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Article

Calculation of Restoring and Repulsive Magnetic Forces between two Non-coaxial Coils of Rectangular Cross-Section with Parallel Axes

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Abstract: In this paper, we introduce new formulas for calculating the restoring and repulsive magnetic forces between two non-coaxial circular loops with parallel axes. These formulas are derived from a modified version of Grover's formula for mutual inductance between the coils in question. Utilizing these formulas, we compute the restoring and repulsive magnetic forces between two non-coaxial thick coils of rectangular cross-sections with parallel axes. In these calculations, we apply the filament method and conduct investigations to determine the optimal number of subdivisions for the coils in terms of computational time and accuracy. The method presented in this paper is also applicable to all conventional non-coaxial coils, such as disks, solenoids, and non-conventional coils like Bitter coils, all with parallel axes. This paper emphasizes the accuracy and computational efficiency of the calculations. Furthermore, the new method is validated according to several previously established methods

Keywords: mutual inductance; restoring force; repulsive force; coils of rectangular cross-section with parallel axes

1. Introduction

The calculation of mutual inductance and magnetic force between coils of various shapes and positions has been the subject of numerous studies [1-4]. Most of these calculations are presented in analytical form for coils with simple configurations.

Analytical and semi-analytical methods for calculating self and mutual inductances of conducting elements in electrical circuits, as well as magnetic force interactions between these elements, have emerged as powerful mathematical tools [5-12]. These methods have been instrumental in advancing power transfer, wireless communication, sensing and actuation technologies, and have found applications across a wide range of scientific disciplines, including electrical and electronic engineering, medicine, physics, nuclear magnetic resonance, mechatronics, and robotics, among others.

While there are several efficient numerical methods available in commercially developed software, analytical and semi-analytical methods offer the advantage of providing calculation results in the form of a final formula with a finite number of input parameters. This feature can significantly reduce computational effort when applicable. In this paper, we present new formulas for calculating the restoring (radial) and repulsive (axial) magnetic forces between two non-coaxial circular loops with parallel axes. Using these formulas and the filament method, we calculate the forces between two non-coaxial coils of rectangular cross-sections with parallel axes. We emphasize the importance of accuracy and computational efficiency by selecting different numbers of subdivisions for the coils.

Our analysis demonstrates that using the same number of subdivisions is not preferable due to significant computational time. Therefore, we choose varying numbers of subdivisions that considerably reduce computational time without compromising accuracy, which is crucial from an engineering perspective.

The results are compared with those obtained and presented in terms of line integrals [13]. These semi-analytical expressions may not be familiar to most engineers, who may require simpler expressions for practical use. In this paper, we provide straightforward formulas that also utilize single integrals for Maxwell coils, as employed in the filament method. This approach is highly accessible and suitable for professionals, including engineers and physicists, working in this field. Utilizing programming languages such as MATLAB, or Mathematica these simple formulas can be easily implemented.

We also employ another method [5], where the single integral is replaced by the summation of its kernel function over very small segments within the integration interval $[0, \pi]$. This method achieves considerably reduced computational time with satisfactory accuracy. All methods are validated according to the methods given in and [14].

2. Basic Expressions

In this paper we use the general formula for calculating the mutual inductance between two circular loops with angular misalignment, whose axes intersect but not at the center either of one, to calculate the restoring and the repulsive magnetic force between them (See Fig1) [1],

M

$$= 2 \frac{\mu_0}{\pi} \sqrt{R_p R_s} \int_0^\pi \frac{\left[\cos\theta - \frac{d}{R_s} \cos\varphi \right] \phi(k)}{k \sqrt{V^3}} d\varphi \quad (1)$$

wherein,

$$\alpha = \frac{R_s}{R_p}, \quad \beta = \frac{c}{R_p}, \quad V =$$

$$= \sqrt{1 - \cos^2\varphi \sin^2\theta - 2 \frac{d}{R_s} \cos\varphi \cos\theta + \frac{d^2}{R_s^2}}$$

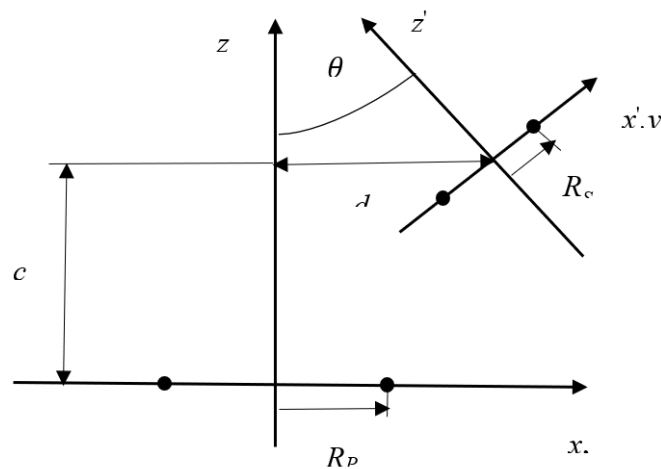


Figure 1. Filamentary circular loops with angular misalignment (axes intersect but not at the center either of one)

$$k^2 = \frac{4\alpha V}{(1 + \alpha V)^2 + \zeta^2}, \quad \zeta = \beta - \alpha \cos \varphi \sin \theta$$

R_P and R_S - are the radii of the primary and secondary loops in (m), respectively.

d - is the perpendicular distance between the axes of coils in (m).

c - is the axial distance between the centers of coils in (m).

θ - is angle in radian, between unit vector normal of the inclined loop positioned at its center and the unit vector of the axis z of the primary loop.

$\mu_0 = 4\pi \cdot 10^{-7} \text{ H/m}$ is the permeability of vacuum.

In (1) angle θ takes the value $0 \leq \theta \leq \pi/2$.

In Song at al. gave the modified Grover's formula (1) (when $\theta = 0$), for calculating the mutual inductance between two non-coaxial loops with the parallel axes

$$M =$$

$$\frac{\mu_0}{\pi} \sqrt{R_P} \int_0^\pi \frac{r^2 + R_S^2 - d^2}{k\sqrt{r^3}} \phi(k) d\varphi$$

wherein,

$$r = \sqrt{R_S^2 - 2R_S d \cos(\varphi) + d^2}, \quad k^2 = \frac{4R_P r}{(R_P + r)^2 + c^2}$$

The restoring (radial) and the repulsive (axial) magnetic force can be obtained by [3],

$$F_{\text{restoring}}$$

$$= F_r = I_P I_S \frac{\partial M}{\partial d} \quad (3)$$

$$F_{\text{propulsive}} = F_a$$

$$= I_P I_S \frac{\partial M}{\partial c} \quad (4)$$

According the force between two filaments is one of the attractions if the currents in them are in the same direction around the common axis. If the currents in the two filaments are in opposite directions around the common axis, the force between them is one of repulsion.

Using (2), (3) and (4) we have,

$$\begin{aligned} F_r = \frac{\mu_0 I_P I_S R_S \sqrt{R_P}}{\pi} \int_0^\pi [R_S d (\cos^2 \varphi - 3) + (R_S^2 \\ + d^2) \cos \varphi] \frac{\phi(k)}{k\sqrt{r^7}} d\varphi - \\ \frac{\mu_0 I_P I_S R_S}{4\pi \sqrt{R_P}} \int_0^\pi (R_S \cos \varphi - d)(R_S - d \cos \varphi)(R_P^2 + c^2 \\ - r^2) \frac{k\psi(k)}{\sqrt{r^9}} d\varphi \end{aligned} \quad (5)$$

$$F_a = -\frac{\mu_0 I_P I_S c R_S}{2\pi \sqrt{R_P}} \int_0^\pi \frac{k(R_S - d \cos \varphi) \psi(k)}{\sqrt{r^5}} d\varphi \quad (6)$$

wherein

$$\psi(k) = \frac{1 - \frac{k^2}{2}}{1 - k^2} E(k) - K(k)$$

$E(k)$ and $K(k)$ are the complete elliptic function of the first and the second kind [17-18].

3. Filament Method for Calculating the Magnetic force F_r and F_a between two non-Coaxial Coils of Rectangular Cross-Section with Parallel Axes

Let us treat two non-coaxial coils of rectangular cross-section with the parallel axes (See Figure 2).

Using the filament method [6], and the formulas for the mutual inductance M as well as for the magnetic forces F_r and F_a are as follows,

$$M = \frac{N_1 N_2}{(2K+1)(2N+1)(2m+1)(2n+1)} \sum_{g=-K}^{g=K} \sum_{h=-N}^{h=N} \sum_{p=-m}^{p=m} \sum_{l=-n}^{l=n} M(g, h, p, l) \quad (7)$$

and

$$F_r = \frac{N_1 N_2}{(2K+1)(2N+1)(2m+1)(2n+1)} \sum_{g=-K}^{g=K} \sum_{h=-N}^{h=N} \sum_{p=-m}^{p=m} \sum_{l=-n}^{l=n} F_r(g, h, p, l) \quad (8)$$

$$F_a = \frac{N_1 N_2}{(2K+1)(2N+1)(2m+1)(2n+1)} \sum_{g=-K}^{g=K} \sum_{h=-N}^{h=N} \sum_{p=-m}^{p=m} \sum_{l=-n}^{l=n} F_a(g, h, p, l) \quad (9)$$

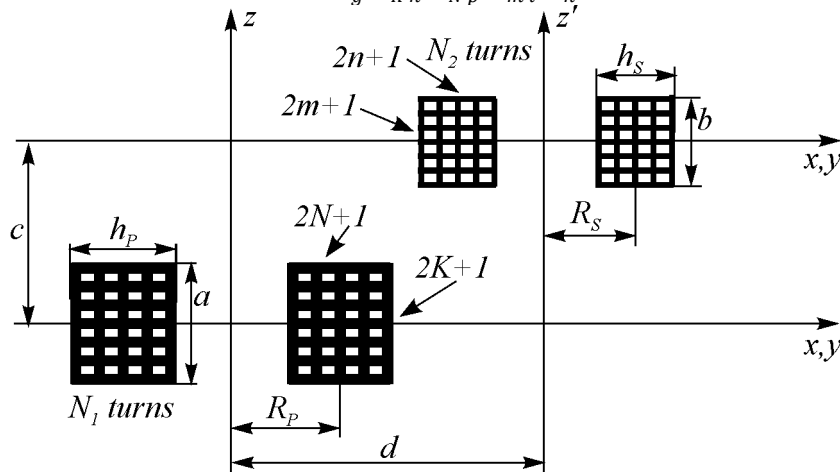


Figure 2. Configuration of mesh matrix: Two non-coaxial coils of rectangular cross-section with the parallel axes.

$$r = \sqrt{R_s^2 - 2R_s d \cos(\varphi) + d^2}$$

$$R_I = \frac{R_1 + R_2}{2}, \quad R_{II} = \frac{R_3 + R_4}{2}, \quad a = h_I = R_2 - R_1, \quad b = h_{II} = R_4 - R_3$$

$$R_p(h) = R_I + \frac{h_I}{2N+1} h, \quad (h = -N, \dots, 0, \dots, N)$$

$$R_s(l) = R_{II} + \frac{h_{II}}{2n+1} l, \quad (l = -n, \dots, 0, \dots, n)$$

$$z(g, p) = c + \frac{a}{2K+1} g - \frac{b}{2m+1} p, \quad (g = -K, \dots, 0, \dots, K; \quad p = -m, \dots, 0, \dots, m)$$

$$k^2(h, l, p, g) = \frac{4R_P(h)r(l)}{(R_P(h) + r(l))^2 + z^2(g, p)}$$

$$r(l) = \sqrt{R_S^2(l) - 2R_S(l)d\cos(\varphi) + d^2}$$

$$M(g, h, p, l) = \frac{\mu_0}{\pi} \sqrt{R_P(h)} \int_0^\pi \frac{r^2(l) + R_S^2(l) - d^2}{k(h, l, p, g)\sqrt{r^3(l)}} \phi(k) d\varphi$$

$$F_r = \frac{\mu_0 I_P I_S}{\pi} \int_0^\pi [R_S(l)d(\cos^2\varphi - 3) + (R_S^2(l) + d^2)\cos\varphi] R_S(l) \sqrt{R_P(h)} \frac{\phi(k)}{k\sqrt{r^7(l)}} d\varphi -$$

$$\frac{\mu_0 I_P I_S}{4\pi} \int_0^\pi (R_S(l)\cos\varphi - d)(R_S(l) - d\cos\varphi)(R_P^2(h) + z^2(g, p) - r^2(l)) \frac{R_S(l)k\psi(k)}{\sqrt{R_P(h)}\sqrt{r^9(l)}} d\varphi$$

$$F_a = -\frac{\mu_0 I_P I_S}{2\pi} \int_0^\pi \frac{z(g, p)R_S(l)k(R_S - d\cos\varphi)\psi(k)}{\sqrt{R_P(h)}\sqrt{r^5(l)}} d\varphi$$

$$\phi(k) = \left(1 - \frac{k^2}{2}\right) K(k) - E(k), \quad \psi(k) = \frac{1 - \frac{k^2}{2}}{(1 - k^2)} E(k) - K(k)$$

I_1 – current in the primary coil.

I_2 – current in the secondary coil.

N_1 – number of turns in the primary coil.

N_2 – number of turns in the secondary coil.

R_1 – inner radius of the primary coil of rectangular cross section.

R_2 – outer radius of the primary coil of rectangular cross section.

R_3 – inner radius of the secondary coil of rectangular cross section.

R_4 – outer radius of the secondary coil of rectangular cross section.

a and b – heights of the primary and the secondary coil, respectively.

d – perpendicular distance between axes of coils.

c – distance between the plans of centers of coils.

$R_P(h)$ – average radius of the primary coil positioned in the plane (x, y) whose axis is 'z' axis.

$R_S(l)$ – inclined average radius of the secondary coil

K, N, n , and m – the number of the subdivisions of thick coils.

4. Examples

In this example we calculate the mutual inductance, the restoring and the propulsive magnetic force as a function of the displacement of two non-coaxial loops with the parallel axes where we have: $R_p = 42.5$ mm, $R_s = 20$ mm. The perpendicular distance between coils axes is $d = 3$ mm, [10].

Clearly, we obtained identical results because the formulas are the same; however, equation (2) is derived without using the angle of inclination. In both cases the single integration is used. The absolute discrepancy AD is zero in each case. Here, we present another numerical approach to solve equation (2) [5], which is particularly interesting from an engineering perspective. Equation (2) is solved using the summation of small segments of the interval over the range $[0, \pi]$, thereby avoiding numerical integration. This assumption allows for a considerable reduction in computational time with very high accuracy. In TABLE II, we provide a comparative calculation of equation (2) using both integration and summation methods.

TABLE I. Mutual inductance as a function of the axial displacement c of two non-coaxial loops with the parallel axes, using single integration.

$c(m)$	$M(nH)$ [1]	$M(nH)$ (2)	$A.D.$ (%)
0.000	20.48852432867563	20.48852432867563	0.0
0.001	20.46296134770761	20.46296134770761	0.0
0.002	20.38668420445834	20.38668420445834	0.0
0.003	20.26090954763262	20.26090954763262	0.0
0.004	20.08760437072583	20.08760437072583	0.0
0.005	19.86940071618871	19.86940071618871	0.0
0.006	19.60948700160182	19.60948700160182	0.0
0.007	19.31148475780680	19.31148475780680	0.0
0.008	18.97931979155518	18.97931979155518	0.0
0.009	18.61709599187806	18.61709599187806	0.0
0.010	18.22897847733389	18.22897847733389	0.0
0.011	17.81909087755921	17.81909087755921	0.0

TABLE II. Mutual inductance as a function of the axial displacement c of two non-coaxial loops with the parallel axes, using single integration and summation.

$c(m)$	$M(nH)$ [1] Single Integration	$M(nH)$ (2), [5] Summation	$A.D.$ (%)
0.000	20.48852432867563	20.49048986995235	0.00959337649
0.001	20.46296134770761	20.46492499834691	0.00959612152
0.002	20.38668420445834	20.38864220081033	0.00960429039
0.003	20.26090954763262	20.26285819617522	0.00961777425
0.004	20.08760437072583	20.08954001265537	0.00963600185
0.005	19.86940071618871	19.87131994910612	0.00965923907
0.006	19.60948700160182	19.61138637150371	0.00968597445
0.007	19.31148475780680	19.31336108410397	0.00971611619
0.008	18.97931979155518	18.98117010170295	0.00974908567
0.009	18.61709599187806	18.61891789353698	0.00978617535
0.010	18.22897847733389	18.23076914790572	0.00982320854
0.011	17.81909087755921	17.82084807899210	0.00986134166

From TABLE II we can see a very good agreement between two numerical approaches with the absolute discrepancy around 0.0097 %. For these examples MATLAB is used to calculate numerically the single integral on the interval $[0; \pi]$ and FORTRAN is used to calculate the summation of the kernel function on the interval $[0; \pi]$.

In TABLES III and IV the restoring and the axial forces calculations are given for (5) and (6) are compared with the results obtained in [10]. The single integration is used in (5) and (6). Clearly, we obtained identical results by two approaches.

Here, we present another numerical approach to solve equations (5) and (6) [5], which is particularly interesting from an engineering perspective. Equations (5) and (6) are solved using the

summation of small intervals over the range $[0, \pi]$, thereby avoiding numerical integration. This assumption allows for a considerable reduction in computational time with very high accuracy. In TABLES V and VI, we provide a comparative calculation of equations (5) and (6) using both integration and summation methods.

TABLE III. Restoring force as a function of the axial displacement c of two non-coaxial loops with the parallel axes, using single integration.

$c(m)$	Restoring force $F_r (\mu N)$	Restoring force (5) $F_r (\mu N)$	$A.D. (%)$
0.000	0.07547749710028973	0.07547749710028973	0.0
0.001	0.07488583327209774	0.07488583327209774	0.0
0.002	0.07313671349198709	0.07313671349198709	0.0
0.003	0.07030546181947066	0.07030546181947066	0.0
0.004	0.06651032498899204	0.06651032498899204	0.0
0.005	0.06190265669550978	0.06190265669550978	0.0
0.006	0.05665516867960698	0.05665516867960698	0.0
0.007	0.05094972551589357	0.05094972551589357	0.0
0.008	0.04496601777908185	0.04496601777908185	0.0
0.009	0.03887211239892259	0.03887211239892259	0.0
0.010	0.03281745285065881	0.03281745285065881	0.0
0.011	0.02692846490789022	0.02692846490789022	0.0

TABLE IV. Repulsive force as a function of the axial displacement c of tw non-coaxial loops with the parallel axes, using single integration.

$c(m)$	Repulsive force $F_a (\mu N)$	Repulsive force (6) $F_a (\mu N)$	$A.D. (%)$
0.000	0.00	0.00	0.0
0.001	-0.0510570118824195	-0.0510570118824195	0.0
0.002	-0.1012935792952149	-0.1012935792952149	0.0
0.003	-0.1499264624141375	-0.1499264624141375	0.0
0.004	-0.1962433850245375	-0.1962433850245375	0.0
0.005	-0.2396304448823946	-0.2396304448823946	0.0
0.006	-0.2795912286657358	-0.2795912286657358	0.0
0.007	-0.3157568384768270	-0.3157568384768270	0.0
0.008	-0.3478871535458957	-0.3478871535458957	0.0
0.009	-0.3758645407463232	-0.3758645407463232	0.0
0.010	-0.3996817851575968	-0.3996817851575968	0.0
0.011	-0.4194262218421366	-0.4194262218421366	0.0

TABLE V. Restoring force as a function of the axial displacement c of two non-coaxial loops with the parallel axes, using single integration and summation.

$c(m)$	Restoring force (5) $F_r (\mu N)$ Single integral,[1]	Restoring force (5) $F_r (\mu N)$ Summation, [5]	$A.D. (%)$
0.000	0.07547749710028973	0.07525745846863121	0.29
0.001	0.07488583327209774	0.07466619941804611	0.29
0.002	0.07313671349198709	0.07291828457836584	0.30
0.003	0.07030546181947066	0.07008901078750480	0.30
0.004	0.06651032498899204	0.06629657942474166	0.32
0.005	0.06190265669550978	0.06169228882577443	0.34
0.006	0.05665516867960698	0.05644877724710618	0.36
0.007	0.05094972551589357	0.05074783460471877	0.40
0.008	0.04496601777908185	0.04476907078064662	0.44
0.009	0.03887211239892259	0.03868047679800412	0.49
0.010	0.03281745285065881	0.03263140496709844	0.57
0.011	0.02692846490789022	0.02674821108617715	0.67

TABLE VI. Repulsive force as a function of the axial displacement c of two non-coaxial loops with the parallel axes, using single integration and summation.

$c(m)$	Repulsive force (6) $F_a(\mu N)$ Single integral	Repulsive force (6) $F_a(\mu N)$ Summation	$A.D.$ (%)
0.000	0.00	0.00	--
0.001	-0.0510570118824195	-0.0510607877912594	0.0074
0.002	-0.1012935792952149	-0.1013010959350985	0.0074
0.003	-0.1499264624141375	-0.1499376442971153	0.0075
0.004	-0.1962433850245375	-0.1962581393878131	0.0075
0.005	-0.2396304448823946	-0.2396486153037206	0.0076
0.006	-0.2795912286657358	-0.2796126687702186	0.0077
0.007	-0.3157568384768270	-0.3157813532493243	0.0078
0.008	-0.3478871535458957	-0.347914523854540	0.0079
0.009	-0.3758645407463232	-0.3758945034502981	0.0080
0.010	-0.3996817851575968	-0.3997141139905325	0.0081
0.011	-0.4194262218421366	-0.4194606522115783	0.0082

The results of the restoring and repulsive forces using the numerical integration (MATLAB) and the summation (FORTRAN) are given in TABLES V and VI. All results are in very good agreement either by the numerical integration or the numerical summation. The average absolute discrepancy is about 0.0097 %.

Example 2.

Parameters for two non-coaxial cylindrical coils with parallel axes given in are as follows:

Coil 1	Coil 2
Inner radius (cm) R_1 =9.69645	R_3 = 7.1247
Outer radius (cm) R_2 =13.84935	R_4 = 8.5217
Length (cm) a = 2.413	b = 14.2748
Turns N_1 = 516	N_2 =1142

The axial distance between coils is $c = 0$.

Here, we will calculate the mutual inductance by the presenting method using the filament method using the same and different numbers of the subdivisions. Our goal is to find the best accuracy and the smallest computational time if possible.

Let us begin with the same number of the subdivisions $K = N = m = n = 20$.

It is obvious from TABLE VII that results obtained by two different approaches are in very good agreement with the absolute average discrepancy about 0.0075 %, but the computational time for the filament method is considerably enormous that is not preferable from the engineering point of consideration. Thus, the same number of subdivisions is not the smart choice in the mutual inductance calculation using the filament method. We can have very good precision of obtained results but with considerably big computational tame. This is why one must find the good compromise between the accuracy and the computational time in the choice of the number of the subdivisions of coils.

TABLE VII. Mutual inductance as a function of the perpendicular displacement d

$d(m)$	$M(mH),$	$M(mH),$ (7)	$Time$ (s)	$A.D.$ (%)
0.000	56.89550827752887	56.89876728224178	856.952685	0.0057
0.003	56.91083176832369	56.91408965245684	867.414842	0.0057
0.005	56.93806682400835	56.94132270480409	2761.818773	0.0057
0.008	57.00441609146245	57.00766703064307	1177.523890	0.0057
0.011	57.10129482231152	57.10453856704491	521.859077	0.0057
0.224	-5.774844679019263	-5.775269172262961	419.295176	0.007
0.250	-4.026004905310310	-4.026291199492929	452.488898	0.007
0.300	-2.194029475697561	-2.194140166466710	361.227363	0.005
0.400	-0.8609762300871849	-0.860995852556949	505.658034	0.002

0.500	-0.4256179797832593	-0.425622573906044	1648.168221	0.001
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Now, let's conduct the following analysis concerning the coil dimensions and the number of subdivisions. We propose the optimization method to minimize the three subdivisions (variables) in the function of one subdivision (variable) for the given coils' dimensions. The relations between two subdivisions are linear. It means that we have the problems of four linear homogenic equations where one depends on the others three. We use the following reasoning to find the minimal number of the subdivisions (variables) to reduce the computational time end keep the good accuracy. Choosing the smallest dimension between the radial and axial coils dimensions ($L_N = R_2 - R_1$, $L_n = R_4 - R_3$, $L_a = z_2 - z_1 = a$, $L_b = z_4 - z_3 = b$), we arbitrary choose the corresponding subdivision. Other three subdivisions will depend on this arbitrarily chosen subdivision. The procedures are as follows:

- A) Find $t = \min\{L_N, L_a, L_n, L_b\} = t_{min}$
- B) For obtained t_{min} , we choose the corresponding variable (subdivision), for example, $n \rightarrow t_{min}$.
- C) Now we have, $K = \frac{a}{R_4 - R_3}n$, $N = \frac{R_2 - R_1}{R_4 - R_3}n$, $m = \frac{b}{R_4 - R_3}n$, $n = n$

In this example we have,
 $R_2 - R_1=4.15485$; $R_4 - R_3=1.397$; $a = 2.413$; $b = 14.2748$. The smallest dimension is $R_4 - R_3=1.397$ which corresponds to the radial subdivision n of the second coil. Let us expresses all subdivisions in the function of n .

$$N/n = [(R_2 - R_1)/ (R_4 - R_3)] = [2.9727] = 3 \text{ or } N= 3n$$

$$K/n = [a/ (R_4 - R_3)] = [1.72727] =2 \text{ or } K = 2n$$

$$m/n = [b/ (R_4 - R_3)] = [10.218218] =10 \text{ or } m = 10n$$

In the TABLE VIII, IX and X we give the calculations of the mutual inductance by the filament method where the values of n subdivisions are different. All other subdivisions K , N and m are in the function of n as previously discussed.

From TABLES VIII, IX and X we can see very good agreement between two approaches. In all calculations we have very high accuracy between two different approaches where the average absolute discrepancy is about 0.0069 %, (TABLE VIII), 0.0043 %, (TABLE IX) and 0.0029 %, (TABLE X). Also, for both methods we have for each calculation the same four significant figures. Moreover, the calculation for the different number of the subdivisions considerably reduced the computational time (TABLES VII and IX) regarding the calculation for the same number of the subdivisions (TABLE VII).

Even though there is not considerable difference between the calculations regarding the accuracy given in TABLES VII and IX, it is recommended to choose $K = 6$, $N = 9$, $m = 30$ and $n = 3$. Also, without any reserve one can take $K = 8$, $N = 12$, $m = 40$ and $n = 4$, because of a very good accuracy and the relatively small computational time.

In TABLE VIII we choose $n = 3$, that gives $K = 6$, $N = 9$ and $m = 30$.

TABLE VIII. Mutual inductance as a function of the perpendicular displacement d

$d(m)$	$M(mH)$,	$M(mH)$, (7)	Time (s)	A.D. (%)
0.000	56.89550827752887	56.89286673907408	11.337791	0.0046
0.003	56.91083176832369	56.90818957928507	10.567271	0.0046
0.005	56.93806682400835	56.93542347847519	10.742036	0.0046
0.008	57.00441609146245	57.00176992753069	10.759457	0.0046
0.011	57.10129482231152	57.09864457245561	10.969503	0.0046
0.224	-5.774844679019263	-5.774288816171031	11.535825	0.0096
0.250	-4.026004905310310	-4.025622477401055	10.564258	0.0095
0.300	-2.194029475697561	-2.193828394461475	10.826811	0.0092
0.400	-0.8609762300871849	-0.860900867409665	10.847333	0.0088
0.500	-0.4256179797832593	-0.425581528154343	10.518138	0.0086

In TABLE IX we choose $n = 4$, that gives $K = 8$, $n = 4$, $N = 12$ and $m = 40$

TABLE IX. Mutual inductance as a function of the perpendicular displacement d

$d(m)$	$M(mH),$	$M(mH), (7)$	$Time (s)$	$A.D. (%)$
0.000	56.89550827752887	56.89385079631084	30.964315	0.003
0.003	56.91083176832369	56.90917390279342	31.016049	0.003
0.005	56.93806682400835	56.93640827549144	31.642019	0.003
0.008	57.00441609146245	57.00275587942618	35.509275	0.003
0.011	57.10129482231152	57.09963217765802	93.660610	0.003
0.224	-5.774844679019263	-5.774507929355869	115.902304	0.006
0.250	-4.026004905310310	-4.025773287164149	161.774595	0.006
0.300	-2.194029475697561	-2.193908170034211	126.383069	0.006
0.400	-0.8609762300871849	-0.860931031003053	114.444533	0.005
0.500	-0.4256179797832593	-0.42559617669354	151.118313	0.005

In TABLE X we choose $n = 5$, that gives $K = 10$, $N = 15$ and $m = 50$.

TABLE X. Mutual inductance as a function of the perpendicular displacement d

$d(m)$	$M(mH),$	$M(mH), (7)$	$Time (s)$	$A.D. (%)$
0.000	56.89550827752887	56.8943751557484	194.760392	0.002
0.003	56.91083176832369	56.9096983928553	72.493173	0.002
0.005	56.93806682400835	56.93693299776592	122.589127	0.002
0.008	57.00441609146245	57.00328116767053	132.458158	0.002
0.011	57.10129482231152	57.10015829724649	152.743919	0.002
0.224	-5.774844679019263	-5.774619127636496	265.588813	0.004
0.250	-4.026004905310310	-4.025849785469264	309.181010	0.004
0.300	-2.194029475697561	-2.193948422607928	128.801917	0.004
0.400	-0.8609762300871849	-0.860946133501390	285.441115	0.0035
0.500	-0.4256179797832593	-0.425603088258163	401.471515	0.0035

TABLE XI. Mutual inductance as a function of the perpendicular displacement d [13], using the summation [5].

$d(m)$	$M(mH),$	$M(mH)$	$Time (s), [5]$	$A.D. (%)$
0.000	56.89550827752887	56.88331624723717	36	0.0214
0.003	56.91083176832369	56.89896818427943	37	0.0208
0.005	56.93806682400835	56.92641707644633	37	0.0205
0.008	57.00441609146245	56.99307960663946	37	0.0199
0.011	57.10129482231152	57.09026280933549	37	0.0193
0.224	-5.774844679019263	-5.764590879596135	37	0.178
0.250	-4.026004905310310	-4.018716712530881	36	0.181
0.300	-2.194029475697561	-2.189813740930188	36	0.192
0.400	-0.8609762300871849	-0.859076834887261	36	0.220
0.500	-0.4256179797832593	-0.424546874039815	36	0.251

Also, we give the mutual inductance calculation obtained by the summation, [5], TABLE XI. These results are expected regarding the accuracy and the computational time because we used the summation instead the integration [1]. The number of the subdivisions is $K = N = m = n = 20$. These results are in the good agreement with those obtained by two previous methods,

In the calculations of the restoring (radial) and the repulsive (axial) forces we will use the same reasoning in the choice of the number of the subdivisions.

Example 3.

From Example 2, let's calculate the restoring and repulsive magnetic forces between the coils in questions.

Here, we utilize the optimal choice of subdivisions, with $K=8$, $N=12$, $m=40$, and $n=4$, as determined in the previous example. This selection ensures both good accuracy and minimal computational time for the integral approach. In contrast, for the summation approach, the number of subdivisions is set to $K=N=m=n=20$.

The comparison will involve using the formulas (8) for the radial magnetic force and (9) for the axial magnetic force, obtained through integration (as presented in this work), alongside the method that employs summation instead of integration, as outlined in reference [5].

TABLE XII. The radial magnetic force as a function of the perpendicular displacement d using the numerical integration and the summation [3].

$d(m)$	$F_r(mN)$, Eq. (8)	$F_r(mN)$, (8),	$A.D.$ (%)
0.000	-3.494310397 e-14~ 0	0.111041384395338	--
0.003	10.21401930788771	10.32269230189135	1.06
0.005	17.01798782490178	17.12512978277371	0.63
0.008	27.20764213665493	27.31257183181061	0.39
0.011	37.36736781567217	37.47032614951112	0.28
0.224	83.08829769865285	82.94606120526218	0.17
0.250	53.55797101390077	53.46734216052616	0.17
0.300	24.21219509927497	24.17102394119574	0.17
0.400	6.87399595335325	6.861513422034208	0.18
0.500	2.660323169441172	2.655031212570641	0.20

TABLE XIII. The axial magnetic force as a function of the perpendicular displacement d using the numerical integration and the summation [3].

$d(m)$	$F_a(mN)$, Eq. (9)	$F_a(mN)$, (9),[5], [15]
0.000	4.7457265094e-15~ 0	0
0.003	8.4129547976e-16~ 0	0
0.005	-2.2009000655e-15~ 0	0
0.008	-6.3471089636e-16~ 0	0
0.011	6.4183753659e-16 ~ 0	0
0.224	-4.2605870746e-17~ 0	0
0.250	1.6914772640e-16~ 0	0
0.300	8.0537633318e-17~ 0	0
0.400	2.5785955007e-17~ 0	0
0.500	1.0128379992e-18~ 0	0

From TABLE XII we have the good agreement between results obtained two methods in which the numerical integration and the numerical summation are used on the interval of the consideration $\theta \in [0; \pi]$. Obviously that the results for the radial magnetic force F_r , (8) obtained by the numerical integration are more precise, but the method is usable as comparative benchmark. The absolute discrepancy is between 0.1% and 1.06%.

From TABLE XIII the axial magnetic force F_a is zero for all points of the calculation that is practically confirmed by the presented method, Eq. (9). The second method doesn't give exactly zero for the axial force F_a because of the positive and negative variations during the summation on the interval of the consideration. The third method give exactly zero due to the axial factor involved in the force expression,

$$f_2(\kappa, c) = -e^{-\kappa(c+h_1+h_2)}(e^{-2c\kappa} - 1)(e^{-2h_1\kappa} - 1)/\kappa$$

where $h_1=a/2$ and $h_2=b/2$, and κ are the eigenvalues due to the introduction of artificial boundary, and it can be concluded that $F_a = 0$ from $f_2(\kappa, 0) = 0$.

Example 4.

Here, we give this example that can be used as the benchmark problem for tasting the different methods that treat the coils in question.

Parameters for two non-coaxial cylindrical coils with parallel axes given in and used in are as follows:

Coil 1	Coil 2
Inner radius (m) 0.071247	0.085217
Outer radius (m) 0.0969645	0.13849
Length (m) 0.142748	0.02413
Turns 1142	516

Let us find the following values.
 $L_N = R_2 - R_1 = 0.01397$, $L_n = R_4 - R_3 = 0.041529$,
 $L_a = z_2 - z_1 = a = 0.142748$, $L_b = z_4 - z_3 = b = 0.02413$

A) Let us find the minimum values between them.

$$t = \min\{L_N, L_a, L_n, L_b\} = L_N = t_{min} = 0.01397$$

B) This minimum value must correspond to following variable (subdivision).

$$t_{min} = 0.01397 \rightarrow N$$

C) Three variables (subdivisions) in the function of this variable are as follows:

$$K = [a / (R_2 - R_1)] N = [10.218181] = 10 \qquad \text{or} \qquad K = 10N$$

$$m / N = [b / (R_2 - R_1)] = [1.72727] = 2 \qquad \text{or} \qquad m = 2N$$

$$n / N = [(R_4 - R_3) / (R_2 - R_1)] = [2.9727] = 3 \qquad \text{or} \qquad n = 3N$$

D) The next step is to find the best choice of subdivisions for the arbitrarily chosen smallest variable.

In TABLE XIV, for one calculation of the mutual inductance given in [13], whit $d = 0.006$ m, $c = 0.059309$ m, $M = 44.7454180199$ mH, we test different values of $N = \{1,2,3,4,5,6,7,8\}$.

TABLE XIV. The best choice of the number of the subdivisions

$K/N/m/n$	$M(\text{mH}), (7)$	$Time (s)$	$A.D. (\%)$
10/1/2/3	44.73698768471552	0.724790	0.01884
20/ 2/4/6	44.74148468303699	4.277472	0.00879
30/ 3/6/9	44.74326891694706	11.439728	0.00480
40/4/8/12	44.74407612900293	36.149399	0.00300
50/5/10/15	44.74450319954718	218.047730	0.00204
60/6/12/18	44.74475522011805	145.106737	0.00148
70/7/14/21	44.74491602293224	873.712863	0.00112
80/8/16/23	44.74502477488414	3226.565	0.00088

Obviously, it is not logical to increase the number of subdivision N beyond 4 because the accuracy doesn't change significantly while the computational time increases enormously. Moreover, it is not practical from the engineering point of view. Thus, the best choice is to take $N = 3$ or even $N = 4$.

A) We can further improve the accuracy and computational time of calculations by adjusting the number of subdivisions based on our previous choices. By minimizing a chosen subdivision (variable), we then select the next smaller dimension with a corresponding subdivision (variable).

These two subdivisions may be increased by 1, 2, or 3, while the other two are decreased by 1, 2 or 3. This approach can significantly improve both accuracy and computational time.

From the previous TABLE XIV we begin with the choice of $N = 3$. Now, $K_1 = 30$, $N_1 = 3$, $m_1 = 6$, $N_1 = 9$.

The following calculation will increase two the two smallest variables by 1 and decrease the two largest variables by 1 1, TABLE XV. This process can be continued using the same logic, successively incrementing, and decrementing the variables by 1. This means $K_2 = 29$, $N_2 = 4$, $m_2 = 7$, $N_2 = 8$ and so on.

TABLE XV. The best choice of the number of the subdivisions

$K_1/N_1/m_1/n_1$	$M(\text{mH}), (7)$	$Time (s)$	$A.D. (\%)$
30/ 3/6/9	44.74326891694706	11.439728	0.0048
29/4/7/8	44.74436616452383	14.218988	0.0024
28/5/8/7	44.74479926514896	16.099376	0.0014
27/6/9/6	44.74487932957280	18.593136	0.0012
26/7/10/5	44.74465342835332	19.083521	0.0017

From TABLE XV, one can see that the previous statement is effective, as the accuracy does not change significantly and neither does the computational time.

Practically, we proposed a new approach in choosing the optimal numbers for the variables (subdivisions) that archives very high accuracy and the smallest possible time of calculation.

Let us choose $N=3$, that gives,

$$K = 30; N = 3; m = 6; n = 9.$$

Now, we calculate by the presented method the mutual inductance given and test the computational time and the accuracy. All comparative results are given in TABLE XVI.

TABLE XVI. Mutual inductance $M(\text{mH})$ as a function of the perpendicular displacement d [14]

$d(\text{m})$	$c(\text{m})$	$M, [14]$	$M, (7)$	$Time(s)$	$A.D(\%)$
0.006	0.01	56.6162643374	56.61363465455378	50.949498	0.0046
0.006	0.02	55.5894956502	55.5869094089611	51.984518	0.0047
0.006	0.03	53.8629343708	53.86042062215667	22.085411	0.0047
0.006	0.04	51.4222297882	51.41981711532675	17.279959	0.0047
0.006	0.05	48.2700337321	48.26774838654966	18.802184	0.0047
0.006	0.059309	44.7454180199	44.74326891694706	11.439728	0.0048
0.006	0.07	40.1814759728	40.17949410516751	10.649205	0.0049
0.006	0.083439	34.2304828323	34.22872135097286	22.910502	0.0051
0.006	0.09	31.4566003446	31.45494874434184	31.219811	0.0053
0.006	0.1	27.5571415229	27.55565306023951	22.247917	0.0054
0.006	0.6	0.4370202882	0.4369853665835998	21.303489	0.0080
0.006	1	0.0979060786	0.09789809468339546	15.960037	0.0082
0.020	0.083439	33.9467350341	33.94496955449947	22.260440	0.0052
0.020	0.09	31.0753970232	31.07375095703897	17.657343	0.0053
0.020	0.1	27.1316988802	27.13022256109052	18.291836	0.0054
0.020	0.6	0.4358068619	0.4357720325526511	16.392457	0.0080
0.020	1	0.0978025141	0.09779453844705811	37.723350	0.0082
0.250	0.01	-3.9917061128	-3.991327116889206	17.017424	0.0095
0.250	0.02	-3.8892645048	-3.888895853203926	17.197398	0.0095
0.250	0.03	-3.7202594312	-3.719908143385208	22.417223	0.0094
0.250	0.04	-3.4880421240	-3.487715260454917	22.161433	0.0094
0.250	0.05	-3.1987715126	-3.198475841930457	16.959263	0.0092
0.250	0.059309	-2.8870312678	-2.8867700832056	17.120941	0.0090
0.250	0.07	-2.4940353652	-2.493817639836896	18.719902	0.0087
0.250	0.083439	-1.9793349081	-1.979173375602797	18.864897	0.0082
0.250	0.09	-1.7317005825	-1.731565514863421	18.237672	0.0078
0.250	0.1	-1.3709099874	-1.370811777058473	10.940643	0.0072
0.250	0.6	0.2768190451	0.2767965107849468	10.749782	0.0081
0.250	1	0.0819441585	0.08193745655136937	11.089051	0.0082

For the previously chosen the number of subdivisions, $K = 30; N = 3; m = 6; n = 9$, we calculate the radial and the axial magnetic force between the coils in questions. The method given in [15,16] is used as the comparative method. The calculation of F_r and F_a can be used as the benchmark problem for tasting other methods for calculating these two magnetic forces for coils in question regarding the accuracy and the computational time.

TABLE XVII. Radial magnetic force F_r (mN) as a function of the perpendicular displacement d [14]

$d(m)$	$c(m)$	$F_r, (8)$	$F_r, [15,16]$
0.006	0.01	20.576008	20.602994
0.006	0.02	20.990200	21.039176
0.006	0.03	21.439313	21.548290
0.006	0.04	21.400845	21.663334
0.006	0.05	19.748335	19.744878
0.006	0.059309	15.020430	15.057457
0.006	0.07	4.9434673	5.075121
0.006	0.083439	-7.967173	-7.932075
0.006	0.09	-11.538574	-11.626505
0.006	0.1	-13.554995	-13.571601
0.006	0.6	-0.040067	-0.042237
0.006	1	-0.003416	-0.003751
0.020	0.083439	-36.535795	-36.780664
0.020	0.09	-45.290737	-45.297743
0.020	0.1	-48.219792	-48.217222
0.020	0.6	-0.133111	-0.135434
0.020	1	-0.011372	-0.012146
0.250	0.01	52.955752	52.724566
0.250	0.02	51.142662	51.096883
0.250	0.03	48.047731	47.986993
0.250	0.04	43.598200	43.504992
0.250	0.05	37.781110	37.632765
0.250	0.059309	31.269094	31.279649
0.250	0.07	22.934243	22.893013
0.250	0.083439	12.369791	12.358394
0.250	0.09	7.635737	7.679720
0.250	0.1	1.330825	1.330104
0.250	0.6	-0.970102	-0.971188
0.250	1	-0.114421	-0.115135

TABLE XVIII. Axial magnetic force F_a (mN) as a function of the perpendicular displacement d [14]

$d(m)$	$c(m)$	$F_a, (9)$	$F_a, [15,16]$
0.006	0.01	-68.191312	-68.205467
0.006	0.02	-137.391008	-137.389334
0.006	0.03	-208.173230	-208.158349
0.006	0.04	-279.963203	-279.898948
0.006	0.05	-349.604001	-349.520901
0.006	0.059309	-405.214825	-405.294053
0.006	0.07	-442.390770	-442.207026
0.006	0.083439	-433.226828	-433.307560
0.006	0.09	-411.017469	-411.060773
0.006	0.1	-367.800478	-367.886034
0.006	0.6	-2.102442	-2.102696
0.006	1	-0.289647	-0.289810
0.020	0.083439	-454.238985	-454.305635
0.020	0.09	-420.626836	-420.633025
0.020	0.1	-368.311148	-368.325780
0.020	0.6	-2.092916	-2.093189
0.020	1	-0.289140	-0.289332
0.250	0.01	6.851848	6.851354
0.250	0.02	13.609163	13.606354
0.250	0.03	20.134301	20.133180
0.250	0.04	26.203735	26.196089
0.250	0.05	31.476643	31.448595
0.250	0.059309	35.281758	35.276902
0.250	0.07	37.883856	37.884650
0.250	0.083439	38.163078	38.169832
0.250	0.09	37.213046	37.211353

0.250	0.1	34.771519	34.771530
0.250	0.6	-0.958292	-0.958551
0.250	1	-0.214685	-0.214975

From TABLES XVII and XVIII one can see very good agreements of results obtained by two different methods even though there are some differences for some points of the calculations. It can be explained by the following facts.

1) The presented method treats two coils of rectangular cross-section with the parallel axes, in the unbounded space libre, which are divided into circular filamentary coils. To account for the finite dimensions of the coils, massive solenoids are subdivided into meshes of filamentary coils as shown at Figure 2. The cross-sectional areas of two coils are divided into $(2K + 1)$ by $(2N + 1)$ cells for the first coil and $(2m + 1)$ by $(2n + 1)$ cells for the second coil, where K , N , m and n are the numbers of the subdivisions of coils, [6], and [12]. Event though we use the analytical Maxwell's formulas for the mutual inductance or the magnetic force between two circular loops, we cannot say that the presented filament method for the massive coils is purely analytical because its precision and the computational time depend on the number of subdivisions. This statement was studded in the previous examples. As it was shown the number of the subdivisions has the influence on the accuracy.

2) The compared method is a boundary value problem of circular coils with parallel axes shielded by a cuboid of high permeability. It means the coils are bounded by the medium of high permeability regarding the free space, in which are coils, where the mixed boundary conditions are satisfied on six surfaces of the artificial cuboid. Thus, this approach is approximate, but it proves to be accurate and efficient enough for practical applications. It means that this method can bring some differences in accuracy.

Even though we compare the results obtained by two different methods one for open space and other for artificial boundaries in bounded space, both gives very satisfactory results for calculating the magnetic force between two coils of rectangular cross section with the parallel axes.

With 1) and 2) we explain the possible differences of accuracy for some cases of calculation.

Example 5.

Finally, we give the rare examples that can find in the literature to calculate the mutual inductance between two non-coaxial coils of the rectangular cross-section with the parallel axes [1]. For this combination, the dimensions and the number of turns is as follows:

$$R_1 = 4 \text{ cm}, R_2 = 6 \text{ cm}, z_2 - z_1 = a = 10 \text{ cm}, N_1 = 150$$

$$R_3 = 2.5 \text{ cm}, R_4 = 3.5 \text{ cm}, z_4 - z_3 = b = 5 \text{ cm}, N_1 = 50$$

The perpendicular displacement of two coil axes is $d = 10 \text{ cm}$, and the axial displacement of the centers of the two coils is $c = 10.5 \text{ cm}$.

In the mutual inductance is,

$$M = 3.144 \text{ } \mu\text{H}$$

According to the optimal minimizing method given by presented approach concerning the high accuracy and the small computational time, after some tests we choose the number of the subdivisions $K = 30$; $N = 6$; $m = 15$ for arbitrarily chosen $n = 9$.

Using the approach presented in this paper the mutual inductance is,

$$M = 3.13606092090 \text{ } \mu\text{H}$$

Elapsed time is 17.133452 seconds (with MATLAB, Intel Core i5-12500H @ 2.5 GHz).

The method of [15, 16] gives,

$$M = 3.136970 \text{ } \mu\text{H}$$

and the elapsed time is 5.2 seconds (with Mathematica, Intel Core i7-8700 @ 3.2 GHz).

The absolute discrepancy regarding the accuracy between the presented method and this one given in [15-16] is around 0.029%.

In the mutual inductance is calculated using the general formula of Dwight an Purssell, arranged as series involving zonal harmonics, Equations (190) and (191) [1]. The convergence of this series is sufficient for most purposes as long as all distances $r_m = \sqrt{d_m^2 + \rho^2}$ are greater than $(A+a)$, where d_m , ρ are the perpendicular displacement of two coil axes the axial displacement of the centers of the

two coils, respectively. A and a are the mean radii of two coils of rectangular cross section respectively, and [19]. Since the general term of the series is known, it should be possible to use over the full range. However, the calculation of higher power terms becomes very tedious and time consuming [1]. This is why Grower took only four terms of this series and obtained $M = 3.144 \mu\text{H}$. It was problematic to take more terms because of mentioned issues as well as very slow convergence.

We did many tests of (190) and (191) from which we found very slow convergence. For two terms more we obtain

$$M = 3.14454842613 \mu\text{H}$$

The absolute discrepancy is 0.27%. Taking still more terms which signs changes alternatively will oscillate without significantly improve the accuracy because of the slow convergence.

However, for the different coils dimensions these formulas are not working correctly that is mentioned in [1]. This is why we consider the approach presented here as general for any coil's dimensions.

Now, let us calculate the radial and the axial magnetic force between the coils in questions respecting all parameters in the previously calculated mutual inductance.

$$F_r = -136.725877825 \mu\text{N}$$

$$F_a = 39.3340997099 \mu\text{N}$$

As a comparison, the method of [15, 16] gives,

$$F_r = -136.753948 \mu\text{N}$$

$$F_a = 39.344618 \mu\text{N}$$

Obviously, all results are in very good agreement.

The calculation provided by the presented method could also serve as a benchmark for other methods addressing this problem. Additionally, this method could be automatically applied to calculate the mutual inductance and the magnetic force between other coil configurations (solenoids, disks) with parallel axes.

5. Conclusions

In this paper we give the new formulas for calculating the restoring (radial) and the repulsive (axial) magnetic forces between two non-coaxial coils of rectangular cross-section with the parallel axes. These formulas are derived from modified Grover's formula for the mutual inductance between two non-coaxial loops with parallel axes. The validity of the presented approach is validated with an already established method. Presented formulas are used for calculating the radial and the axial force between two non-coaxial coils of rectangular cross-section with parallel axes using the filament method. Also, we presented the method to minimize the variables (subdivisions) in the filament method to find the compromise between the satisfactory accuracy and the corresponding small time of the calculation. We mention this method is applicable between non-coaxial conventional coils with parallel axes (massive-loop; massive-disk; massive-solenoid; two disks; disk-loop; disk-solenoid; two solenoids and solenoid-loop). This method can be useful for the engineers which are working in this domain because of its simplicity. The proposed method is comprehensible, fast, and very precise.

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