Let $Z = \{z_1, z_2\}$ and define $\omega, \rho, \lambda \in I^Z$ as follows: $\omega = \{\frac{z_1}{0.3}, \frac{z_2}{0.2}\}$, $\rho = \{\frac{z_1}{0.7}, \frac{z_2}{0.8}\}$, $\lambda = \{\frac{z_1}{0.5}, \frac{z_2}{0.4}\}$. Define $\zeta, \mathcal{I} : I^Z \longrightarrow I$ as follows:

$$
\zeta(\nu) = \begin{cases} 1, & \text{if} \quad \nu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{3}, & \text{if} \quad \nu = \omega, \\ \frac{1}{2}, & \text{if} \quad \nu = \rho, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{I}(\mu) = \begin{cases} 1, & \text{if} \quad \mu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \mu < \underline{0.5}, \\ 0, & \text{otherwise}. \end{cases}
$$

Thus, λ is an $\frac{1}{3}$-F$\gamma\mathcal{I}$-open set, but it is neither $\frac{1}{3}$-FS$\mathcal{I}$-open nor $\frac{1}{3}$-F$\alpha\mathcal{I}$-open.

Example 3.4. Let $Z = \{z_1, z_2\}$ and define $\omega, \lambda \in I^Z$ as follows: $\omega = \{\frac{z_1}{0.5}, \frac{z_2}{0.4}\}$, $\lambda = \{\frac{z_1}{0.4}, \frac{z_2}{0.5}\}$. Define $\zeta, \mathcal{I} : I^Z \longrightarrow I$ as follows:

$$
\zeta(\nu) = \begin{cases} 1, & \text{if} \quad \nu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if} \quad \nu = \omega, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{I}(\mu) = \begin{cases} 1, & \text{if} \quad \mu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \mu < \underline{0.4}, \\ 0, & \text{otherwise}. \end{cases}
$$

Thus, λ is an $\frac{1}{3}$-FS$\beta\mathcal{I}$-open set, but it is not $\frac{1}{3}$-F$\gamma\mathcal{I}$-open.

Corollary 3.1. In an $\mathcal{FITS}$ $(Z, \zeta, \mathcal{I})$ and $k \in I_\circ$. Then
(1) the union of k-F$\gamma\mathcal{I}$-open sets is k-F$\gamma\mathcal{I}$-open;
(2) the intersection of k-F$\gamma\mathcal{I}$-closed sets is k-F$\gamma\mathcal{I}$-closed.

Proof. This is easily proved by Definition 3.1 and Remark 3.1. □

Corollary 3.2. In an $\mathcal{FITS}$ $(Z, \zeta, \mathcal{I})$, for each k-F$\gamma\mathcal{I}$-open set $\omega \in I^Z$.
(1) If ω is an k-FR$\mathcal{I}$-open set, then ω is k-FS$\mathcal{I}$-open.
(2) If ω is an k-FR$\mathcal{I}$-closed set, then ω is k-FP$\mathcal{I}$-open.
(3) If $I_\zeta(\omega, k) = \underline{0}$, then ω is k-FP$\mathcal{I}$-open.
(4) If $C^*_\zeta(\omega, k) = \underline{0}$, then ω is k-FS$\mathcal{I}$-open.

Proof. The proof follows by Definitions 2.10 and 3.1. □

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 12 March 2025 doi:10.20944/preprints202503.0839.v1

Corollary 3.3. In an $\mathcal{FITS}$ $(Z,\zeta,\mathcal{I})$, for each k-F$\gamma\mathcal{I}$-closed set $\omega \in I^Z$.

(1) If ω is an k-FR$\mathcal{I}$-open set, then ω is k-FP$\mathcal{I}$-closed.

(2) If ω is an k-FR$\mathcal{I}$-closed set, then ω is k-FS$\mathcal{I}$-closed.

(3) If $I_\zeta(\omega,k) = \underline{0}$, then ω is k-FS$\mathcal{I}$-closed.

(4) If $C^*_\zeta(\omega,k) = \underline{0}$, then ω is k-FP$\mathcal{I}$-closed.

Proof. The proof follows by Definition 2.10 and Remark 3.1. □

Definition 3.2. In an $\mathcal{FITS}$ $(Z,\zeta,\mathcal{I})$, for each $\omega \in I^Z$ and $k \in I_\circ$, we define a fuzzy γ-$\mathcal{I}$-closure operator $\gamma C^*_\zeta : I^Z \times I_\circ \longrightarrow I^Z$ as follows:

$$\gamma C^*_\zeta(\omega,k) = \bigwedge \{\rho \in I^Z : \omega \leq \rho,\ \rho \text{ is } k\text{-F}\gamma\mathcal{I}\text{-closed}\}.$$

Proposition 3.2. In an $\mathcal{FITS}$ $(Z,\zeta,\mathcal{I})$, for each $\omega \in I^Z$ and $k \in I_\circ$. A fuzzy set ω is k-F$\gamma\mathcal{I}$-closed iff $\gamma C^*_\zeta(\omega,k) = \omega$.

Proof. This is easily proved from Definition 3.2. □

Theorem 3.1. In an $\mathcal{FITS}$ $(Z,\zeta,\mathcal{I})$, for each $\omega,\rho \in I^Z$ and $k \in I_\circ$. A fuzzy γ-$\mathcal{I}$-closure operator γC^*_ζ : $I^Z \times I_\circ \longrightarrow I^Z$ satisfies the following properties.

(1) $\gamma C^*_\zeta(\underline{0},k) = \underline{0}$.

(2) $\omega \leq \gamma C^*_\zeta(\omega,k) \leq C_\zeta(\omega,k)$.

(3) $\gamma C^*_\zeta(\omega,k) \leq \gamma C^*_\zeta(\rho,k)$ if $\omega \leq \rho$.

(4) $\gamma C^*_\zeta(\gamma C^*_\zeta(\omega,k),k) = \gamma C^*_\zeta(\omega,k)$.

(5) $\gamma C^*_\zeta(\omega \vee \rho,k) \geq \gamma C^*_\zeta(\omega,k) \vee \gamma C^*_\zeta(\rho,k)$.

Proof. (1), (2), and (3) are easily proved by Definition 3.2.

(4) From (2) and (3), $\gamma C^*_\zeta(\omega,k) \leq \gamma C^*_\zeta(\gamma C^*_\zeta(\omega,k),k)$. Now, we show $\gamma C^*_\zeta(\omega,k) \geq \gamma C^*_\zeta(\gamma C^*_\zeta(\omega,k),k)$. If $\gamma C^*_\zeta(\omega,k)$ does not contain $\gamma C^*_\zeta(\gamma C^*_\zeta(\omega,k),k)$, there is $z \in Z$ and $s \in (0,1)$ with

$$\gamma C^*_\zeta(\omega,k)(z) < s < \gamma C^*_\zeta(\gamma C^*_\zeta(\omega,k),k)(z). \qquad (\mathcal{N})$$

Since $\gamma C^*_\zeta(\omega,k)(z) < s$, by Definition 3.2, there is $\mu \in I^Z$ as an k-F$\gamma\mathcal{I}$-closed set and $\omega \leq \mu$ with $\gamma C^*_\zeta(\omega,k)(z) \leq \mu(z) < s$. Since $\omega \leq \mu$, then $\gamma C^*_\zeta(\omega,k) \leq \mu$. Again, by the definition of γC^*_ζ, then $\gamma C^*_\zeta(\gamma C^*_\zeta(\omega,k),k) \leq \mu$. Hence, $\gamma C^*_\zeta(\gamma C^*_\zeta(\omega,k),k)(z) \leq \mu(z) < s$, which is a contradiction for $(\mathcal{N})$. Thus, $\gamma C^*_\zeta(\omega,k) \geq \gamma C^*_\zeta(\gamma C^*_\zeta(\omega,k),k)$. Therefore, $\gamma C^*_\zeta(\gamma C^*_\zeta(\omega,k),k) = \gamma C^*_\zeta(\omega,k)$.

(5) Since $\omega \leq \omega \vee \rho$ and $\rho \leq \omega \vee \rho$, hence by (3), $\gamma C^*_\zeta(\omega,k) \leq \gamma C^*_\zeta(\omega \vee \rho,k)$ and $\gamma C^*_\zeta(\rho,k) \leq \gamma C^*_\zeta(\omega \vee \rho,k)$. Thus, $\gamma C^*_\zeta(\omega \vee \rho,k) \geq \gamma C^*_\zeta(\omega,k) \vee \gamma C^*_\zeta(\rho,k)$. □

Definition 3.3. In an $\mathcal{FITS}$ $(Z,\zeta,\mathcal{I})$, for each $\omega \in I^Z$ and $k \in I_\circ$, we define a fuzzy γ-$\mathcal{I}$-interior operator $\gamma I^*_\zeta : I^Z \times I_\circ \longrightarrow I^Z$ as follows: $\gamma I^*_\zeta(\omega,k) = \bigvee \{\rho \in I^Z : \rho \leq \omega,\ \rho \text{ is } r\text{-F}\gamma\mathcal{I}\text{-open}\}$.

Proposition 3.3. Let $(Z,\zeta,\mathcal{I})$ be an $\mathcal{FITS}$, $\omega \in I^Z$, and $k \in I_\circ$. Then

(1) $\gamma C^*_\zeta(\omega^c,k) = (\gamma I^*_\zeta(\omega,k))^c$;

(2) $\gamma I^*_\zeta(\omega^c,k) = (\gamma C^*_\zeta(\omega,k))^c$.

Proof. (1) For each $\omega \in I^Z$, we have $\gamma C^*_\zeta(\omega^c,k) = \bigwedge\{\rho \in I^Z : \omega^c \leq \rho,\ \rho \text{ is } k\text{-F}\gamma\mathcal{I}\text{-closed}\} = [\bigvee\{\rho^c \in I^Z : \rho^c \leq \omega,\ \rho^c \text{ is } k\text{-F}\gamma\mathcal{I}\text{-open}\}]^c = (\gamma I^*_\zeta(\omega,k))^c$.

(2) This is similar to that of (1). □

Proposition 3.4. In an $\mathcal{FITS}$ $(Z,\zeta,\mathcal{I})$, for each $\omega \in I^Z$ and $k \in I_\circ$. A fuzzy set ω is k-F$\gamma\mathcal{I}$-open iff $\gamma I^*_\zeta(\omega,k) = \omega$.

Proof. This is easily proved from Definition 3.3. □

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 12 March 2025 doi:10.20944/preprints202503.0839.v1

Theorem 3.2. In an $\mathcal{FITS}$ $(Z,\zeta,\mathcal{I})$, for each $\omega,\rho \in I^Z$ and $k \in I_\circ$. A fuzzy γ-$\mathcal{I}$-interior operator γI^*_ζ : $I^Z \times I_\circ \longrightarrow I^Z$ satisfies the following properties.

(1) $\gamma I^*_\zeta(\underline{1},k) = \underline{1}$.

(2) $I_\zeta(\omega,k) \leq \gamma I^*_\zeta(\omega,k) \leq \omega$.

(3) $\gamma I^*_\zeta(\omega,k) \leq \gamma I^*_\zeta(\rho,k)$ if $\omega \leq \rho$.

(4) $\gamma I^*_\zeta(\gamma I^*_\zeta(\omega,k),k) = \gamma I^*_\zeta(\omega,k)$.

(5) $\gamma I^*_\zeta(\omega,k) \wedge \gamma I^*_\zeta(\rho,k) \geq \gamma I^*_\zeta(\omega \wedge \rho,k)$.

Proof. The proof is similar to that of Theorem 3.1. □

Definition 3.4. Let $z_s \in P_s(Z)$, $\omega \in I^Z$, and $k \in I_\circ$. An $\mathcal{FITS}$ $(Z,\zeta,\mathcal{I})$ is said to be an k-F$\gamma\mathcal{I}$-regular space if $z_s \overline{\mathcal{Q}} \omega$ for each k-F$\gamma\mathcal{I}$-closed set ω, there is $\mu_i \in I^Z$ with $\zeta(\mu_i) \geq k$ for $i = 1,2$, such that $z_s \in \mu_1$, $\omega \leq \mu_2$, and $\mu_1 \overline{\mathcal{Q}} \mu_2$.

Definition 3.5. Let $\omega,\rho \in I^Z$ and $k \in I_\circ$. An $\mathcal{FITS}$ $(Z,\zeta,\mathcal{I})$ is said to be an k-F$\gamma\mathcal{I}$-normal space if $\omega \overline{\mathcal{Q}} \rho$ for each k-F$\gamma\mathcal{I}$-closed sets ω and ρ, there is $\mu_i \in I^Z$ with $\zeta(\mu_i) \geq k$ for $i = 1,2$, such that $\omega \leq \mu_1$, $\rho \leq \mu_2$, and $\mu_1 \overline{\mathcal{Q}} \mu_2$.

Theorem 3.3. Let $(Z,\zeta,\mathcal{I})$ be an $\mathcal{FITS}$, $z_s \in P_s(Z)$, $\omega \in I^Z$, and $k \in I_\circ$. The following statements are equivalent.

(1) $(Z,\zeta,\mathcal{I})$ is an k-F$\gamma\mathcal{I}$-regular space.

(2) If $z_s \in \omega$ for each k-F$\gamma\mathcal{I}$-open set ω, there is $\rho \in I^Z$ with $\zeta(\rho) \geq k$, and

$$z_s \in \rho \leq C_\zeta(\rho,k) \leq \omega.$$

(3) If $z_s \overline{\mathcal{Q}} \omega$ for each k-F$\gamma\mathcal{I}$-closed set ω, there is $\mu_i \in I^Z$ with $\zeta(\mu_i) \geq k$ for $i = 1,2$, such that $z_s \in \mu_1$, $\omega \leq \mu_2$, and $C_\zeta(\mu_1,k) \overline{\mathcal{Q}} C_\zeta(\mu_2,k)$.

Proof. (1) ⇒ (2) Let $z_s \in \omega$ for each k-F$\gamma\mathcal{I}$-open set ω, then $z_s \overline{\mathcal{Q}} \omega^c$. Since $(Z,\zeta,\mathcal{I})$ is k-F$\gamma\mathcal{I}$-regular, then there is $\rho,\nu \in I^Z$ with $\zeta(\rho) \geq k$ and $\zeta(\nu) \geq k$, such that $z_s \in \rho$, $\omega^c \leq \nu$, and $\rho \overline{\mathcal{Q}} \nu$. Thus, $z_s \in \rho \leq \nu^c \leq \omega$, so $z_s \in \rho \leq C_\zeta(\rho,k) \leq \omega$.

(2) ⇒ (3) Let $z_s \overline{\mathcal{Q}} \omega$ for each k-F$\gamma\mathcal{I}$-closed set ω, then $z_s \in \omega^c$. By (2), there is $\nu \in I^Z$ with $\zeta(\nu) \geq k$ and $z_s \in \nu \leq C_\zeta(\nu,k) \leq \omega^c$. Since $\zeta(\nu) \geq k$, then ν is an k-F$\gamma\mathcal{I}$-open set and $z_s \in \nu$. Again, by (2), there is $\mu \in I^Z$ such that $\zeta(\mu) \geq k$, and $z_s \in \mu \leq C_\zeta(\mu,k) \leq \nu \leq C_\zeta(\nu,k) \leq \omega^c$. Hence, $\omega \leq (C_\zeta(\nu,k))^c = I_\zeta(\nu^c,k) \leq \nu^c$. Set $\lambda = I_\zeta(\nu^c,k)$, and thus $\zeta(\lambda) \geq k$. Then, $C_\zeta(\lambda,k) \leq \nu^c \leq (C_\zeta(\mu,k))^c$. Therefore, $C_\zeta(\mu,k) \overline{\mathcal{Q}} C_\zeta(\lambda,k)$.

(3) ⇒ (1) This is easily proved by Definition 3.4. □

Theorem 3.4. Let $(Z,\zeta,\mathcal{I})$ be an $\mathcal{FITS}$, $\omega,\rho \in I^Z$, and $k \in I_\circ$. The following statements are equivalent.

(1) $(Z,\zeta,\mathcal{I})$ is an k-F$\gamma\mathcal{I}$-normal space.

(2) If $\rho \leq \omega$ for each k-F$\gamma\mathcal{I}$-closed set ρ and k-F$\gamma\mathcal{I}$-open set ω, there is $\nu \in I^Z$ with $\zeta(\nu) \geq k$, and $\rho \leq \nu \leq C_\zeta(\nu,k) \leq \omega$.

(3) If $\omega \overline{\mathcal{Q}} \rho$ for each k-F$\gamma\mathcal{I}$-closed sets ω and ρ, there is $\mu_i \in I^Z$ with $\zeta(\mu_i) \geq k$ for $i = 1,2$, such that $\omega \leq \mu_1$, $\rho \leq \mu_2$, and $C_\zeta(\mu_1,k) \overline{\mathcal{Q}} C_\zeta(\mu_2,k)$.

Proof. The proof is similar to that of Theorem 3.3. □

4. On Fuzzy $\gamma\mathcal{I}$-Continuity

Definition 4.1. A fuzzy mapping $\mathbb{P} : (Z,\zeta,\mathcal{I}) \longrightarrow (Y,\Im)$ is called F$\gamma\mathcal{I}$-continuous if $\mathbb{P}^{-1}(\omega)$ is an k-F$\gamma\mathcal{I}$-open set, for any $\omega \in I^Y$ with $\Im(\omega) \geq k$ and $k \in I_\circ$.

Lemma 4.1. Every F$\gamma\mathcal{I}$-continuity is an Fγ-continuity [26].

Proof. The proof follows from Definitions 2.5, 4.1, and Lemma 3.1. □

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 12 March 2025 doi:10.20944/preprints202503.0839.v1

Remark 4.1. If we take $\mathcal{I} = \mathcal{I}_0$, then F$\gamma\mathcal{I}$-continuity and F$\gamma$-continuity [26] are equivalent.

Remark 4.2. The converse of Lemma 4.1 fails as Example 4.1 will show.

Example 4.1. Define $\zeta, \mathcal{I}, \Im : I^Z \longrightarrow I$ as follows:

$$\zeta(\rho) = \begin{cases} 1, & \text{if} \quad \rho \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if} \quad \rho = \underline{0.7}, \\ \frac{1}{3}, & \text{if} \quad \rho = \underline{0.3}, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{I}(\nu) = \begin{cases} 1, & \text{if} \quad \nu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \nu \leq \underline{0.6}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\Im(\theta) = \begin{cases} 1, & \text{if} \quad \theta \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \theta = \underline{0.6}, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, the identity fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Z, \Im)$ is Fγ-continuous, but it is not F$\gamma\mathcal{I}$-continuous.

Remark 4.3. From the previous definitions, we have the following diagram.

$$\begin{array}{ccccc} & & \textbf{FP}\mathcal{I}\textbf{-continuity} & & \\ & \nearrow & \downarrow & & \\ \textbf{Fff}\mathcal{I}\textbf{-continuity} & \longrightarrow & \textbf{Ffl}\mathcal{I}\textbf{-continuity} & \longrightarrow & \textbf{FSfi}\mathcal{I}\textbf{-continuity} \\ & \searrow & \uparrow & & \\ & & \textbf{FS}\mathcal{I}\textbf{-continuity} & & \end{array}$$

Remark 4.4. The converse of the above diagram fails as Examples 4.2, 4.3, and 4.4 will show.

Example 4.2. Let $Z = \{z_1, z_2\}$ and define $\omega, \rho, \lambda \in I^Z$ as follows: $\omega = \{\frac{z_1}{0.4}, \frac{z_2}{0.3}\}$, $\rho = \{\frac{z_1}{0.5}, \frac{z_2}{0.4}\}$, $\lambda = \{\frac{z_1}{0.4}, \frac{z_2}{0.5}\}$. Define $\zeta, \mathcal{I}, \Im : I^Z \longrightarrow I$ as follows:

$$\zeta(\mu) = \begin{cases} 1, & \text{if} \quad \mu \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{4}, & \text{if} \quad \mu = \rho, \\ \frac{1}{2}, & \text{if} \quad \mu = \omega, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{I}(\nu) = \begin{cases} 1, & \text{if} \quad \nu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \nu < \underline{0.3}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\Im(\theta) = \begin{cases} 1, & \text{if} \quad \theta \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{4}, & \text{if} \quad \theta = \lambda, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, the identity fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Z, \Im)$ is F$\gamma\mathcal{I}$-continuous, but it is not FP$\mathcal{I}$-continuous.

Example 4.3. Let $Z = \{z_1, z_2\}$ and define $\omega, \rho, \lambda \in I^Z$ as follows: $\omega = \{\frac{z_1}{0.3}, \frac{z_2}{0.2}\}$, $\rho = \{\frac{z_1}{0.7}, \frac{z_2}{0.8}\}$, $\lambda = \{\frac{z_1}{0.5}, \frac{z_2}{0.4}\}$. Define $\zeta, \mathcal{I}, \Im : I^Z \longrightarrow I$ as follows:

$$\zeta(\nu) = \begin{cases} 1, & \text{if} \quad \nu \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \nu = \omega, \\ \frac{1}{2}, & \text{if} \quad \nu = \rho, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{I}(\mu) = \begin{cases} 1, & \text{if} \quad \mu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \mu < \underline{0.5}, \\ 0, & \text{otherwise}, \end{cases}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 12 March 2025 doi:10.20944/preprints202503.0839.v1

$$\Im(\theta)=\begin{cases}1, & \text{if} \quad \theta\in\{\underline{1},\underline{0}\},\\ \frac{1}{3}, & \text{if} \quad \theta=\lambda,\\ 0, & \text{otherwise.}\end{cases}$$

Thus, the identity fuzzy mapping $\mathbb{P}:(Z,\zeta,\mathcal{I})\longrightarrow(Z,\Im)$ is F$\gamma\mathcal{I}$-continuous, but it is neither FS$\mathcal{I}$-continuous nor F$\alpha\mathcal{I}$-continuous.

Example 4.4. Let $Z=\{z_1,z_2\}$ and define $\omega,\lambda\in I^Z$ as follows: $\omega=\{\frac{z_1}{0.5},\frac{z_2}{0.4}\}$, $\lambda=\{\frac{z_1}{0.4},\frac{z_2}{0.5}\}$. Define $\zeta,\mathcal{I},\Im:I^Z\longrightarrow I$ as follows:

$$\zeta(\nu)=\begin{cases}1, & \text{if} \quad \nu\in\{\underline{1},\underline{0}\},\\ \frac{1}{2}, & \text{if} \quad \nu=\omega,\\ 0, & \text{otherwise,}\end{cases} \qquad \mathcal{I}(\mu)=\begin{cases}1, & \text{if} \quad \mu=\underline{0},\\ \frac{1}{2}, & \text{if} \quad \underline{0}<\mu<\underline{0.4},\\ 0, & \text{otherwise,}\end{cases}$$

$$\Im(\theta)=\begin{cases}1, & \text{if} \quad \theta\in\{\underline{1},\underline{0}\},\\ \frac{1}{3}, & \text{if} \quad \theta=\lambda,\\ 0, & \text{otherwise.}\end{cases}$$

Thus, the identity fuzzy mapping $\mathbb{P}:(Z,\zeta,\mathcal{I})\longrightarrow(Z,\Im)$ is FS$\beta\mathcal{I}$-continuous, but it is not F$\gamma\mathcal{I}$-continuous.

Theorem 4.1. A fuzzy mapping $\mathbb{P}:(Z,\zeta,\mathcal{I})\longrightarrow(Y,\Im)$ is F$\gamma\mathcal{I}$-continuous iff for any $z_s\in P_s(Z)$ and any $\rho\in I^Y$ with $\Im(\rho)\geq k$ containing $\mathbb{P}(z_s)$, there is $\omega\in I^Z$ that is k-F$\gamma\mathcal{I}$-open containing z_s with $\mathbb{P}(\omega)\leq\rho$ and $k\in I_\circ$.

Proof. ($\Rightarrow$) Let $z_s\in P_s(Z)$ and $\rho\in I^Y$ with $\Im(\rho)\geq k$ containing $\mathbb{P}(z_s)$, and then $\mathbb{P}^{-1}(\rho)\leq\gamma I^*_\zeta(\mathbb{P}^{-1}(\rho),k)$. Since $z_s\in\mathbb{P}^{-1}(\rho)$, then we obtain $z_s\in\gamma I^*_\zeta(\mathbb{P}^{-1}(\rho),k)=\omega$ (say). Hence, $\omega\in I^Z$ is k-F$\gamma\mathcal{I}$-open containing z_s with $\mathbb{P}(\omega)\leq\rho$.

($\Leftarrow$) Let $z_s\in P_s(Z)$ and $\rho\in I^Y$ with $\Im(\rho)\geq k$ containing $\mathbb{P}(z_s)$. According to the assumption there is $\omega\in I^Z$ that is k-F$\gamma\mathcal{I}$-open containing z_s with $\mathbb{P}(\omega)\leq\rho$. Hence, $z_s\in\omega\leq\mathbb{P}^{-1}(\rho)$ and $z_s\in\gamma I^*_\zeta(\mathbb{P}^{-1}(\rho),k)$. Thus, $\mathbb{P}^{-1}(\rho)\leq\gamma I^*_\zeta(\mathbb{P}^{-1}(\rho),k)$, so $\mathbb{P}^{-1}(\rho)$ is an k-F$\gamma\mathcal{I}$-open set. Then, $\mathbb{P}$ is Fγ-$\mathcal{I}$-continuous. □

Theorem 4.2. Let $\mathbb{P}:(Z,\zeta,\mathcal{I})\longrightarrow(Y,\Im)$ be a fuzzy mapping and $k\in I_\circ$. Then the following statements are equivalent for every $\omega\in I^Z$ and $\rho\in I^Y$:

(1) $\mathbb{P}$ is F$\gamma\mathcal{I}$-continuous.

(2) $\mathbb{P}^{-1}(\rho)$ is k-F$\gamma\mathcal{I}$-closed, for every $\rho\in I^Y$ with $\Im(\rho^c)\geq k$.

(3) $\mathbb{P}(\gamma C^*_\zeta(\omega,k))\leq C_\Im(\mathbb{P}(\omega),k)$.

(4) $\gamma C^*_\zeta(\mathbb{P}^{-1}(\rho),k)\leq\mathbb{P}^{-1}(C_\Im(\rho,k))$.

(5) $\mathbb{P}^{-1}(I_\Im(\rho,k))\leq\gamma I^*_\zeta(\mathbb{P}^{-1}(\rho),k)$.

Proof. (1) $\Leftrightarrow$ (2) The proof follows by $\mathbb{P}^{-1}(\rho^c)=(\mathbb{P}^{-1}(\rho))^c$ and Definition 4.1.

(2) $\Rightarrow$ (3) Let $\omega\in I^Z$. By (2), we have $\mathbb{P}^{-1}(C_\Im(\mathbb{P}(\omega),k))$ is k-F$\gamma\mathcal{I}$-closed. Thus,

$$\gamma C^*_\zeta(\omega,k)\leq\gamma C^*_\zeta(\mathbb{P}^{-1}(\mathbb{P}(\omega)),k)\leq\gamma C^*_\zeta(\mathbb{P}^{-1}(C_\Im(\mathbb{P}(\omega),k)),k)=\mathbb{P}^{-1}(C_\Im(\mathbb{P}(\omega),k)).$$

Therefore, $\mathbb{P}(\gamma C^*_\zeta(\omega,k))\leq C_\Im(\mathbb{P}(\omega),k)$.

(3) $\Rightarrow$ (4) Let $\rho\in I^Y$. By (3), $\mathbb{P}(\gamma C^*_\zeta(\mathbb{P}^{-1}(\rho),k))\leq C_\Im(\mathbb{P}(\mathbb{P}^{-1}(\rho)),k)\leq C_\Im(\rho,k)$. Thus, $\gamma C^*_\zeta(\mathbb{P}^{-1}(\rho),k)\leq\mathbb{P}^{-1}(\mathbb{P}(\gamma C^*_\zeta(\mathbb{P}^{-1}(\rho),k)))\leq\mathbb{P}^{-1}(C_\Im(\rho,k))$.

(4) $\Leftrightarrow$ (5) The proof follows by $\mathbb{P}^{-1}(\rho^c)=(\mathbb{P}^{-1}(\rho))^c$ and Proposition 3.3.

(5) $\Rightarrow$ (1) Let $\rho\in I^Y$ with $\Im(\rho)\geq k$. By (5), we obtain $\mathbb{P}^{-1}(\rho)=\mathbb{P}^{-1}(I_\Im(\rho,k))\leq\gamma I^*_\zeta(\mathbb{P}^{-1}(\rho),k)\leq\mathbb{P}^{-1}(\rho)$. Then, $\gamma I^*_\zeta(\mathbb{P}^{-1}(\rho),k)=\mathbb{P}^{-1}(\rho)$. Thus, $\mathbb{P}^{-1}(\rho)$ is k-F$\gamma\mathcal{I}$-open, so $\mathbb{P}$ is F$\gamma\mathcal{I}$-continuous. □

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 12 March 2025 doi:10.20944/preprints202503.0839.v1

Definition 4.2. A fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Y, \Im)$ is called FA$\gamma\mathcal{I}$-continuous if $\mathbb{P}^{-1}(\omega) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(I_\Im(C_\Im(\omega, k), k)), k)$, for any $\omega \in I^Y$ with $\Im(\omega) \geq k$ and $k \in I_\circ$.

Lemma 4.2. Every F$\gamma\mathcal{I}$-continuity is an FA$\gamma\mathcal{I}$-continuity.

Proof. The proof follows by Definitions 4.1 and 4.2. □

Remark 4.5. The converse of Lemma 4.2 fails as Example 4.5 will show.

Example 4.5. Let $Z = \{z_1, z_2, z_3\}$ and define $\omega, \rho, \lambda \in I^Z$ as follows: $\omega = \{\frac{z_1}{0.4}, \frac{z_2}{0.2}, \frac{z_3}{0.4}\}$, $\rho = \{\frac{z_1}{0.5}, \frac{z_2}{0.5}, \frac{z_3}{0.4}\}$, $\lambda = \{\frac{z_1}{0.3}, \frac{z_2}{0.2}, \frac{z_3}{0.6}\}$. Define $\zeta, \mathcal{I}, \Im : I^Z \longrightarrow I$ as follows:

$$\zeta(\nu) = \begin{cases} 1, & \text{if} \quad \nu \in \{\underline{0}, \underline{1}\}, \\ \frac{2}{3}, & \text{if} \quad \nu = \omega, \\ \frac{1}{2}, & \text{if} \quad \nu = \rho, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{I}(\mu) = \begin{cases} 1, & \text{if} \quad \mu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \mu \leq \underline{0.6}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\Im(\theta) = \begin{cases} 1, & \text{if} \quad \theta \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \theta = \lambda, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, the identity fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Z, \Im)$ is FA$\gamma\mathcal{I}$-continuous, but it is not F$\gamma\mathcal{I}$-continuous.

Theorem 4.3. A fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Y, \Im)$ is FA$\gamma\mathcal{I}$-continuous iff for any $z_s \in P_s(Z)$ and any $\rho \in I^Y$ with $\Im(\rho) \geq k$ containing $\mathbb{P}(z_s)$, there is $\omega \in I^Z$ that is k-F$\gamma\mathcal{I}$-open containing z_s with $\mathbb{P}(\omega) \leq I_\Im(C_\Im(\rho, k), k)$ and $k \in I_\circ$.

Proof. ($\Rightarrow$) Let $z_s \in P_s(Z)$ and $\rho \in I^Y$ with $\Im(\rho) \geq k$ containing $\mathbb{P}(z_s)$, and then

$$\mathbb{P}^{-1}(\rho) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(I_\Im(C_\Im(\rho, k), k)), k).$$

Since $z_s \in \mathbb{P}^{-1}(\rho)$, then $z_s \in \gamma I^*_\zeta(\mathbb{P}^{-1}(I_\Im(C_\Im(\rho, k), k)), k) = \omega$ (say). Therefore, $\omega \in I^Z$ is k-F$\gamma\mathcal{I}$-open containing z_s with $\mathbb{P}(\omega) \leq I_\Im(C_\Im(\rho, k), k)$.

($\Leftarrow$) Let $z_s \in P_s(Z)$ and $\rho \in I^Y$ with $\Im(\rho) \geq k$ such that $z_s \in \mathbb{P}^{-1}(\rho)$. According to the assumption there is $\omega \in I^Z$ that is k-F$\gamma\mathcal{I}$-open containing z_s with $\mathbb{P}(\omega) \leq I_\Im(C_\Im(\rho, k), k)$. Hence, $z_s \in \omega \leq \mathbb{P}^{-1}(I_\Im(C_\Im(\rho, k), k))$ and

$$z_s \in \gamma I^*_\zeta(\mathbb{P}^{-1}(I_\Im(C_\Im(\rho, k), k)), k).$$

Thus, $\mathbb{P}^{-1}(\rho) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(I_\Im(C_\Im(\rho, k), k)), k)$. Therefore, $\mathbb{P}$ is FA$\gamma\mathcal{I}$-continuous. □

Theorem 4.4. Let $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Y, \Im)$ be a fuzzy mapping, $\rho \in I^Y$, and $k \in I_\circ$. Then the following statements are equivalent:

(1) $\mathbb{P}$ is FA$\gamma\mathcal{I}$-continuous.
(2) $\mathbb{P}^{-1}(\rho)$ is k-F$\gamma\mathcal{I}$-open, for every k-FR-open set ρ.
(3) $\mathbb{P}^{-1}(\rho)$ is k-F$\gamma\mathcal{I}$-closed, for every k-FR-closed set ρ.
(4) $\gamma C^*_\zeta(\mathbb{P}^{-1}(\rho), k) \leq \mathbb{P}^{-1}(C_\Im(\rho, k))$, for every k-Fγ-open set ρ.
(5) $\gamma C^*_\zeta(\mathbb{P}^{-1}(\rho), k) \leq \mathbb{P}^{-1}(C_\Im(\rho, k))$, for every k-FS-open set ρ.

Proof. (1) $\Rightarrow$ (2) Let $z_s \in P_s(Z)$ and $\rho \in I^Y$ be an k-FR-open set with $z_s \in \mathbb{P}^{-1}(\rho)$. Hence, by (1), there is $\omega \in I^Z$ that is k-F$\gamma\mathcal{I}$-open with $z_s \in \omega$ and $\mathbb{P}(\omega) \leq I_\Im(C_\Im(\rho, k), k)$. Thus, $\omega \leq \mathbb{P}^{-1}(I_\Im(C_\Im(\rho, k), k)) = \mathbb{P}^{-1}(\rho)$ and $z_s \in \gamma I^*_\zeta(\mathbb{P}^{-1}(\rho), k)$. Therefore, $\mathbb{P}^{-1}(\rho) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(\rho), k)$, so $\mathbb{P}^{-1}(\rho)$ is k-F$\gamma\mathcal{I}$-open.

(2) $\Rightarrow$ (3) If $\rho \in I^Y$ is k-FR-closed, then by (2), $\mathbb{P}^{-1}(\rho^c) = (\mathbb{P}^{-1}(\rho))^c$ is k-F$\gamma\mathcal{I}$-open. Thus, $\mathbb{P}^{-1}(\rho)$ is k-F$\gamma\mathcal{I}$-closed.

(3) $\Rightarrow$ (4) If $\rho \in I^Y$ is k-Fγ-open and since $C_\Im(\rho, k)$ is k-FR-closed, then by (3), $\mathbb{P}^{-1}(C_\Im(\rho, k))$ is k-F$\gamma\mathcal{I}$-closed. Since $\mathbb{P}^{-1}(\rho) \leq \mathbb{P}^{-1}(C_\Im(\rho, k))$, hence

$$\gamma C^*_\zeta(\mathbb{P}^{-1}(\rho), k) \leq \mathbb{P}^{-1}(C_\Im(\rho, k)).$$

(4) $\Rightarrow$ (5) The proof follows from the fact that any k-FS-open set is k-Fγ-open.

(5) $\Rightarrow$ (3) If $\rho \in I^Y$ is k-FR-closed, and then ρ is k-FS-open. By (5),

$$\gamma C^*_\zeta(\mathbb{P}^{-1}(\rho), k) \leq \mathbb{P}^{-1}(C_\Im(\rho, k)) = \mathbb{P}^{-1}(\rho).$$

Hence, $\mathbb{P}^{-1}(\rho)$ is k-F$\gamma\mathcal{I}$-closed.

(3) $\Rightarrow$ (1) If $z_s \in P_s(Z)$ and $\rho \in I^Y$ with $\Im(\rho) \geq k$ such that $z_s \in \mathbb{P}^{-1}(\rho)$, and then $z_s \in \mathbb{P}^{-1}(I_\Im(C_\Im(\rho, k), k))$. Since $[I_\Im(C_\Im(\rho, k), k)]^c$ is k-FR-closed, by (3), $\mathbb{P}^{-1}([I_\Im(C_\Im(\rho, k), k)]^c)$ is k-F$\gamma\mathcal{I}$-closed. Hence, $\mathbb{P}^{-1}(I_\Im(C_\Im(\rho, k), k))$ is k-F$\gamma\mathcal{I}$-open and $z_s \in \gamma I^*_\zeta(\mathbb{P}^{-1}(I_\Im(C_\Im(\rho, k), k)), k)$. Thus, $\mathbb{P}^{-1}(\rho) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(I_\Im(C_\Im(\rho, k), k)), k)$. Therefore, $\mathbb{P}$ is FA$\gamma\mathcal{I}$-continuous. □

Definition 4.3. A fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Y, \Im)$ is called FW$\gamma\mathcal{I}$-continuous if $\mathbb{P}^{-1}(\omega) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(C_\Im(\omega, k)), k)$, for any $\omega \in I^Y$ with $\Im(\omega) \geq k$ and $k \in I_\circ$.

Lemma 4.3. Every F$\gamma\mathcal{I}$-continuity is an FW$\gamma\mathcal{I}$-continuity.

Proof. The proof follows by Definitions 4.1 and 4.3. □

Remark 4.6. The converse of Lemma 4.3 fails as Example 4.6 will show.

Example 4.6. Let $Z = \{z_1, z_2, z_3\}$ and define $\omega, \rho, \lambda \in I^Z$ as follows: $\omega = \{\frac{z_1}{0.4}, \frac{z_2}{0.2}, \frac{z_3}{0.4}\}$, $\rho = \{\frac{z_1}{0.5}, \frac{z_2}{0.5}, \frac{z_3}{0.4}\}$, $\lambda = \{\frac{z_1}{0.3}, \frac{z_2}{0.2}, \frac{z_3}{0.6}\}$. Define $\zeta, \mathcal{I}, \Im : I^Z \longrightarrow I$ as follows:

$$\zeta(\nu) = \begin{cases} 1, & \text{if} \quad \nu \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \nu = \omega, \\ \frac{1}{2}, & \text{if} \quad \nu = \rho, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{I}(\mu) = \begin{cases} 1, & \text{if} \quad \mu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \mu \leq \underline{0.6}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\Im(\theta) = \begin{cases} 1, & \text{if} \quad \theta \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \theta = \lambda, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, the identity fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Z, \Im)$ is FW$\gamma\mathcal{I}$-continuous, but it is not F$\gamma\mathcal{I}$-continuous.

Theorem 4.5. A fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Y, \Im)$ is FW$\gamma\mathcal{I}$-continuous iff for any $z_s \in P_s(Z)$ and any $\rho \in I^Y$ with $\Im(\rho) \geq k$ containing $\mathbb{P}(z_s)$, there is $\omega \in I^Z$ that is k-F$\gamma\mathcal{I}$-open containing z_s with $\mathbb{P}(\omega) \leq C_\Im(\rho, k)$ and $k \in I_\circ$.

Proof. ($\Rightarrow$) Let $z_s \in P_s(Z)$ and $\rho \in I^Y$ with $\Im(\rho) \geq k$ containing $\mathbb{P}(z_s)$, and then

$$\mathbb{P}^{-1}(\rho) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(C_\Im(\rho, k)), k).$$

Since $z_s \in \mathbb{P}^{-1}(\rho)$, then $z_s \in \gamma I^*_\zeta(\mathbb{P}^{-1}(C_\Im(\rho, k)), k) = \omega$ (say). Hence, $\omega \in I^Z$ is k-F$\gamma\mathcal{I}$-open containing z_s with $\mathbb{P}(\omega) \leq C_\Im(\rho, k)$.

($\Leftarrow$) Let $z_s \in P_s(Z)$ and $\rho \in I^Y$ with $\Im(\rho) \geq k$ such that $z_s \in \mathbb{P}^{-1}(\rho)$. According to the assumption there is $\omega \in I^Z$ that is k-F$\gamma\mathcal{I}$-open containing z_s with $\mathbb{P}(\omega) \leq C_\Im(\rho, k)$. Hence, $z_s \in \omega \leq \mathbb{P}^{-1}(C_\Im(\rho, k))$ and $z_s \in \gamma I^*_\zeta(\mathbb{P}^{-1}(C_\Im(\rho, k)), k)$. Thus, $\mathbb{P}^{-1}(\rho) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(C_\Im(\rho, k)), k)$. Therefore, $\mathbb{P}$ is FWγ-$\mathcal{I}$-continuous. □

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 12 March 2025 doi:10.20944/preprints202503.0839.v1

Theorem 4.6. Let $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Y, \Im)$ be a fuzzy mapping, $\rho \in I^Y$, and $k \in I_\circ$. Then the following statements are equivalent:

(1) $\mathbb{P}$ is FW$\gamma\mathcal{I}$-continuous.
(2) $\mathbb{P}^{-1}(\rho) \geq \gamma C^*_\zeta(\mathbb{P}^{-1}(I_\Im(\rho, k)), k)$, if $\Im(\rho^c) \geq k$.
(3) $\gamma I^*_\zeta(\mathbb{P}^{-1}(C_\Im(\rho, k)), k) \geq \mathbb{P}^{-1}(I_\Im(\rho, k))$.
(4) $\gamma C^*_\zeta(\mathbb{P}^{-1}(I_\Im(\rho, k)), k) \leq \mathbb{P}^{-1}(C_\Im(\rho, k))$.

Proof. (1) $\Leftrightarrow$ (2) The proof follows by Proposition 3.3 and Definition 4.3.

(2) $\Rightarrow$ (3) Let $\rho \in I^Y$. Hence by (2),

$$\gamma C^*_\zeta(\mathbb{P}^{-1}(I_\Im(C_\Im(\rho^c, k), k)), k) \leq \mathbb{P}^{-1}(C_\Im(\rho^c, k)).$$

Thus, $\mathbb{P}^{-1}(I_\Im(\rho, k)) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(C_\Im(\rho, k)), k)$.

(3) $\Leftrightarrow$ (4) The proof follows from Proposition 3.3.

(4) $\Rightarrow$ (1) Let $\rho \in I^Y$ with $\Im(\rho) \geq k$. Hence by (4), $\gamma C^*_\zeta(\mathbb{P}^{-1}(I_\Im(\rho^c, k)), k) \leq \mathbb{P}^{-1}(C_\Im(\rho^c, k)) = \mathbb{P}^{-1}(\rho^c)$. Thus, $\mathbb{P}^{-1}(\rho) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(C_\Im(\rho, k)), k)$, so $\mathbb{P}$ is FW$\gamma\mathcal{I}$-continuous. □

Lemma 4.4. Every FA$\gamma\mathcal{I}$-continuity is an FW$\gamma\mathcal{I}$-continuity.

Proof. The proof follows by Definitions 4.2 and 4.3. □

Remark 4.7. The converse of Lemma 4.4 fails as Example 4.7 will show.

Example 4.7. Let $Z = \{z_1, z_2, z_3\}$ and define $\omega, \lambda, \rho \in I^Z$ as follows: $\omega = \{\frac{z_1}{0.6}, \frac{z_2}{0.2}, \frac{z_3}{0.4}\}$, $\lambda = \{\frac{z_1}{0.3}, \frac{z_2}{0.2}, \frac{z_3}{0.5}\}$, $\rho = \{\frac{z_1}{0.3}, \frac{z_2}{0.2}, \frac{z_3}{0.4}\}$. Define $\zeta, \mathcal{I}, \Im : I^Z \longrightarrow I$ as follows:

$$\zeta(\nu) = \begin{cases} 1, & \text{if } \nu \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{4}, & \text{if } \nu = \omega, \\ \frac{1}{2}, & \text{if } \nu = \rho, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{I}(\mu) = \begin{cases} 1, & \text{if } \mu = \underline{0}, \\ \frac{1}{2}, & \text{if } \underline{0} < \mu \leq \underline{0.5}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\Im(\theta) = \begin{cases} 1, & \text{if } \theta \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{4}, & \text{if } \theta = \lambda, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, the identity fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Z, \Im)$ is FW$\gamma\mathcal{I}$-continuous, but it is not FA$\gamma\mathcal{I}$-continuous.

Remark 4.8. From the previous discussions and definitions, we have the following diagram.

Ffl$\mathcal{I}$-continuity $\longrightarrow$ **FAfl$\mathcal{I}$-continuity** $\longrightarrow$ **FWfl$\mathcal{I}$-continuity**

Proposition 4.1. Let $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (X, \eta)$ and $\mathbb{Y} : (X, \eta) \longrightarrow (Y, \Im)$ be two fuzzy mappings. Then the composition $\mathbb{Y} \circ \mathbb{P}$ is FA$\gamma\mathcal{I}$-continuous if $\mathbb{P}$ is F$\gamma\mathcal{I}$-continuous and $\mathbb{Y}$ is fuzzy continuous.

Proof. The proof follows by the previous definitions. □

5. On Fuzzy $\gamma\mathcal{I}$-Irresoluteness

Definition 5.1. A fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Y, \Im)$ is called F$\gamma\mathcal{I}$-irresolute if $\mathbb{P}^{-1}(\omega)$ is an k-F$\gamma\mathcal{I}$-open set, for any k-Fγ-open set $\omega \in I^Y$ and $k \in I_\circ$.

Lemma 5.1. Every F$\gamma\mathcal{I}$-irresolute mapping is F$\gamma\mathcal{I}$-continuous.

Proof. The proof follows from Definitions 4.1, 5.1, and Remark 2.1. □

 doi:10.20944/preprints202503.0839.v1

Remark 5.1. The converse of Lemma 5.1 fails as Example 5.1 will show.

Example 5.1. Let $Z = \{z_1, z_2\}$ and define $\lambda, \rho \in I^Z$ as follows: $\lambda = \{\frac{z_1}{0.5}, \frac{z_2}{0.5}\}$, $\rho = \{\frac{z_1}{0.5}, \frac{z_2}{0.4}\}$. Define $\zeta, \mathcal{I}, \Im : I^Z \longrightarrow I$ as follows:

$$\zeta(\nu) = \begin{cases} 1, & \text{if} \quad \nu \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{2}, & \text{if} \quad \nu = \rho, \\ 0, & \text{otherwise,} \end{cases} \qquad \mathcal{I}(\mu) = \begin{cases} 1, & \text{if} \quad \mu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \mu < \underline{0.5}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\Im(\theta) = \begin{cases} 1, & \text{if} \quad \theta \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{3}, & \text{if} \quad \theta = \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the identity fuzzy mapping $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Z, \Im)$ is F$\gamma\mathcal{I}$-continuous, but it is not F$\gamma\mathcal{I}$-irresolute.

Theorem 5.1. Let $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (Y, \Im)$ be a fuzzy mapping and $k \in I_\circ$. Then the following statements are equivalent for every $\omega \in I^Z$ and $\rho \in I^Y$:

(1) $\mathbb{P}$ is F$\gamma\mathcal{I}$-irresolute.
(2) $\mathbb{P}^{-1}(\rho)$ is k-F$\gamma\mathcal{I}$-closed, for every k-Fγ-closed set ρ.
(3) $\mathbb{P}(\gamma C^*_\zeta(\omega, k)) \leq \gamma C_\Im(\mathbb{P}(\omega), k)$.
(4) $\gamma C^*_\zeta(\mathbb{P}^{-1}(\rho), k) \leq \mathbb{P}^{-1}(\gamma C_\Im(\rho, k))$.
(5) $\mathbb{P}^{-1}(\gamma I_\Im(\rho, k)) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(\rho), k)$.

Proof. (1) $\Leftrightarrow$ (2) The proof follows by $\mathbb{P}^{-1}(\rho^c) = (\mathbb{P}^{-1}(\rho))^c$ and Definition 5.1.

(2) $\Rightarrow$ (3) Let $\omega \in I^Z$. By (2), we have $\mathbb{P}^{-1}(\gamma C_\Im(\mathbb{P}(\omega), k))$ is k-F$\gamma\mathcal{I}$-closed. Thus,

$$\gamma C^*_\zeta(\omega, k) \leq \gamma C^*_\zeta(\mathbb{P}^{-1}(\mathbb{P}(\omega)), k) \leq \gamma C^*_\zeta(\mathbb{P}^{-1}(\gamma C_\Im(\mathbb{P}(\omega), k)), k) = \mathbb{P}^{-1}(\gamma C_\Im(\mathbb{P}(\omega), k)).$$

Therefore, $\mathbb{P}(\gamma C^*_\zeta(\omega, k)) \leq \gamma C_\Im(\mathbb{P}(\omega), k)$.

(3) $\Rightarrow$ (4) Let $\rho \in I^Y$. By (3), $\mathbb{P}(\gamma C^*_\zeta(\mathbb{P}^{-1}(\rho), k)) \leq \gamma C_\Im(\mathbb{P}(\mathbb{P}^{-1}(\rho)), k) \leq \gamma C_\Im(\rho, k)$. Thus, $\gamma C^*_\zeta(\mathbb{P}^{-1}(\rho), k) \leq \mathbb{P}^{-1}(\mathbb{P}(\gamma C^*_\zeta(\mathbb{P}^{-1}(\rho), k))) \leq \mathbb{P}^{-1}(\gamma C_\Im(\rho, k))$.

(4) $\Leftrightarrow$ (5) The proof follows by $\mathbb{P}^{-1}(\rho^c) = (\mathbb{P}^{-1}(\rho))^c$ and Proposition 3.3.

(5) $\Rightarrow$ (1) Let $\rho \in I^Y$ be an k-Fγ-open set. By (5),

$$\mathbb{P}^{-1}(\rho) = \mathbb{P}^{-1}(\gamma I_\Im(\rho, k)) \leq \gamma I^*_\zeta(\mathbb{P}^{-1}(\rho), k) \leq \mathbb{P}^{-1}(\rho).$$

Thus, $\gamma I^*_\zeta(\mathbb{P}^{-1}(\rho), k) = \mathbb{P}^{-1}(\rho)$. Therefore, $\mathbb{P}^{-1}(\rho)$ is k-F$\gamma\mathcal{I}$-open, so $\mathbb{P}$ is F$\gamma\mathcal{I}$-irresolute. □

Proposition 5.1. Let $\mathbb{P} : (Z, \zeta, \mathcal{I}) \longrightarrow (X, \eta)$ and $\mathbb{Y} : (X, \eta) \longrightarrow (Y, \Im)$ be two fuzzy mappings. Then the composition $\mathbb{Y} \circ \mathbb{P}$ is F$\gamma\mathcal{I}$-irresolute (resp. F$\gamma\mathcal{I}$-continuous) if $\mathbb{P}$ is F$\gamma\mathcal{I}$-irresolute and $\mathbb{Y}$ is Fγ-irresolute (resp. fuzzy continuous).

Proof. The proof follows by the previous definitions. □

Definition 5.2. A fuzzy mapping $\mathbb{P} : (Z, \zeta) \longrightarrow (Y, \Im, \mathcal{I})$ is called F$\gamma\mathcal{I}$-open if $\mathbb{P}(\omega)$ is an k-F$\gamma\mathcal{I}$-open set, for any $\omega \in I^Z$ with $\zeta(\omega) \geq k$ and $k \in I_\circ$.

Definition 5.3. A fuzzy mapping $\mathbb{P} : (Z, \zeta) \longrightarrow (Y, \Im, \mathcal{I})$ is called F$\gamma\mathcal{I}$-irresolute open if $\mathbb{P}(\omega)$ is an k-F$\gamma\mathcal{I}$-open set, for any k-Fγ-open set $\omega \in I^Z$ and $k \in I_\circ$.

Lemma 5.2. Each F$\gamma\mathcal{I}$-irresolute open mapping is F$\gamma\mathcal{I}$-open.

Proof. The proof follows from Definitions 5.2, 5.3, and Remark 2.1. □

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 12 March 2025 doi:10.20944/preprints202503.0839.v1

Remark 5.2. The converse of Lemma 5.2 fails as Example 5.2 will show.

Example 5.2. Let $Z = \{z_1, z_2\}$ and define $\omega, \lambda \in I^Z$ as follows: $\omega = \{\frac{z_1}{0.5}, \frac{z_2}{0.5}\}$, $\lambda = \{\frac{z_1}{0.5}, \frac{z_2}{0.4}\}$. Define $\zeta, \Im, \mathcal{I} : I^Z \longrightarrow I$ as follows:

$$\zeta(\nu) = \begin{cases} 1, & \text{if} \quad \nu \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{5}, & \text{if} \quad \nu = \omega, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathcal{I}(\mu) = \begin{cases} 1, & \text{if} \quad \mu = \underline{0}, \\ \frac{1}{2}, & \text{if} \quad \underline{0} < \mu < \underline{0.5}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\Im(\theta) = \begin{cases} 1, & \text{if} \quad \theta \in \{\underline{1}, \underline{0}\}, \\ \frac{1}{5}, & \text{if} \quad \theta = \lambda, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, the identity fuzzy mapping $\mathbb{P} : (Z, \zeta) \longrightarrow (Z, \Im, \mathcal{I})$ is F$\gamma\mathcal{I}$-open, but it is not F$\gamma\mathcal{I}$-irresolute open.

Theorem 5.2. Let $\mathbb{P} : (Z, \zeta) \longrightarrow (Y, \Im, \mathcal{I})$ be a fuzzy mapping and $k \in I_\circ$. Then the following statements are equivalent for every $\omega \in I^Z$ and $\rho \in I^Y$:

(1) $\mathbb{P}$ is F$\gamma\mathcal{I}$-open.

(2) $\mathbb{P}(I_\zeta(\omega, k)) \leq \gamma I^*_\Im(\mathbb{P}(\omega), k)$.

(3) $I_\zeta(\mathbb{P}^{-1}(\rho), k) \leq \mathbb{P}^{-1}(\gamma I^*_\Im(\rho, k))$.

(4) For every ρ and every ω with $\zeta(\omega^c) \geq k$ and $\mathbb{P}^{-1}(\rho) \leq \omega$, there is $\mu \in I^Y$ is k-F$\gamma\mathcal{I}$-closed with $\rho \leq \mu$ and $\mathbb{P}^{-1}(\mu) \leq \omega$.

Proof. (1) $\Rightarrow$ (2) Since $\mathbb{P}(I_\zeta(\omega, k)) \leq \mathbb{P}(\omega)$, hence by (1), $\mathbb{P}(I_\zeta(\omega, k))$ is k-F$\gamma\mathcal{I}$-open. Thus,

$$\mathbb{P}(I_\zeta(\omega, k)) \leq \gamma I^*_\Im(\mathbb{P}(\omega), k).$$

(2) $\Rightarrow$ (3) Set $\omega = \mathbb{P}^{-1}(\rho)$, and hence by (2), $\mathbb{P}(I_\zeta(\mathbb{P}^{-1}(\rho), k)) \leq \gamma I^*_\Im(\mathbb{P}(\mathbb{P}^{-1}(\rho)), k) \leq \gamma I^*_\Im(\rho, k)$. Thus, $I_\zeta(\mathbb{P}^{-1}(\rho), k) \leq \mathbb{P}^{-1}(\gamma I^*_\Im(\rho, k))$.

(3) $\Rightarrow$ (4) Let $\rho \in I^Y$ and $\omega \in I^Z$ with $\zeta(\omega^c) \geq k$ such that $\mathbb{P}^{-1}(\rho) \leq \omega$. Since $\omega^c \leq \mathbb{P}^{-1}(\rho^c)$, $\omega^c = I_\zeta(\omega^c, k) \leq I_\zeta(\mathbb{P}^{-1}(\rho^c), k)$. Hence by (3), $\omega^c \leq I_\zeta(\mathbb{P}^{-1}(\rho^c), k) \leq \mathbb{P}^{-1}(\gamma I^*_\Im(\rho^c, k))$. Then, we have

$$\omega \geq (\mathbb{P}^{-1}(\gamma I^*_\Im(\rho^c, k)))^c = \mathbb{P}^{-1}(\gamma C^*_\Im(\rho, k)).$$

Thus, $\gamma C^*_\Im(\rho, k) \in I^Y$ is k-F$\gamma\mathcal{I}$-closed with $\rho \leq \gamma C^*_\Im(\rho, k)$ and $\mathbb{P}^{-1}(\gamma C^*_\Im(\rho, k)) \leq \omega$.

(4) $\Rightarrow$ (1) Let $\nu \in I^Z$ with $\zeta(\nu) \geq k$. Set $\rho = (\mathbb{P}(\nu))^c$ and $\omega = \nu^c$, $\mathbb{P}^{-1}(\rho) = \mathbb{P}^{-1}((\mathbb{P}(\nu))^c) \leq \omega$. Hence by (4), there is $\mu \in I^Y$ is k-F$\gamma\mathcal{I}$-closed with $\rho \leq \mu$ and $\mathbb{P}^{-1}(\mu) \leq \omega = \nu^c$. Thus, $\mathbb{P}(\nu) \leq \mathbb{P}(\mathbb{P}^{-1}(\mu^c)) \leq \mu^c$. On the other hand, since $\rho \leq \mu$, $\mathbb{P}(\nu) = \rho^c \geq \mu^c$. Hence, $\mathbb{P}(\nu) = \mu^c$, so $\mathbb{P}(\nu)$ is an k-F$\gamma\mathcal{I}$-open set. Therefore, $\mathbb{P}$ is F$\gamma\mathcal{I}$-open. □

Theorem 5.3. Let $\mathbb{P} : (Z, \zeta) \longrightarrow (Y, \Im, \mathcal{I})$ be a fuzzy mapping and $k \in I_\circ$. Then the following statements are equivalent for every $\omega \in I^Z$ and $\rho \in I^Y$:

(1) $\mathbb{P}$ is F$\gamma\mathcal{I}$-irresolute open.

(2) $\mathbb{P}(\gamma I_\zeta(\omega, k)) \leq \gamma I^*_\Im(\mathbb{P}(\omega), k)$.

(3) $\gamma I_\zeta(\mathbb{P}^{-1}(\rho), k) \leq \mathbb{P}^{-1}(\gamma I^*_\Im(\rho, k))$.

(4) For every ρ and every ω is an k-Fγ-closed set with $\mathbb{P}^{-1}(\rho) \leq \omega$, there is $\mu \in I^Y$ is k-F$\gamma\mathcal{I}$-closed with $\rho \leq \mu$ and $\mathbb{P}^{-1}(\mu) \leq \omega$.

Proof. The proof is similar to that of Theorem 5.2. □

Definition 5.4. A fuzzy mapping $\mathbb{P} : (Z, \zeta) \longrightarrow (Y, \Im, \mathcal{I})$ is called F$\gamma\mathcal{I}$-closed if $\mathbb{P}(\omega)$ is an k-F$\gamma\mathcal{I}$-closed set, for any $\omega \in I^Z$ with $\zeta(\omega^c) \geq k$ and $k \in I_\circ$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 12 March 2025 doi:10.20944/preprints202503.0839.v1

Definition 5.5. A fuzzy mapping $\mathbb{P} : (Z, \zeta) \longrightarrow (Y, \Im, \mathcal{I})$ is called F$\gamma\mathcal{I}$-irresolute closed if $\mathbb{P}(\omega)$ is an k-F$\gamma\mathcal{I}$-closed set, for any k-Fγ-closed set $\omega \in I^Z$ and $k \in I_\circ$.

Lemma 5.3. Each F$\gamma\mathcal{I}$-irresolute closed mapping is F$\gamma\mathcal{I}$-closed.

Proof. The proof follows from Definitions 5.4 and 5.5. □

Theorem 5.4. Let $\mathbb{P} : (Z, \zeta) \longrightarrow (Y, \Im, \mathcal{I})$ be a fuzzy mapping and $k \in I_\circ$. Then the following statements are equivalent for every $\omega \in I^Z$ and $\rho \in I^Y$:

(1) $\mathbb{P}$ is F$\gamma\mathcal{I}$-closed.

(2) $\gamma C^*_\Im(\mathbb{P}(\omega), k) \leq \mathbb{P}(C_\zeta(\omega, k))$.

(3) $\mathbb{P}^{-1}(\gamma C^*_\Im(\rho, k)) \leq C_\zeta(\mathbb{P}^{-1}(\rho), k)$.

(4) For every ρ and every ω with $\zeta(\omega) \geq k$ and $\mathbb{P}^{-1}(\rho) \leq \omega$, there is $\mu \in I^Y$ is k-F$\gamma\mathcal{I}$-open with $\rho \leq \mu$ and $\mathbb{P}^{-1}(\mu) \leq \omega$.

Proof. The proof is similar to that of Theorem 5.2. □

Theorem 5.5. Let $\mathbb{P} : (Z, \zeta) \longrightarrow (Y, \Im, \mathcal{I})$ be a fuzzy mapping and $k \in I_\circ$. Then the following statements are equivalent for every $\omega \in I^Z$ and $\rho \in I^Y$:

(1) $\mathbb{P}$ is F$\gamma\mathcal{I}$-irresolute closed.

(2) $\gamma C^*_\Im(\mathbb{P}(\omega), k) \leq \mathbb{P}(\gamma C_\zeta(\omega, k))$.

(3) $\mathbb{P}^{-1}(\gamma C^*_\Im(\rho, k)) \leq \gamma C_\zeta(\mathbb{P}^{-1}(\rho), k)$.

(4) For every ρ and every ω is an k-Fγ-open set with $\mathbb{P}^{-1}(\rho) \leq \omega$, there is $\mu \in I^Y$ is k-F$\gamma\mathcal{I}$-open with $\rho \leq \mu$ and $\mathbb{P}^{-1}(\mu) \leq \omega$.

Proof. The proof is similar to that of Theorem 5.2. □

Proposition 5.2. Let $\mathbb{P} : (Z, \zeta) \longrightarrow (Y, \Im, \mathcal{I})$ be a bijective fuzzy mapping. Then $\mathbb{P}$ is F$\gamma\mathcal{I}$-irresolute open iff $\mathbb{P}$ is F$\gamma\mathcal{I}$-irresolute closed.

Proof. The proof follows from:

$$\mathbb{P}^{-1}(\gamma C^*_\Im(\nu, k)) \leq \gamma C_\zeta(\mathbb{P}^{-1}(\nu), k) \iff \mathbb{P}^{-1}(\gamma I^*_\Im(\nu^c, k)) \leq \gamma I_\zeta(\mathbb{P}^{-1}(\nu^c), k).$$

□

6. Conclusions

In this work, a novel class of fuzzy sets, called k-F$\gamma\mathcal{I}$-open sets, has been introduced in $\mathcal{FITS}$s based on Šostak′s sense. Some characterizations of k-F$\gamma\mathcal{I}$-open sets along with their mutual relationships have been investigated with the help of some examples. Moreover, the notions of F$\gamma\mathcal{I}$-interior operators and F$\gamma\mathcal{I}$-closure operators have been presented and discussed. Also, we defined and investigated new types of fuzzy $\mathcal{I}$-separation axioms,called k-F$\gamma\mathcal{I}$-regular spaces and k-F$\gamma\mathcal{I}$-normal spaces using k-F$\gamma\mathcal{I}$-closed sets. After that, the notion of F$\gamma\mathcal{I}$-continuity has been explored and discussed. Additionally, the notions of FA$\gamma\mathcal{I}$-continuous mappings and FW$\gamma\mathcal{I}$-continuous mappings, which are weaker forms of F$\gamma\mathcal{I}$-continuous mappings, have been defined and characterized. Finally, we defined and studied some new fuzzy $\gamma\mathcal{I}$-mappings via k-F$\gamma\mathcal{I}$-open sets and k-F$\gamma\mathcal{I}$-closed sets, called F$\gamma\mathcal{I}$-open mappings, F$\gamma\mathcal{I}$-closed mappings, F$\gamma\mathcal{I}$-irresolute mappings, F$\gamma\mathcal{I}$-irresolute open mappings, and F$\gamma\mathcal{I}$-irresolute closed mappings. In the next works, we intend to explore the following topics:

• Defining fuzzy upper and lower $\gamma\mathcal{I}$-continuous multifunctions and k-fuzzy $\gamma\mathcal{I}$-connected sets.

• Extending these notions given here in the frame of fuzzy soft topological (k-minimal) spaces as defined in [34–39].

• Finding a use for these notions given here to include double fuzzy topological spaces as defined in [40,41].

 doi:10.20944/preprints202503.0839.v1

References

1. Zadeh, L.A. Fuzzy Sets. *Inform. Control* **1965**, 8, 338-353.
2. Ahmad, B.; Kharal, A. On fuzzy soft sets. *Adva. Fuzzy Syst.* **2009**, 2009, 586507.
3. Cagman, N.; Enginoglu, S.; Citak, F. Fuzzy soft set theory and its application. *Iran. J. Fuzzy Syst.* **2011**, 8(3), 137-147.
4. Atef, M.; Ali, M.I.; Al-shami, T.M. Fuzzy soft covering based multi-granulation fuzzy rough sets and their applications. *Comput. Appl. Math.* **2021**, 40(4), 115.
5. Chang, C.L. Fuzzy topological spaces. *J. Math. Anal. Appl.* **1968**, 24, 182-190.
6. Šostak, A.P. On a fuzzy topological structure. *In: Proceedings of the 13th winter school on abstract analysis, Section of topology, Palermo: Circolo Matematico di Palermo* **1985**, 89-103.
7. Ramadan, A.A. Smooth topological spaces. *Fuzzy Set. Syst.* **1992**, 48, 371-375.
8. Chattopadhyay, K.C.; Samanta, S.K. Fuzzy topology: fuzzy closure operator, fuzzy compactness and fuzzy connectedness. *Fuzzy Set. Syst.* **1993**, 54(2), 207-212.
9. El-Gayyar, M.K.; Kerre, E.E.; Ramadan, A.A. Almost compactness and near compactness in smooth topological spaces. *Fuzzy Set. Syst.* **1994**, 62(2), 193-202.
10. Höhle, U.; Šostak, A.P. A general theory of fuzzy topological spaces. *Fuzzy Set. Syst.* **1995**, 73, 131-149.
11. Ramadan, A.A.; Abbas, S.E.; Kim, Y.C. Fuzzy irresolute mappings in smooth fuzzy topological spaces. *J. Fuzzy Math.* **2001**, 9(4), 865-877.
12. Kim, Y.C.; Ramadan A.A.; Abbas, S.E. Weaker forms of continuity in Šostak's fuzzy topology. *Indian J. Pure Appl. Math.* **2003**, 34(2), 311-333.
13. Abbas, S.E. Fuzzy super irresolute functions. *Inter. J. Math. Mathematical Sci.* **2003**, 42, 2689-2700.
14. Abbas, S.E. Fuzzy β-irresolute functions. *Appl. Math. Comp.* **2004**, 157, 369-380.
15. Kim, Y.C.; Abbas, S.E. On several types of r-fuzzy compactness. *J. Fuzzy Math.* **2004**, 12(4), 827-844.
16. Aygün, H.; Abbas, S.E. On characterization of some covering properties in L-fuzzy topological spaces in Šostak sense. *Inform. Sciences* **2004**, 165, 221-233,
17. Aygün, H.; Abbas, S.E. Some good extensions of compactness in Šostak's L-fuzzy topology. *Hacett. J. Math. Stat.* **2007**, 36(2), 115-125,
18. Li, H.Y.; Shi., F.G. Some separation axioms in I-fuzzy topological spaces. *Fuzzy Set. Syst.* **2008**, 159, 573-587.
19. Li, H.Y.; Shi, F.G. Measures of fuzzy compactness in L-fuzzy topological spaces. *Comput. Math. Appl.* **2010**, 59, 941-947.
20. Shi, F.G.; Li, R.X. Compactness in L-fuzzy topological spaces. *Hacet. J. Math. Stat.* **2011**, 40(6), 767-774,
21. Fang, J.; Guo, Y. Quasi-coincident neighborhood structure of relative I-fuzzy topology and its applications. *Fuzzy Set. Syst.* **2012**, 190, 105-117.
22. El-Dardery, M.; Ramadan, A.A.; Kim, Y.C. L-fuzzy topogenous orders and L-fuzzy topologies. *J. Intell. Fuzzy Syst.* **2013**, 24(4), 685-691.
23. Kalaivani, C.; Roopkumar, R. Fuzzy Perfect Mappings and Q-Compactness in Smooth Fuzzy Topological Spaces. *Fuzzy Inform. Eng.* **2014**, 6(1), 115-131.
24. Solovyov, S.A. On fuzzification of topological categories. *Fuzzy Set. Syst.* **2014**, 238, 1-25.
25. Minana, J.J.; Šostak, A.P. Fuzzifying topology induced by a strong fuzzy metric. *Fuzzy Set. Syst.* **2016**, 300, 24-39,
26. Alsharari, F.; Saleh, H.Y.; Taha, O.; Taha, I.M. Fuzzy topological approaches via r-fuzzy γ-open sets in the sense of Šostak. *Preprints* **2025**, 2025021193, 1-23.
27. Taha, I.M.; Abbas, S.E. A new notion of fuzzy local function and some applications. *Adv. Fuzzy Sys.* **2022**, 8954163, 1-10.
28. Taha, I.M. On r-fuzzy ℓ-open sets and continuity of fuzzy multifunctions via fuzzy ideals. *J. Math. Comput. Sci.* **2020**, 10(6), 2613-2633.
29. Taha, I.M. On r-generalized fuzzy ℓ-closed sets: properties and applications. *J. Math.* **2021**, 2021, 1-8.
30. Taha, I.M. r-fuzzy δ-ℓ-open sets and fuzzy upper (lower) δ-ℓ-continuity via fuzzy idealization. *J. Math. Comput. Sci.* **2022**, 25(1), 1-9.
31. Taha, I.M. On fuzzy upper and lower α-ℓ-continuity and their decomposition. *J. Math. Comput. Sci.* **2021**, 11(1), 427-441.
32. Ramadan, A.A.; Kim, Y.C.; Ei-Gayyar, M.K. Smooth ideals. *J. Korean Instit. Intell. Sys.* **2002**, 12(1), 90-95.

33. Kandil, A.; El-Shafei, M.E. Regularity axioms in fuzzy topological spaces and FRi-proximities. *Fuzzy Set. Syst.* **1988**, 27, 217-231.
34. Aygünoğlu, A.; Çetkin, V.; Aygün, H. An introduction to fuzzy soft topological spaces. *Hacet. J. Math. Stat.* **2014**, 43, 193-208.
35. Taha, I.M. Compactness on fuzzy soft r-minimal spaces. *Int. J. Fuzzy Logic Intell. Syst.* **2021**, 21, 251-258.
36. Taha, I.M. Some new results on fuzzy soft r-minimal spaces. *AIMS Math.* **2022**, 7, 12458-12470.
37. Alqurashi, W.; Taha, I.M. On fuzzy soft α-open sets, α-continuity, and α-compactness: some novel results. *Eur. J. Pure Appl. Math.* **2024**, 17, 4112-4134.
38. Alshammari, I.; Taha, I.M. On fuzzy soft β-continuity and β-irresoluteness: some new results. *AIMS Math.* **2024**, 9, 11304-11319.
39. Alshammari, I.; Taha, O.; El-Bably, M.K.; Taha, I.M. On r-fuzzy soft δ-open sets with applications in fuzzy soft topological spaces. *Eur. J. Pure Appl. Math.* **2025**, 18(1), 1-21.
40. Taha, I.M. Some properties of (r,s)-generalized fuzzy semi-closed sets and some applications. *J. Math. Comput. Sci.* **2022**, 27(2), 164-175.
41. Alsharari, F.; Taha, O.M.; Taha, I.M. Some new types of fuzzy closed sets, separation axioms, and compactness via double fuzzy topologies. *Eur. J. Pure Appl. Math.* **2024**, 17(4), 4093-4111.