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Article

A Characterization of Lie Type Higher Derivations on von Neumann Algebras with Local Actions

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Abstract: Let m and n be the fixed positive integers. Suppose $\mathcal{A}$ is a von Neuman algebra with no central summands of type I_1 and L_m be a Lie type higher derivation i.e., an additive (linear) map $L_m : \mathcal{A} \to \mathcal{A}$ such that $L_m(p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)) = \sum_{l_1+l_2+\cdots+l_n=m} p_n\big(L_{l_1}(\mathfrak{S}_1), L_{l_2}(\mathfrak{S}_2), \cdots, L_{l_n}(\mathfrak{S}_n)\big)$ for all $\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n \in \mathcal{A}$. In the present paper, we study Lie type higher derivations on von Neuman algebras and prove that every additive Lie type higher derivation on $\mathcal{A}$ has a standard form at zero product as well as at projection product. Further, we discuss some more related results.

Keywords: Lie derivation; Multiplicative Lie type-derivation; multiplicative Lie type-higher derivation; von Neumann algebra

MSC: 47B47; 16W25; 46K15

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1. Introduction

Let $\mathcal{R}$ be a commutative ring with unity, $\mathcal{A}$ be an algebra over $\mathcal{R}$ and $\mathcal{Z}(\mathcal{A})$ be the center of $\mathcal{A}$. Recall that an $\mathcal{R}$-linear map $L : \mathcal{A} \to \mathcal{A}$ is called a derivation on $\mathcal{A}$ if for all $\mathfrak{S}, \mathfrak{T} \in \mathcal{A}$, $L(\mathfrak{S}\mathfrak{T}) = L(\mathfrak{S})\mathfrak{T} + \mathfrak{S}L(\mathfrak{T})$. An $\mathcal{R}$-linear map $L : \mathcal{A} \to \mathcal{A}$ is called a Lie derivation (resp. Lie triple derivation) on $\mathcal{A}$ if for all $\mathfrak{S}, \mathfrak{T}, \mathcal{W} \in \mathcal{A}$, $L([\mathfrak{S}, \mathfrak{T}]) = [L(\mathfrak{S}), \mathfrak{T}] + [\mathfrak{S}, L(\mathfrak{T})]\big(resp. L([[\mathfrak{S}, \mathfrak{T}], \mathcal{W}]) = [[L(\mathfrak{S}), \mathfrak{T}], \mathcal{W}] + [[\mathfrak{S}, L(\mathfrak{T})], \mathcal{W}] + [[\mathfrak{S}, \mathfrak{T}], L(\mathcal{W})]\big)$, where $[\mathfrak{S}, \mathfrak{T}] = \mathfrak{S}\mathfrak{T} - \mathfrak{T}\mathfrak{S}$ is the usual Lie product. Let $\mathbb{N}$ be the set of non-negative integers and $\mathcal{D} = \{d_m\}_{m \in \mathbb{N}}$ be a family of additive mappings $d_m : \mathcal{A} \to \mathcal{A}$ such that $d_0 = id_{\mathcal{A}}$, the identity map on $\mathcal{A}$. Then $\mathcal{D}$ is called :

(*i*) a *higher derivation* on $\mathcal{A}$, if for every $m \in \mathbb{N}$, $d_m(\mathfrak{S}\mathfrak{T}) = \sum\limits_{r+s=m} d_r(\mathfrak{S})d_s(\mathfrak{T})$ for all $\mathfrak{S}, \mathfrak{T} \in \mathcal{A}$.

(*ii*) a *Lie higher derivation* on $\mathcal{A}$ if for every $m \in \mathbb{N}$, $d_m([\mathfrak{S}, \mathfrak{T}]) = \sum\limits_{r+s=m} [d_r(\mathfrak{S}), d_s(\mathfrak{T})]$ for all $\mathfrak{S}, \mathfrak{T} \in \mathcal{A}$.

(*iii*) a *triple higher derivation* on $\mathcal{A}$ if for every $m \in \mathbb{N}$, $d_m([[\mathfrak{S}, \mathfrak{T}], \mathcal{W}]) = \sum\limits_{r+s+k=m} [[d_r(\mathfrak{S}), d_s(\mathfrak{T})], d_k(\mathcal{W})]$ for all $\mathfrak{S}, \mathfrak{T}, \mathcal{W} \in \mathcal{A}$.

In [1], Abdullaev initiated the study of Lie n-derivations. Define the sequence of polynomials: $p_1(\mathfrak{S}) = \mathfrak{S}$ and $p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n) = [p_{n-1}(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_{n-1}), \mathfrak{S}_n]$ for all $n \in \mathbb{Z}$ and $n \geq 2$. Here $p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)$ is known as the $(n-1)$-th commutator. An additive (linear) map $L_m : \mathcal{A} \to \mathcal{A}$ is called a Lie n-higher derivation if

$$L_m(p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)) = \sum_{l_1+l_2+\cdots+l_n=m} p_n(L_{l_1}(\mathfrak{S}_1), L_{l_2}(\mathfrak{S}_2), \cdots, L_{l_n}(\mathfrak{S}_n))$$

for all $\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n \in \mathcal{A}$. In particular, by giving different values to n, we obtain Lie higher derivation, Lie triple higher derivation and Lie n-higher derivations. These derivations collectedly are referred as Lie type higher derivations.

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Since last few decades, examining the various properties of derivations defined through the most well known rule given by Leibniz under the influence of various algebraic structures is a vast topic of study among the algebraists. In [5], Bresar characterized an additive Lie derivation as the sum of a derivation and an additive map on a prime ring $\mathcal{R}$ with $ch(\mathcal{R}) \neq 2$. In [10], Johnson worked on Lie derivation on $\mathbb{C}^*$-algebras and proved that every continous linear Lie derivation from a $\mathbb{C}^*$-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule $\mathcal{M}$ can be written as $\tau + h$ (i.e., every continous linear Lie derivation from a $\mathbb{C}^*$-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule $\mathcal{M}$ is standard), where $\tau : \mathcal{A} \to \mathcal{M}$ is a derivation and $h : \mathcal{A} \to Z(\mathcal{M})$ vanishing at each commutators. In [15], Mathieu and Villena proved that on $\mathbb{C}^*$-algebra, every linear Lie derivation is standard. Qi and Hou [20], worked on nest algebras and proved that additive Lie derivations of nest algebras on Banach spaces is standard.
During the recent question of finding the condition under which a linear map becomes a Lie derivation or simply a derivation influenced observations of so many researchers (see [4], [12],[2],[9],[18],[19] and references therein). The purpose of the above studies in the most of cases, was to obtain the restrictions under which Lie derivations or derivations can be completely determined by the action on some subsets of the algebras. There are several articles on the study of local actions of Lie derivations of operator algebras. In [[14]], Lu and Jing proved that for a Banach space $\mathcal{X}$ of dimension greater than two and a linear map $L : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ such that $L([\mathfrak{S}, \mathfrak{T}]) = [L(\mathfrak{S}), \mathfrak{T}] + [\mathfrak{S}, L(\mathfrak{T})]$ for all $\mathfrak{S}, \mathfrak{T} \in \mathcal{B}(\mathcal{X})$ with $\mathfrak{S}\mathfrak{T} = 0$ (resp. $\mathfrak{S}\mathfrak{T} = P$, where $\mathfrak{Q}$ is a fixed nontrivial idempotent), then there exist an operator $\mathcal{T} \in \mathcal{B}(\mathcal{X})$ and a linear map $\phi : \mathcal{B}(\mathcal{X}) \to \mathbb{C}I$ vanishing at all the commutators $[\mathfrak{S}, \mathfrak{T}]$ with $\mathfrak{S}\mathfrak{T} = 0$(resp. $\mathfrak{S}\mathfrak{T} = P$) such that $L(\mathfrak{S}) = \mathcal{T}\mathfrak{S} + \phi(\mathfrak{S})$ for all $\mathfrak{S} \in \mathcal{B}(\mathcal{X})$. In [8], Ji and Qi proved that if $\mathcal{T}$ is a triangular algebra over a commutative ring $\mathcal{R}$ then under certain restrictions on $\mathcal{T}$, if $L : \mathcal{T} \to \mathcal{T}$ is an $\mathcal{R}$-linear map satisfying $L([\mathfrak{S}, \mathfrak{T}]) = [L(\mathfrak{S}), V] + [\mathfrak{S}, L(V)]$ for all $\mathfrak{S}, \mathfrak{T} \in \mathcal{T}$ with $\mathfrak{S}\mathfrak{T} = 0$(resp. $\mathfrak{S}\mathfrak{T} = \mathfrak{Q}$, where $\mathfrak{Q}$ is the standard idempotent of $\mathcal{T}$), then $L = d + \phi$, where $d : \mathcal{T} \to \mathcal{T}$ is a derivation and $\phi : \mathcal{T} \to Z(\mathcal{T})$ is an $\mathcal{R}$-linear map vanishing at all the commutators $[\mathfrak{S}, \mathfrak{T}]$ with$\mathfrak{S}\mathfrak{T} = 0$(resp. $\mathfrak{S}\mathfrak{T} = \mathfrak{Q}$). Qi and Hou [19] characterized Lie derivation on von Neumann algebra $\mathcal{A}$ without central summands of type I_1. In [21], Qi and Ji proved the same result for $\mathfrak{S}\mathfrak{T} = \mathfrak{Q}$, where $\mathfrak{Q}$ is a core-free projection. In [18], Qi characterized Lie derivation on $\mathcal{J}$-subspace lattice algebras and proved the same result due to Lu and Jing [14] on $\mathcal{J}$-subspace lattice algebra Alg$\mathcal{L}$, where $\mathcal{L}$ is a $\mathcal{J}$-subspace lattice on a Banach space $\mathcal{X}$ over the real or complex field with dimension greater than 2. Liu [13] studied the characterization of Lie triple derivation on von Neumann algebra with no central abelian projections. For further references (see [7],[23],[24], and references therein). Recently, M. Ashraf and A. Jabeen[3] characterized the Lie type derivations on von Neumann Algebra with no central summands of type I_1, where they showed that every Lie type derivation on von Neumann algebra has standard form at zero product as well as at projection product.

The objective of this paper is to investigate Lie type higher derivations on von Neumann algebras with no central summands of type I_1 and prove that on a von Neumann algebra every Lie type higher derivation has standard form at zero product as well as at projection product. Precisely, we prove that every additive map $L_m : \mathcal{A} \to \mathcal{A}$ satisfying $L_m(p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)) = \sum_{l_1+l_2++l_n=m} p_n\Big(L_{l_1}(\mathfrak{S}_1), L_{l_2}(\mathfrak{S}_2), L_{l_3}(\mathfrak{S}_3), \cdots, L_{l_n}(\mathfrak{S}_n)\Big)$ for all $\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n \in \mathcal{A}$ with $\mathfrak{S}_1\mathfrak{S}_2 = 0$ is of the form $L_m(\mathfrak{S}) = \phi_m(\mathfrak{S}) + \zeta_m(\mathfrak{S})$ for all $\mathfrak{S} \in \mathcal{A}$, where $\phi_m : \mathcal{A} \to \mathcal{A}$ is an additive higher derivation and $\zeta_m : \mathcal{A} \to Z(\mathcal{A})$ an additive higher map whose range is in $Z(\mathcal{A})$. Further, we discuss some more related results.

2. Main Results

In this section, we discuss the characterization of Lie type higher derivation on von Neumann algebras having no central summands of type I_1 at zero product. In proving our main results, we use the following known lemmas:

Lemma 1. *([16], Lemma 5). For projections $P, Q \in \mathcal{A}$ with $\bar{P} = \bar{Q} \neq 0$, if $T \in \mathcal{A}$ commutes with PXQ and QXP for all $X \in \mathcal{A}$, then T commutes with PXP and QXQ for all $X \in \mathcal{A}$.*

Lemma 2. *([6], Lemma 5). Let $\mathcal{A}$ be a von Neumann algebra with no central summands of type I_1, if $Z \in Z(\mathcal{A})$ is such that $Z\mathcal{A} \subseteq Z(\mathcal{A})$, then $Z = 0$.*

Lemma 3. *([16], Lemma 14). Let $\mathcal{A}$ be a von Neumann algebra and assume that $P \in \mathcal{A}$ is a core free projection in $\mathcal{A}$. Then $P\mathcal{A}P \cap Z(\mathcal{A}) = 0$.*

Lemma 4. *([3], Lemma 2.5). Let $\mathfrak{S}_{ii} \in \mathcal{A}_{ii}, i = 1,2$. If $\mathfrak{S}_{11}\mathfrak{T}_{12} = \mathfrak{T}_{12}\mathfrak{S}_{11}$ for all $\mathfrak{T}_{12} \in \mathcal{A}_{12}$, then $\mathfrak{S}_{11} + \mathfrak{S}_{22} \in Z(\mathcal{A})$.*

Lemma 5. *([16], Lemma 4). If $\mathcal{A}$ is a von Neumann algebra with no central summands of type I_1, then each non zero central projection of $\mathcal{A}$ is the central carrier of a core-free projection of $\mathcal{A}$.*

The first main result of this paper is the following theorem.

Theorem 1. *Let $\mathcal{A}$ be a von Neumann algebra with no central summands of type I_1 and an additive map $L_m : \mathcal{A} \to \mathcal{A}$ satisfying $L_m(p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)) = \sum_{l_1+l_2++l_n=m} p_n\Big(L_{l_1}(\mathfrak{S}_1),$ $L_{l_2}(\mathfrak{S}_2), L_{l_3}(\mathfrak{S}_3), \cdots, L_{l_n}(\mathfrak{S}_n)\Big)$ for all $\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n \in \mathcal{A}$ with $\mathfrak{S}_1\mathfrak{S}_2 = 0$. Then there exists an additive higher derivation $\phi_m : \mathcal{A} \to \mathcal{A}$ and an additive higher map $\zeta_m : \mathcal{A} \to Z(\mathcal{A})$ which annihilates every $(n-1)$th commutator $p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)$ with $\mathfrak{S}_1\mathfrak{S}_2 = 0$, such that $L_m(\mathfrak{S}) = \phi_m(\mathfrak{S}) + \zeta_m(\mathfrak{S})$ for all $\mathfrak{S} \in \mathcal{A}$.*

For projections $\mathfrak{Q}_p, \mathfrak{Q}_q \in \mathcal{A}$, let $\mathfrak{Q}_0 = \mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{Q}_q - \mathfrak{Q}_q L_m(\mathfrak{Q}_p)\mathfrak{Q}_p = 0$ and let us define a map $\pi_m : \mathcal{A} \to \mathcal{A}$ as an inner higher derivation $\pi_m(\mathfrak{S}) = [\mathfrak{S}, \mathfrak{Q}_0]$ for all $\mathfrak{S} \in \mathcal{A}$. Clearly $L_m = L'_m - \pi_m$ is a Lie n-higher derivation. Since

$$\begin{aligned} L'_m(\mathfrak{Q}_p) &= L_m(\mathfrak{Q}_p) - [\mathfrak{Q}_p, \mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{Q}_q - \mathfrak{Q}_q L_m(\mathfrak{Q}_p)\mathfrak{Q}_p] \\ &= L_m(\mathfrak{Q}_p) - \mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{Q}_q - \mathfrak{Q}_q L_m(\mathfrak{Q}_p)\mathfrak{Q}_p \\ &= \mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{Q}_p + \mathfrak{Q}_q L_m(\mathfrak{Q}_p)\mathfrak{Q}_q. \end{aligned}$$

One easily gets $\mathfrak{Q}_p L'_m(\mathfrak{Q}_p)\mathfrak{Q}_q = \mathfrak{Q}_q L'_m(\mathfrak{Q}_p)\mathfrak{Q}_p = 0$. Accordingly, it suffices to consider only those Lie n-higher derivations, which satisfy $\mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{Q}_q = \mathfrak{Q}_q L_m(\mathfrak{Q}_p)\mathfrak{Q}_p = 0$.
We give proof of Theorem 2.1 in series of lemmas. We begin with the following lemmas:

Lemma 6. $L_m(\mathfrak{Q}_p), L_m(\mathfrak{Q}_q) \in Z(\mathfrak{T})$.

Proof. To prove this lemma, we use the principle of mathematical induction on m, for $m = 1$, the result is true by [3]. Assume that the result holds for all $k \le m$. We will show that it also holds for $k = m$. Since $\mathfrak{S}_{12}\mathfrak{Q}_p = \mathfrak{Q}_p\mathfrak{S}\mathfrak{Q}_q\mathfrak{Q}_p = 0$ for all $\mathfrak{S}_{12} \in \mathcal{A}_{12}$, we have

$$\begin{aligned} &L_m\Big(p_n(\mathfrak{S}_{12}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p)\Big) \\ &\quad = p_n\Big(L_m(\mathfrak{S}_{12}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) + p_n(\mathfrak{S}_{12}, L_m(\mathfrak{Q}_p), \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p) \\ &\quad + p_n\Big(\mathfrak{S}_{12}, \mathfrak{Q}_p, L_m(\mathfrak{Q}_p), \cdots, \mathfrak{Q}_p\Big) + \cdots + p_n(\mathfrak{S}_{12}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, L_m(\mathfrak{Q}_p)) \\ &\quad + \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0 \le l_1, l_2, \cdots, l_n \le m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{12}), L_{l_2}(\mathfrak{Q}_p), L_{l_3}(\mathfrak{Q}_p), \cdots, L_{l_n}(\mathfrak{Q}_p)\Big). \end{aligned}$$

Which implies

$$\begin{aligned} L_m((-1)^{n-1}\mathfrak{S}_{12}) &= (-1)^{n-1}\mathfrak{Q}_p L_m(\mathfrak{S}_{12})\mathfrak{Q}_q + \mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{Q}_p \\ &\quad + (-1)^{n-2}(n-1)[\mathfrak{S}_{12}, L_m(\mathfrak{Q}_p)]. \end{aligned} \tag{1}$$

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Premultiplying by $\mathfrak{Q}_p$ to the above equation, we get

$$(-1)^{n-1}\mathfrak{Q}_p L_m(\mathfrak{S}_{12}) = (-1)^{n-1}\mathfrak{Q}_p L_m(\mathfrak{S}_{12})\mathfrak{Q}_q + (-1)^{n-2}(n-1)\big(\mathfrak{S}_{12}L_m(\mathfrak{Q}_p) - \mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{S}_{12}\big)$$

and by postmultiplying $\mathfrak{Q}_q$ to the same equation, we get $\mathfrak{S}_{12}L_m(\mathfrak{Q}_p)\mathfrak{Q}_q = \mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{S}_{12}$. Then by using lemma 4, we have $L_m(\mathfrak{Q}_p) \in Z(\mathcal{A})$.
Knowing the fact that $\mathfrak{Q}_q\mathfrak{Q}_p = 0$, one can write

$$\begin{aligned}
0 &= L_m\Big(p_n\big(\mathfrak{Q}_q, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\big)\Big)\\
&= p_n\Big(L_m(\mathfrak{Q}_q), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) + p_n\Big(\mathfrak{Q}_q, L_m(\mathfrak{Q}_p), \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big)\\
&+ p_n\Big(\mathfrak{Q}_q, \mathfrak{Q}_p, L_m(\mathfrak{Q}_p), \cdots, \mathfrak{Q}_p\Big) + \cdots + p_n\Big(\mathfrak{Q}_q, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, L_m(\mathfrak{Q}_p)\Big)\\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m\\ 0\le l_1,l_2,\cdots,l_n\le m-1}} p_n\Big(L_{l_1}(\mathfrak{Q}_q), L_{l_2}(\mathfrak{Q}_p), L_{l_3}(\mathfrak{Q}_p), \cdots, L_{l_n}(\mathfrak{Q}_p)\Big)\\
&= p_n\Big(L_m(\mathfrak{Q}_q), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big)\\
&= (-1)^{n-1}\mathfrak{Q}_p L_m(\mathfrak{Q}_q)\mathfrak{Q}_q + \mathfrak{Q}_q L_m(\mathfrak{Q}_q)\mathfrak{Q}_p.
\end{aligned}$$

Which implies $\mathfrak{Q}_p L_m(\mathfrak{Q}_q)\mathfrak{Q}_q = \mathfrak{Q}_q L_m(\mathfrak{Q}_q)\mathfrak{Q}_p = 0$. Now using $p_n(\mathfrak{Q}_q, \mathfrak{S}_{12}, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p) = 0$ and applying the similar calculations as above, we obtain that $L_m(\mathfrak{Q}_q) \in Z(\mathcal{A})$. Therefore $L_m(\mathfrak{Q}_p), L_m(\mathfrak{Q}_q) \in Z(\mathcal{A})$ and hence, the lemma holds for all $m \in \mathbb{N}$. □

Lemma 7. $L_m(\mathcal{A}_{ij}) \subseteq \mathcal{A}_{ij}$, $(1 \le i \ne j \le 2)$.

Proof. We will show that $L_m(\mathcal{A}_{12}) \subseteq \mathcal{A}_{12}$. The other case i.e, $L_m(\mathcal{A}_{21}) \subseteq \mathcal{A}_{21}$ can be shown similarly. For $m = 1$, it is true by [3]. Now suppose that it holds for all $k \le m-1$. We will show that it also holds for $k = m$. Using Lemma 6 and equation (2.1), we have
$L_m(\mathfrak{S}_{12}) = \mathfrak{Q}_p L_m(\mathfrak{S}_{12})\mathfrak{Q}_q + (-1)^{n-1}\mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{Q}_p$. From this equation one can easily obtain $\mathfrak{Q}_p L_m(\mathfrak{S}_{12})\mathfrak{Q}_p = \mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{Q}_q = 0$ and if n is even, then $2\mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{Q}_p = 0$. But when n is odd, then for all $\mathfrak{S}_{12}, \mathfrak{T}_{12} \in \mathcal{A}_{12}$, as $\mathfrak{S}_{12}\mathfrak{T}_{12} = 0$, one can easily see that

$$\begin{aligned}
0 &= L_m\Big(p_n\big(\mathfrak{S}_{12}, \mathfrak{T}_{12}, \mathcal{W}_{12}, -\mathfrak{Q}_p, \cdots, -\mathfrak{Q}_p\big)\Big)\\
&= p_n\Big(L_m(\mathfrak{S}_{12}), \mathfrak{T}_{12}, \mathcal{W}_{12}, -\mathfrak{Q}_p, \cdots, -\mathfrak{Q}_p\Big) + p_n\Big(\mathfrak{S}_{12}, L_m(\mathfrak{T}_{12}), \mathcal{W}_{12}, -\mathfrak{Q}_p, \cdots, -\mathfrak{Q}_p\Big)\\
&+ p_n\Big(\mathfrak{S}_{12}, \mathfrak{T}_{12}, L_m(\mathcal{W}_{12}), -\mathfrak{Q}_p, \cdots, -\mathfrak{Q}_p\Big) + \cdots + p_n\Big(\mathfrak{S}_{12}, \mathfrak{T}_{12}, \mathcal{W}_{12}, -\mathfrak{Q}_p, \cdots, L_m(-\mathfrak{Q}_p)\Big)\\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m\\ 0\le l_1,l_2,\cdots,l_n\le m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{12}), L_{l_2}(\mathfrak{T}_{12}), L_{l_3}(\mathcal{W}_{12}), L_{l_4}(-\mathfrak{Q}_p), \cdots, L_{l_n}(-\mathfrak{Q}_p)\Big)\\
&= p_n\Big(L_m(\mathfrak{S}_{12}), \mathfrak{T}_{12}, \mathcal{W}_{12}, -\mathfrak{Q}_p, \cdots, -\mathfrak{Q}_p\Big) + p_n\Big(\mathfrak{S}_{12}, L_m(\mathfrak{T}_{12}), \mathcal{W}_{12}, -\mathfrak{Q}_p, \cdots, -\mathfrak{Q}_p\Big).
\end{aligned}$$

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Which can be written as $0 = \big[[L_m(\mathfrak{S}_{12}), \mathfrak{T}_{12}], \mathcal{W}_{12}\big] + \big[[\mathfrak{S}_{12}, L_m(\mathfrak{T}_{12})], \mathcal{W}_{12}\big]$. From this equation, we get $[L_m(\mathfrak{S}_{12}), \mathfrak{T}_{12}] + [\mathfrak{S}_{12}, L_m(\mathfrak{T}_{12})] \in Z(\mathcal{A})$. Now put $z = [L_m(\mathfrak{S}_{12}), \mathfrak{T}_{12}]$ $+ [\mathfrak{S}_{12}, L_m(\mathfrak{T}_{12})]$. Then

$$\begin{aligned}
[L_m(\mathfrak{S}_{12}), \mathfrak{T}_{12}] &= Z - [\mathfrak{S}_{12}, L_m(\mathfrak{T}_{12})] \\
&= Z - p_n\Big(\mathfrak{S}_{12}, -\mathfrak{Q}_p, \cdots, -\mathfrak{Q}_p, L_m(\mathfrak{T}_{12})\Big) \\
&= Z + L_m\Big(p_n\big(\mathfrak{S}_{12}, -\mathfrak{Q}_p, \cdots, -\mathfrak{Q}_p, \mathfrak{T}_{12}\big)\Big) - p_n\Big(L_m(\mathfrak{S}_{12}), -\mathfrak{Q}_p, \cdots, -\mathfrak{Q}_p, \mathfrak{T}_{12}\Big) \\
&= Z - p_n\Big(L_m(\mathfrak{S}_{12}), -\mathfrak{Q}_p, \cdots, -\mathfrak{Q}_p, \mathfrak{T}_{12}\Big) \\
&= z - [\mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{Q}_p, \mathfrak{T}_{12}].
\end{aligned}$$

This implies that $[\mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{Q}_p, \mathfrak{T}_{12}] \in Z(\mathcal{A})$ and therefore $\mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{Q}_p \mathfrak{T}_{12} = 0$. Since $\bar{\mathfrak{Q}}_q = I$, we have $\mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{Q}_p = 0$. Therefore, $L_m(\mathcal{A}_{12}) \subseteq \mathcal{A}_{12}$. Hence for all $m \in \mathbb{N}$, $L_m(\mathcal{A}_{12}) \subseteq \mathcal{A}_{12}$. □

Lemma 8. *There exists maps ζ_{m_i} on $\mathcal{A}_{ii}$, such that $L_m(\mathfrak{S}_{ii}) - \zeta_{m_i}(\mathfrak{S}_{ii})I \in \mathcal{A}_{ii}$ for any $\mathfrak{S}_{ii} \in \mathcal{A}, i = 1, 2$.*

Proof. Using Lemma 6 and knowing the fact that $\mathfrak{S}_{11}\mathfrak{Q}_q = 0$. We have

$$\begin{aligned}
0 &= L_m\Big(p_n\big(\mathfrak{S}_{11}, \mathfrak{Q}_q, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\big)\Big) \\
&= p_n\Big(L_m(\mathfrak{S}_{11}), \mathfrak{Q}_q, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + p_n\Big(\mathfrak{S}_{11}, L_m(\mathfrak{Q}_q), \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) \\
&+ p_n\Big(\mathfrak{S}_{11}, \mathfrak{Q}_q, L_m(\mathfrak{Q}_q), \cdots, \mathfrak{Q}_q\Big) + \cdots + p_n\Big(\mathfrak{S}_{11}, \mathfrak{Q}_q, \mathfrak{Q}_q, \cdots, L_m(\mathfrak{Q}_q)\Big) \\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0\le l_1, l_2, \cdots, l_n \le m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{11}), L_{l_2}(\mathfrak{Q}_q), L_{l_3}(\mathfrak{Q}_q), \cdots, L_{l_n}(\mathfrak{Q}_q)\Big) \\
&= p_n\Big(L_m(\mathfrak{S}_{11}), \mathfrak{Q}_q, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) \\
&= L_m(\mathfrak{S}_{11})\mathfrak{Q}_q + (-1)^{n-1}\mathfrak{Q}_q L_m(\mathfrak{S}_{11}) \\
&= \mathfrak{Q}_p L_m(\mathfrak{S}_{11})\mathfrak{Q}_q + (-1)^{n-1}\mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_p.
\end{aligned}$$

From which, we get $\mathfrak{Q}_p L_m(\mathfrak{S}_{11})\mathfrak{Q}_q = \mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_p = 0$. To complete the proof of lemma, we need to show that $\mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_q = 0$. For this take any $\mathfrak{S}_{22} \in \mathcal{A}_{22}$ and $\mathfrak{T}_{12} \in \mathcal{A}_{12}$, we have

$$\begin{aligned}
0 &= L_m\Big(p_n\big(\mathfrak{S}_{11}, \mathfrak{S}_{22}, \mathcal{W}_{12}, \mathfrak{Q}_q, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\big)\Big) \\
&= p_n\Big(L_m(\mathfrak{S}_{11}), \mathfrak{S}_{22}, \mathcal{W}_{12}, \mathfrak{Q}_q, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + p_n\Big(\mathfrak{S}_{11}, L_m(\mathfrak{S}_{22}), \mathcal{W}_{12}, \mathfrak{Q}_q, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) \\
&+ p_n\Big(\mathfrak{S}_{11}, \mathfrak{S}_{22}, L_m(\mathcal{W}_{12}), \mathfrak{Q}_q, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + \cdots + p_n\Big(\mathfrak{S}_{11}, \mathfrak{S}_{22}, \mathcal{W}_{12}, \mathfrak{Q}_q, \cdots, L_m(\mathfrak{Q}_q)\Big) \\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0\le l_1, l_2, \cdots, l_n \le m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{11}), L_{l_2}(\mathfrak{S}_{22}), L_{l_3}(\mathfrak{T}_{12}), L_{l_4}(\mathfrak{Q}_q), L_{l_5}(\mathfrak{Q}_q), \cdots, L_{l_n}(\mathfrak{Q}_q)\Big) \\
&= p_n\Big(L_m(\mathfrak{S}_{11}), \mathfrak{S}_{22}, \mathcal{W}_{12}, \mathfrak{Q}_q, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + p_n\Big(\mathfrak{S}_{11}, L_m(\mathfrak{S}_{22}), \mathcal{W}_{12}, \mathfrak{Q}_q, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) \\
&= \Big[[L_m(\mathfrak{S}_{11}), \mathfrak{S}_{22}], \mathfrak{T}_{12}\Big] + \Big[[\mathfrak{S}_{11}, L_m(\mathfrak{S}_{22})], \mathfrak{T}_{12}\Big].
\end{aligned}$$

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Which implies that $\left[L_m(\mathfrak{S}_{11}),\mathfrak{S}_{22}\right]+\left[\mathfrak{S}_{11},L_m(\mathfrak{S}_{22})\right]\in Z(\mathcal{A})$. By pre and post-multiplying $\mathfrak{Q}_q$, we get $\left[\mathfrak{Q}_qL_m(\mathfrak{S}_{11})\mathfrak{Q}_q,\mathfrak{S}_{22}\right]\in Z(\mathcal{A})\mathfrak{Q}_q$. This implies $\left[\mathfrak{Q}_qL_m(\mathfrak{S}_{11})\mathfrak{Q}_q,\mathfrak{S}_{22}\right]=0$. Which means there exists some $Z\in Z(\mathcal{A})$, such that $\mathfrak{Q}_qL_m(\mathfrak{S}_{11})\mathfrak{Q}_q=Z\mathfrak{Q}_q$ and therefore

$$L_m(\mathfrak{S}_{11})=\mathfrak{Q}_pL_m(\mathfrak{S}_{11})\mathfrak{Q}_p+\mathfrak{Q}_qL_m(\mathfrak{S}_{11})\mathfrak{Q}_q \tag{2}$$
$$=\mathfrak{Q}_pL_m(\mathfrak{S}_{11})\mathfrak{Q}_p-Z\mathfrak{Q}_p+Z. \tag{3}$$

Since, $Z\in Z(\mathcal{A})$, we have $\mathfrak{Q}_qZ\mathfrak{Q}_p=\mathfrak{Q}_pZ\mathfrak{Q}_q=0$. From the above equations $Z-Z^{'}=(\mathfrak{Q}_pZ\mathfrak{Q}_p+\mathfrak{Q}_qL_m(\mathfrak{S}_{11})\mathfrak{Q}_q)-(\mathfrak{Q}_pZ^{'}\mathfrak{Q}_p+\mathfrak{Q}_qL_m(\mathfrak{S}_{11}^{'})\mathfrak{Q}_q)$. Then by Lemma 3, $\mathfrak{Q}_p\mathcal{A}\mathfrak{Q}_p\cap Z(\mathcal{A})=\{0\}$ and thus $Z=Z^{'}$. One can also define a map ζ_{m_1} on $\mathcal{A}_{11}$ by $\zeta_{m_1}(\mathfrak{S}_{11})=Z\in Z(\mathcal{A})$. Then by comparing it with equation (2.3), we get $L_m(\mathfrak{S}_{11})-\zeta_{m_1}(\mathfrak{S}_{11})=\mathfrak{Q}_pL_m(\mathfrak{S}_{11})\mathfrak{Q}_p-\mathfrak{Q}_pZ\mathfrak{Q}_p\in\mathcal{A}_{11}$ for all $\mathfrak{S}_{11}\in\mathcal{A}_{11}$. With the similar steps there exists a map ζ_{m_2} on $\mathcal{A}_{22}$, such that $\zeta_{m_2}(\mathfrak{S}_{22})=Z\in Z(\mathcal{A})$ and $L_m(\mathfrak{S}_{22})-\zeta_{m_2}(\mathfrak{S}_{22})\in\mathcal{A}_{22}$ for all $\mathfrak{S}_{22}\in\mathcal{A}_{22}$. □

Now define two maps $\phi_m:\mathcal{A}\to\mathcal{A}$ by $\phi_m(\mathfrak{S})=L_m(\mathfrak{S})-\zeta_m(\mathfrak{S})$ and $\zeta_m:\mathcal{A}\to Z(\mathcal{A})$ by $\zeta_m(\mathfrak{S})=\zeta_{m_1}(\mathfrak{Q}_p\mathfrak{S}\mathfrak{Q}_p)+\zeta_{m_2}(\mathfrak{Q}_q\mathfrak{S}\mathfrak{Q}_q)$ for all $\mathfrak{S}\in\mathcal{A}$. Then one can easily observe that $\phi_m(\mathfrak{Q}_i)=0$, $\phi_m(\mathcal{A}_{ij})\subseteq\mathcal{A}_{ij}$; $i,j=1,2$ *and* $\phi_m(\mathfrak{S}_{ij})=L_m(\mathfrak{S}_{ij})$; *for all* $\mathfrak{S}_{ij}\in\mathcal{A}_{ij}$; $i,j=1,2(i\neq j)$.

Lemma 9. ϕ_m *is an additive map.*

Proof. As $\phi_m=L_m-\zeta_m$ *and* $\zeta_m=\zeta_{m_1}+\zeta_{m_2}$, we need to show that ζ_{m_1} *and* ζ_{m_2} are additive. For this take any $\mathfrak{S}_{11},\mathfrak{T}_{11}\in\mathcal{A}_{11}$, we have

$$\zeta_{m_1}(\mathfrak{S}_{11})=\mathfrak{Q}_p\zeta_{m_1}(\mathfrak{S}_{11})\mathfrak{Q}_p+\mathfrak{Q}_qL_m(\mathfrak{S}_{11})\mathfrak{Q}_q,$$

$$\zeta_{m_1}(\mathfrak{T}_{11})=\mathfrak{Q}_p\zeta_{m_1}(\mathfrak{T}_{11})\mathfrak{Q}_p+\mathfrak{Q}_qL_m(\mathfrak{T}_{11})\mathfrak{Q}_q$$

and

$$\zeta_{m_1}(\mathfrak{S}_{11}+\mathfrak{T}_{11})=\mathfrak{Q}_p\zeta_{m_1}(\mathfrak{S}_{11}+\mathfrak{T}_{11})\mathfrak{Q}_p+\mathfrak{Q}_qL_m(\mathfrak{S}_{11}+\mathfrak{T}_{11})\mathfrak{Q}_q.$$

By combining all the above three expressions, one can easily find that $\zeta_{m_1}(\mathfrak{S}_{11})+\zeta_{m_1}(\mathfrak{T}_{11})-\zeta_{m_1}(\mathfrak{S}_{11}+\mathfrak{T}_{11})=\mathfrak{Q}_p\zeta_{m_1}(\mathfrak{S}_{11})\mathfrak{Q}_p+\mathfrak{Q}_p\zeta_{m_1}(\mathfrak{T}_{11})\mathfrak{Q}_p-\mathfrak{Q}_p\zeta_{m_1}(\mathfrak{S}_{11}+\mathfrak{T}_{11})\mathfrak{Q}_p$. Since $\mathfrak{Q}_p\zeta_{m_1}(\mathfrak{S}_{11})\mathfrak{Q}_p+\mathfrak{Q}_p\zeta_{m_1}(\mathfrak{T}_{11})\mathfrak{Q}_p-\mathfrak{Q}_p\zeta_{m_1}(\mathfrak{S}_{11}+\mathfrak{T}_{11})\mathfrak{Q}_p\in Z(\mathcal{A})$ and $\mathfrak{Q}_p\zeta_{m_1}(\mathfrak{S}_{11})\mathfrak{Q}_p+\mathfrak{Q}_p\zeta_{m_1}(\mathfrak{T}_{11})\mathfrak{Q}_p-\mathfrak{Q}_p\zeta_{m_1}(\mathfrak{S}_{11}+\mathfrak{T}_{11})\mathfrak{Q}_p\in\mathfrak{Q}_p\mathcal{A}\mathfrak{Q}_p$, we know that $Z(\mathcal{A})\cap\mathfrak{Q}_p\mathcal{A}\mathfrak{Q}_p=\{0\}$, by Lemma 3. We have $\mathfrak{Q}_p\zeta_{m_1}(\mathfrak{S}_{11})\mathfrak{Q}_p+\mathfrak{Q}_p\zeta_{m_1}(\mathfrak{T}_{11})\mathfrak{Q}_p-\mathfrak{Q}_p\zeta_{m_1}(\mathfrak{S}_{11}+\mathfrak{T}_{11})\mathfrak{Q}_p=0$. Hence, $\zeta_{m_1}(\mathfrak{S}_{11}+\mathfrak{T}_{11})=\zeta_{m_1}(\mathfrak{S}_{11})+\zeta_{m_1}(\mathfrak{T}_{11})$. This implies ζ_{m_1} is additive. Similarly, we can show that ζ_{m_2} is additive. Therefore, ϕ_m is an additive map. □

Lemma 10. *For any* $\mathfrak{S}_{ii}\in\mathcal{A}_{ii}$, $\mathfrak{T}_{ij}\in\mathcal{A}_{ij}$, $\mathfrak{S}_{ij}\in\mathcal{A}_{ij}$ *and* $\mathfrak{T}_{jj}\in\mathcal{A}_{jj}$; $i,j=1,2,(i\neq j)$. *We have*
(a) $\phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ij})=\sum_{s+t=m}\phi_s(\mathfrak{S}_{ii})\phi_t(\mathfrak{T}_{ij})$,
(b) $\phi_m(\mathfrak{S}_{ij}\mathfrak{T}_{jj})=\sum_{s+t=m}\phi_s(\mathfrak{S}_{ij})\phi_t(\mathfrak{T}_{jj})$.

Proof. We will give the proof of (a) and second can be proved similarly. Lets us prove the lemma with the help of mathematical induction on m. For $m=1$ it is true by [3]. Suppose it is true for all $k\leq m-1$. We will show that it also holds for $k=m$. Since for $i\neq j,\mathfrak{T}_{ij}\mathfrak{S}_{ii}=0$. We have

$$\phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ij})=L_m(\mathfrak{S}_{ii}\mathfrak{T}_{ij})$$
$$=L_m\Big(p_n\big(\mathfrak{S}_{ii},\mathfrak{T}_{ij},P_j,P_j,\cdots,P_j\big)\Big)$$

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$$
\begin{aligned}
&= p_n\Big(L_m(\mathfrak{S}_{ii}), \mathfrak{T}_{ij}, P_j, P_j, \cdots, P_j\Big) + p_n\Big(\mathfrak{S}_{ii}, L_m(\mathfrak{T}_{ij}), P_j, P_j, \cdots, P_j\Big) \\
&+ p_n\Big(\mathfrak{S}_{ii}, \mathfrak{T}_{ij}, L_m(P_j), P_j, \cdots, P_j\Big) + \cdots + p_n\Big(\mathfrak{S}_{ii}, \mathfrak{T}_{ij}, P_j, \cdots, L_m(P_j)\Big) \\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0\le l_1,l_2,\cdots,l_n\le m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{ii}), L_{l_2}(\mathfrak{T}_{ij}), L_{l_3}(P_j), \cdots, L_{l_n}(P_j)\Big) \\
&= p_n\Big(L_m(\mathfrak{S}_{ii}), \mathfrak{T}_{ij}, P_j, P_j, \cdots, P_j\Big) + p_n\Big(\mathfrak{S}_{ii}, L_m(\mathfrak{T}_{ij}), P_j, P_j, \cdots, P_j\Big) \\
&+ \sum_{\substack{s+t=m \\ 0<s,t\le m-1}} L_s(\mathfrak{S}_{ii})L_t(\mathfrak{T}_{ij}) \\
&= \phi_m(\mathfrak{S}_{ii})\mathfrak{T}_{ij} + \mathfrak{S}_{ii}\phi_m(\mathfrak{T}_{ij}) + \sum_{\substack{s+t=m \\ 0<s,t\le m-1}} \phi_s(\mathfrak{S}_{ii})\phi_t(\mathfrak{T}_{ij}) \\
&= \sum_{s+t=m} \phi_s(\mathfrak{S}_{ii})\phi_t(\mathfrak{T}_{ij}).
\end{aligned}
$$

Similarly, one can prove $\phi_m(\mathfrak{S}_{ij}\mathfrak{T}_{jj}) = \sum_{s+t=m} \phi_s(\mathfrak{S}_{ij})\phi_t(\mathfrak{T}_{jj})$. □

Lemma 11. *For any* $\mathfrak{S}_{ii}, \mathfrak{T}_{ii} \in \mathcal{A}_{ii}$. *We have* $\phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ii}) = \sum_{s+t=m} \phi_s(\mathfrak{S}_{ii})\phi_t(\mathfrak{T}_{ii})$; $i = 1, 2$.

Proof. For any $\mathfrak{S}_{ii}, \mathfrak{T}_{ii} \in \mathcal{A}_{ii}$ *and* $\mathfrak{T}_{ij} \in \mathcal{A}_{ij}$, and using Lemma 10, we have

$$
\begin{aligned}
&\phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ii}\mathfrak{T}_{ij}) \\
&\quad = \mathfrak{S}_{ii}\mathfrak{T}_{ii}\phi_m(\mathfrak{T}_{ij}) + \phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ii})\mathfrak{T}_{ij} + \sum_{\substack{r+s=m \\ 0<r,s\le m-1}} \phi_r(\mathfrak{S}_{ii}\mathfrak{T}_{ii})\phi_s(\mathfrak{T}_{ij}) \\
&\quad = \mathfrak{S}_{ii}\mathfrak{T}_{ii}\phi_m(\mathfrak{T}_{ij}) + \phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ii})\mathfrak{T}_{ij} + \sum_{\substack{r+s=m \\ 0<r,s\le m-1}} \Bigg(\sum_{\substack{p+q=r \\ 0\le p,q\le r}} \phi_p(\mathfrak{S}_{ii})\phi_q(\mathfrak{T}_{ii})\Bigg)\phi_s(\mathfrak{T}_{ij})
\end{aligned}
$$

$$
\begin{aligned}
&= \mathfrak{S}_{ii}\mathfrak{T}_{ii}\phi_m(\mathfrak{T}_{ij}) + \phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ii})\mathfrak{T}_{ij} + \sum_{\substack{r+s=m \\ 0<r,s\le m-1}} \Bigg(\mathfrak{S}_{ii}\phi_r(\mathfrak{T}_{ii}) + \phi_r(\mathfrak{S}_{ii})\mathfrak{T}_{ii} \\
&+ \sum_{\substack{p+q=r \\ 0<p,q\le r-1}} \phi_p(\mathfrak{S}_{ii})\phi_q(\mathfrak{T}_{ii})\Bigg)\phi_s(\mathfrak{T}_{ij}) \\
&= \mathfrak{S}_{ii}\mathfrak{T}_{ii}\phi_m(\mathfrak{T}_{ij}) + \phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ii})\mathfrak{T}_{ij} + \sum_{\substack{r+s=m \\ 0<r,s\le m-1}} \mathfrak{S}_{ii}\phi_r(\mathfrak{T}_{ii})\phi_s(\mathfrak{T}_{ij}) \\
&+ \sum_{\substack{r+s=m \\ 0<r,s\le m-1}} \phi_r(\mathfrak{S}_{ii})\mathfrak{T}_{ii}\phi_s(\mathfrak{T}_{ij}) + \sum_{\substack{r+s=m \\ p+q=s \\ 0<r,s\le m-1 \\ 0<p,q\le r-1}} \phi_p(\mathfrak{S}_{ii})\phi_q(\mathfrak{T}_{ii})\phi_r(\mathfrak{T}_{ij}). \qquad (4)
\end{aligned}
$$

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On the otherhand, we have

$$\begin{aligned}
\phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ii}\mathfrak{T}_{ij}) &= \mathfrak{S}_{ii}\phi_m(\mathfrak{T}_{ii}\mathfrak{T}_{ij}) + \phi_m(\mathfrak{S}_{ii})\mathfrak{T}_{ii}\mathfrak{T}_{ij} + \sum_{\substack{r+s=m\\0<r,s\leq m-1}} \phi_r(\mathfrak{S}_{ii})\phi_s(\mathfrak{T}_{ii}\mathfrak{T}_{ij})\\
&= \mathfrak{S}_{ii}\phi_m(\mathfrak{T}_{ii}\mathfrak{T}_{ij}) + \phi_m(\mathfrak{S}_{ii})\mathfrak{T}_{ii}\mathfrak{T}_{ij} + \sum_{\substack{r+s=m\\0<r,s\leq m-1}} \phi_r(\mathfrak{S}_{ii})\Bigg(\sum_{\substack{p+q=s\\0\leq p,q\leq s}} \phi_p(\mathfrak{T}_{ii})\phi_q(\mathfrak{T}_{ij})\Bigg)\\
&= \mathfrak{S}_{ii}\phi_m(\mathfrak{T}_{ii}\mathfrak{T}_{ij}) + \phi_m(\mathfrak{S}_{ii})\mathfrak{T}_{ii}\mathfrak{T}_{ij} + \sum_{\substack{r+s=m\\0<r,s\leq m-1}} \phi_r(\mathfrak{S}_{ii})\Bigg(\mathfrak{T}_{ii}\phi_s(\mathfrak{T}_{ij}) + \phi_s(\mathfrak{T}_{ii})\mathfrak{T}_{ij}\\
&+ \sum_{\substack{p+q=s\\0<p,q\leq s-1}} \phi_p(\mathfrak{T}_{ii})\phi_q(\mathfrak{T}_{ij})\Bigg)\\
&= \mathfrak{S}_{ii}\phi_m(\mathfrak{T}_{ii}\mathfrak{T}_{ij}) + \phi_m(\mathfrak{S}_{ii})\mathfrak{T}_{ii}\mathfrak{T}_{ij} + \sum_{\substack{r+s=m\\0<r,s\leq m-1}} \phi_r(\mathfrak{S}_{ii})\mathfrak{T}_{ii}\phi_s(\mathfrak{T}_{ij})\\
&+ \sum_{\substack{r+s=m\\0<r,s\leq m-1}} \phi_r(\mathfrak{S}_{ii})\phi_s(\mathfrak{T}_{ii})\mathfrak{T}_{ij} + \sum_{\substack{r+s=m\\p+q=s\\0<r,s\leq m-1\\0<p,q\leq s-1}} \phi_r(\mathfrak{S}_{ii})\phi_p(\mathfrak{T}_{ii})\phi_q(\mathfrak{T}_{ij})\\
&= \mathfrak{S}_{ii}\phi_m(\mathfrak{T}_{ii})\mathfrak{T}_{ij} + \mathfrak{S}_{ii}\mathfrak{T}_{ii}\phi_m(\mathfrak{T}_{ij}) + \sum_{\substack{r+s=m\\0<r,s\leq m-1}} \mathfrak{S}_{ii}\phi_r(\mathfrak{T}_{ii})\phi_s(\mathfrak{T}_{ij})\\
&+ \phi_m(\mathfrak{S}_{ii})\mathfrak{T}_{ii}\mathfrak{T}_{ij} + \sum_{\substack{r+s=m\\0<r,s\leq m-1}} \phi_r(\mathfrak{S}_{ii})\mathfrak{T}_{ii}\phi_s(\mathfrak{T}_{ij})\\
&+ \sum_{\substack{r+s=m\\0<r,s\leq m-1}} \phi_r(\mathfrak{S}_{ii})\phi_s(\mathfrak{T}_{ii})\mathfrak{T}_{ij} + \sum_{\substack{r+s=m\\p+q=s\\0<r,s\leq m-1\\0<p,q\leq s-1}} \phi_r(\mathfrak{S}_{ii})\phi_p(\mathfrak{T}_{ii})\phi_q(\mathfrak{T}_{ij}). \qquad (5)
\end{aligned}$$

From equations (2.4) and (2.5), we obtain $\phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ii})\mathfrak{T}_{ij} = \mathfrak{S}_{ii}\phi_m(\mathfrak{T}_{ii})\mathfrak{T}_{ij} + \phi_m(\mathfrak{S}_{ii})\mathfrak{T}_{ii}\mathfrak{T}_{ij} + \sum_{\substack{r+s=m\\0<r,s\leq m-1}} \phi_r(\mathfrak{S}_{ii})\phi_s(\mathfrak{T}_{ii})\mathfrak{T}_{ij}$. Since $\bar{P}_i = I$, this follows from the fact $\{\mathcal{A}P_i(h) : h \in \mathcal{H}\}$ is dense in $\mathcal{H}$. Hence $\phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ii}) = \sum_{s+t=m} \phi_s(\mathfrak{S}_{ii})\phi_t(\mathfrak{T}_{ii})$ for all $\mathfrak{S}_{ii}, \mathfrak{T}_{ii} \in \mathcal{R};\ i = 1,2$. This completes the proof of the lemma. □

Lemma 12. *For any $\mathfrak{S}_{ij} \in \mathcal{A}_{ij}$ and $\mathfrak{T}_{ji} \in \mathcal{A}_{ji}$. We have* $\phi_m(\mathfrak{S}_{ij}\mathfrak{T}_{ji}) = \sum_{s+t=m} \phi_s(\mathfrak{S}_{ij})\phi_t(\mathfrak{T}_{ji})$; $i, j = 1,2, (i \neq j)$.

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Proof. To prove our lemma, we use the principle of mathematical induction. For $m = 1$ it is true by [3]. Suppose it holds for all $k \leq m$. We show that it also holds for $k = m$.
Since, for any $\mathfrak{S}_{12} \in \mathcal{A}_{12}$, $\mathfrak{S}_{12}\mathfrak{Q}_p = 0$ and $L_m(\mathfrak{Q}_q) \in Z(\mathfrak{T})$, so we have

$$\begin{aligned}
&L_m\Big(p_n\big(\mathfrak{S}_{12}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \mathfrak{T}_{21}\big)\Big)\\
&= p_n\Big(L_m(\mathfrak{S}_{12}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \mathfrak{T}_{21}\Big) + p_n\Big(\mathfrak{S}_{12}, L_m(\mathfrak{Q}_p), \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \mathfrak{T}_{21}\Big)\\
&+ p_n\Big(\mathfrak{S}_{12}, \mathfrak{Q}_p, L_m(\mathfrak{Q}_p), \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \mathfrak{T}_{21}\Big) + \cdots + p_n\Big(\mathfrak{S}_{12}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, L_m(\mathfrak{T}_{21})\Big)\\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m\\ 0<l_1,l_2,\cdots,l_n\leq m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{12}), L_{l_2}(\mathfrak{Q}_p), \cdots, L_{l_{n-1}}(\mathfrak{Q}_p), L_{l_n}(\mathfrak{T}_{21})\Big)\\
&= p_n\Big(L_m(\mathfrak{S}_{12}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \mathfrak{T}_{21}\Big) + p_n\Big(\mathfrak{S}_{12}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, L_m(\mathfrak{T}_{21})\Big)\\
&+ \sum_{\substack{s+t=m\\ 0<s,t\leq m-1}} \big[L_s(\mathfrak{S}_{12}), L_t(\mathfrak{T}_{21})\big]\\
&= p_n\Big(\phi_m(\mathfrak{S}_{12}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \mathfrak{T}_{21}\Big) + p_n\Big(\mathfrak{S}_{12}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \phi_m(\mathfrak{T}_{21})\Big)\\
&+ \sum_{\substack{s+t=m\\ 0<s,t\leq m-1}} \big[\phi_s(\mathfrak{S}_{12}), \phi_t(\mathfrak{T}_{21})\big].
\end{aligned}$$

This implies that $L_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12}) = \phi_m(\mathfrak{S}_{12})\mathfrak{T}_{21} - \mathfrak{T}_{21}\phi_m(\mathfrak{S}_{12}) + \mathfrak{S}_{12}\phi_m(\mathfrak{T}_{21})$ $- \phi_m(\mathfrak{T}_{21})\mathfrak{S}_{12} + \sum_{\substack{s+t=m\\ 0<s,t\leq m-1}} \big[\phi_s(\mathfrak{S}_{12}), \phi_t(\mathfrak{T}_{21})\big]$.
But, we know that for all $\mathfrak{S} \in \mathcal{A}$, $\phi_m(\mathfrak{S}) = L_m(\mathfrak{S}) - \zeta_m(\mathfrak{S})$. Therefore, one can easily arrive at

$$\begin{aligned}
&\phi_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12}) + \zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})\\
&\quad= \phi_m(\mathfrak{S}_{12})\mathfrak{T}_{21} - \mathfrak{T}_{21}\phi_m(\mathfrak{S}_{12}) + \mathfrak{S}_{12}\phi_m(\mathfrak{T}_{21}) - \phi_m(\mathfrak{T}_{21})\mathfrak{S}_{12} + \sum_{\substack{s+t=m\\ 0<s,t\leq m-1}} \phi_s(\mathfrak{S}_{12})\phi_t(\mathfrak{T}_{21})\\
&\quad- \sum_{\substack{s+t=m\\ 0<s,t\leq m-1}} \phi_t(\mathfrak{T}_{21})\phi_s(\mathfrak{S}_{12}).
\end{aligned}$$

Premultiplying the above equation by $\mathfrak{S}_{12}$ and usind Lemma 9, we get

$$\begin{aligned}
\mathfrak{S}_{12}\phi_m(\mathfrak{T}_{21}\mathfrak{S}_{12}) - \zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})\mathfrak{S}_{12} &= \mathfrak{S}_{12}\phi_m(\mathfrak{T}_{21})\mathfrak{S}_{12} + \mathfrak{S}_{12}\mathfrak{T}_{21}\phi_m(\mathfrak{S}_{12})\\
&+ \mathfrak{S}_{12} \sum_{\substack{s+t=m\\ 0<s,t\leq m-1}} \phi_t(\mathfrak{T}_{21})\phi_s(\mathfrak{S}_{12}) \qquad (6)
\end{aligned}$$

and post multiplying the same equation by $\mathfrak{S}_{12}$, we get

$$\begin{aligned}
\phi_m(\mathfrak{S}_{12}\mathfrak{T}_{21})\mathfrak{S}_{12} + \zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})\mathfrak{S}_{12} &= \phi_m(\mathfrak{S}_{12})\mathfrak{T}_{21}\mathfrak{S}_{12} + \mathfrak{S}_{12}\phi_m(\mathfrak{T}_{21})\mathfrak{S}_{12}\\
&+ \sum_{\substack{s+t=m\\ 0<s,t\leq m-1}} \phi_s(\mathfrak{S}_{12})\phi_t(\mathfrak{T}_{21})\mathfrak{S}_{12}. \qquad (7)
\end{aligned}$$

By comparing equations (2.6) and (2.7), we get

$$\begin{aligned}
&\mathfrak{S}_{12}\phi_m(\mathfrak{T}_{21}\mathfrak{S}_{12}) - \zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})\mathfrak{S}_{12} - \mathfrak{S}_{12}\mathfrak{T}_{21}\phi_m(\mathfrak{S}_{12})\\
&\qquad= \phi_m(\mathfrak{S}_{12}\mathfrak{T}_{21})\mathfrak{S}_{12} + \zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})\mathfrak{S}_{12} - \phi_m(\mathfrak{S}_{12})\mathfrak{T}_{21}\mathfrak{S}_{12}\\
&\qquad- \sum_{\substack{s+t=m\\ 0\leq s,t\leq m-1}} \phi_s(\mathfrak{S}_{12})\phi_t(\mathfrak{T}_{21})\mathfrak{S}_{12}. \qquad (8)
\end{aligned}$$

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Then, the application of Lemma 10, we get

$$\begin{aligned}\mathfrak{S}_{12}\phi_m(\mathfrak{T}_{21}\mathfrak{S}_{12}) + \phi_m(\mathfrak{S}_{12})\mathfrak{T}_{21}\mathfrak{S}_{12} &= \phi_m(\mathfrak{S}_{12}\mathfrak{T}_{21}\mathfrak{S}_{12})\\ &= \phi_m(\mathfrak{S}_{12}\mathfrak{T}_{21})\mathfrak{S}_{12} + \mathfrak{S}_{12}\phi_m(\mathfrak{T}_{21}\mathfrak{S}_{12}).\end{aligned}$$

Now, we prove that $\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12}) = 0$. For any $\mathfrak{S}_{12} \in \mathcal{A}_{12}$, let $\mathfrak{S}_{12} = V|\mathfrak{S}_{12}|$ be its polar decomposition. This implies that $\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})|\mathfrak{S}_{12}| = 0$ and thus $|\mathfrak{S}_{12}|\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})^* = 0$. Which follows that

$$\mathfrak{S}_{12}\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12}) = |\mathfrak{S}_{12}|\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})^* = 0. \tag{9}$$

On the other hand we similarly can show that

$$\mathfrak{T}_{21}\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12}) = 0. \tag{10}$$

Then by multiplying $\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})^*$ to equation (2.6) and using equations (2.9) and (2.10), we get

$$\phi_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})^* + \zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})^* = 0. \tag{11}$$

Now, by using Lemma 8 and equations (2.10) and (2.11), one can find that

$$\begin{aligned}&\phi_m(\mathfrak{S}_{12}\mathfrak{T}_{21})\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})^*\\ &= \phi_m\big(\mathfrak{S}_{12}\mathfrak{T}_{21}\mathfrak{Q}_p\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})^*\mathfrak{Q}_p\big) - \mathfrak{S}_{12}\mathfrak{T}_{21}\phi_m\big(\mathfrak{Q}_p\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})^*\mathfrak{Q}_p\big)\\ &= -\mathfrak{S}_{12}\mathfrak{T}_{21}\phi_m\big(\mathfrak{Q}_p(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})^*\mathfrak{Q}_p\big)\end{aligned}$$

and

$$\begin{aligned}&\phi_m(\mathfrak{T}_{21}\mathfrak{S}_{12})\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})^*\\ &= \phi_m\big(\mathfrak{T}_{12}\mathfrak{S}_{21}\mathfrak{Q}_q\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})^*\mathfrak{Q}_q\big) - \mathfrak{T}_{21}\mathfrak{S}_{12}\phi_m\big(\mathfrak{Q}_q\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})^*\mathfrak{Q}_q\big)\\ &= -\mathfrak{T}_{21}\mathfrak{S}_{12}\phi_m\big(\mathfrak{Q}_q(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})^*\mathfrak{Q}_q\big).\end{aligned}$$

Thus equation (2.13), implies that $\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})^*\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})^* = 0$ and hence $\phi_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12}) = 0$. Therefore from equations (2.6) and (2.7) and using Lemma 11, we get $\phi_m(\mathfrak{S}_{12}\mathfrak{T}_{21}) = \phi_m(\mathfrak{S}_{12})\mathfrak{T}_{21} + \mathfrak{S}_{12}\phi_m(\mathfrak{T}_{21}) + \sum_{\substack{s+t=m\\ 0<s,t\leq m-1}} \phi_s(\mathfrak{S}_{12})\phi_t(\mathfrak{T}_{21})$ i.e, $\phi_m(\mathfrak{S}_{12}\mathfrak{T}_{21}) = \sum_{\substack{s+t=m\\ 0\leq s,t\leq m}} \phi_s(\mathfrak{S}_{12})\phi_t(\mathfrak{T}_{21})$ and $\phi_m(\mathfrak{T}_{21}\mathfrak{S}_{12}) = \sum_{\substack{s+t=m\\ 0\leq s,t\leq m}} \phi_s(\mathfrak{T}_{21})\phi_t(\mathfrak{S}_{12})$ for all $\mathfrak{S}_{12} \in \mathcal{A}_{12}$, $\mathfrak{T}_{21} \in \mathcal{A}_{21}$. This proves that the lemma is also true for $k = m$. Hence, the lemma is true for all $m \in \mathbb{N}$. □

We have all the pieces to carry the proof of our first main result of this paper.

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Proof of Theorem 1. In view of Lemmas 10 - 12, one can easily, see that ϕ_m is an additive higher derivation and it can observe that $\zeta_m(\mathfrak{S}_{jj}) \in Z(\mathcal{A})$ for $j = 1, 2$ and $\zeta_m(\mathfrak{S}_{ji}) = 0$ for $j \neq i$. We now show that $\zeta_m\Big(p_n(\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \cdots, \mathfrak{S}_n)\Big) = 0$ for all $\mathfrak{S}_i \in \mathcal{A}$; $1 \leq i \leq n$.

$$\begin{aligned}
\zeta_m\Big(p_n(\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \cdots, \mathfrak{S}_n)\Big) \\
&= L_m\Big(p_n(\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \cdots, \mathfrak{S}_n)\Big) - \phi_m\Big(p_n(\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \cdots, \mathfrak{S}_n)\Big) \\
&= p_n(L_m(\mathfrak{S}_1), \mathfrak{S}_2, \mathfrak{S}_3, \cdots, \mathfrak{S}_n) + \cdots + p_n(\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \cdots, L_m(\mathfrak{S}_n)) \\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0\leq l_1, l_2, \cdots, l_n \leq m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_1, L_{l_2}(\mathfrak{S}_2), L_{l_3}(\mathfrak{S}_3), \cdots, L_{l_n}(\mathfrak{S}_n)\Big) \\
&- p_n(\phi_m(\mathfrak{S}_1), \mathfrak{S}_2, \mathfrak{S}_3, \cdots, \mathfrak{S}_n) - \cdots - p_n(\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \cdots, \phi_m(\mathfrak{S}_n)) \\
&- \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0\leq l_1, l_2, \cdots, l_n \leq m-1}} p_n\Big(\phi_{l_1}(\mathfrak{S}_1, \phi_{l_2}(\mathfrak{S}_2), \phi_{l_3}(\mathfrak{S}_3), \cdots, \phi_{l_n}(\mathfrak{S}_n)\Big)
\end{aligned}$$

$$\begin{aligned}
&= p_n(\phi_m(\mathfrak{S}_1), \mathfrak{S}_2, \mathfrak{S}_3, \cdots, \mathfrak{S}_n) + \cdots + p_n(\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \cdots, \phi_m(\mathfrak{S}_n)) \\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0\leq l_1, l_2, \cdots, l_n \leq m-1}} p_n\Big(\phi_{l_1}(\mathfrak{S}_1, \phi_{l_2}(\mathfrak{S}_2), \phi_{l_3}(\mathfrak{S}_3), \cdots, \phi_{l_n}(\mathfrak{S}_n)\Big) \\
&- p_n(\phi_m(\mathfrak{S}_1), \mathfrak{S}_2, \mathfrak{S}_3, \cdots, \mathfrak{S}_n) - \cdots - p_n(\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \cdots, \phi_m(\mathfrak{S}_n)) \\
&- \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0\leq l_1, l_2, \cdots, l_n \leq m-1}} p_n\Big(\phi_{l_1}(\mathfrak{S}_1, \phi_{l_2}(\mathfrak{S}_2), \phi_{l_3}(\mathfrak{S}_3), \cdots, \phi_{l_n}(\mathfrak{S}_n)\Big) \\
&= 0.
\end{aligned}$$

We can now conclude from the above observations if $L_m : \mathcal{A} \to \mathcal{A}$ is an additive Lie n-higher derivation, there exists an additive higher derivation ϕ_m of $\mathcal{A}$ and a map $\zeta_m : \mathcal{A} \to Z(\mathcal{A})$ that vanishes at $p_n\big(\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \cdots, \mathfrak{S}_n\big)$ with $\mathfrak{S}_1\mathfrak{S}_2 = 0$ for all $\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \cdots, \mathfrak{S}_n \in \mathcal{A}$, such that $L_m = \phi_m + \zeta_m$. □

Note that every additive derivation $d : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is an inner derivation [22]. Nowicki [17] proved that if every additive(linear) derivation of $\mathcal{A}$ is inner, then every additive (linear) higher derivation of $\mathcal{A}$ is inner [25]. Hence, by *Theorem* 2.1, the following corollary is immediate.

Corollary 1. *Let $\mathcal{A}$ be a von Neuman algebra with no central summands of type I_1 and a linear map $L_m : \mathcal{A} \to \mathcal{A}$ satisfying*

$$L_m\big(p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)\big) = \sum_{l_1+l_2+\cdots+l_n=m} p_n\big(L_{l_1}(\mathfrak{S}_1), L_{l_2}(\mathfrak{S}_2), \cdots, L_{l_n}(\mathfrak{S}_n)\big)$$

for all $\mathfrak{S}_i \in \mathcal{A}$; $1 \leq i \leq n$, with $\mathfrak{S}_1\mathfrak{S}_2 = 0$. Then there exists an operator $\mathcal{T} \in \mathcal{A}$ and a linear map ζ_m : $\mathcal{A} \to Z(\mathcal{A})$ which annihilates every $(n-1)$th commutator $p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)$ with $\mathfrak{S}_1\mathfrak{S}_2 = 0$ such that $L_m(\mathfrak{S}) = \mathfrak{S}\mathcal{T} - \mathcal{T}\mathfrak{S} + \zeta_m(\mathfrak{S})$ for all $\mathfrak{S} \in \mathcal{A}$.

Corollary 2. *Let $\mathcal{A}$ be a von Neuman algebra with no central summands of type I_1 and a linear map $L_m : \mathcal{A} \to \mathcal{A}$ satisfying*

$$L_m\big(p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)\big) = \sum_{l_1+l_2+\cdots+l_n=m} p_n\big(L_{l_1}(\mathfrak{S}_1), L_{l_2}(\mathfrak{S}_2), \cdots, L_{l_n}(\mathfrak{S}_n)\big)$$

for all $\mathfrak{S}_i \in \mathcal{A}$; $1 \leq i \leq n$. Then L_m is an additive lie higher derivation if and only if there exists an additive higher derivation $\phi_m : \mathcal{A} \to \mathcal{A}$ and an additive map $\zeta_m : \mathcal{A} \to Z(\mathcal{A})$ which annihilates every $(n-1)$th commutator $p_n(\mathfrak{S}_1, \mathfrak{S}_1, \cdots, \mathfrak{S}_1)$ such that $L_m(\mathfrak{S}) = \phi_m(\mathfrak{S}) + \zeta_m(\mathfrak{S})$ for all $\mathfrak{S} \in \mathcal{A}$.

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In the next segment, we study the characterization of Lie derivations on general von Neuman algebras having no central summands of type I_1 by action at projection product. Now, we state and prove second main result of this paper.

Theorem 2. *Let $\mathcal{A}$ be a von Neumann algebra with no central summands of type I_1 and an additive higher map $L_m : \mathcal{A} \to \mathcal{A}$ satisfying $L_m(p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)) = \sum_{l_1+l_2++l_n=m} p_n\Big(p_n(L_{l_1}(\mathfrak{S}_1), L_{l_2}(\mathfrak{S}_2), L_{l_3}(\mathfrak{S}_3), \cdots, L_{l_n}(\mathfrak{S}_n))\Big)$ for all $\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n \in \mathcal{A}$ with $\mathfrak{S}_1\mathfrak{S}_2 = P$, where P is a core-free projection with the central carrier I. Then there exists an additive higher derivation $\phi_m : \mathcal{A} \to \mathcal{A}$ and an additive higher map $\zeta_m : \mathcal{A} \to Z(\mathcal{A})$ that annihilates every $(n-1)$th commutator $p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)$ with $\mathfrak{S}_1\mathfrak{S}_2 = P$, such that $L_m(\mathfrak{S}) = \phi_m(\mathfrak{S}) + \zeta_m(\mathfrak{S})$ for all $\mathfrak{S} \in \mathcal{A}$.*

Let $\mathfrak{Q}_0 = \mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{Q}_q - \mathfrak{Q}_q L_m(\mathfrak{Q}_p)\mathfrak{Q}_p$ and let us define a map $\pi_m : \mathcal{A} \to \mathcal{A}$ as an inner higher derivation $\pi_m(\mathfrak{S}) = [\mathfrak{S}, \mathfrak{Q}_0]$ for all $\mathfrak{S} \in \mathcal{A}$. Clearly $L_m = L_m^{'} - \pi_m$ is also a Lie n-higher derivation. Since

$$\begin{aligned} L_m^{'}(\mathfrak{Q}_p) &= L_m(\mathfrak{Q}_p) - [\mathfrak{Q}_p, \mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{Q}_q - \mathfrak{Q}_q L_m(\mathfrak{Q}_p)\mathfrak{Q}_p] \\ &= L_m(\mathfrak{Q}_p) - \mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{Q}_q - \mathfrak{Q}_q L_m(\mathfrak{Q}_p)\mathfrak{Q}_p \\ &= \mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{Q}_p + \mathfrak{Q}_q L_m(\mathfrak{Q}_p)\mathfrak{Q}_q. \end{aligned}$$

One easily gets $\mathfrak{Q}_p L_m^{'}(\mathfrak{Q}_p)\mathfrak{Q}_q = \mathfrak{Q}_q L_m^{'}(\mathfrak{Q}_p)\mathfrak{Q}_p = 0$. Accordingly, it suffices to consider only those Lie n-higher derivations $L_m : \mathcal{A} \to \mathcal{A}$ which satisfy $\mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{Q}_q = \mathfrak{Q}_q L_m(\mathfrak{Q}_p)\mathfrak{Q}_p = 0$.

Lemma 13. $L_m(\mathfrak{Q}_p), L_m(\mathfrak{Q}_q) \in Z(\mathcal{A})$.

Proof. For $k = 1$, it is true by [3]. Suppose that it holds for all $k \le m-1$. We show it also holds for $k = m$. Since $(\mathfrak{S}_{12} + \mathfrak{Q}_p)\mathfrak{Q}_p = \mathfrak{Q}_p$ for all $\mathfrak{S}_{12} \in \mathcal{A}_{12}$. Therefore, we can write

$$\begin{aligned} &L_m\Big(p_n(\mathfrak{S}_{12} + \mathfrak{Q}_p), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) \\ &\quad = p_n\Big(L_m(\mathfrak{S}_{12} + \mathfrak{Q}_p), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) + p_n\Big(\mathfrak{S}_{12} + \mathfrak{Q}_p, L_m(\mathfrak{Q}_p), \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) \\ &\quad + p_n\Big(\mathfrak{S}_{12} + \mathfrak{Q}_p, \mathfrak{Q}_p, L_m(\mathfrak{Q}_p), \cdots, \mathfrak{Q}_p\Big) + \cdots + p_n\Big(\mathfrak{S}_{12} + \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, L_m(\mathfrak{Q}_p)\Big) \\ &\quad + \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0<l_1,l_2,\cdots,l_n\le m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{12} + \mathfrak{Q}_p), L_{l_2}(\mathfrak{Q}_p), L_{l_3}(\mathfrak{Q}_p), \cdots, L_{l_n}(\mathfrak{Q}_p)\Big) \\ &\quad = (-1)^{n-1}\mathfrak{Q}_p L_m(\mathfrak{S}_{12})\mathfrak{Q}_q + \mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{Q}_p \\ &\quad + (-1)^{n-2}(n-1)[\mathfrak{S}_{12}, L_m(\mathfrak{Q}_p)]. \end{aligned}$$

From which we get

$$\begin{aligned} L_m((-1)^{n-1}\mathfrak{S}_{12}) = (-1)^{n-1}\mathfrak{Q}_p L_m(\mathfrak{S}_{12})\mathfrak{Q}_q + \mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{Q}_p \\ + (-1)^{n-2}(n-1)[\mathfrak{S}_{12}, L_m(\mathfrak{Q}_p)]. \end{aligned} \quad (12)$$

Now by premultiplying $\mathfrak{Q}_p$ and postmultiplying $\mathfrak{Q}_q$ to above equation, one finds that

$$\mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{S}_{12} = \mathfrak{S}_{12} L_m(\mathfrak{Q}_p)\mathfrak{Q}_q.$$

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Since $\mathfrak{Q}_p L_m(\mathfrak{Q}_p)\mathfrak{Q}_q = \mathfrak{Q}_q L_m(\mathfrak{Q}_p)\mathfrak{Q}_p = 0$, by using above equation and lemma 4, we get $L_m(\mathfrak{Q}_p) \in Z(\mathcal{A})$. Now by using $(\mathfrak{Q}_q + \mathfrak{Q}_p)\mathfrak{Q}_p = \mathfrak{Q}_p$, it follows that

$$\begin{aligned} 0 &= L_m\Big(p_n\big(\mathfrak{Q}_q + \mathfrak{Q}_p, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\big)\Big) \\ &= p_n\Big(L_m(\mathfrak{Q}_q + \mathfrak{Q}_p), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) + p_n\Big(\mathfrak{Q}_q + \mathfrak{Q}_p, L_m(\mathfrak{Q}_p), \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) \\ &+ \cdots + p_n\Big(\mathfrak{Q}_q + \mathfrak{Q}_p, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, L_m(\mathfrak{Q}_p)\Big) + \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0<l_1,l_2,\cdots,l_n\leq m-1}} p_n\Big(L_{l_1}(\mathfrak{Q}_q + \mathfrak{Q}_p) \\ &, L_{l_2}(\mathfrak{Q}_p), L_{l_3}(\mathfrak{Q}_p), \cdots, L_{l_n}(\mathfrak{Q}_p)\Big). \end{aligned}$$

Which gives $0 = (-1)^{n-1}\mathfrak{Q}_p L_m(\mathfrak{Q}_q)\mathfrak{Q}_q + \mathfrak{Q}_q L_m(\mathfrak{Q}_q)\mathfrak{Q}_p$. It follows that $\mathfrak{Q}_p L_m(\mathfrak{Q}_q)\mathfrak{Q}_q = \mathfrak{Q}_q L_m(\mathfrak{Q}_q)\mathfrak{Q}_p = 0$. On the otherhand by using $p_n(\mathfrak{Q}_p + \mathfrak{S}_{12}, \mathfrak{Q}_q + \mathfrak{Q}_p - \mathfrak{S}_{12}, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p) = 0$ and making the similar calculations as above, one obtains that $L_m(\mathfrak{Q}_q) \in Z(\mathcal{A})$. Hence $L_m(\mathfrak{Q}_p), L_m(\mathfrak{Q}_q) \in Z(\mathcal{A})$ for all $m \in \mathbb{N}$. □

Lemma 14. $L_m(A_{ij}) \subseteq \mathcal{A}_{ij}$, $1 \leq i \neq j \leq 2$.

Proof. We prove the lemma with the help of principle of mathematical induction. For $m = 1$, it is true by [3]. Suppose that the lemma holds for all $k \leq m-1$. We will prove that it is also true for $k = m$. First consider the case for $i = 1$ *and* $j = 2$, the other case $i = 2$ *and* $j = 1$ will be proved in the similar way. By using equation (2.12) and $L_m(\mathfrak{Q}_p) \in Z(\mathcal{A})$, we have $L_m(\mathfrak{S}_{12}) = \mathfrak{Q}_p L_m(\mathfrak{S}_{12})\mathfrak{Q}_q + (-1)^{n-1}\mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{Q}_p$. By pre and post-multiplying $\mathfrak{Q}_p$ and $\mathfrak{Q}_q$ to the above equatioin, we get $\mathfrak{Q}_p L_m(\mathfrak{S}_{12})\mathfrak{Q}_p = 0$ *and* $\mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{Q}_q = 0$ respectively. Hence $\mathfrak{Q}_p L_m(\mathfrak{S}_{12})\mathfrak{Q}_p = \mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{Q}_q = 0$. Since $(\mathfrak{Q}_p + \mathfrak{S}_{12})\mathfrak{Q}_p = \mathfrak{Q}_p$, we can write

$$\begin{aligned} 0 &= L_m\Big(p_n\big(\mathfrak{Q}_p + \mathfrak{S}_{12}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \mathfrak{T}_{12}\big)\Big) \\ &= p_n\Big(L_m(\mathfrak{Q}_p + \mathfrak{S}_{12}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \mathfrak{T}_{12}\Big) + p_n\Big(\mathfrak{Q}_p + \mathfrak{S}_{12}, L_m(\mathfrak{Q}_p), \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \mathfrak{T}_{12}\Big) \\ &+ \cdots + \Big(\mathfrak{Q}_p + \mathfrak{S}_{12}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, L_m(\mathfrak{Q}_p), \mathfrak{T}_{12}\Big) + p_n\Big(\mathfrak{Q}_p + \mathfrak{T}_{12}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, L_m(\mathfrak{T}_{12})\Big) \\ &+ \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0\leq l_1,l_2,\cdots,l_n\leq m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{12}), L_{l_2}(\mathfrak{Q}_p), L_{l_3}(\mathfrak{Q}_p), \cdots, L_{l_{n-1}}(\mathfrak{Q}_p), L_{l_n}(\mathfrak{T}_{12})\Big) \\ &= p_n\Big(L_m(\mathfrak{Q}_p + \mathfrak{S}_{12}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, \mathfrak{T}_{12}\Big) + p_n\Big(\mathfrak{Q}_p + \mathfrak{T}_{12}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p, L_m(\mathfrak{T}_{12})\Big) \\ &+ \sum_{\substack{s+t=m \\ 0<s,t\leq m-1}} L_s(\mathfrak{S}_{12})L_t(\mathfrak{T}_{12}). \end{aligned}$$

This follows that

$$0 = \mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{T}_{12} - \mathfrak{T}_{12}L_m(\mathfrak{S}_{12})\mathfrak{Q}_p + (-1)^{n-1}L_m(\mathfrak{T}_{12})\mathfrak{S}_{12} - (-1)^{n-1}\mathfrak{S}_{12}L_m(\mathfrak{T}_{12}).$$

Then by multiplying $\mathfrak{Q}_q$ on both sides to the above equation, one obtains $\mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{T}_{12} = 0$ and by multiplying $\mathfrak{T}_{12}$ from right hand side and using $\mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{T}_{12} = 0$, we find that $\mathfrak{T}_{12}L_m(\mathfrak{S}_{12})\mathfrak{T}_{12} = 0$. Then by linearizing, we get $\mathfrak{T}_{12}L_m(\mathfrak{S}_{12})\mathcal{T}_{12} + \mathcal{T}_{12}L_m(\mathfrak{S}_{12})\mathfrak{T}_{12} = 0$ for all $\mathcal{T}_{12}, \mathfrak{T}_{12} \in \mathcal{A}_{12}$. Now it can be easily observed that

$$\begin{aligned} &\mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{T}_{12}L_m(\mathfrak{S}_{12})\Big[\mathfrak{T}_{12}L_m(\mathfrak{S}_{12})\mathcal{T}_{12}\Big]L_m(\mathfrak{S}_{12})\mathfrak{Q}_p \\ &+ \mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{T}_{12}L_m(\mathfrak{S}_{12})\Big[\mathcal{T}_{12}L_m(\mathfrak{S}_{12})\mathfrak{T}_{12}\Big]L_m(\mathfrak{S}_{12})\mathfrak{Q}_p = 0. \end{aligned}$$

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Which implies

$$\mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{T}_{12}L_m(\mathfrak{S}_{12})\mathcal{T}_{12}L_m(\mathfrak{S}_{12})\mathfrak{T}_{12}L_m(\mathfrak{S}_{12})\mathfrak{Q}_p = 0.$$

As $\mathcal{A}$ is semi prime, one can easily see that $\mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{T}_{12}L_m(\mathfrak{S}_{12})\mathfrak{Q}_p = 0$ and therefore $\mathfrak{Q}_q L_m(\mathfrak{S}_{12})\mathfrak{Q}_p = 0$. Hence $L_m(\mathcal{A}_{12}) \subseteq \mathcal{A}_{12}$, which shows that the lemma also holds for $k = m$. Therefore, $L_m(\mathcal{A}_{12}) \subseteq \mathcal{A}_{12}$ holds for all $m \in \mathbb{N}$. Similarly, we can easily prove that $L_m(\mathcal{A}_{21}) \subseteq \mathcal{A}_{21}$. □

Lemma 15. *There exists maps ζ_{m_i} on $\mathcal{A}_{ii}$ such that $L_m(\mathfrak{S}_{ii}) - \zeta_{m_i}(\mathfrak{S}_{ii}) \in \mathcal{A}_{ii}$ and $L_m(\mathfrak{S}_{ii}) \subseteq \mathcal{A}_{ii} + \mathcal{A}_{jj}$, for any $\mathfrak{S}_{ii} \in \mathcal{A}_{ii}$; $i = 1,2$ and $i \neq j$.*

Proof. We will prove the lemma with the help of principle of mathematical induction. For $m = 1$, it is true by [3]. Suppose that the lemma holds for all $k \leq m-1$. We will show that it also holds for $k = m$. Here, we give the proof for the case $i = 1$ and the proof for the case $i = 2$ follows the similar steps. Suppose $\mathfrak{S}_{11} \in \mathcal{A}_{11}$ is invertible, this implies that there exists $\mathfrak{S}_{11}^{-1} \in \mathcal{A}_{11}$, such that $\mathfrak{S}_{11}\mathfrak{S}_{11}^{-1} = \mathfrak{S}_{11}^{-1}\mathfrak{S}_{11} = \mathfrak{Q}_p$. Therefore, we can write

$$\begin{aligned}
0 &= L_m\Big(p_n(\mathfrak{S}_{11}^{-1}, \mathfrak{S}_{11}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p)\Big) \\
&= p_n\Big(L_m(\mathfrak{S}_{11}^{-1}), \mathfrak{S}_{11}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) + p_n\Big(\mathfrak{S}_{11}^{-1}, L_m(\mathfrak{S}_{11}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) \\
&+ p_n\Big(\mathfrak{S}_{11}^{-1}, \mathfrak{S}_{11}, L_m(\mathfrak{Q}_p), \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) + \cdots + p_n\Big(\mathfrak{S}_{11}^{-1}, \mathfrak{S}_{11}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, L_m(\mathfrak{Q}_p)\Big) \\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0\leq l_1,l_2,\cdots,l_n\leq m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{11}^{-1}), L_{l_2}(\mathfrak{S}_{11}), L_{l_3}(\mathfrak{Q}_p), L_{l_4}(\mathfrak{Q}_p), \cdots, L_{l_n}(\mathfrak{Q}_p)\Big).
\end{aligned}$$

Also since, $(\mathfrak{S}_{11}^{-1} + \mathfrak{Q}_q)\mathfrak{S}_{11} = \mathfrak{Q}_p$ and using Lemma 13, we have

$$\begin{aligned}
0 &= L_m\Big(p_n(\mathfrak{S}_{11}^{-1} + \mathfrak{Q}_q, \mathfrak{S}_{11}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p)\Big) \\
&= p_n\Big(L_m(\mathfrak{S}_{11}^{-1} + \mathfrak{Q}_q), \mathfrak{S}_{11}, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) + p_n\Big(\mathfrak{S}_{11}^{-1} + \mathfrak{Q}_q, L_m(\mathfrak{S}_{11}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) \\
&+ p_n\Big(\mathfrak{S}_{11}^{-1} + \mathfrak{Q}_q, \mathfrak{S}_{11}, L_m(\mathfrak{Q}_p), \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) + \cdots + p_n\Big(\mathfrak{S}_{11}^{-1} + \mathfrak{Q}_q, \mathfrak{S}_{11}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, L_m(\mathfrak{Q}_p)\Big) \\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0\leq l_1,l_2,\cdots,l_n\leq m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{11}^{-1} + \mathfrak{Q}_q), L_{l_2}(\mathfrak{S}_{11}), L_{l_3}(\mathfrak{Q}_p), L_{l_4}(\mathfrak{Q}_p), \cdots, L_{l_n}(\mathfrak{Q}_p)\Big) \\
&= p_n\Big(L_m(\mathfrak{S}_{11}^{-1}) + L_m(\mathfrak{Q}_q), \mathfrak{S}_{11}, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) + p_n\Big(\mathfrak{S}_{11}^{-1} + \mathfrak{Q}_q, L_m(\mathfrak{S}_{11}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) \\
&+ \sum_{l=3}^{n} p_n\Big(\mathfrak{S}_{11}^{-1} + \mathfrak{Q}_q, \mathfrak{S}_{11}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, L_m(\mathfrak{Q}_p)_l, \cdots, \mathfrak{Q}_p\Big) \\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0\leq l_1,l_2,\cdots,l_n\leq m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{11}^{-1}) + L_{l_1}(\mathfrak{Q}_q), L_{l_2}(\mathfrak{S}_{11}), L_{l_3}(\mathfrak{Q}_p), L_{l_4}(\mathfrak{Q}_p), \cdots, L_{l_n}(\mathfrak{Q}_p)\Big).
\end{aligned}$$

On comparing above two equations, we have
$0 = p_n\Big(\mathfrak{Q}_q, L_m(\mathfrak{S}_{11}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big) = \mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_p + (-1)^{n-1}\mathfrak{Q}_p L_m(\mathfrak{S}_{11})\mathfrak{Q}_q$.
It can easily observe from the above equation that $\mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_p = \mathfrak{Q}_p L_m(\mathfrak{S}_{11})\mathfrak{Q}_q = 0$, from which

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we get $L_m(\mathfrak{S}_{ii}) \subseteq \mathcal{A}_{ii} + \mathcal{A}_{jj}$. As $(\mathfrak{S}_{11}^{-1} + \mathfrak{T}_{22})\mathfrak{S}_{11} = \mathfrak{Q}_p$ and $(\mathfrak{S}_{11}^{-1})\mathfrak{S}_{11} = \mathfrak{Q}_p$. Then for any $\mathfrak{T}_{22} \in \mathcal{A}_{22}$ and $\mathcal{W}_{12} \in \mathcal{A}_{12}$, it can be easily seen that

$$\begin{aligned}
0 &= L_m\Big(p_n\big(\mathfrak{S}_{11}^{-1}, \mathfrak{S}_{11}, \mathcal{W}_{12}, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\big)\Big) \\
&= p_n\Big(L_m(\mathfrak{S}_{11}^{-1}), \mathfrak{S}_{11}, \mathcal{W}_{12}, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + p_n\Big(\mathfrak{S}_{11}^{-1}, L_m(\mathfrak{S}_{11}), \mathcal{W}_{12}, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) \\
&+ p_n\Big(\mathfrak{S}_{11}^{-1}, \mathfrak{S}_{11}, L_m(\mathcal{W}_{12}), \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + \cdots + p_n\Big(\mathfrak{S}_{11}^{-1}, \mathfrak{S}_{11}, \mathcal{W}_{12}, \mathfrak{Q}_q, \cdots, L_m(\mathfrak{Q}_q)\Big) \\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0\le l_1,l_2,\cdots,l_n \le m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{11}^{-1}), L_{l_2}(\mathfrak{S}_{11}), L_{l_3}(\mathcal{W}_{12}), L_{l_4}(\mathfrak{Q}_q), \cdots, L_{l_n}(\mathfrak{Q}_q)\Big). \qquad (13)
\end{aligned}$$

and

$$\begin{aligned}
0 &= L_m\Big(p_n\big(\mathfrak{S}_{11}^{-1} + \mathfrak{T}_{22}, \mathfrak{S}_{11}, \mathcal{W}_{12}, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\big)\Big) \\
&= p_n\Big(L_m(\mathfrak{S}_{11}^{-1} + \mathfrak{T}_{22}), \mathfrak{S}_{11}, \mathcal{W}_{12}, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + p_n\Big(\mathfrak{S}_{11}^{-1} + \mathfrak{T}_{22}, L_m(\mathfrak{S}_{11}), \mathcal{W}_{12}, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) \\
&+ p_n\Big(\mathfrak{S}_{11}^{-1} + \mathfrak{T}_{22}, \mathfrak{S}_{11}, L_m(\mathcal{W}_{12}), \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + \cdots + p_n\Big(\mathfrak{S}_{11}^{-1} + \mathfrak{T}_{22}, \mathfrak{S}_{11}, \mathcal{W}_{12}, \mathfrak{Q}_q, \cdots, L_m(\mathfrak{Q}_q)\Big) \\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0\le l_1,l_2,\cdots,l_n \le m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{11}^{-1} + \mathfrak{T}_{22}), L_{l_2}(\mathfrak{S}_{11}), L_{l_3}(\mathcal{W}_{12}), L_{l_4}(\mathfrak{Q}_q), \cdots, L_{l_n}(\mathfrak{Q}_q)\Big) \\
&= p_n\Big(L_m(\mathfrak{S}_{11}^{-1}) + L_m(\mathfrak{T}_{22}), \mathfrak{S}_{11}, \mathcal{W}_{12}, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + p_n\Big(\mathfrak{S}_{11}^{-1} + \mathfrak{T}_{22}, L_m(\mathfrak{S}_{11}), \mathcal{W}_{12}, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) \\
&+ p_n\Big(\mathfrak{S}_{11}^{-1} + \mathfrak{T}_{22}, \mathfrak{S}_{11}, L_m(\mathcal{W}_{12}), \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + \cdots + p_n\Big(\mathfrak{S}_{11}^{-1} + \mathfrak{T}_{22}, \mathfrak{S}_{11}, \mathcal{W}_{12}, \mathfrak{Q}_q, \cdots, L_m(\mathfrak{Q}_q)\Big) \\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0\le l_1,l_2,\cdots,l_n \le m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{11}^{-1}) + L_{l_1}(\mathfrak{T}_{22}), L_{l_2}(\mathfrak{S}_{11}), L_{l_3}(\mathcal{W}_{12}), L_{l_4}(\mathfrak{Q}_q), \cdots, L_{l_n}(\mathfrak{Q}_q)\Big) \qquad (14)
\end{aligned}$$

Comparing equations (2.13) and (2.14), we obtain

$$\begin{aligned}
0 &= p_n\Big(L_m(\mathfrak{T}_{22}), \mathfrak{S}_{11}, \mathcal{W}_{12}, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + p_n\Big(\mathfrak{T}_{22}, L_m(\mathfrak{S}_{11}), \mathcal{W}_{12}, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) \\
&= p_{n-1}\Big(\big[L_m(\mathfrak{T}_{22}), \mathfrak{S}_{11}\big], \mathcal{W}_{12}, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + p_{n-1}\Big(\big[\mathfrak{T}_{22}, L_m(\mathfrak{S}_{11})\big], \mathcal{W}_{12}, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) \\
&= \Big[\big[L_m(\mathfrak{T}_{22}), \mathfrak{S}_{11}\big], \mathcal{W}_{12}\Big] + \Big[\big[\mathfrak{T}_{22}, L_m(\mathfrak{S}_{11})\big], \mathcal{W}_{12}\Big].
\end{aligned}$$

Which leads to $\big[L_m(\mathfrak{T}_{22}), \mathfrak{S}_{11}\big] + \big[\mathfrak{T}_{22}, L_m(\mathfrak{S}_{11})\big] \in Z(\mathcal{A})$. Then multiplying the above equation both sides by $\mathfrak{Q}_q$, one arrives at $\big[\mathfrak{T}_{22}, \mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_q\big] \in Z(\mathcal{A})\mathfrak{Q}_q$ and therefore $\big[\mathfrak{T}_{22}, \mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_q\big] = 0$. This implies that there exists some $Z \in Z(\mathcal{A})$ such that $\mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_q = Z\mathfrak{Q}_q$. If $\mathfrak{S}_{11}$ is not invertible in $\mathcal{A}_{11}$, then one can find a sufficiently large number say r in a way such that $r\mathfrak{Q}_p - \mathfrak{S}_{11}$ is invertible in $\mathcal{A}_{11}$. Following the preceeding case $\mathfrak{Q}_p L_m(r\mathfrak{Q}_p - \mathfrak{S}_{11})\mathfrak{Q}_q + \mathfrak{Q}_q L_m(r\mathfrak{Q}_p - \mathfrak{S}_{11})\mathfrak{Q}_p = 0$ and $\mathfrak{Q}_q L_m(r\mathfrak{Q}_p - \mathfrak{S}_{11})\mathfrak{Q}_q = Z\mathfrak{Q}_q$. As $L_m(\mathfrak{Q}_p) \in Z(\mathcal{A})$, we have $\mathfrak{Q}_p L_m(\mathfrak{S}_{11})\mathfrak{Q}_q + \mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_p = 0$ and $\mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_q \in Z(\mathcal{A})\mathfrak{Q}_q$. Without loss of generality, we denote $\mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_q = Z\mathfrak{Q}_q$. Therefore for any $\mathfrak{S}_{11} \in \mathcal{A}_{11}$, we have

$$\begin{aligned}
L_m(\mathfrak{S}_{11}) &= \mathfrak{Q}_p L_m(\mathfrak{S}_{11})\mathfrak{Q}_p + \mathfrak{Q}_q L_m(\mathfrak{S}_{11})\mathfrak{Q}_q \\
&= \mathfrak{Q}_p L_m(\mathfrak{S}_{11})\mathfrak{Q}_p - Z\mathfrak{Q}_p + Z.
\end{aligned}$$

We define a map say ζ_{m_1} on $\mathcal{A}_{11}$ by $\zeta_{m_1}(\mathfrak{S}_{11}) = Z$ and then by combining it with the above equation, we get $L_m(\mathfrak{S}_{11}) - \zeta_{m_1}(\mathfrak{S}_{11}) = \mathfrak{Q}_p L_m(\mathfrak{S}_{11})\mathfrak{Q}_p - \mathfrak{Q}_p \zeta_{m_1}(\mathfrak{S}_{11})\mathfrak{Q}_p \in \mathcal{A}_{11}$ for any $\mathfrak{S}_{11} \in \mathcal{A}_{11}$. Hence, the lemma is true for $k = m$. Therefore, the lemma is true for all $m \in \mathbb{N}$. For the case when $i = 2$, we take

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$(\mathfrak{Q}_p + \mathfrak{T}_{22})\mathfrak{Q}_p = \mathfrak{Q}_p$ to get $\mathfrak{Q}_q L_m(\mathfrak{T}_{22})\mathfrak{Q}_p + (-1)^{n-1}\mathfrak{Q}_p L_m(\mathfrak{T}_{22})\mathfrak{Q}_q = 0$ and then following the similar steps as that for $i = 1$, we find that

$$L_m(\mathfrak{T}_{22}) - \zeta_{m_2}(\mathfrak{T}_{22}) = \mathfrak{Q}_q L_m(\mathfrak{T}_{22})\mathfrak{Q}_q - \mathfrak{Q}_q\zeta_{m_1}(\mathfrak{T}_{22})\mathfrak{Q}_q \in \mathcal{A}_{22}$$

for any $\mathfrak{T}_{22} \in \mathcal{A}_{22}$, and completes the proof of the lemma. □

We now define maps $\phi_m : \mathcal{A} \to \mathcal{A}$ and $\zeta_m : \mathcal{A} \to Z(\mathcal{A})$ by $\phi_m\mathfrak{S} = L_m\mathfrak{S} - \zeta_m\mathfrak{S}$ and $\zeta_m\mathfrak{S} = \zeta_{m_1}(\mathfrak{Q}_p\mathfrak{S}\mathfrak{Q}_p) + \zeta_{m_2}(\mathfrak{Q}_q\mathfrak{S}\mathfrak{Q}_q)$ for all $\mathfrak{S} \in \mathcal{A}$. One can easily observe that $\phi_m(P_i) = 0$, $\phi_m(\mathcal{A}_{ij}) \subseteq \mathcal{A}_{ij}$, $i, j = 1, 2$ *and* $\phi_m(\mathfrak{S}_{ij}) = L_m(\mathfrak{S}_{ij})$ *for all* $\mathfrak{S}_{ij} \in \mathcal{A}_{ij}, 1 \leq i \neq j \leq 2$.

Lemma 16. *ϕ_m is an additive map.*

Proof. The proof is similar to that of Lemma 9. □

Lemma 17. *For any $\mathfrak{S}_{ii} \in \mathcal{A}_{ii}$, $\mathfrak{S}_{ij}, \mathfrak{T}_{ij} \in \mathcal{A}_{ij}$ and $\mathfrak{T}_{jj} \in \mathcal{A}_{jj}$, $i, j = 1, 2$, $(i \neq j)$, we have,*
(a) $\phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ij}) = \sum_{\substack{s+t=m \\ 0\leq s,t\leq m}} \phi_s(\mathfrak{S}_{ii})\phi_t(\mathfrak{T}_{ij})$,
(b) $\phi_m(\mathfrak{S}_{ij}\mathfrak{T}_{jj}) = \sum_{\substack{s+t=m \\ 0\leq s,t\leq m}} \phi_s(\mathfrak{S}_{ij})\phi_t(\mathfrak{T}_{jj})$.

Proof. (a) We prove it with the help of principle of mathematical induction. For $m = 1$, it is true by [3]. Suppose that it holds for all $k \leq m - 1$. We will prove that it is also true for $k = m$. We take the case for $i = 1$ *and* $j = 2$. If $\mathfrak{S}_{11} \in \mathcal{A}_{11}$ is invertible, then for any $\mathcal{W}_{12} \in \mathcal{A}_{12}$, we have $(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12} + \mathfrak{S}_{11}^{-1})\mathfrak{S}_{11} = \mathfrak{Q}_p$. Therefore, we have

$$\begin{aligned}
&\phi_m(\mathcal{W}_{12}) = L_m\Big(p_n\big(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12} + \mathfrak{S}_{11}^{-1}, \mathfrak{S}_{11}, \mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\big)\Big)\\
&= p_n\Big(L_m(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12} + \mathfrak{S}_{11}^{-1}), \mathfrak{S}_{11}, \mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + p_n\Big(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12} + \mathfrak{S}_{11}^{-1}, L_m(\mathfrak{S}_{11})\\
&, \mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + p_n\Big(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12} + \mathfrak{S}_{11}^{-1}, \mathfrak{S}_{11}, L_m(\mathfrak{Q}_p), \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + \cdots\\
&+ p_n\Big(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12} + \mathfrak{S}_{11}^{-1}, \mathfrak{S}_{11}, \mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, L_m(\mathfrak{Q}_q)\Big) + \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0\leq l_1,l_2,\cdots,l_n\leq m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12} + \mathfrak{S}_{11}^{-1}),\\
&L_{l_2}(\mathfrak{S}_{11}), L_{l_3}(\mathfrak{Q}_p), L_{l_4}(\mathfrak{Q}_q), \cdots, L_{l_n}(\mathfrak{Q}_q)\Big)\\
&= p_n\Big(L_m(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12}) + L_m(\mathfrak{S}_{11}^{-1}), \mathfrak{S}_{11}, \mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + p_n\Big(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12} + \mathfrak{S}_{11}^{-1}, L_m(\mathfrak{S}_{11}), \mathfrak{Q}_p\\
&, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + \cdots + p_n\Big(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12} + \mathfrak{S}_{11}^{-1}, \mathfrak{S}_{11}, \mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, L_m(\mathfrak{Q}_q)\Big)\\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0\leq l_1,l_2,\cdots,l_n\leq m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12}) + L_{l_1}(\mathfrak{S}_{11}^{-1}), L_{l_2}(\mathfrak{S}_{11}), L_{l_3}(\mathfrak{Q}_p), L_{l_4}(\mathfrak{Q}_q), \cdots, L_{l_n}(\mathfrak{Q}_q)\Big),
\end{aligned}$$

and

$$\begin{aligned}
0 &= p_n\Big(L_m(\mathfrak{S}_{11}^{-1}), \mathfrak{S}_{11}, \mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + p_n\Big(\mathfrak{S}_{11}^{-1}, L_m(\mathfrak{S}_{11}), \mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big)\\
&+ p_n\Big(\mathfrak{S}_{11}^{-1}, \mathfrak{S}_{11}, L_m(\mathfrak{Q}_p), \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + \cdots + p_n\Big(\mathfrak{S}_{11}^{-1}, \mathfrak{S}_{11}, \mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, L_m(\mathfrak{Q}_q)\Big)\\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m \\ 0\leq l_1,l_2,\cdots,l_n\leq m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{11}^{-1}), L_{l_2}(\mathfrak{S}_{11}), L_{l_3}(\mathfrak{Q}_p), L_{l_4}(\mathfrak{Q}_q), \cdots, L_{l_n}(\mathfrak{Q}_q)\Big)
\end{aligned}$$

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Since, $L_m(\mathfrak{Q}_p), L_m(\mathfrak{Q}_q) \in Z(\mathcal{A})$ and ϕ_m is additive. From the above two equations we obtain

$$\begin{aligned}
&\phi_m(\mathcal{W}_{12})\\
&= p_n\Big(L_m(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12}), \mathfrak{S}_{11}, \mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + p_n\Big(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12}, L_m(\mathfrak{S}_{11}), \mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big)\\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m\\ 0\leq l_1,l_2,\cdots,l_n\leq m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12}), L_{l_2}(\mathfrak{S}_{11}), L_{l_3}(\mathfrak{Q}_p), L_{l_4}(\mathfrak{Q}_q), \cdots, L_{l_n}(\mathfrak{Q}_q)\Big)\\
&= p_n\Big(\phi_m(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12}), \mathfrak{S}_{11}, \mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big) + p_n\Big(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12}, \phi_m(\mathfrak{S}_{11}), \mathfrak{Q}_p, \mathfrak{Q}_q, \cdots, \mathfrak{Q}_q\Big)\\
&+ \sum_{\substack{s+t=m\\ 0<s,t\leq m-1}} \phi_t(\mathfrak{S}_{11})\phi_s(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12})\\
&= \phi_m(\mathfrak{S}_{11})\mathfrak{S}_{11}^{-1}\mathcal{W}_{12} + \mathfrak{S}_{11}\phi_m(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12}) + \sum_{\substack{s+t=m\\ 0<s,t\leq m-1}} \phi_t(\mathfrak{S}_{11})\phi_s(\mathfrak{S}_{11}^{-1}\mathcal{W}_{12}).
\end{aligned}$$

By replacing $\mathcal{W}_{12}$ with $\mathfrak{S}_{11}\mathfrak{T}_{12}$ in the above equation, we get

$$\begin{aligned}
\phi_m(\mathfrak{S}_{11}\mathfrak{T}_{12}) &= \phi_m(\mathfrak{S}_{11})\mathfrak{T}_{12} + \mathfrak{S}_{11}\phi_m(\mathfrak{T}_{12}) + \sum_{\substack{s+t=m\\ 0<s,t\leq m-1}} \phi_s(\mathfrak{T}_{12})\phi_t(\mathfrak{S}_{11})\\
&= \sum_{\substack{s+t=m\\ 0\leq s,t\leq m}} \phi_t(\mathfrak{S}_{11})\phi_s(\mathfrak{T}_{12}).
\end{aligned}$$

Now, if $\mathfrak{S}_{11}$ is not invertible in $\mathcal{A}_{11}$, we can find a sufficiently large number say r such that $r\mathfrak{Q}_p - \mathfrak{S}_{11}$ is invertible in $\mathcal{A}_{11}$. Then $\phi_m\big((r\mathfrak{Q}_p - \mathfrak{S}_{11})\mathfrak{T}_{12}\big) = (r\mathfrak{Q}_p - \mathfrak{S}_{11})\phi_m(\mathfrak{T}_{12}) + \phi_m(r\mathfrak{Q}_p - \mathfrak{S}_{11})\mathfrak{T}_{12}$. Since $\mathfrak{Q}_p$ is invertible in $\mathcal{A}_{11}$, therefore from the above equation we get

$$\phi_m(\mathfrak{S}_{11}\mathfrak{T}_{12}) = \sum_{\substack{s+t=m\\ 0\leq s,t\leq m-1}} \phi_t(\mathfrak{S}_{11})\phi_s(\mathfrak{T}_{12}).$$

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For $i = 2$ *and* $j = 1$, sinse $(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21})(\mathfrak{Q}_p + \mathfrak{T}_{21}) = \mathfrak{Q}_p$. We have

$$\begin{aligned}
-\phi_m(\mathfrak{T}_{21}) &= L_m\Big(p_n\big(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21}, \mathfrak{Q}_p + \mathfrak{T}_{21}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\big)\Big)\\
&= p_n\Big(L_m(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21}), \mathfrak{Q}_p + \mathfrak{T}_{21}, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big)\\
&+ p_n\Big(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21}, L_m(\mathfrak{Q}_p + \mathfrak{T}_{21}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big)\\
&+ \cdots + p_n\Big(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21}, \mathfrak{Q}_p + \mathfrak{T}_{21}, \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, L_m(\mathfrak{Q}_p)\Big)\\
&+ \sum_{\substack{l_1+l_2+\cdots+l_n=m\\ 0\le l_1,l_2,\cdots,l_n\le m-1}} p_n\Big(L_{l_1}(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21}), L_{l_2}(\mathfrak{Q}_p + \mathfrak{T}_{21}), L_{l_3}(\mathfrak{Q}_p), L_{l_4}(\mathfrak{Q}_p),\\
&\cdots, L_{l_n}(\mathfrak{Q}_p)\Big)\\
&= p_n\Big(L_m(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21}), \mathfrak{Q}_p + \mathfrak{T}_{21}, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big)\\
&+ p_n\Big(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21}, L_m(\mathfrak{Q}_p + \mathfrak{T}_{21}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big)\\
&+ \sum_{\substack{s+t=m\\ 0<s,t\le m-1}} p_n\Big(L_s(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21}), L_t(\mathfrak{Q}_p + \mathfrak{T}_{21}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big)\\
&= p_n\Big(\phi_m(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21}), \mathfrak{Q}_p + \mathfrak{T}_{21}, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big)\\
&+ p_n\Big(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21}, \phi_m(\mathfrak{Q}_p + \mathfrak{T}_{21}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big)\\
&+ \sum_{\substack{s+t=m\\ 0<s,t\le m-1}} p_n\Big(\phi_s(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21}), \phi_t(\mathfrak{Q}_p + \mathfrak{T}_{21}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big).
\end{aligned}$$

Since ϕ_m is additive. Therefore from above equation we get

$$\begin{aligned}
&-\phi_m(\mathfrak{T}_{21}) = p_n\Big(\phi_m(\mathfrak{Q}_p) + \phi_m(\mathfrak{S}_{22}) - \phi_m(\mathfrak{S}_{22}\mathfrak{T}_{21}), \mathfrak{Q}_p + \mathfrak{T}_{21}, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big)\\
&+ p_n\Big(\mathfrak{Q}_p + \mathfrak{S}_{22} - \mathfrak{S}_{22}\mathfrak{T}_{21}, \phi_m(\mathfrak{Q}_p) + \phi_m(\mathfrak{T}_{21}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big)\\
&+ \sum_{\substack{s+t=m\\ 0<s,t\le m-1}} p_n\Big(\phi_s(\mathfrak{Q}_p) + \phi_s(\mathfrak{S}_{22}) - \phi_s(\mathfrak{S}_{22}\mathfrak{T}_{21}), \phi_t(\mathfrak{Q}_p) + \phi_t(\mathfrak{T}_{21}), \mathfrak{Q}_p, \mathfrak{Q}_p, \cdots, \mathfrak{Q}_p\Big)\\
&= -\phi_m(\mathfrak{S}_{22}\mathfrak{T}_{21}) + \phi_m(\mathfrak{S}_{22})\mathfrak{T}_{21} + \mathfrak{S}_{22}\phi_m(\mathfrak{T}_{21}) - \phi_m(\mathfrak{T}_{21}) + \sum_{\substack{s+t=m\\ 0<s,t\le m-1}} \phi_s(\mathfrak{S}_{22})\phi_t(\mathfrak{T}_{21}).
\end{aligned}$$

Which follows that

$$\phi_m(\mathfrak{S}_{22}\mathfrak{T}_{21}) = \phi_m(\mathfrak{S}_{22})\mathfrak{T}_{21} + \mathfrak{S}_{22}\phi_m(\mathfrak{T}_{21}) + \sum_{\substack{s+t=m\\ 0<s,t\le m-1}} \phi_s(\mathfrak{S}_{22})\phi_t(\mathfrak{T}_{21})$$

i.e, $\phi_m(\mathfrak{S}_{22}\mathfrak{T}_{21}) = \sum_{\substack{s+t=m\\ 0\le s,t\le m}} \phi_s(\mathfrak{S}_{22})\phi_t(\mathfrak{T}_{21})$ for all $\mathfrak{S}_{22} \in \mathcal{A}_{22}$ and $\mathfrak{T}_{21} \in \mathcal{A}_{21}$.

(b) For $i = 1$, $j = 2$ by considering $(\mathfrak{Q}_p + \mathfrak{S}_{12})(\mathfrak{Q}_p - \mathfrak{T}_{22} + \mathfrak{S}_{12}\mathfrak{T}_{22}) = \mathfrak{Q}_p$ and using the same approach as above, one can easily obtain $\phi_m(\mathfrak{S}_{12}\mathfrak{T}_{22}) = \sum_{\substack{s+t=m\\ 0\le s,t\le m}} \phi_s(\mathfrak{S}_{12})\phi_t(\mathfrak{T}_{22})$ for all $\mathfrak{S}_{12} \in \mathcal{A}_{12}$ and $\mathfrak{T}_{22} \in \mathcal{A}_{22}$ and for the case when $i = 2$, $j = 1$, by considering $\mathfrak{S}_{11}(\mathcal{W}_{21}\mathfrak{S}_{11}^{-1} + \mathfrak{S}_{11}^{-1}) = \mathfrak{Q}_p$, we can easily prove that $\phi_m(\mathfrak{S}_{21}\mathfrak{T}_{11}) = \sum_{\substack{s+t=m\\ 0\le s,t\le m}} \phi_s(\mathfrak{S}_{21})\phi_t(\mathfrak{T}_{11})$ for all $\mathfrak{S}_{21} \in \mathcal{A}_{21}$ *and* $\mathfrak{T}_{11} \in \mathcal{A}_{11}$. □

Lemma 18. *For any* $\mathfrak{S}_{ii}$, $\mathfrak{T}_{ii} \in \mathcal{A}_{ii}$, *we have,* $\phi_m(\mathfrak{S}_{ii}\mathfrak{T}_{ii}) = \phi_m(\mathfrak{S}_{ii})\mathfrak{T}_{ii} + \mathfrak{S}_{ii}\phi_m(\mathfrak{T}_{ii})$, $i = 1, 2$.

Proof. The proof of this lemma is same as that of Lemma 11. □

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Lemma 19. *For any* $\mathfrak{S}_{ij} \in \mathcal{A}_{ij}$, $\mathfrak{T}_{ji} \in \mathcal{A}_{ji}$, *we have*, $\phi_m(\mathfrak{S}_{ij}\mathfrak{T}_{ji}) = \phi_m(\mathfrak{S}_{ij})\mathfrak{T}_{ji} + \mathfrak{S}_{ij}\phi_m(\mathfrak{T}_{ji})$; $1 \leq i \neq j \leq 2$.

Proof. We prove the lemma with the help of principle of mathematical induction. For $m = 1$, it is true by [3]. Suppose that the lemma holds for all $k \leq m-1$. We will prove that it is true for $k = m$. Take any $\mathfrak{S}_{12} \in \mathcal{A}_{12}$, since $(\mathfrak{S}_{12} + \mathfrak{Q}_p)\mathfrak{Q}_p = \mathfrak{Q}_p$. Then

$$\begin{aligned}
&L_m\Big(p_n\big(\mathfrak{S}_{12}+\mathfrak{Q}_p,\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p,\mathfrak{T}_{21}\big)\Big)\\
&= p_n\Big(L_m(\mathfrak{S}_{12}+\mathfrak{Q}_p),\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p,\mathfrak{T}_{21}\Big) + p_n\Big(\mathfrak{S}_{12}+\mathfrak{Q}_p,L_m(\mathfrak{Q}_p),\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p,\mathfrak{T}_{21}\Big)\\
&+\cdots+p_n\Big(\mathfrak{S}_{12}+\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,L_m(\mathfrak{Q}_p),\mathfrak{T}_{21}\Big) + p_n\Big(\mathfrak{S}_{12}+\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p,L_m(\mathfrak{T}_{21})\Big)\\
&+\sum_{\substack{l_1+l_2+\cdots+l_n=m\\0\leq l_1,l_2,\cdots,l_n\leq m-1}} p_n\Big(L_{l_1}(\mathfrak{S}_{12}+\mathfrak{Q}_p),L_{l_2}(\mathfrak{Q}_p),L_{l_3}(\mathfrak{Q}_p),\cdots,L_{l_{n-1}}(\mathfrak{Q}_p),L_{l_n}(\mathfrak{T}_{21})\Big)\\
&= p_n\Big(L_m(\mathfrak{S}_{12}+\mathfrak{Q}_p),\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p,\mathfrak{T}_{21}\Big) + p_n\Big(\mathfrak{S}_{12}+\mathfrak{Q}_p,\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p,L_m(\mathfrak{T}_{21})\Big)\\
&+\sum_{\substack{s+t=m\\0<s,t\leq m-1}} L_s(\mathfrak{S}_{12}+\mathfrak{Q}_p)L_t(\mathfrak{T}_{21})\\
&= p_n\Big(\phi_m(\mathfrak{S}_{12}+\mathfrak{Q}_p),\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p,\mathfrak{T}_{21}\Big) + p_n\Big(\mathfrak{S}_{12}+\mathfrak{Q}_p,\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p,\phi_m(\mathfrak{T}_{21})\Big)\\
&+\sum_{\substack{s+t=m\\0<s,t\leq m-1}} \phi_s(\mathfrak{S}_{12}+\mathfrak{Q}_p)\phi_t(\mathfrak{T}_{21})\\
&= p_n\Big(\phi_m(\mathfrak{S}_{12}),\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p,\mathfrak{T}_{21}\Big) + p_n\Big(\mathfrak{S}_{12},\mathfrak{Q}_p,\mathfrak{Q}_p,\cdots,\mathfrak{Q}_p,\phi_m(\mathfrak{T}_{21})\Big)\\
&+\sum_{\substack{s+t=m\\0<s,t\leq m-1}} \phi_s(\mathfrak{S}_{12})\phi_t(\mathfrak{T}_{21}).
\end{aligned}$$

From this we get, $L_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12}) = \phi_m(\mathfrak{S}_{12})\mathfrak{T}_{21} + \mathfrak{S}_{12}\phi_m(\mathfrak{T}_{21}) - \phi_m(\mathfrak{T}_{21})\mathfrak{S}_{12} - \mathfrak{T}_{21}\phi_m(\mathfrak{S}_{12}) + \sum_{\substack{s+t=m\\0<s,t\leq m-1}} \phi_s(\mathfrak{S}_{12})\phi_t(\mathfrak{T}_{21})$. Since $\phi_m(\mathfrak{S}) = L_m(\mathfrak{S}) - \zeta_{m_1}(\mathfrak{Q}_p\mathfrak{S}\mathfrak{Q}_p) - \zeta_{m_2}(\mathfrak{Q}_q\mathfrak{S}\mathfrak{Q}_q)$ for all $\mathfrak{S} \in \mathcal{A}$. We have

$$\begin{aligned}
&\phi_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12}) + \zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12})\\
&= \phi_m(\mathfrak{S}_{12})\mathfrak{T}_{21} + \mathfrak{S}_{12}\phi_m(\mathfrak{T}_{21}) - \phi_m(\mathfrak{T}_{21})\mathfrak{S}_{12} - \mathfrak{T}_{21}\phi_m(\mathfrak{S}_{12})\\
&+\sum_{\substack{s+t=m\\0<s,t\leq m-1}} \phi_s(\mathfrak{S}_{12})\phi_t(\mathfrak{T}_{21}).
\end{aligned}$$

Using Lemma 12 and applying the similar steps to obtain $\zeta_m(\mathfrak{S}_{12}\mathfrak{T}_{21} - \mathfrak{T}_{21}\mathfrak{S}_{12}) = 0$. Then from the above relation, we get

$$\begin{aligned}
\phi_m(\mathfrak{S}_{12}\mathfrak{T}_{21}) &= \phi_m(\mathfrak{S}_{12})\mathfrak{T}_{21} + \mathfrak{S}_{12}\phi_m(\mathfrak{T}_{21}) + \sum_{\substack{s+t=m\\0<s,t\leq m-1}} \phi_s(\mathfrak{S}_{12})\phi_t(\mathfrak{T}_{21})\\
&= \sum_{\substack{s+t=m\\0\leq s,t\leq m}} \phi_s(\mathfrak{S}_{12})\phi_t(\mathfrak{T}_{21})
\end{aligned}$$

and

$$\phi_m(\mathfrak{T}_{21}\mathfrak{S}_{12}) = \phi_m(\mathfrak{T}_{21})\mathfrak{S}_{12} + \mathfrak{T}_{21}\phi_m(\mathfrak{S}_{12}) + \sum_{\substack{s+t=m \\ 0<s,t\leq m-1}} \phi_s(\mathfrak{T}_{21})\phi_t(\mathfrak{S}_{12})$$
$$= \sum_{\substack{s+t=m \\ 0\leq s,t\leq m}} \phi_s(\mathfrak{T}_{21})\phi_t(\mathfrak{S}_{12})$$

for all $\mathfrak{S}_{12} \in \mathcal{A}_{12}$, $\mathfrak{T}_{21} \in \mathcal{A}_{12}$. This shows that the lemma is utrue for all $m \in \mathbb{N}$. □

Proof of Theorem 2. is similar as that of Theorem 1. □

As a direct consequence of Theorem 2, we have the corollary :

Corollary 3. *Let $\mathcal{A}$ be a von Neuman algebra with no central summands of type i_1 and a linear map $L_m : \mathcal{A} \to \mathcal{A}$ satisfying*

$$L_m\big(p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)\big) = \sum_{l_1+l_2+\cdots+l_n=m} p_n\big(L_{l_1}(\mathfrak{S}_1), L_{l_2}(\mathfrak{S}_2), \cdots, L_{l_n}(\mathfrak{S}_n)\big)$$

for all $\mathfrak{S}_i \in \mathcal{A}$; $1 \leq i \leq n$, with $\mathfrak{S}_1\mathfrak{S}_2 = \mathfrak{Q}_p$, where $\mathfrak{Q}_p$ is a core free projection with thje central carrier I. Then there exists an operator $\mathcal{T} \in \mathcal{A}$ and a linear map $\zeta_m : \mathcal{A} \to Z(\mathcal{A})$ which annihilates every $(n-1)$th commutator $p_n(\mathfrak{S}_1, \mathfrak{S}_2, \cdots, \mathfrak{S}_n)$ with $\mathfrak{S}_1\mathfrak{S}_2 = \mathfrak{Q}_p$ such that $L_m(\mathfrak{S}) = \mathfrak{S}\mathcal{T} - \mathcal{T}\mathfrak{S} + \zeta_m(\mathfrak{S})$ for all $\mathfrak{S} \in \mathcal{A}$.

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