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Article

The Action Cost of Distinction: A Phase-Topological Foundation for Multi-Scale Complexity

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Abstract

This work proposes a structural framework for the universal laws of multi-scale complexity introduced by Siegenfeld and Bar-Yam (Entropy, 2025). Their formulation presents the complexity profile, the multi-scale law of requisite variety, and the sum rule as phenomenological descriptors, but leaves their physical basis unspecified. Here we propose to interpret these laws as grounded in a phase-topological principle: the minimal action bound for distinguishability, $\delta S \geq \hbar$. In this interpretation, Planck's constant acts as the minimal quantum of action required to sustain coherent distinctions. Within this framework three structural results follow: (i) a capacity bound limiting the number of coherent supports in a region by its available action budget, corresponding to the requisite variety condition; (ii) a balance principle showing that the total action budget provides a fixed bound that can be redistributed across scales, underlying the sum rule; and (iii) a projection rule expressing the phenomenological complexity profile as the per-scale entropy increments of a structural measure of distinction. Together, these results suggest that the phenomenological complexity laws emerge as consequences of a single structural action principle, in close analogy to the way thermodynamic irreversibility is grounded in statistical mechanics. By drawing an explicit parallel with Landauer's bound $k_B T \ln 2$ for erasure, we propose to interpret \hbar as setting the minimal action cost of distinction. This framework provides a possible route toward unifying complexity science with the broader physical context of quantized action and finite informational capacity.

Keywords: complexity profile; requisite variety (multi-scale); sum rule; action principle; Planck's constant; Landauer bound; symplectic topology; quantum speed limits; structural constraints

1. Introduction: From the Cost of Erasure to the Cost of Distinction

1.1. Conceptual Anchor: Maxwell's Demon

The deep link between information and physics was crystallized by Maxwell's Demon and its resolution via Landauer's principle [1]: erasing one bit of information has an irreducible thermodynamic cost of $k_B T \ln 2$. Subsequent developments in the thermodynamics of computation [2,3] established a rigorous association between *erasure* and energetic expenditure. Independent lines of work—from Bekenstein's geometric entropy bounds [4] to Zurek's decoherence program [5]—further reinforced that information is constrained by physical law.

Here we propose to interpret Planck's constant \hbar as the minimal quantum of *action* required for a physical *distinction* to be sustainably realized,¹ in contrast to the statistical cost of erasure. This lower bound is structural and originates in phase-topological consistency conditions. In geometric quantization, these conditions appear as integrality of the symplectic flux,

$$\frac{1}{2\pi\hbar} \int_{\Sigma} \omega \in \mathbb{Z} \quad \text{for every closed two-surface } \Sigma,$$

¹ By this we mean that a state maintains phase coherence through at least one characteristic dynamical cycle, remaining robust against immediate decoherence.

with ω the symplectic form on phase space [6–8]. Equivalently, the cohomology class $[\omega/2\pi\hbar]$ is integral, ensuring globally consistent $U(1)$ phase holonomies. We note that the canonical flux quantum is $2\pi\hbar$; throughout this paper we adopt \hbar as the operative unit for sustainable distinction. In this sense, the unit that protects coherent distinctions is anchored in topology rather than in statistical mechanics.

1.2. The Problem of Complexity and a Proposed Framework

Recent advances in complexity science identify robust multi-scale regularities. The framework of Siegenfeld and Bar-Yam [9,10] formalizes a complexity profile, a multi-scale law of requisite variety, and a sum rule. These are presented as phenomenological regularities, leaving their physical origin unspecified.

To address this gap, we develop a two-level architecture that links phenomenology to structural constraints:

- **Phenomenological level:** information-theoretic descriptors (complexity profile, multi-scale requisite variety, sum rule) [11];
- **Structural level:** a symplectic-topological formulation in which constraints follow from a minimal-action requirement

$$\delta S[\gamma] \geq \hbar, \quad \delta S[\gamma] := \oint_{\gamma} \Lambda,$$

where $\Lambda = p dq - H dt$ is the Poincaré–Cartan one-form and γ is a closed cycle in phase space.

This architecture parallels the relation between Thermodynamics and Statistical Mechanics: macroscopic laws emerge from microscopic structure [12]. In our formulation, the inequality $\delta S \geq \hbar$ is taken as an operational formulation of the underlying quantization dictated by geometric integrality.

2. Formalism of the Phase-Topological Framework: Foundations

2.1. Coherence Principle and Minimal Units

Definition 1 (Coherence Principle). *For any physically distinguishable closed cycle γ in phase space,*

$$\delta S[\gamma] = \oint_{\gamma} \Lambda \geq \hbar,$$

where $\Lambda = p dq - H dt$ is the canonical Poincaré–Cartan one-form of the action. A minimal coherent unit is any configuration saturating this inequality, $\delta S = \hbar$.

Note. Here $\delta S[\gamma]$ denotes the *line integral* of Λ over a closed trajectory, not a variational derivative in the Euler–Lagrange sense.

This condition is adopted here as a *working structural postulate* of the framework. The interpretation is operational: every sustainably realized distinction requires at least one quantum of action, meaning that it preserves phase coherence sufficiently to correspond to an (approximately) orthogonal support in Hilbert space.

The structural origin of this postulate lies in geometric quantization. The holonomy of the $U(1)$ prequantum bundle, together with the Maslov index correction, imposes the Bohr–Sommerfeld quantization rule,

$$\oint_{\gamma} p dq = 2\pi\hbar \left(n + \frac{\mu(\gamma)}{4} \right), \quad n \in \mathbb{Z},$$

for any closed curve γ on a Lagrangian submanifold of phase space, where $\mu(\gamma)$ is the Maslov index [8,13,14]. This quantization condition ensures the global consistency of phase holonomies. In our reformulation, it is interpreted as a minimal-action requirement for the sustainability of distinct physical configurations.

The role of the dynamical component $-H dt$ in the Poincaré–Cartan form will be analyzed in later sections and appendices, where explicit Hamiltonian evolutions and quantum speed-limit bounds are discussed (Appendix A). For now, the principle is established at the level of structural consistency.

2.2. Measure of Distinction μ_Ω

To quantify the effective number of reliably distinguishable configurations, we introduce the structural measure:

- For a finite set of pure states U :

$$\mu_\Omega(U) := \dim \text{span}(U),$$

i.e. the dimension of the subspace spanned by U . For non-orthogonal sets this counts the linear span, not the number of reliably distinguishable states.

- For a mixed state with density operator ρ :

$$\mu_\Omega(\rho) := \exp(S_{\text{vN}}(\rho)),$$

where $S_{\text{vN}}(\rho) = -\text{Tr}(\rho \log \rho)$ is the von Neumann entropy. This definition, sometimes referred to as the *exponential of the entropy*, interprets S_{vN} as an *effective Hilbert space dimension*, consistent with approaches to typical entanglement and effective dimension in quantum statistical mechanics [15, 16].

Throughout we use the natural logarithm, so μ_Ω is expressed in nats; using \log_2 would instead yield bits. A fundamental property of this measure is subadditivity under composition of subsystems: for two subsystems A and B within a fixed context,

$$\mu_\Omega(\rho_{AB}) \leq \mu_\Omega(\rho_A) \mu_\Omega(\rho_B).$$

This inequality follows directly from the subadditivity of the von Neumann entropy:

$$S_{\text{vN}}(\rho_{AB}) \leq S_{\text{vN}}(\rho_A) + S_{\text{vN}}(\rho_B),$$

and therefore

$$\exp(S_{\text{vN}}(\rho_{AB})) \leq \exp(S_{\text{vN}}(\rho_A)) \exp(S_{\text{vN}}(\rho_B)).$$

Equality holds for product states, while strict inequality arises in the presence of correlations or entanglement. This expresses the structural limitation that the number of jointly sustainable distinctions cannot exceed the available resource budget. In later sections we show how this information-theoretic property is connected to the coherence bound $\delta S \geq \hbar$; the two are not independent assumptions but complementary expressions of the same structural limitation, linked explicitly in subsequent theorems.

3. Results: Structural Derivation of Complexity Laws

3.1. Multi-Scale Law of Requisite Variety

Theorem 1 (Capacity Bound). *The number of mutually orthogonal coherent units that can be sustained in a region R is bounded above by the available action budget:*

$$M_{\max}(R) \leq \left\lfloor \frac{\delta S[R]}{\hbar} \right\rfloor.$$

Here $\delta S[R]$ is defined as the supremum of the action integral over all relevant closed cycles γ contained within the spacetime region R ,

$$\delta S[R] := \sup_{\gamma \subset R} \oint_{\gamma} \Lambda.$$

This definition ensures that $\delta S[R]$ is well-posed and consistent with the geometric setting. The inequality follows directly from the coherence principle: each coherent unit requires at least one

quantum of action to remain stable. Hence the number of reliably distinguishable configurations cannot exceed the ratio $\delta S[R]/\hbar$.

Clarification. In geometric quantization the elementary flux quantum is $2\pi\hbar$; here we adopt \hbar as the operational threshold for sustainable distinction. This should be understood as the minimal *action cost of distinguishability*, not as a literal Bohr–Sommerfeld spacing. The bound of Theorem 1 may be further restricted by the finite Hilbert-space dimension of the system, as discussed in Appendix A.

Connection to Phenomenological Laws.

In the framework of Ref. [9], the multi-scale law of requisite variety requires that the system complexity $C_X(n)$ must not fall below that of the environment, $C_Y(n)$, for all scales n . The capacity bound provides the corresponding structural resource constraint: if system X has a smaller action budget than needed to encode the states of its environment Y , then necessarily $C_X(n) < C_Y(n)$ for some scales. This is best interpreted as a structural limitation underlying Ashby’s law, not a literal proof of its full phenomenological formulation.

3.2. Sum Rule

Principle 2 (Balance of the Action Budget). *The total action budget $\delta S[R]$ associated with a region R provides a fixed upper bound on the system’s informational capacity. This budget remains constant under redistribution of coherent structures, provided the region and its boundary conditions are fixed. While the distribution of complexity may shift between microscopic and macroscopic scales, the overall budget $\delta S[R]$ does not change.*

This expresses that action functions as a finite resource: a system may allocate its budget to many small-scale distinctions or to fewer large-scale correlated structures, but the total capacity remains constrained by $\delta S[R]$.

Connection to Phenomenological Laws.

The sum rule of Ref. [9],

$$\sum_n C_X(n) = \sum_{x \in X} H(x),$$

is interpreted here as the phenomenological projection of this balance principle. The right-hand side, $\sum_{x \in X} H(x)$, represents the total information capacity of the system’s components, structurally bounded by the fixed action budget $\delta S[R]$.

Illustrative Example.

In a spin chain, the same budget δS can be allocated in two limiting ways: (i) into N independent coherent units (a microscopic peak of complexity), or (ii) into a single highly entangled GHZ-type configuration (a macroscopic peak). Both respect the same resource bound, though the distribution of $C_X(n)$ across scales differs. In both cases one finds $\sum_n C_X(n) = N$, consistent with the budget constraint.

3.3. Complexity Profile and the Distinction Measure μ_Ω

Theorem 3 (Projection of Distinction). *For a coarse-grained system, the phenomenological complexity profile $C_X(n)$ can be expressed as the per-scale increment of the structural distinction measure:*

$$C_X(n) = \ln \mu_\Omega(\Pi_{n+1}\rho) - \ln \mu_\Omega(\Pi_n\rho),$$

where Π_n denotes the coarse-graining operator at resolution scale n , modeled as an entropy non-decreasing channel such as a conditional expectation (pinching) onto the subalgebra corresponding to scale n . Here Π_{\max} refers to the coarsest-graining operator, corresponding to the maximal available scale.

This definition guarantees $C_X(n) \geq 0$ whenever each coarse-graining step is entropy non-decreasing. In typical systems the profile decays with scale, consistent with phenomenological observations, though local fluctuations may occur. The telescoping sum

$$\sum_n C_X(n) = \ln \mu_\Omega(\Pi_{\max}\rho) - \ln \mu_\Omega(\Pi_1\rho)$$

ensures consistency with the action budget constraint.

We emphasize that the entropic formulation given above is taken as the primary definition of the complexity profile. The combinatorial model of Appendix B serves as an illustrative discrete analogue; their equivalence holds under canonical coarse-graining, as discussed in Appendix C.

3.4. Conclusion

Within this formulation, the phenomenological laws of complexity—the multi-scale law of requisite variety, the sum rule, and the scale-dependent complexity profile—can be interpreted as emergent consequences of a single structural action principle, rather than as independent postulates.

Schematic Correspondence of Structural Results

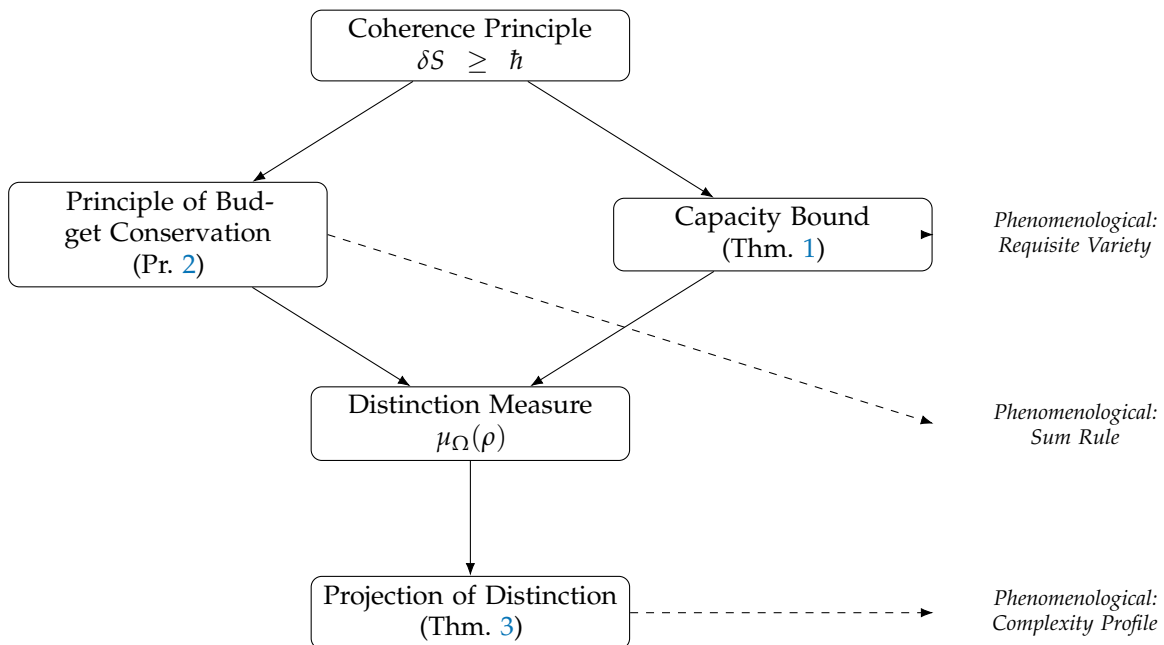


Figure 1. Correspondence between structural principles (left/middle) and phenomenological laws of multi-scale complexity (right). Dashed arrows indicate the projection of structural results into phenomenological formulations.

This diagram highlights the logical structure: the Coherence Principle generates both the Capacity Bound and the Balance principle. Together with the distinction measure, they underlie the projection theorem, from which the phenomenological complexity laws follow.

4. Discussion

The results presented above suggest that the phenomenological laws of multi-scale complexity can be interpreted as emergent quantum-informational consequences of a single structural principle of action. This section outlines broader implications of this finding, ranging from reinterpretations of foundational concepts in physics to potential applications in quantum information, cybernetics, and the philosophy of science.

4.1. The Phase-Topological Framework as the “Statistical Mechanics” of Complexity

The analogy between the two-level framework proposed here and the relationship between Statistical Mechanics and Thermodynamics, introduced in Section 1, can now be made more precise. The results in Section 3 provide a formal bridge between these two levels. The laws of multi-scale complexity, as formulated by Siegenfeld and Bar-Yam [9], function as macroscopic, phenomenological rules governing the distribution of information in complex systems. The framework developed here provides a microscopic, structural foundation for these rules.

The Capacity Bound and the Principle of Budget Conservation are not postulated information-theoretic laws but physical constraints grounded in the symplectic-topological properties of the underlying phase space. Thus, the universal laws of complexity may be viewed as emergent quantum-informational laws, in parallel with how the laws of thermodynamics arise from the microscopic mechanics of constituent systems [10,12,17,18].

Earlier formulations of multiscale complexity and requisite variety [19,20] introduced these ideas descriptively. The present framework provides a structural origin, suggesting that what has been observed phenomenologically as “laws of complexity” may be interpreted as consequences of a more primitive constraint: the minimal action bound $\delta S \geq \hbar$, where $\delta S[R]$ is defined as in Section 3.

4.2. Reinterpreting Maxwell’s Demon through the Action Cost

The classic thought experiment of Maxwell’s Demon illustrates the deep connection between information and thermodynamics. Landauer’s principle resolved the paradox by identifying the thermodynamic cost of erasing information [1–3]. Within the present framework, we interpret each act of establishing a physical distinction as requiring at least one quantum of action, \hbar . Therefore, the demon’s ability to decrease entropy is limited not only by the thermodynamics of its memory but, more fundamentally, by the available *action budget* required to perform distinctions. This limitation may be regarded as phase-topological in nature, representing the cost of establishing information rather than merely erasing it. From this perspective, Ashby’s law of requisite variety [21] can be situated in a broader physical context.

4.3. Broader Implications

Quantum Information and Holography.

The Capacity Bound

$$M_{\max}(R) \leq \left\lfloor \frac{\delta S[R]}{\hbar} \right\rfloor$$

establishes a fundamental limit on the number of distinguishable states that can be supported within a region defined by its available action. This resonates with holographic principles, such as the Bekenstein bound [4], which constrain the information content of a region in terms of its geometric properties. The present work suggests a possible connection between the action budget of a spacetime region and its geometric information bounds, though this analogy remains speculative and requires further investigation.

Biology and Cybernetics.

Ashby’s Law of Requisite Variety, a cornerstone of cybernetics [21], states that an effective regulator must have at least as much *variety* as the system it controls. The multiscale version of this law, as articulated by Siegenfeld and Bar-Yam [9], may find a possible physical grounding in the action framework. The requirement for a system to possess sufficient variety corresponds to the requirement of having a sufficient action budget to sustain the needed distinctions. This aligns with recent studies of complexity matching and eco-evolutionary dynamics [22,23], suggesting that cybernetic principles may be physically constrained by available action resources.

Philosophy of Science.

The framework developed here is consistent with structural realist interpretations of scientific theories [24,25]. What appears fundamental, within this framework, are not individual objects (particles, fields) but the structural constraints—action principles, topological quantization rules—from which observable entities emerge as stable, distinguishable configurations. This perspective resonates with Anderson’s dictum that “more is different” [17] and with the view that emergent macroscopic laws require structural explanation rather than independent postulation.

Dynamics versus Constraints.

It is important to note that the present framework primarily defines the structural constraints and the space of possibilities for complexity profiles, as dictated by the action budget. The specific dynamics, governed by a system’s Hamiltonian, determine which particular profile is realized within these bounds. In other words, the constraints define the feasible set, while the dynamics select the actual trajectory. A detailed study of this interplay represents a promising direction for future research.

The structural principles derived above can be summarized in a compact chain of correspondences:

Table 1. Correspondence between the phenomenological laws of Siegenfeld–Bar-Yam (Entropy, 2025) and the structural phase-topological framework developed here.

Phenomenological Law	Structural Foundation (This Work)	Interpretation
<i>Multi-scale law of requisite variety:</i> $C_X(n) \geq C_Y(n)$ for all n .	<i>Capacity Bound:</i> $M_{\max}(R) \leq \lfloor \delta S[R] / \hbar \rfloor$.	The system cannot sustain more distinctions than allowed by its action budget. If $\delta S_X < \delta S_Y$, then $C_X(n) < C_Y(n)$ necessarily follows for some scales.
<i>Sum rule:</i> $\sum_{x \in X} C_X(n) = \sum_{x \in X} H(x)$.	<i>Principle of Budget Conservation:</i> $\delta S[R]$ provides a fixed upper bound under redistribution across scales (with fixed boundary conditions).	Redistribution of complexity across scales reflects the balance of the total action budget, while the total informational capacity remains constrained by $\delta S[R]$.
<i>Complexity profile</i> $C_X(n)$.	<i>Projection of Distinction:</i> $C_X(n) = \ln \mu_{\Omega}(\Pi_{n+1}\rho) - \ln \mu_{\Omega}(\Pi_n\rho)$.	The profile can be interpreted as a finite-difference projection of the structural measure μ_{Ω} under hierarchical coarse-graining, rather than as an independent fundamental law.

This correspondence emphasizes that the complexity laws described in Ref. [9] should not be treated as independent postulates. Instead, they can be interpreted as quantum-informational consequences of a deeper structural principle: the minimal action cost of distinction.

5. Conclusions

The results of this work can be summarized as follows: the minimal action bound $\delta S \geq \hbar$ defines a bounded capacity for distinctions; this bounded capacity can be interpreted as giving rise to structural contextuality in the relations between system and environment; and from this structural contextuality the universal laws of multi-scale complexity can be understood to emerge.

Within this framework, Planck’s constant may be understood as the minimal quantum of action required to sustain distinguishable configurations. The coherence bound enforces a finite capacity for distinctions, providing, within this framework, a structural account of contextuality in system–

environment interactions and supporting the appearance of the phenomenological laws of multi-scale complexity as formulated by Siegenfeld and Bar-Yam.

The framework developed here suggests a foundation for complexity theory analogous to the role of statistical mechanics in thermodynamics. What Siegenfeld and Bar-Yam formulated phenomenologically — the multi-scale law of requisite variety, the sum rule, and the complexity profile — can be interpreted as emergent consequences of a single structural principle: the minimal action cost of distinction.

By grounding these laws in a phase-topological principle, the present formulation offers a possible structural route toward integrating complexity science with the broader physical framework of quantized action and finite informational capacity.

Appendix A. Action Budget for a Spin Chain

Explicit Calculation of $M_{\max}(R)$ for a System of N Qubits

We specialize the coherence principle to a spin chain. Consider a chain of N qubits, each represented by a localized coherent unit \mathcal{O}_j with minimal action cost

$$\delta S[\mathcal{O}_j] \geq \hbar. \quad (\text{A1})$$

The total chain is described by the tensor product Hilbert space

$$\mathcal{H}_N = \bigotimes_{j=1}^N \mathbb{C}^2, \quad \dim \mathcal{H}_N = 2^N.$$

Action budget.

Let R be a finite spacetime region (a world-tube) in which the spin chain is embedded. We define the available action budget as the supremum of the action line integral over all relevant closed cycles γ contained within R :

$$\delta S[R] := \sup_{\gamma \subset R} \oint_{\gamma} \Lambda, \quad \Lambda = p dq - H dt. \quad (\text{A2})$$

With fixed boundary conditions (no external change of drive/controls at ∂R), the coherence principle implies

$$M(R) \hbar \leq \delta S[R], \quad (\text{A3})$$

where $M(R)$ is the maximal number of mutually orthogonal states sustainable in R . Hence,

$$M_{\max}(R) \leq \left\lfloor \frac{\delta S[R]}{\hbar} \right\rfloor. \quad (\text{A4})$$

Consistency with Hilbert space capacity.

Since the Hilbert space dimension is finite,

$$M_{\max}(R) \leq 2^N. \quad (\text{A5})$$

Therefore,

$$M_{\max}(R) \leq \min\left(2^N, \left\lfloor \frac{\delta S[R]}{\hbar} \right\rfloor\right). \quad (\text{A6})$$

Entropy bound.

For any mixed state ρ supported on the chain,

$$S(\rho) \leq \log M_{\max}(R), \quad (\text{A7})$$

where the logarithm is taken in base e (nats). Using \log_2 instead would yield the bound in bits. Thus the distinguishability capacity is limited by the available action budget.

Dimensional check.

$\delta S[R]$ and \hbar have the same units of action, so $\delta S[R]/\hbar$ is dimensionless, consistent with its interpretation as a maximal count.

Operational Estimate via the Hamiltonian

Consider a uniform nearest-neighbor Hamiltonian

$$H = J \sum_{i=1}^{N-1} \sigma_i^z \sigma_{i+1}^z, \quad (\text{A8})$$

with coupling constant J (energy scale). For evolution over $[0, T]$, when spatial boundary terms are negligible, the dominant contribution to $\oint \Lambda$ arises from the temporal component:

$$\delta S[R] \lesssim \int_0^T \langle H(t) \rangle dt \leq \int_0^T \|H(t)\| dt. \quad (\text{A9})$$

For the ZZ Hamiltonian the local terms commute and

$$\|H\| \leq |J|(N-1), \quad \Rightarrow \quad \delta S[R] \leq |J|(N-1)T. \quad (\text{A10})$$

Combining with (A4) gives the operational bound

$$M_{\max}(R) \leq \min\left(2^N, \left\lfloor \frac{|J|(N-1)T}{\hbar} \right\rfloor\right). \quad (\text{A11})$$

Variants.

- For XX/XY nearest-neighbor models one still has the triangle-inequality estimate $\|H\| \leq \sum_i \|h_i\|$, yielding $\|H\| \lesssim |J|(N-1)$ under bounded local strengths. - For a homogeneous field $H = h \sum_{i=1}^N \sigma_i^z$, $\|H\| = |h|N$, hence

$$M_{\max}(R) \leq \min\left(2^N, \left\lfloor \frac{|h|NT}{\hbar} \right\rfloor\right).$$

- More generally, for any local Hamiltonian with bounded interaction strength, the action budget scales at most linearly with N and with T .

Interpretation.

Two independent ceilings control the number of distinct coherent states producible in time T : (i) the Hilbert-space ceiling 2^N ; (ii) the action ceiling $\sim (\text{energy scale}) \times T/\hbar$. Thus the coherence condition $\delta S \geq \hbar$ manifests as a bound on the number of orthogonalizations achievable within a finite window.

Connection to Quantum Speed Limits

For a time-dependent Hamiltonian $H(t)$, the minimal orthogonalization time τ_{\perp} obeys the Mandelstam–Tamm and Margolus–Levitin bounds [26,27]:

$$\tau_{\perp} \geq \frac{\pi\hbar}{2\Delta E'}, \quad (\text{Mandelstam–Tamm}), \quad (\text{A12})$$

$$\tau_{\perp} \geq \frac{\pi\hbar}{2E'}, \quad (\text{Margolus–Levitin}), \quad (\text{A13})$$

where ΔE is the instantaneous energy variance and $E(t) = \langle H(t) \rangle - E_0$ is the mean energy above the ground state. Integrating over time,

$$M_{\max}(T) \leq 1 + \left\lfloor \frac{2}{\pi\hbar} \int_0^T \Delta E(t) dt \right\rfloor, \quad M_{\max}(T) \leq 1 + \left\lfloor \frac{2}{\pi\hbar} \int_0^T E(t) dt \right\rfloor. \quad (\text{A14})$$

Heuristically, each orthogonalization carries an action cost of order $\sim \frac{\pi}{2}\hbar$; the precise constants depend on the fidelity threshold adopted in projective Hilbert space. These bounds should therefore be read as asymptotic ceilings, consistent with the resource inequality $M(R)\hbar \leq \delta S[R]$.

Schematic Bound Comparison

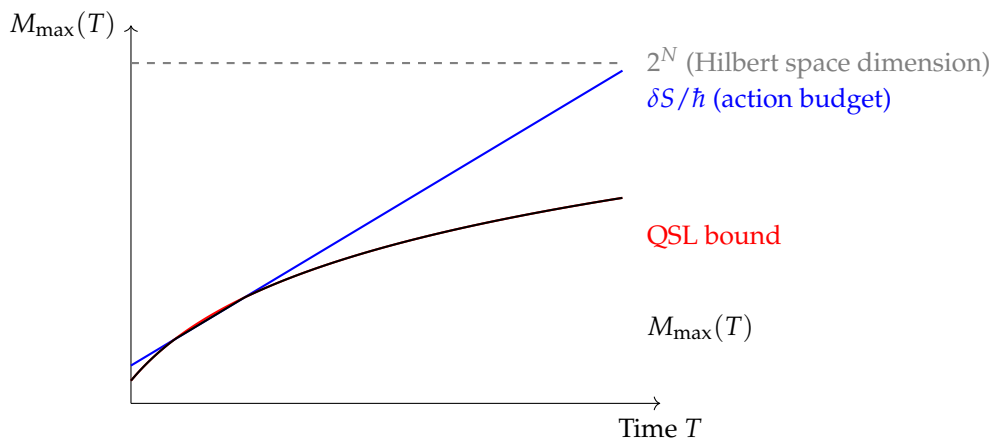


Figure A2. Three ceilings limiting the number of orthogonal states in a spin chain. The Hilbert-space ceiling 2^N (gray dashed), the action-budget ceiling $\delta S/\hbar$ (blue), and the quantum-speed-limit ceiling (red) jointly constrain $M_{\max}(T)$ (black). Curves illustrative, not to scale.

Appendix B. Demonstration of the Sum Rule

This appendix illustrates the structural content of the Sum Rule (Principle 2) on explicit many-body states.

Profiles $C_X(n)$ for Product vs. GHZ States Under the Same δS Budget

Setting and budget of action.

Consider a chain $X = \{X_1, \dots, X_N\}$ of N qubits realized as N localized *coherent units*, each corresponding to one sustainable degree of freedom in Hilbert space. The coherence threshold implies that each stable support requires at least one quantum of action:

$$\delta S[\mathcal{O}_j] \geq \hbar.$$

Fix a spacetime region R sustaining the chain with total action budget $\delta S[R]$, defined invariantly through the Poincaré–Cartan one-form Λ as

$$\delta S[R] := \sup_{\gamma \subset R} \oint_{\gamma} \Lambda,$$

taken over all *admissible* closed cycles under fixed boundary conditions and process duration T (cf. Appendix A). Set

$$B := \left\lfloor \frac{\delta S[R]}{\hbar} \right\rfloor.$$

We assume $B \geq N$, so that all N units can be sustained.

Multiscale profile (combinatorial definition).

Define a *dependency hypergraph* \mathcal{G} on X by placing a (hyper)edge on any subset $A \subseteq X$ ($|A| \geq 2$) whenever the joint state exhibits nontrivial correlation across A (e.g. nonzero mutual information $I(A; \bar{A}) > 0$ in the chosen ensemble/context). Let $\mathcal{C} = \{C_1, \dots, C_m\}$ be the connected components of \mathcal{G} with sizes s_1, \dots, s_m (so $\sum_{i=1}^m s_i = N$). For $n = 1, \dots, N$ define the *complexity profile*

$$C_X(n) := \#\{i \in \{1, \dots, m\} : s_i \geq n\}. \quad (\text{A15})$$

Intuitively, $C_X(n)$ counts how many independent coherent components extend to scale $\geq n$. This discrete, graph-based definition is used here as an illustrative analogue; the entropic formulation given in Appendix C is the primary structural definition.

Sum Rule (discrete form).

The profile (A15) satisfies the exact identity

$$\sum_{n=1}^N C_X(n) = \sum_{i=1}^m s_i = N. \quad (\text{A16})$$

Proof. Each component C_i of size s_i contributes 1 to $C_X(n)$ for every $n \leq s_i$ and contributes 0 otherwise; summing over n yields s_i , then summing over i yields N .

Action–profile correspondence.

Since each coherent unit costs at least \hbar , one has

$$\sum_{j=1}^N \delta S[\mathcal{O}_j] \geq N\hbar.$$

Let S_{proc} denote the action accumulated along the realized closed trajectory $\Gamma \subset R$. By budget additivity for the realized process,

$$\sum_{j=1}^N \delta S[\mathcal{O}_j] \leq S_{\text{proc}} \leq \delta S[R].$$

Hence the chain of inequalities

$$N \leq \frac{1}{\hbar} \sum_{j=1}^N \delta S[\mathcal{O}_j] \leq \frac{\delta S[R]}{\hbar}. \quad (\text{A17})$$

Thus, when the sustained number of units equals N , the budget version of the sum rule reads

$$\sum_{n=1}^N C_X(n) = N \leq \left\lfloor \frac{\delta S[R]}{\hbar} \right\rfloor.$$

If the available budget only sustains $B < N$ units (e.g. sparse activation), the same construction restricted to the active units yields $\sum_n C_X(n) = B$.

Two extremal architectures under the same δS budget

(A) Product register (independent local DOF).

The dependency hypergraph has no edges; all vertices are isolated. Hence the connected components are $m = N$ with sizes $s_i = 1$. By (A15),

$$C_X^{\text{prod}}(1) = N, \quad C_X^{\text{prod}}(n) = 0 \quad (n \geq 2),$$

and the sum rule gives $\sum_{n=1}^N C_X^{\text{prod}}(n) = N$.

(B) GHZ architecture (global DOF).

For the GHZ ensemble, the dependency hypergraph is fully connected ($m = 1, s_1 = N$). Thus,

$$C_X^{\text{GHZ}}(n) = 1 \quad \text{for all } n = 1, \dots, N,$$

and $\sum_{n=1}^N C_X^{\text{GHZ}}(n) = N$.

Same action, different profiles.

Both architectures use the *same unit budget* $N\hbar$, but allocate distinguishability across scales differently:

$$\{C_X^{\text{prod}}(n)\} = \{N, 0, \dots, 0\}, \quad \{C_X^{\text{GHZ}}(n)\} = \underbrace{\{1, 1, \dots, 1\}}_{N \text{ entries}}.$$

Quantum-Informational Check

For the product register, $I(A; \bar{A})_\rho = 0$ for all $A \subset X$. For the GHZ ensemble, $I(A; \bar{A})_\rho = \log 2$ (in nats; equals 1 bit under \log_2) for every nontrivial bipartition. Thus the combinatorial profile above is consistent with the quantum mutual-information criterion used to define edges.

Units and dimensional check.

$C_X(n)$ is dimensionless (a pure count). The budget constraint enters only through the number of sustained units,

$$\sum_n C_X(n) \leq \frac{\delta S[R]}{\hbar},$$

providing a *dimensionless bound*, consistent with $C_X(n)$ being a pure count.

General Remark: Redistributing a Fixed δS

Given a fixed action budget $\delta S[R]$, one may trade *local variety* for *global coordination* without changing $\sum_n C_X(n)$: independent registers concentrate C_X at $n = 1$, while globally entangled architectures (like GHZ) spread C_X flatly across all n . This is the operational content of the Sum Rule under the coherence budget.

(C) Block-GHZ (“clustered”) architecture.

Partition X into m blocks of size L , so $N = mL$. On each block prepare an L -qubit GHZ state:

$$|\Psi_{\text{block-GHZ}}\rangle = \bigotimes_{k=1}^m \frac{|0\rangle_{B_k}^{\otimes L} + |1\rangle_{B_k}^{\otimes L}}{\sqrt{2}}.$$

The hypergraph has m connected components of sizes $s_i = L$. Therefore,

$$C_X^{\text{block-GHZ}}(n) = \begin{cases} m, & 1 \leq n \leq L, \\ 0, & n > L, \end{cases} \quad \sum_{n=1}^N C_X^{\text{block-GHZ}}(n) = mL = N,$$

which interpolates between product ($L = 1$) and GHZ ($L = N$).

Remark on graph and cluster states.

For a connected graph state, the hypergraph defined by unconditional correlations is connected, yielding $C_X(n) \equiv 1$. In a specific measurement context (stabilizer readout), one obtains effective

finite correlation length, producing a plateau in $C_X(n)$ followed by decay. This context dependence is consistent with structural contextuality, while the Sum Rule and the action budget constraint

$$\sum_n C_X(n) \leq \frac{\delta S[R]}{\hbar}$$

remain intact.

Schematic Profiles (Product / Block-GHZ / GHZ)

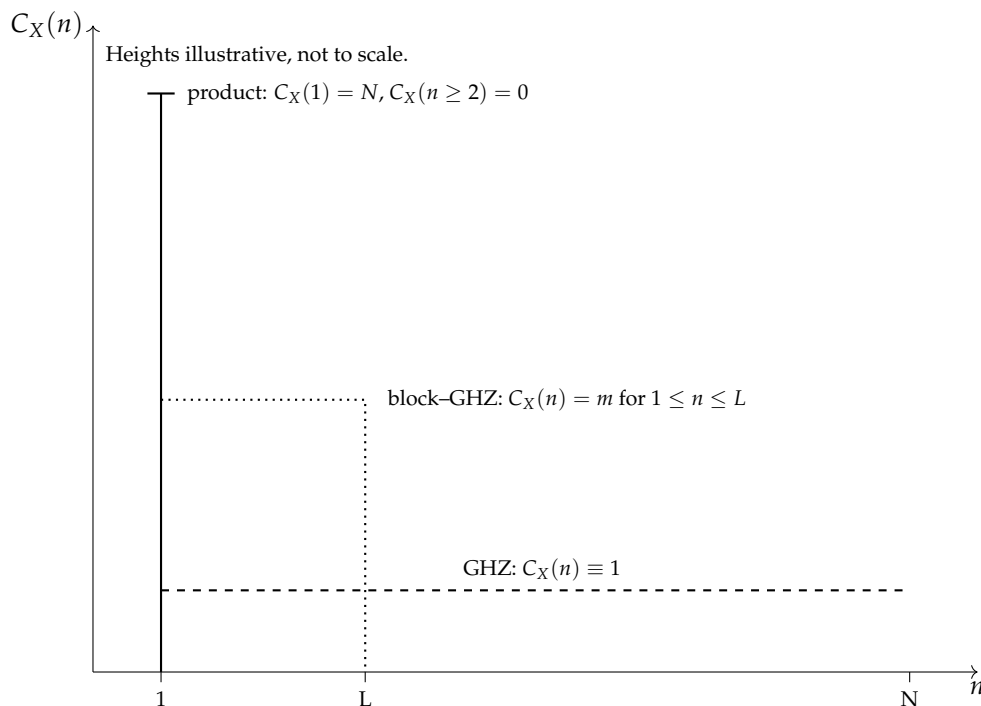


Figure A3. Complexity profiles $C_X(n)$ for product (“needle”), block-GHZ (“plateau”), and GHZ (flat line at 1). Different allocations of the same action budget correspond to different shapes of the profile, but the discrete area under each profile is invariant and equals N .

Note on summation convention.

In the combinatorial formulation used here, the sum in Eq. (A16) runs to N , while in the entropic formulation of Appendix C it runs to $N - 1$ (since the last coarse-graining step to Π_N contributes trivially). The two conventions are equivalent up to this endpoint.

Appendix C. Proof of Theorem 3

Statement of Theorem 3

Theorem 3 asserts that for a coarse-grained system, the phenomenological complexity profile $C_X(n)$ can be identified with the per-scale logarithmic increment of the structural measure of distinction:

$$C_X(n) = \ln \mu_\Omega(\Pi_{n+1}\rho) - \ln \mu_\Omega(\Pi_n\rho),$$

where Π_n denotes the coarse-graining operator acting at scale n , and ρ is the density operator of the system. Throughout we use the natural logarithm (units: nats); using \log_2 yields the same statements in bits up to a constant factor. We work with a finite chain of scales $n = 1, \dots, N$, where Π_1 is the finest and Π_N is the coarsest coarse-graining.

Hierarchical decomposition

Consider a system $X = \{X_1, \dots, X_N\}$ organized into nested scales of resolution:

$$\mathcal{H}_X = \bigotimes_{j=1}^N \mathcal{H}_{X_j}.$$

A coarse-graining at scale n is represented by a map

$$\Pi_n : \mathcal{B}(\mathcal{H}_X) \rightarrow \mathcal{A}^{(n)} \subset \mathcal{B}(\mathcal{H}_X),$$

modeled as a *conditional expectation* (a pinching / entropy non-decreasing channel) onto a von Neumann subalgebra $\mathcal{A}^{(n)}$ that captures observables resolvable at scale n . The algebras form a decreasing chain

$$\mathcal{A}^{(N)} \subset \dots \subset \mathcal{A}^{(n+1)} \subset \mathcal{A}^{(n)} \subset \dots \subset \mathcal{A}^{(1)} = \mathcal{B}(\mathcal{H}_X),$$

so that Π_{n+1} is strictly coarser than Π_n . Operationally, Π_n discards distinctions below scale n , ensuring a consistent coarse-graining chain

$$\rho \mapsto \Pi_1 \rho \mapsto \Pi_2 \rho \mapsto \dots \mapsto \Pi_N \rho.$$

Structural measure of distinction

For a mixed state ρ , define

$$\mu_\Omega(\rho) := \exp(S_{\text{vN}}(\rho)), \quad S_{\text{vN}}(\rho) = -\text{Tr}(\rho \log \rho).$$

Under coarse-graining,

$$\mu_\Omega(\Pi_n \rho) = \exp(S_{\text{vN}}(\Pi_n \rho)).$$

Monotonicity

For conditional expectations (and, more generally, pinching maps / entropy non-decreasing channels), the von Neumann entropy is monotone non-decreasing (Uhlmann, Araki):

$$S_{\text{vN}}(\Pi_{n+1} \rho) \geq S_{\text{vN}}(\Pi_n \rho).$$

Consequently,

$$\ln \mu_\Omega(\Pi_{n+1} \rho) \geq \ln \mu_\Omega(\Pi_n \rho),$$

and the per-scale increments

$$C_X(n) = \ln \mu_\Omega(\Pi_{n+1} \rho) - \ln \mu_\Omega(\Pi_n \rho)$$

are non-negative.

Equivalence Under Canonical Coarse-Graining

The phenomenological profile of Siegenfeld–Bar–Yam counts distinctions persisting at scale $\geq n$. Under *canonical* coarse-graining procedures that respect the dependency partition of the system (cf. Appendix B)—i.e., conditional expectations that collapse each dependency component to a single effective degree of freedom for n below its size and to a trivial one above—the operational content of “persisting distinctions” coincides with the finite-difference loss of distinguishability:

$$C_X(n) \equiv \ln \mu_\Omega(\Pi_{n+1} \rho) - \ln \mu_\Omega(\Pi_n \rho)$$

This provides the claimed identification in the class of canonical coarse-grainings used throughout the paper.

Entropy–Action Bound and Completion of the Proof

We now complete the bridge to the action budget $\delta S[R]$ and \hbar .

Lemma C.1 (Entropy bounded by capacity).

Let R be a spacetime region with action budget $\delta S[R]$ and let $M_{\max}(R)$ be the maximal number of mutually orthogonal states that can be sustained in R (Theorem 1). Then for any state σ supported in R and for any scale n ,

$$S_{\text{vN}}(\Pi_n \sigma) \leq \log M_{\max}(R).$$

Proof. Write $\Pi_n \sigma = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$, $\sum_i \lambda_i = 1$, with $\{|\psi_i\rangle\}$ orthonormal. By definition of $M_{\max}(R)$, no ensemble of more than $M_{\max}(R)$ mutually orthogonal states can be sustained in R . Hence $\text{rank}(\Pi_n \sigma) \leq M_{\max}(R)$, giving $S_{\text{vN}}(\Pi_n \sigma) \leq \log \text{rank}(\Pi_n \sigma) \leq \log M_{\max}(R)$. \square

Theorem C.2 (Action–profile bound).

For any canonical coarse-graining chain $\{\Pi_n\}$ as above and any initial state ρ supported in R ,

$$\sum_{n=1}^{N-1} C_X(n) = S_{\text{vN}}(\Pi_N \rho) - S_{\text{vN}}(\Pi_1 \rho) \leq \log M_{\max}(R) \leq \log \left(\max\{1, \lfloor \frac{\delta S[R]}{\hbar} \rfloor\} \right) \leq \frac{\delta S[R]}{\hbar}.$$

Proof. The first equality is the telescoping sum of per-scale increments. By Lemma C.1, $S_{\text{vN}}(\Pi_N \rho) \leq \log M_{\max}(R)$ and $S_{\text{vN}}(\Pi_1 \rho) \geq 0$, giving the first inequality. The second follows from Theorem 1. The last step uses $\log(1+x) \leq x$ for $x \geq 0$, with $\max\{1, \lfloor \delta S[R]/\hbar \rfloor\} \leq 1 + \delta S[R]/\hbar$. \square

Remark (Tightness).

The logarithmic bound $\sum_n C_X(n) \leq \log \max\{1, \lfloor \delta S[R]/\hbar \rfloor\}$ is tighter; the final linear bound $\sum_n C_X(n) \leq \delta S[R]/\hbar$ is a convenient but coarse estimate.

Corollary C.3 (Bits version).

With \log_2 ,

$$\sum_{n=1}^{N-1} C_X(n) \leq \log_2 \left(\max\{1, \lfloor \frac{\delta S[R]}{\hbar} \rfloor\} \right).$$

Conclusion

Theorem 3 identifies $C_X(n)$ with the per-scale entropy increments of the structural measure μ_Ω under canonical coarse-graining. Lemma C.1 and Theorem C.2 complete the proof by bounding the *total* profile $\sum_n C_X(n)$ in terms of the structural action budget $\delta S[R]$: the sum cannot exceed $\log \max\{1, \lfloor \delta S[R]/\hbar \rfloor\}$ (and therefore is $\leq \delta S[R]/\hbar$). Hence the phenomenological profile is not only equivalent to entropy increments but also *structurally* constrained by the minimal action principle.

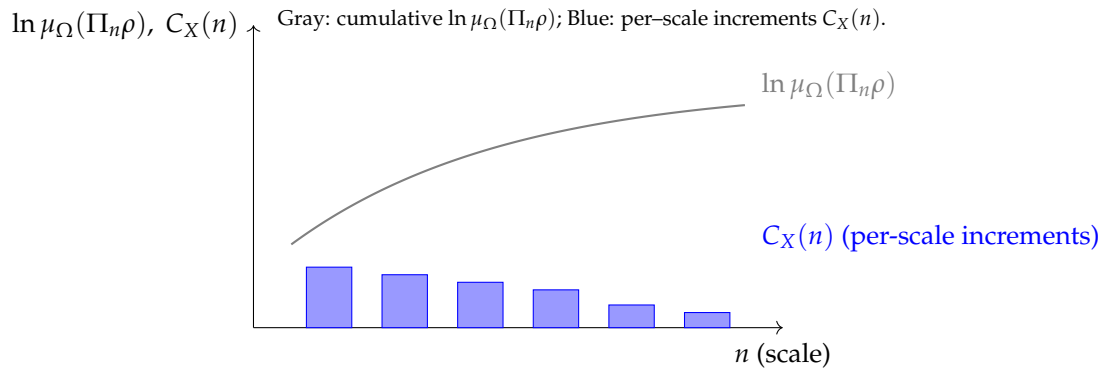


Figure A4. Schematic illustration of Theorem 3. The gray curve shows the cumulative distinction measure $\ln \mu_{\Omega}(\Pi_n \rho)$, which increases monotonically with the scale n under canonical coarse-graining. The blue bars represent the finite differences $C_X(n)$, which are nonnegative and whose discrete sum $\sum_{n=1}^{N-1} C_X(n)$ is bounded by $\log \max\{1, \lfloor \delta S[R]/\hbar \rfloor\}$. Heights illustrative, not to scale.

Supplementary Note: Monotonicity of Entropy under Canonical Coarse-Graining

At first sight, there appears to be a tension between two statements: on the one hand, the von Neumann entropy under canonical coarse-graining is known to be monotonically increasing (Uhlmann [28], Araki [29]); on the other hand, the phenomenological complexity profile of Siegenfeld-Bar-Yam is monotonically decreasing with scale. This is not a contradiction, because the two quantities refer to different levels of description. The entropy $S_{\text{vN}}(\Pi_n \rho)$ is a *cumulative measure*, growing as distinctions are discarded, while the profile $C_X(n)$ is defined by the *finite differences* of this entropy between successive scales. The construction below makes this relation explicit and identifies the profile as the sequence of per-scale entropy gains (equivalently, information losses).

The correspondence established in Theorem 3 relies on the monotonicity of the quantity

$$\ln \mu_{\Omega}(\Pi_n \rho) = S_{\text{vN}}(\Pi_n \rho)$$

with respect to the scale parameter n . Since the data-processing inequality for relative entropy does not guarantee monotonicity of the von Neumann entropy under *arbitrary* CPTP maps, we here specify the class of coarse-grainings used.

Canonical coarse-graining.

We restrict attention to the canonical family $(\Pi_n)_{n \geq 1}$, where each Π_n is the CPTP conditional expectation onto a decreasing chain of von Neumann subalgebras

$$\mathcal{A}^{(n+1)} \subset \mathcal{A}^{(n)} \subset \mathcal{B}(\mathcal{H}_X),$$

followed by the induced Stinespring compression to $\mathcal{H}_X^{(n)}$. Operationally, this corresponds to discarding all distinctions below scale n .

Monotonicity result.

For conditional expectations onto von Neumann subalgebras, the von Neumann entropy is non-decreasing (Uhlmann [28], Araki [29]):

$$S_{\text{vN}}(\Pi_{n+1} \rho) \geq S_{\text{vN}}(\Pi_n \rho).$$

Equivalently,

$$\ln \mu_{\Omega}(\Pi_{n+1} \rho) \geq \ln \mu_{\Omega}(\Pi_n \rho).$$

Profile as finite differences.

We now define the phenomenological complexity profile as the per-scale entropy gain (equivalently, information loss) from the coarse-graining process. This is given by the finite difference:

$$C_X(n) := \ln \mu_\Omega(\Pi_{n+1}\rho) - \ln \mu_\Omega(\Pi_n\rho) \geq 0.$$

This defines a nonnegative sequence $\{C_X(n)\}$. Its sum is a telescoping series that measures the total gain in entropy from the finest to the coarsest scale:

$$\sum_{n=1}^{N-1} C_X(n) = \ln \mu_\Omega(\Pi_N\rho) - \ln \mu_\Omega(\Pi_1\rho).$$

Interpretation.

Thus the structural distinction measure $\ln \mu_\Omega(\Pi_n\rho)$ is nondecreasing with the coarse-graining scale n , while the phenomenological complexity profile $C_X(n)$ is given by its discrete, per-scale increments. This provides the rigorous justification for identifying the profiles of Siegenfeld–Bar–Yam as the finite-difference shadows of μ_Ω under hierarchical coarse-graining.

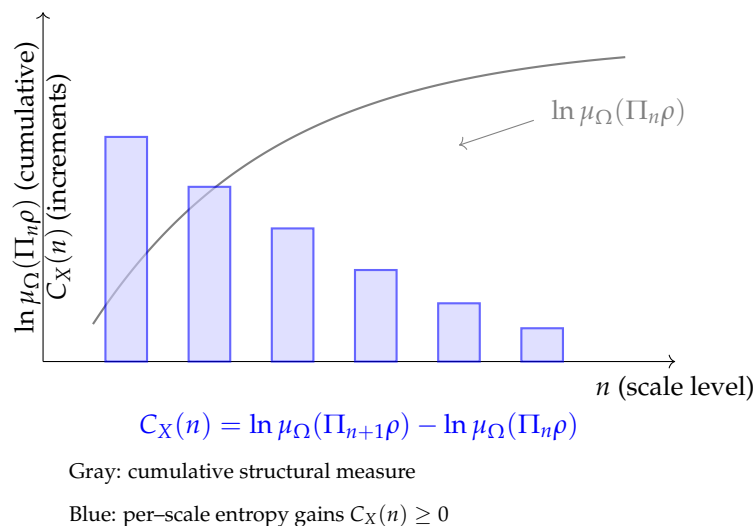


Figure A5. Canonical coarse-graining increases the cumulative entropy (gray curve), while the per-scale finite differences $C_X(n)$ (blue bars) are nonnegative and typically decrease with n . The telescoping sum $\sum_{n=1}^{N-1} C_X(n) = \ln \mu_\Omega(\Pi_N\rho) - \ln \mu_\Omega(\Pi_1\rho)$ is consistent with the action budget constraint. Heights illustrative, not to scale.

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