Preprints.org

Article

Not peer-reviewed version

A Proof of Conjecture 1 in Kulenović, Ladas and Overdeep (2003)

Rafik Zeraoulia *

Posted Date: 15 June 2026

doi: 10.20944/preprints202606.1105.v1

Keywords: rational difference equation; global convergence; monotone solutions; asymptotic behaviour

Article

A Proof of Conjecture 1 in Kulenović, Ladas and Overdeep (2003)

Rafik Zeraoulia [1,2]

[1] Department of Mathematics, Faculty of Matter Sciences and Computer Science, University of Djilali Bounaama Khemis Miliana, Khemis Miliana 44225, Algeria; zeraoulia@univ-dbkm.dz

[2] Laboratory of Pure and Applied Mathematics, Amar Teledji University, Laghouat 03000, Algeria

Abstract

In [1], Kulenović, Ladas and Overdeep posed a conjecture asserting that every positive solution of the rational second-order difference equation $y_{n+1} = \frac{y_n(1+y_n)^2}{y_n(1+y_n)+(1+y_{n-1})}$, $\quad n = 0, 1, \ldots$, converges to a finite limit. We confirm this conjecture by deriving a short identity showing that the sign of $y_{n+1} - y_n$ is invariant with respect to n, so every positive solution is monotone. A simple estimate then gives an explicit initial-data-dependent upper bound in the increasing case, while the decreasing case is bounded below by positivity. Hence every positive solution converges. In addition, we introduce the auxiliary sequence $t_n := \frac{y_n(1+y_n)}{1+y_{n-1}}$, which is monotone in the direction opposite to that of y_n. It yields nested two-sided enclosures of the limit and an exact invariant-series formula. Writing $g_n = t_n - y_n$ and $\rho_n = t_n/(1+t_n)^2$, we prove that $I_n = y_n + g_n \sum_{j=0}^{\infty} \frac{1}{1+t_{n+j}} \prod_{m=0}^{j-1} \rho_{n+m}$ is independent of n and satisfies $I_n = L = \lim_{k\to\infty} y_k$. Hence the limiting equilibrium selected by the initial data is determined by the invariant value I_0.

Keywords: rational difference equation; global convergence; monotone solutions; asymptotic behaviour

MSC: 39A10; 39A20

---

1. Introduction

The qualitative behaviour of rational difference equations has been studied extensively because such equations arise in discrete-time modelling and often exhibit rich global dynamics. A systematic treatment of many second-order rational recurrences can be found in the monograph [2]. Further background on nonlinear difference equations, higher-order rational recurrences, and standard qualitative methods may be found in [3,4]. In a short note of open problems and conjectures, Kulenović, Ladas and Overdeep [1] proposed, among other questions, the following conjecture.

Remark 1 (Conjecture 1 in [1]). *Show that every positive solution of*

$$y_{n+1} = \frac{y_n(1+y_n)^2}{y_n(1+y_n)+(1+y_{n-1})}, \qquad n = 0, 1, \ldots, \tag{1}$$

with initial conditions $y_{-1}, y_0 > 0$ has a finite limit.

A special feature of (1) is that every positive number is an equilibrium. Therefore, a direct passage to the limit in (1) gives only the identity

$$L = \frac{L(1+L)^2}{L(1+L)+(1+L)} = L,$$

and does not by itself determine which equilibrium is selected by the initial data. This makes it important to keep track of monotonicity and of quantities that enclose the limiting value along the orbit.

We note that Conjecture 1 asks for global convergence without requiring a closed-form expression for the limit. The argument below is based on a short difference identity that shows the sign of successive increments $y_{n+1} - y_n$ cannot change with n. This immediately forces monotonicity and, together with a simple estimate on the increments in the increasing case, yields boundedness and convergence. After proving convergence, we introduce an auxiliary sequence that provides nested two-sided enclosures of the limit and, more importantly, an exact invariant-series formula whose value is precisely the selected limiting equilibrium. This gives a rigorous determination of the limit from the initial orbit and supplies a certified stopping criterion for numerical computation.

The purpose of this note is to confirm the conjecture. The proof is elementary: we first show that the sign of $y_{n+1} - y_n$ equals the sign of $y_n - y_{n-1}$ for all $n \geq 0$, which forces monotonicity. The decreasing case then converges immediately by monotone convergence. In the increasing case we prove that the increments decay at least geometrically, which provides an explicit upper bound and yields convergence to a finite limit.

2. Main Result

Throughout we assume $y_{-1}, y_0 > 0$ and define $\{y_n\}_{n=-1}^{\infty}$ by (1). Note that positivity is forward invariant: if $y_{n-1}, y_n > 0$, then the right-hand side of (1) is positive, so $y_{n+1} > 0$ for all $n \geq 0$. Set

$$D_n := y_n(1 + y_n) + (1 + y_{n-1}) > 0.$$

Lemma 1 (Difference identity)**.** *For every $n \geq 0$,*

$$y_{n+1} - y_n = \frac{y_n(y_n - y_{n-1})}{D_n}. \tag{2}$$

Consequently,

$$\operatorname{sgn}(y_{n+1} - y_n) = \operatorname{sgn}(y_n - y_{n-1}).$$

Proof. Starting from (1),

$$y_{n+1} - y_n = y_n \left(\frac{(1 + y_n)^2}{D_n} - 1 \right) = y_n \frac{(1 + y_n)^2 - D_n}{D_n}.$$

Since

$$(1 + y_n)^2 - D_n = (1 + 2y_n + y_n^2) - (y_n + y_n^2 + 1 + y_{n-1}) = y_n - y_{n-1},$$

we obtain (2). The sign statement follows because $y_n > 0$ and $D_n > 0$. □

Lemma 2 (Monotonicity)**.** *Every positive solution of (1) is monotone. More precisely: if $y_0 > y_{-1}$, then $\{y_n\}_{n \geq 0}$ is strictly increasing; if $y_0 < y_{-1}$, then it is strictly decreasing; and if $y_0 = y_{-1}$, then $y_n \equiv y_0$.*

Proof. If $y_0 = y_{-1}$, then (2) gives $y_1 - y_0 = 0$, and by induction $y_{n+1} - y_n = 0$ for all n, so the solution is constant.

Assume $y_0 \neq y_{-1}$. Then $y_1 - y_0$ has the same sign as $y_0 - y_{-1}$ by Lemma 2.1. Applying Lemma 2.1 iteratively shows that the sign of $y_{n+1} - y_n$ is constant in n, hence the sequence is strictly monotone. □

Lemma 3 (An initial-data-dependent upper bound in the increasing case)**.** *Assume $y_0 > y_{-1}$ so that $\{y_n\}_{n \geq 0}$ is increasing. Let $d_n := y_n - y_{n-1}$ for $n \geq 0$ (so $d_0 = y_0 - y_{-1} > 0$). Then*

$$0 < d_n \leq \frac{d_0}{(1 + y_0)^n}, \qquad n = 0, 1, 2, \ldots, \tag{3}$$

 doi:10.20944/preprints202606.1105.v1

and, in particular,

$$y_n \le y_0 + \frac{y_0 - y_{-1}}{y_0} = y_0 + 1 - \frac{y_{-1}}{y_0} \qquad \text{for all } n \ge 0. \tag{4}$$

Proof. From (2),

$$d_{n+1} = y_{n+1} - y_n = \frac{y_n d_n}{D_n} < \frac{y_n d_n}{y_n(1+y_n)} = \frac{d_n}{1+y_n},$$

where we used $D_n = y_n(1+y_n) + (1+y_{n-1}) > y_n(1+y_n)$. Since $\{y_n\}$ is increasing, $y_n \ge y_0$ for all $n \ge 0$, hence

$$d_{n+1} \le \frac{d_n}{1+y_0}.$$

Iterating yields (3). Summing (3) over $k \ge 1$ gives

$$y_n = y_0 + \sum_{k=1}^{n} d_k \le y_0 + \sum_{k=1}^{\infty} \frac{d_0}{(1+y_0)^k} = y_0 + \frac{d_0}{y_0},$$

which is (4). □

Remark 2. *The estimate (4) is not uniform over the whole set of positive initial data. It is an explicit orbitwise bound depending on the chosen initial values y_{-1} and y_0. This is the only meaning in which boundedness is used below: each individual positive solution is bounded.*

We now state and prove the main theorem, which confirms Conjecture 1 in [1].

Theorem 1 (Global convergence)**.** *Every positive solution of (1) converges to a finite limit.*

Proof. By Lemma 2.2, the sequence $\{y_n\}_{n\ge 0}$ is monotone.

If it is decreasing, then $y_n \ge 0$ for all n and therefore $\{y_n\}$ is bounded below; hence it converges to a finite limit by the monotone convergence theorem.

If it is increasing, then Lemma 2.3 provides the explicit upper bound (4), so $\{y_n\}$ is increasing and bounded above, and therefore converges to a finite limit. □

Remark 3. *Theorems 3.2 and 3.4 below provide two complementary descriptions of the selected limiting equilibrium. The first gives nested two-sided bounds and an effective stopping criterion. The second gives an exact invariant-series expression for L from the initial orbit. This addresses the limit-selection issue caused by the continuum of equilibria of (1).*

3. A Two-Sided Characterisation and an Invariant Formula for the Limit

The proof of Theorem 2.4 guarantees the existence of a finite limit

$$L := \lim_{n\to\infty} y_n \in [0, \infty).$$

Because (1) has a continuum of equilibria, the value of L cannot be recovered by solving an equilibrium equation alone. The following auxiliary sequence gives an orbit-based replacement. It first produces nested intervals whose unique common point is precisely L, and then yields an invariant-series formula whose value is exactly the selected equilibrium.

Lemma 4 (An auxiliary sequence)**.** *For $n \ge 0$ define*

$$t_n := \frac{y_n(1+y_n)}{1+y_{n-1}}. \tag{5}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 June 2026 doi:10.20944/preprints202606.1105.v1

Then, for every $n \geq 0$,

$$t_n - y_n = \frac{y_n(y_n - y_{n-1})}{1 + y_{n-1}}, \tag{6}$$

and, for every $n \geq 0$,

$$t_{n+1} - t_n = \frac{y_n^3(1 + y_n)^2(y_{n-1} - y_n)}{(1 + y_{n-1})(1 + y_{n-1} + y_n + y_n^2)^2}. \tag{7}$$

Consequently,

$$\operatorname{sgn}(t_{n+1} - t_n) = \operatorname{sgn}(y_{n-1} - y_n) = -\operatorname{sgn}(y_n - y_{n-1}).$$

Proof. Identity (6) follows immediately from (5):

$$t_n - y_n = y_n\left(\frac{1 + y_n}{1 + y_{n-1}} - 1\right) = \frac{y_n(y_n - y_{n-1})}{1 + y_{n-1}}.$$

To prove (7), write (1) as

$$y_{n+1} = \frac{y_n(1 + y_n)^2}{1 + y_{n-1} + y_n + y_n^2}.$$

Then

$$t_{n+1} = \frac{y_{n+1}(1 + y_{n+1})}{1 + y_n} = \frac{y_{n+1}}{1 + y_n}(1 + y_{n+1}).$$

Substituting the above expression for y_{n+1} and simplifying yields (7). The sign statement follows because all factors in the denominator of (7) are positive. □

Theorem 2 (Two-sided bounds for the limit)**.** *Let $\{y_n\}$ be any positive solution of (1) and let t_n be defined by (5). Then:*

- *If $y_0 > y_{-1}$, then $\{y_n\}_{n \geq 0}$ is strictly increasing, $\{t_n\}_{n \geq 0}$ is strictly decreasing, and*

$$y_n < L < t_n \qquad \text{for all } n \geq 0.$$

  Moreover,

$$L = \sup_{n \geq 0} y_n = \inf_{n \geq 0} t_n = \lim_{n \to \infty} t_n.$$

- *If $y_0 < y_{-1}$, then $\{y_n\}_{n \geq 0}$ is strictly decreasing, $\{t_n\}_{n \geq 0}$ is strictly increasing, and*

$$t_n < L < y_n \qquad \text{for all } n \geq 0,$$

  with

$$L = \inf_{n \geq 0} y_n = \sup_{n \geq 0} t_n = \lim_{n \to \infty} t_n.$$

Proof. Assume $y_0 > y_{-1}$. By Lemma 2.2, $\{y_n\}$ is strictly increasing and converges to L by Theorem 2.4. Identity (6) implies $t_n > y_n$ for all n. By (7), $\{t_n\}$ is strictly decreasing. Since $t_n > y_n \geq y_0$, it is bounded below and hence has a finite limit τ. Passing to the limit in (5) using $y_{n-1} \to L$ and $y_n \to L$ gives

$$\tau = \lim_{n \to \infty} \frac{y_n(1 + y_n)}{1 + y_{n-1}} = \frac{L(1 + L)}{1 + L} = L.$$

Thus $t_n \searrow L$ and $y_n \nearrow L$, proving the claimed inequalities and the extremal characterisation of L.

The case $y_0 < y_{-1}$ is analogous: $\{y_n\}$ decreases to L, (6) yields $t_n < y_n$, and (7) implies that $\{t_n\}$ increases. Taking limits in (5) again gives $\lim_{n \to \infty} t_n = L$. □

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 June 2026 doi:10.20944/preprints202606.1105.v1

Theorem 3 (Nested-interval determination of the limit). *For $n \geq 0$, set*

$$J_n := \big[\min\{y_n, t_n\}, \max\{y_n, t_n\}\big].$$

Then the intervals J_n are nested, their lengths are

$$|J_n| = |t_n - y_n|,$$

and

$$\bigcap_{n=0}^{\infty} J_n = \{L\}.$$

Equivalently, L is the unique real number lying in every interval J_n.

Proof. If $y_0 > y_{-1}$, then $J_n = [y_n, t_n]$. By Theorem 3.2, y_n increases to L and t_n decreases to L, so $J_{n+1} \subset J_n$ and the intersection is $\{L\}$. If $y_0 < y_{-1}$, then $J_n = [t_n, y_n]$, with $t_n \nearrow L$ and $y_n \searrow L$, and the same conclusion follows. If $y_0 = y_{-1}$, then the solution is constant and the assertion is immediate. The formula for $|J_n|$ follows from the definition of J_n. □

Theorem 4 (An invariant-series formula for the selected equilibrium). *Let $\{y_n\}$ be a positive solution of (1), and let*

$$t_n := \frac{y_n(1+y_n)}{1+y_{n-1}}, \qquad n \geq 0.$$

Set

$$g_n := t_n - y_n, \qquad \rho_n := \frac{t_n}{(1+t_n)^2}.$$

Then

$$y_{n+1} = \frac{t_n(1+y_n)}{1+t_n}, \qquad t_{n+1} = \frac{t_n(1+y_{n+1})}{1+t_n}, \tag{8}$$

and consequently

$$y_{n+1} - y_n = \frac{g_n}{1+t_n}, \qquad g_{n+1} = \rho_n g_n. \tag{9}$$

Since $t_n > 0$,

$$0 < \rho_n = \frac{t_n}{(1+t_n)^2} \leq \frac{1}{4}.$$

For $n \geq 0$, define

$$I_n := y_n + g_n \sum_{j=0}^{\infty} \frac{1}{1+t_{n+j}} \prod_{m=0}^{j-1} \rho_{n+m}, \tag{10}$$

where the empty product is understood to be 1. Then the series is absolutely convergent,

$$I_{n+1} = I_n \qquad \text{for all } n \geq 0,$$

and

$$I_n = \lim_{k\to\infty} y_k = L.$$

In particular, the limiting equilibrium selected by the initial values (y_{-1}, y_0) is given by

$$L = y_0 + \left(\frac{y_0(1+y_0)}{1+y_{-1}} - y_0\right) \sum_{j=0}^{\infty} \frac{1}{1+t_j} \prod_{m=0}^{j-1} \frac{t_m}{(1+t_m)^2}. \tag{11}$$

Proof. By the definition of t_n,

$$y_n(1+y_n) = t_n(1+y_{n-1}).$$

doi:10.20944/preprints202606.1105.v1

Substituting this identity into (1) gives

$$y_{n+1} = \frac{t_n(1+y_n)}{1+t_n}.$$

Furthermore,

$$t_{n+1} = \frac{y_{n+1}(1+y_{n+1})}{1+y_n} = \frac{t_n(1+y_{n+1})}{1+t_n},$$

because $y_{n+1}/(1+y_n) = t_n/(1+t_n)$. Thus (8) holds. It follows that

$$y_{n+1} - y_n = \frac{t_n(1+y_n)}{1+t_n} - y_n = \frac{t_n - y_n}{1+t_n} = \frac{g_n}{1+t_n}.$$

Also,

$$g_{n+1} = t_{n+1} - y_{n+1} = \frac{t_n}{1+t_n}(y_{n+1} - y_n) = \frac{t_n}{(1+t_n)^2} g_n = \rho_n g_n.$$

Since $t_n > 0$, the elementary inequality

$$\frac{t_n}{(1+t_n)^2} \leq \frac{1}{4}$$

shows that the series in (10) is absolutely convergent; indeed, each product is bounded above by 4^{-j}.

Let

$$S_n := \sum_{j=0}^{\infty} \frac{1}{1+t_{n+j}} \prod_{m=0}^{j-1} \rho_{n+m}.$$

Then

$$S_n = \frac{1}{1+t_n} + \rho_n S_{n+1}.$$

Using (9), we obtain

$$I_n = y_n + g_n S_n = y_n + \frac{g_n}{1+t_n} + g_n \rho_n S_{n+1} = y_{n+1} + g_{n+1} S_{n+1} = I_{n+1}.$$

Thus I_n is invariant along the orbit.

Finally, (9) and $0 < \rho_n \leq 1/4$ give

$$|g_n| \leq |g_0| 4^{-n}.$$

Moreover,

$$0 < S_n \leq \sum_{j=0}^{\infty} 4^{-j} = \frac{4}{3}.$$

Hence $I_n - y_n = g_n S_n \to 0$. Since I_n is independent of n and $y_n \to L$ by Theorem 2.4, it follows that $I_n = L$ for every $n \geq 0$. Taking $n = 0$ in (10) gives (11). □

Remark 4 (A practical stopping criterion)**.** *Theorem 3.3 implies that, for any $n \geq 0$,*

$$L \in \left[\min\{y_n, t_n\}, \max\{y_n, t_n\}\right], \qquad \max\{y_n, t_n\} - \min\{y_n, t_n\} = |t_n - y_n|.$$

Thus one may approximate L together with an a posteriori error bound by iterating until $|t_n - y_n|$ is below a prescribed tolerance.

Remark 5 (On the nature of the invariant)**.** *The invariant in Theorem 3.4 is an exact convergent-series invariant. It plays the same limit-selection role as the elementary invariant $2y_n + y_{n-1}$ for the linear averaging equation $y_{n+1} = (y_n + y_{n-1})/2$: after passing to the limit in the invariant identity, one obtains the selected*

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 June 2026 doi:10.20944/preprints202606.1105.v1

equilibrium. The formula above is not a finite algebraic first integral; finding an even simpler finite expression would be a further refinement beyond the convergence assertion of Conjecture 1.

4. Numerical Experiments

We briefly illustrate the theoretical conclusions with numerical simulations.

4.1. Trajectories and Convergence of Increments

Figure 1 shows representative trajectories for several initial conditions, including a "large-data" example $(y_{-1}, y_0) = (100, 1000)$. In each case the solution is monotone and converges quickly. Figure 2(a) plots the successive increments $|y_n - y_{n-1}|$ on a semilogarithmic scale for the increasing case $(y_{-1}, y_0) = (1, 2)$, illustrating the geometric decay predicted by Lemma 2.3.

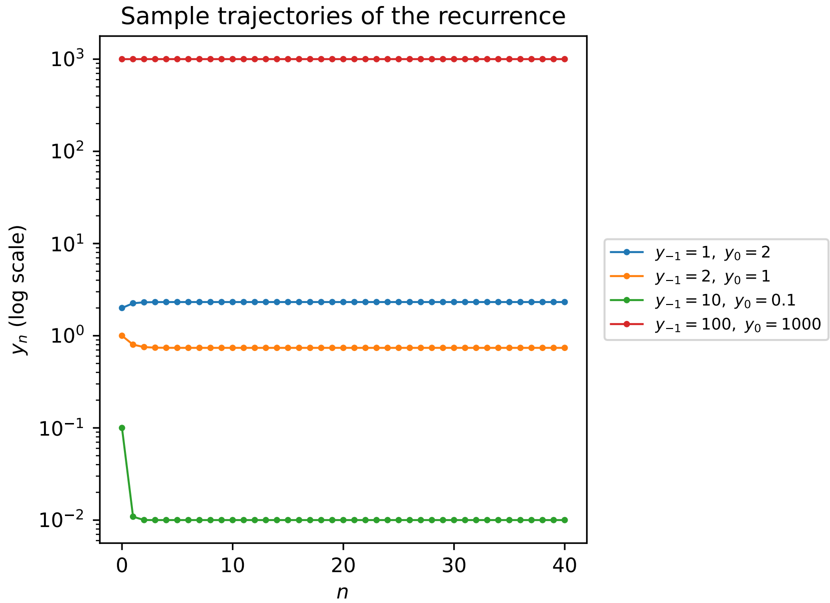

Figure 1. Sample trajectories for (1) (plots of y_n versus n for $n \geq 0$).

4.2. Two-Sided Bounds and Initial-Value Sweep

Figure 2(b) plots y_n together with the auxiliary sequence t_n for both an increasing example $(y_{-1}, y_0) = (1, 2)$ and a decreasing example $(y_{-1}, y_0) = (2, 1)$. In agreement with Theorem 3.2, the two sequences are monotone in opposite directions and rapidly squeeze the common limit. Figure 2(c) shows the gap $|t_n - y_n|$, which is the width of the rigorous enclosure interval for L.

Although equation (1) has no external parameter, one may view the initial value y_0 as a varying input when y_{-1} is fixed. For each y_0 in a logarithmic grid on $[10^{-3}, 10^3]$ (400 points) and fixed $y_{-1} = 1$, we computed an approximation of $L(y_{-1}, y_0) = \lim_{n\to\infty} y_n$ by iterating until the enclosure width $|t_n - y_n| < 10^{-12}$ (see Remark 3.5) was satisfied, and then taking the midpoint of the resulting enclosure interval. Figure 2(d) shows the resulting initial-value sweep $y_0 \mapsto L(1, y_0)$. No qualitative changes are observed over six orders of magnitude, consistent with the global convergence established in Theorem 2.4. All simulations and plots were generated in Python (NumPy/Matplotlib); the script `numerics.py` included with the source reproduces the figures.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 June 2026 doi:10.20944/preprints202606.1105.v1

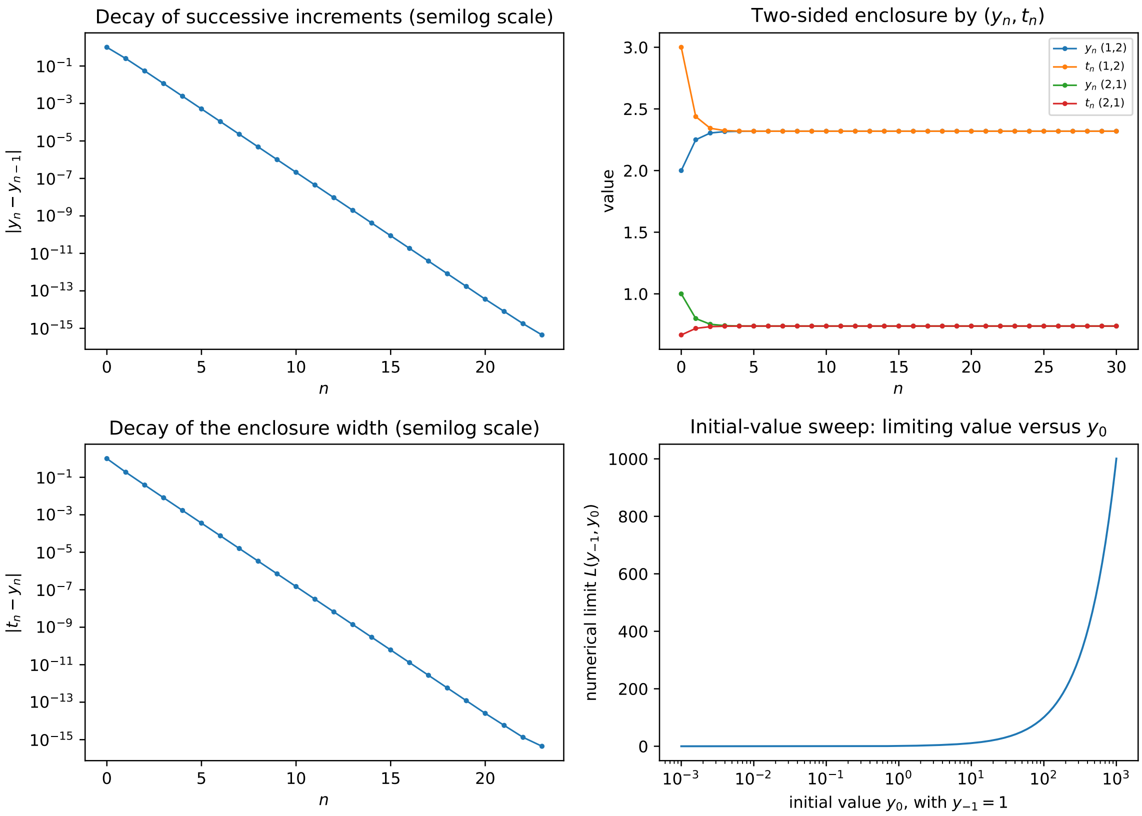

Figure 2. Compact presentation of the numerical experiments. Panel (a) shows decay of successive increments for $(y_{-1}, y_0) = (1, 2)$. Panel (b) shows the two-sided bounds y_n and t_n in increasing and decreasing cases. Panel (c) shows the enclosure width $|t_n - y_n|$ for $(y_{-1}, y_0) = (1, 2)$. Panel (d) shows the computed limit $L(1, y_0)$ versus y_0.

5. Conclusion

We confirmed Conjecture 1 of Kulenović, Ladas and Overdeep [1] by showing that every positive solution of (1) is monotone and bounded, and hence convergent. The boundedness in the increasing case is supplied by an explicit initial-data-dependent estimate. Moreover, the auxiliary sequence t_n provides nested two-sided enclosures whose unique common point is the limit, and Theorem 3.4 supplies an exact invariant-series formula for the selected equilibrium. Thus the limiting value is determined by

$$\{L\} = \bigcap_{n=0}^{\infty} [\min\{y_n, t_n\}, \max\{y_n, t_n\}]$$

and also by the invariant identity

$$L = I_0.$$

This completes the convergence result and gives an explicit orbit-based determination of the limiting value selected by the initial data.

6. Consequences for Dynamical Systems and Chaos Theory

Equation (1) defines the continuous planar map

$$F(x, y) = \left(y, \frac{y(1+y)^2}{y(1+y) + (1+x)} \right)$$

on the positive quadrant. This is the standard state-space representation of a second-order difference equation [5]. Theorem 2.4 and Lemma 2.2 imply that every positive forward orbit

$$F^n(y_{-1}, y_0) = (y_{n-1}, y_n)$$

converges to a point (L, L) on the equilibrium diagonal

$$\mathcal{E} = \{(s, s) : s \geq 0\}$$

in the closure of the positive quadrant.

Corollary 1. *For the positive dynamics generated by (1):*

(i) *the omega-limit set of every orbit is the singleton $\{(L, L)\}$;*
(ii) *there is no positive periodic orbit of prime period greater than one;*
(iii) *there is no Li–Yorke scrambled pair, and therefore no Li–Yorke chaos in the positive phase space in the sense introduced in [6];*
(iv) *the map cannot be Devaney chaotic on any compact invariant subset of the positive quadrant containing more than one point, in the standard sense of [7,8].*

Proof. The first assertion follows directly from $(y_{n-1}, y_n) \to (L, L)$. A periodic orbit that converges must be constant, which proves the second assertion. For two positive initial states whose limits are L and $\widetilde{L}$, respectively, the distance between their nth states converges to $\sqrt{2}\,|L - \widetilde{L}|$. Hence the lower and upper limiting distances coincide, so no pair can satisfy the defining separation condition of a Li–Yorke scrambled pair. Finally, the periodic points of F are precisely the fixed points on the equilibrium diagonal. If periodic points were dense in a compact invariant set K, then, because the equilibrium diagonal is closed, one would have K contained in that diagonal. The restriction $F|_K$ would then be the identity map, which is not topologically transitive when K contains more than one point. Thus the defining conditions for Devaney chaos cannot hold on such a set. □

Thus the proved conjecture has a direct dynamical interpretation: although (1) is nonlinear and rational and possesses a continuum of non-hyperbolic equilibria, its positive dynamics are globally ordered and asymptotically non-chaotic. These conclusions concern positive initial data only; they do not address sign-changing extensions, parameter perturbations, or nonautonomous variants of the recurrence.

Author Contributions: The proof and solution of the conjecture, the theoretical development, the new section on consequences for dynamical systems and chaos theory, and the preparation of this preprint are due to Rafik Zeraoulia. Pedro Caceres and Simeón Casanova Trujillo contributed to selected numerical computations and to the supporting software used for the numerical experiments.

Funding: Not applicable.

Data Availability Statement: No new data were created or analysed in this study. The numerical experiments and figures are reproducible using the accompanying Python script `numerics.py`.

Acknowledgments: Rafik Zeraoulia thanks Allah for helping him prove this 20-year-old conjecture. He dedicates this proof to his family and friends and, especially, to his two sons, Zeraoulia Taha AbdelJalil and Taki Abdesalaem, who are his heart and his life. The author also thanks Gerry Ladas and collaborators for compiling open problems that motivated this work, and the referees for comments that improved the clarity of the exposition.

Conflicts of Interest: The author declares no conflict of interest.

References

1. M. R. S. Kulenović, G. Ladas and C. B. Overdeep, Open Problems and Conjectures: Edited by Gerry Ladas, *Journal of Difference Equations and Applications* **9**(11) (2003), 1053–1056. doi:10.1080/1023619031000154644.
2. M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations*, Chapman & Hall/CRC, Boca Raton, 2001.
3. V. L. Kocić and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers, Dordrecht, 1993.
4. S. Elaydi, *An Introduction to Difference Equations*, 3rd ed., Springer, New York, 2005.

 doi:10.20944/preprints202606.1105.v1

5. M. R. S. Kulenović and O. Merino, *Discrete Dynamical Systems and Difference Equations with Mathematica*, Chapman & Hall/CRC, Boca Raton, 2002. doi:10.1201/9781420035353.
6. T.-Y. Li and J. A. Yorke, Period three implies chaos, *The American Mathematical Monthly* **82**(10) (1975), 985–992. doi:10.1080/00029890.1975.11994008.
7. R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*, 2nd ed., Addison-Wesley, Redwood City, CA, 1989.
8. J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stacey, On Devaney's definition of chaos, *The American Mathematical Monthly* **99**(4) (1992), 332–334. doi:10.1080/00029890.1992.11995856.