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Weighted Prime Number Theorem on Arithmetic Progressions with Refinements

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Abstract

We extend Dirichlet's prime number theorem on arithmetic progressions to the weighted counting function $\pi_s(x;\, q,\, a)$ for $(a, q) = 1$ with the weight p^{-w} for prime p with $w \geq 0$. We also prove that when $0 \leq w < 1/2$, the difference $\pi_w(x;\, q,\, a) - \pi_w(x;\, q,\, b)$ changes its sign infinitely many times as x grows for any coprime a, b $(a \neq b)$ with p under the assumption that Dirichlet L-functions have no real nontrivial zeros.. This result gives a justification of the theory of Aoki-Koyama that Chebyshev's bias is formulated by the asymptotic behavior of $\pi_w(x;\, q,\, a) - \pi_w(x;\, q,\, b)$ at $s = 1/2$.

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1. Introduction

We define the weighted counting function of primes in arithmetic progressions as

$$\pi_w(x;\, q,\, a) = \sum_{\substack{p<x:\,\text{prime} \\ p\equiv a \pmod q}} \frac{1}{p^w} \qquad (w \geq 0)$$

for $q \in \mathbb{Z}$. This is a generalization of the classical counting function $\pi(x;\, q,\, a) = \pi_0(x;\, q,\, a)$. In 1837 Dirichlet proved that

$$\pi(x;\, q,\, a) \sim \pi(x;\, q,\, b) \qquad (x \to \infty)$$

for any a, b which are coprime with q, where $f(x) \sim g(x)$ $(x \to \infty)$ means that $f(x)/g(x) \to 1$ as $x \to \infty$.

On the other hand, in 1853 Chebyshev found the phenomenon that there tend to be more primes p satisfying $p \equiv 3 \pmod 4$ than those with $p \equiv 1 \pmod 4$. In fact, the inequality

$$\pi(x;\, 4,\, 3) \geq \pi(x;\, 4,\, 1) \tag{1.1}$$

holds for any x less than 26861, which is the first prime number violating the inequality (1.1). However, the both sides draw equal at the next prime 26863, and $\pi(x;\, 4,\, 3)$ gets ahead again until 616841. It is computed that more than 97% of $x < 10^{11}$ satisfy the inequality (1.1). This phenomenon is called Chebyshev's bias, and is one of major unsolved mysteries in number theory.

Later, Littlewood [1] proved that the difference $\pi(x;\, 4,\, 3) - \pi(x;\, 4,\, 1)$ changes its sign infinitely many times. In 2023, Aoki and Koyama [2] suggested that Littlewood's phenomenon no longer holds if we put the weight $p^{-1/2}$. They actually obtained the asymptotic under the assumption of the Deep Riemann Hypothesis that

$$\pi_{\frac{1}{2}}(x;\, 4,\, 3) - \pi_{\frac{1}{2}}(x;\, 4,\, 1) \sim \frac{1}{2}\log\log x \quad (x \to \infty). \tag{1.2}$$

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The Deep Riemann Hypothesis (DRH) is a conjecture asserting the convergence of the Euler product of the corresponding L-function on the critical line. More generally they defined Chebyshev bias towards $p \equiv b \pmod q$ (or against $p \equiv a \pmod q$) as the asymptotic

$$\pi_{\frac{1}{2}}(x;\,q,\,b) - \pi_{\frac{1}{2}}(x;\,q,\,a) \sim C \log\log x \quad (x \to \infty)$$

with a constant $C > 0$ depending on q. Under DRH, it is also proved for any $w > 1/2$ that

$$\pi_w(x;\,q,\,b) - \pi_w(x;\,q,\,q) = O(1) \quad (x \to \infty). \tag{1.3}$$

This was one of the reasons why Aoki-Koyama chose $w = 1/2$ for formulating the bias. But the behavior of the left hand side for $0 < w < 1/2$ was unknown.

In this paper, we will solve this problem by generalizing Littlewood's theorem to $0 \le w < 1/2$. In what follows we use the standard notation

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}.$$

and $\varphi(q) = \#(\mathbb{Z}/q\mathbb{Z})^{\times}$. Our first main theorem is as follows.

Theorem 1.1 (Weighted prime number theorem)**.** Let $a \in \mathbb{Z}$ be coprime with $q \in \mathbb{Z}$. For $0 \le w < 1$, it holds that

$$\pi_w(x;\,q,\,a) \sim \frac{1}{\varphi(q)} \mathrm{Li}(x^{1-w}) \quad (x \to \infty).$$

We write $f(x) = \Omega(g(x))$ $(x \to \infty)$ if the following holds: There exists a constant $C > 0$ such that for any $x_0 > 0$, there exist $x_1, x_2 > x_0$ such that $f(x_1) > Cg(x_1)$ and $f(x_2) < -Cg(x_2)$.

It is known by Stark ([3] Theorem 2) that for any coprime a, b $(a \neq b)$ with q

$$\pi(x;\,q,\,a) - \pi(x;\,q,\,b) = \Omega\left(\frac{x^{\frac{1}{2}}}{\log x}\right) \quad (x \to \infty) \tag{1.4}$$

under the assumption that the Dirichlet L-function $L(s,\chi)$ has no zeros in the interval $0 < s < 1$ for any Dirichlet character χ modulo q. Actually Stark obtained the estimate (1.4) for more general cases $\varphi(q)\pi(x;\,q,\,a) - \varphi(q')\pi(x;\,q',\,b)$, but for our purpose the case $q = q'$ is sufficient.

Our second main theorem is a generalization of it.

Theorem 1.2. Suppose that $L(s,\chi)$ has no zeros in the interval $0 < s < 1$ for any Dirichlet character χ modulo q. For $0 \le w < 1/2$ and any coprime a, b $(a \neq b)$ with q, it holds that

$$\pi_w(x;\,q,\,a) - \pi_w(x;\,q,\,b) = \Omega\left(\frac{x^{\frac{1}{2}-w}}{\log x}\right) \quad (x \to \infty).$$

In particular, $\pi_w(x;\,q,\,a) - \pi_w(x;\,q,\,b)$ $(0 \le w < 1/2)$ changes its sign infinitely many times.

Theorem 1.2 together with the estimate (1.3) justifies the choice of $w = 1/2$ for the formulation of Chebyshev's bias.

2. Weighted Prime Number Theorem

The following lemma was proved in the previous paper [4].

Lemma 2.1. Assume $f(x)$ and $g(x)$ are positive valued function on $x \ge 0$. If $\lim_{x\to\infty} \int_0^x f(t)\,dt = \infty$ and $g(t) = o(1)$, it holds that

$$\int_0^x f(t)g(t)\,dt = o\left(\int_0^x f(t)\,dt\right) \quad (x \to \infty).$$

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For $q, a \in \mathbb{Z}$ with $(a, q) = 1$ and $0 \le w < 1$, we denote

$$\psi_w(x;\, q,\, a) = \sum_{\substack{p \equiv a \pmod q \\ p^m \le x}} \frac{\log p}{p^{mw}},$$

where the sum is taken over pairs $(p,\, m)$ with m a positive integer and p a prime satisfying $p \equiv a \pmod q$ and $p^m \le x$.

Lemma 2.2.

$$\psi_w(x;\, q,\, a) \sim \frac{x^{1-w}}{\varphi(q)(1-w)} \quad (x \to \infty)$$

Proof. Putting $\psi(x;\, q,\, a) = \psi_0(x;\, q,\, a)$, we have by partial summation that

$$\psi_w(x;\, q,\, a) = \frac{\psi(x;\, q,\, a)}{x^w} + w \int_2^x \frac{\psi(t;\, q,\, a)}{t^{w+1}}\, dt.$$

From the classical prime number theorem $\psi(x;\, q,\, a) = x/\varphi(q) + o(x)$ $(x \to \infty)$, we compute

$$\psi_w(x;\, q,\, a) = \frac{1}{\varphi(q)}\left(x^{1-w} + o\left(x^{1-w}\right) + w\int_2^x \frac{1}{t^w}\, dt + \int_2^x o\left(\frac{1}{t^w}\right) dt\right).$$

Here the last integral is estimated by Lemma 2.1 as

$$\int_2^x o\left(\frac{1}{t^w}\right) dt = o\left(\int_2^x \frac{1}{t^w}\, dt\right).$$

Thus we have

$$\begin{aligned}\psi_w(x;\, q,\, a) &= \frac{1}{\varphi(q)}\left(x^{1-w} + o\left(x^{1-w}\right) + w\int_2^x \frac{1}{t^w}\, dt + o\left(\int_2^x \frac{1}{t^w}\, dt\right)\right) \\ &= \frac{1}{\varphi(q)} \frac{x^{1-w}}{1-w} + o\left(x^{1-w}\right).\end{aligned}$$

□

For $0 \le w < 1$ we put

$$\theta_w(x;\, q,\, a) = \sum_{\substack{p \equiv a \pmod q \\ p \le x}} \frac{\log p}{p^w},$$

where the sum is taken over primes p satisfying $p \equiv a \pmod q$ and $p \le x$.

Lemma 2.3. It holds that

$$\psi_w(x;\, q,\, a) - \frac{\sqrt{x}\log x}{(1-2w)x^w} \le \theta_w(x;\, q,\, a) \le \psi_w(x;\, q,\, a).$$

Proof. From the definitions the right iequality is immediate. The left one is deduced as follows.

$$\begin{aligned}\psi_w(x;\, q,\, a) - \theta_w(x;\, q,\, a) &\le \log x \sum_{\substack{p \equiv a \pmod q \\ p \le \sqrt{x}}} \frac{1}{p^{2w}} \le \log x \sum_{2 \le n \le \sqrt{x}} \frac{1}{n^{2w}} \\ &\le \log x \int_1^{\sqrt{x}} \frac{dt}{t^{2w}} \le \frac{x^{\frac{1}{2}-w}\log x}{1-2w}.\end{aligned}$$

□

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Proof of Theorem 1.1. From Lemmas 2.2 and 2.3 we have

$$\theta_w(x;\, q,\, a) \sim \psi_w(x;\, q,\, a) \sim \frac{x^{1-w}}{\varphi(q)(1-w)} \quad (x \to \infty). \tag{2.1}$$

Now we will establish the relation between $\theta_w(x;\, q,\, a)$ and $\pi_w(x;\, q,\, a)$. Clearly it holds that

$$\theta_w(x;\, q,\, a) \le \sum_{\substack{p \equiv a \pmod q \\ p \le \sqrt{x}}} \frac{\log x}{p^w} = (\log x)\pi_w(x;\, q,\, a).$$

Hence

$$\frac{\theta_w(x;\, q,\, a)}{\log x} \le \pi_w(x;\, q,\, a).$$

On the other hand, for $0 \le \varepsilon < 1$ we have $1 < x^{1-\varepsilon} \le x$ and

$$\begin{aligned}
\theta_w(x;\, q,\, a) \ge \sum_{\substack{p \equiv a \pmod q \\ x^{1-\varepsilon} \le p \le x}} \frac{\log p}{p^w} &\ge \sum_{\substack{p \equiv a \pmod q \\ x^{1-\varepsilon} \le p \le x}} \frac{\log x^{1-\varepsilon}}{p^w} \\
&= \log x^{1-\varepsilon} \sum_{\substack{p \equiv a \pmod q \\ x^{1-\varepsilon} \le p \le x}} \frac{1}{p^w} \\
&= (\log x^{1-\varepsilon})(\pi_w(x;\, q,\, a) - \pi_w(x^{1-\varepsilon};\, q,\, a)).
\end{aligned}$$

Therefore

$$\pi_w(x;\, q,\, a) \le \frac{1}{1-\varepsilon} \cdot \frac{\theta_w(x;\, q,\, a)}{\log x} + \pi_w(x^{1-\varepsilon};\, q,\, a). \tag{2.2}$$

Since the last term is estimated as

$$\pi_w(x^{1-\varepsilon};\, q,\, a) = \sum_{\substack{p \equiv a \pmod q \\ p \le x^{1-\varepsilon}}} \frac{1}{p^w} < \sum_{n \le x^{1-\varepsilon}} \frac{1}{n^w} = O((x^{1-w})^{1-\varepsilon}),$$

We conclude that

$$\frac{\theta_w(x;\, q,\, a)}{\log x} \le \pi_w(x;\, q,\, a) \le \frac{1}{1-\varepsilon} \cdot \frac{\theta_w(x;\, q,\, a)}{\log x} + O((x^{1-\varepsilon})^{1-w}). \tag{2.3}$$

Now combining with (2.1) we reach

$$\pi_w(x;\, q,\, a) \sim \frac{\theta_w(x;\, q,\, a)}{\log x} \sim \frac{x^{1-w}}{\varphi(q)(1-w)\log x} \sim \frac{\mathrm{Li}(x^{1-w})}{\varphi(q)}.$$

□

3. Refinements

In this section we prove Threom 1.2. The following lemma plays the role for reducing the genral weight w to the case of $w = 0$.

Lemma 3.1.
$$\int_0^\infty \pi_w(e^u;\, q,\, a)e^{-us}du = \frac{s+w}{s}\int_0^\infty \pi(e^u;\, q,\, a)e^{-u(s+w)}du.$$

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Proof. Denote by p_n the n-th smallest element in the set of primes p such that $p \equiv a \pmod q$. Putting $x = e^u$, we compute

$$\begin{aligned}
\int_0^\infty \pi_w(e^u; q, a)e^{-us}du &= \int_1^\infty \pi_w(x; q, a)x^{-s-1}dx \\
&= \sum_{n=1}^\infty \int_{p_n}^{p_{n+1}} \sum_{\substack{p\equiv a \pmod q \\ p\le p_n}} p^{-w}x^{-s-1}dx \\
&= \sum_{n=1}^\infty \sum_{\substack{p\equiv a \pmod q \\ p\le p_n}} p^{-w}\frac{p_n^{-s} - p_{n+1}^{-s}}{s} \\
&= \frac{1}{s}\left(\sum_{n=1}^\infty \sum_{\substack{p\equiv a \pmod q \\ p\le p_n}} p^{-w}p_n^{-s} - \sum_{n=2}^\infty \sum_{\substack{p\equiv a \pmod q \\ p\le p_{n-1}}} p^{-w}p_n^{-s} \right) \\
&= \frac{1}{s}\left(p_1^{-s-w} + \sum_{n=2}^\infty p_n^{-s-w} \right) \\
&= \frac{1}{s}\sum_{n=1}^\infty p_n^{-s-w}.
\end{aligned}$$

Hence for $w = 0$ we have

$$\int_0^\infty \pi(e^u; q, a)e^{-us}du = \frac{1}{s}\sum_{n=1}^\infty p_n^{-s}.$$

By substituting $s + w$ in the variable s, we obtain

$$\begin{aligned}
\int_0^\infty \pi(e^u; q, a)e^{-u(s+w)}du &= \frac{1}{s+w}\sum_{n=1}^\infty p_n^{-s-w} \\
&= \frac{s}{s+w}\int_0^\infty \pi_w(e^u; q, a)e^{-us}du.
\end{aligned}$$

□

We will apply the theory of Tauberian theorems, which was developed by Ingham in the following theorem.

Theorem 3.2. ([5] Theorem 1) Let

$$F(s) = \int_0^\infty A(u)e^{-su}du,$$

where $A(u)$ is real valued and absolutely integrable on every interval $0 \le u \le U$, and the integral is absolutely convergent in some half plane $\sigma > \sigma_1 \ge 0$.

Let $A^*(u)$ be a real trigonometrical polynomial

$$A^*(u) = \sum_{n=-N}^{N} \alpha_n e^{i\gamma_n u} \quad (\gamma_n \in \mathbb{R},\ \gamma_n = \gamma_{-n},\ \alpha_n = \overline{\alpha_{-n}}),$$

and let

$$F^*(s) = \int_0^\infty A^*(u)e^{-su}du = \sum_{n=-N}^{N} \frac{\alpha_n}{s - i\gamma_n} \quad (\sigma > 0).$$

Suppose that $F(s) - F^*(s)$ is regular in the region $\sigma \ge 0$, $-T \le t \le T$ for some $T > 0$.

Then, it holds that

$$\liminf_{u\to\infty} A(u) \le \lim_{u\to\infty} A_T^*(u) \le \limsup_{u\to\infty} A(u),$$

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where

$$A_T^*(u) = \sum_{|\gamma_n| \le T} \left(1 - \frac{|\gamma_n|}{T}\right) \alpha_n e^{i\gamma_n u}$$
$$= \alpha_0 + 2\mathrm{Re}\left(\sum_{0<\gamma_n<T} \left(1 - \frac{\gamma_n}{T}\right) \alpha_n e^{i\gamma_n u}\right).$$

This theorem was reformulated by Stark for the purpose of refining the estimate of $\pi(x; q, a) - \pi(x; q, b)$ as follows:

Theorem 3.3. ([3] p.314) Put $s = \sigma + it \in \mathbb{C}$. Let

$$\frac{F(s)}{s} = \int_0^\infty A(u) e^{-su} du,$$

where $A(u)$ is real valued and absolutely integrable on every interval $0 \le u \le U$, and the integral is absolutely convergent for $\sigma > 1$. For any real sequence γ_n $(n \in \mathbb{Z})$ with $\gamma_{-n} = \gamma_n$ and coefficients $\alpha_n \in \mathbb{C}$ with $\alpha_{-n} = \overline{\alpha_n}$, set

$$F_1(s) = \sum_{n=-N}^{N} \alpha_n \log\left(s - \frac{1}{2} - i\gamma_n\right).$$

Suppose that for some $T > 0$, $F'(s) - F_1'(s)$ is continuous in the region $\sigma \ge 1/2$, $-T \le t \le T$ and analytic in the interior of this region. Then for any u_0,

$$\limsup_{u\to\infty} \frac{uA(u)}{e^{u/2}} \ge -\sum_{|\gamma_n| \le T} \frac{\alpha_n}{\frac{1}{2} + i\gamma_n} \left(1 - \frac{|\gamma_n|}{T}\right) e^{i\gamma_n u_0}.$$

Theorem 3.3 follows from Theorem 3.2 applied to the derivative with respect to s of

$$\frac{F(s)}{s} = \int_1^\infty \left(A(u) + \frac{1}{u} \sum_{n=-N}^{N} \frac{\alpha_n}{\frac{1}{2} + i\gamma_n} e^{(\frac{1}{2} + i\gamma_n)u}\right) e^{-su} du.$$
$$= \sum_{n=-N}^{N} \frac{\alpha_n}{\frac{1}{2} + i\gamma_n} \log\left(s - \frac{1}{2} - i\gamma_n\right) + (\text{an entire function in } s) \tag{3.1}$$

The estimate (1.4) is an immediate consequence of Theorem 3.3 and the fact that the function

$$\int_0^\infty (\pi(e^u; q, a) - \pi(e^u; q, b)) e^{-us} du$$

is analytic in $\mathrm{Re}(s) > 1/2$ ([3] Lemmas 1 and 2).

Proof of Theorem 1.2. From Lemma 3.1 and the above discussion, the function

$$\int_0^\infty (\pi_w(e^u; q, a) - \pi_w(e^u; q, b)) e^{-us} du$$

is analytic in $\mathrm{Re}(s) > \frac{1}{2} - w$. For $0 < w < \frac{1}{2}$, we replace (3.1) with the following function:

$$\frac{F_w(s)}{s+w} = \sum_{n=-N}^{N} \frac{\alpha_n}{\frac{1}{2} + i\gamma_n} \log\left(s + w - \frac{1}{2} - i\gamma_n\right) + (\text{an entire function in } s)$$
$$= \int_1^\infty \left(A_w(u) + \frac{1}{u} \sum_{n=-N}^{N} \frac{\alpha_n}{\frac{1}{2} + i\gamma_n} e^{(\frac{1}{2} - w + i\gamma_n)u}\right) e^{-su} du,$$

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where $A_w(u) = \pi_w(e^u; q, a) - \pi_w(e^u; q, b)$. Then for any u_0,

$$\limsup_{u\to\infty} \frac{uA_w(u)}{e^{(\frac{1}{2}-w)u}} \geq -\sum_{|\gamma_n|\leq T} \frac{\alpha_n}{\frac{1}{2}+i\gamma_n}\left(1-\frac{|\gamma_n|}{T}\right)e^{i\gamma_n u_0}.$$

Consequently it holds that

$$A_w(u) = \Omega\left(\frac{e^{(\frac{1}{2}-w)u}}{u}\right),$$

which is the desired result. □

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