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Inequalities with Some Arithmetic Functions

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Abstract: In the paper are formulated and proved some new inequalities with the classical arithmetic functions φ (of Euler) and ψ (of Dedekind).

Keywords: arithmetic functions; inequalities; Euler function; Dedekind function

1. Introduction

For a positive integer $n = \prod_{i=1}^{r} p_i^{\alpha_i} > 1$, let $\varphi(n)$ and $\psi(n)$, respectively, denote the Euler and Dedekind totient function values, i.e.,

$$\varphi(n) = n\prod_{p|n}\left(1 - \frac{1}{p}\right) = \prod_{i=1}^{r} p_i^{\alpha_i - 1}(p_i - 1), \tag{1}$$

$$\psi(n) = n\prod_{p|n}\left(1 + \frac{1}{p}\right) = \prod_{i=1}^{r} p_i^{\alpha_i - 1}(p_i + 1), \tag{2}$$

where p runs through the prime divisors of n, and for any i $(1 \le i \le r)$ p_i are different primes and α_i are positive integers (see, e.g., [1]).

Let

$$\varphi(1) = \psi(1) = 1.$$

Another arithmetic function, which will be used, is the "core" function of n:

$$\gamma(n) = \prod_{p|n} p = \prod_{i=1}^{r} p_i. \tag{3}$$

Let us also denote

$$\omega(n) = r,$$

i.e., the number of the distinct prime factors of n, and

$$\Omega(n) = \sum_{i=1}^{r} \alpha_i,$$

i.e., the total number of prime factors of n (see [1]).

The aim of this paper is to obtain certain new inequalities for these functions.

2. Main results

Theorem 1. *For $n > 1$ one has*

$$\psi(n) - \varphi(n) \ \ge\ 2^{\omega(n)}\frac{n}{\gamma(n)} \ >\ 2^{\omega(n)}\frac{\varphi(n)}{\gamma(n)}. \tag{4}$$

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Proof. The first inequality of (4) is proved in [2], where the arithmetic inequality

$$\prod_{i=1}^{r}(y_i+1)-\prod_{i=1}^{r}(y_i-1)\geq 2^r$$

is used for $y_i \geq 2$. By putting $y_i = p_i$ $(1 \leq i \leq r)$ and using (1) and (2), the result follows. The second inequality of (4) follows by the classical inequality $n > \varphi(n)$ for $n \geq 2$. □

Corollary 1. *For* $n > 1$,

$$\frac{\psi(n)}{\varphi(n)}-1>\frac{2^{\omega(n)}}{\gamma(n)}. \tag{5}$$

This follows by the weaker inequality in (4), by dividing both terms with $\varphi(n)$.

Theorem 2. *For $n > 1$ one has*

$$\frac{\psi(n)}{n}\geq 1+\frac{\omega(n)}{\gamma(n)}\geq\frac{\varphi(n)}{n}+\frac{2^{\omega(n)}}{\gamma(n)}. \tag{6}$$

Proof. By (1), (2) and (3), the first inequality of (6) can be rewritten as

$$\prod_{i=1}^{r}(p_i+1)\ \geq\ \left(\prod_{i=1}^{r}p_i\right)\left(1+\frac{r}{\prod\limits_{i=1}^{r}p_i}\right)\ =\prod_{i=1}^{r}p_i+r=\gamma(n)+r. \tag{7}$$

Relation (7) follows from the fact that

$$\prod_{i=1}^{r}(p_i+1)\ \geq\ \prod_{i=1}^{r}p_i+\sum_{i=1}^{r}p_i\ \geq\ \prod_{i=1}^{r}p_i+r$$

with an equality only for $r = 1$. The second inequality of (6) can be rewritten as

$$\prod_{i=1}^{r}p_i-\prod_{i=1}^{r}(p_i-1)+r-2^r\geq 0. \tag{8}$$

For $r = 1$ the statement (8) is obvious, for $r = 2$ we obtain:

$$p_1p_2-(p_1-1)(p_2-1)+2-4=p_1+p+2-3>0.$$

For $r \geq 3$ we have that $p_1 \geq 2, p_2 \geq 3, p_3, \ldots, p_r \geq 5$ and

$$\begin{aligned}\prod_{i=1}^{r}p_i-\prod_{i=1}^{r}(p_i-1)+r-2^r\ &>\ \prod_{i=1}^{r}(p_i-1)\left(\prod_{i=1}^{r}\left(1+\frac{1}{p_i-1}\right)-\frac{2^r}{\prod\limits_{i=1}^{r}(p_i-1)}\right)\\ &\geq\ 8\prod_{i=4}^{r}(p_i-1)\left(1-\frac{2^r}{8\prod\limits_{i=4}^{r}(p_i-1)}\right)\geq 0,\end{aligned}$$

which proves the theorem. □

We can note that the relation (6) offers an improvement of the first inequality of (4).

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Corollary 2. *For $n > 1$*

$$\varphi(n) \leq n - \frac{n\omega(n)}{\gamma(n)}. \tag{9}$$

Indeed, the second inequality of (6) can be written as

$$\varphi(n) \leq n - \frac{(2^{\omega(n)} - \omega(n))n}{\gamma(n)}. \tag{10}$$

Obviously, relation (10) is stronger than (9), as $2^r - r \geq r$, i.e., $2^r \geq 2r$ for each $r \geq 1$. Relation (9) can be rewritten also as

$$\frac{n}{\varphi(n)} \geq \frac{\gamma(n)}{\gamma(n) - \omega(n)}.$$

The following refinement of this inequality holds true:

Theorem 3. *For $n > 1$ so that $\omega(n) \geq 2$,*

$$\frac{n}{\varphi(n)} \geq 1 + \frac{2\omega(n)}{\gamma(n)} \geq \frac{\gamma(n)}{\gamma(n) - \omega(n)}. \tag{11}$$

Proof. The first inequality of (11) can be written as

$$\prod_{i=1}^{r} p_i \left(\prod_{i=1}^{r} p_i - \prod_{i=1}^{r} (p_i - 1) \right) \geq 2r \prod_{i=1}^{r} (p_i - 1). \tag{12}$$

Obviously, for $r = 1$, (12) is valid only for $p_1 = 2$.

Let $r = \omega(n) \geq 2$. As

$$\prod_{i=1}^{r} p_i > \prod_{i=1}^{r} (p_i - 1),$$

it will be sufficient to prove that

$$\prod_{i=1}^{r} p_i - \prod_{i=1}^{r} (p_i - 1) > 2r.$$

Since $p_1 \geq 2$, $p_2 \geq 3$ we obtain that:

$$\begin{aligned} \prod_{i=1}^{r} p_i - \prod_{i=1}^{r} (p_i - 1) - 2r &\geq 6 \prod_{i=3}^{r} p_i - 2 \prod_{i=3}^{r} (p_i - 1) - 2r \\ &> 4 \prod_{i=3}^{r} p_i - 2r \begin{cases} = 0, & \text{if } r = 2 \\ > 0, & \text{if } r \geq 3 \end{cases}. \end{aligned}$$

The second inequality of (11) can be written as

$$\gamma(n)^2 \leq \gamma(n) + 2\omega(n)\gamma(n) - \omega(n)\gamma(n) - 2\omega(n)^2$$

or

$$\gamma(n) \geq 2\omega(n),$$

which is true, because

$$\gamma(n) = \prod_{i=1}^{r} p_i \geq 2^r \geq 2r. \quad \square$$

Theorem 4. *For $n > 1$*

$$\psi(n) \geq \frac{n}{2}\left(1 + \frac{n}{\varphi(n)}\right) \geq n\left(1 + \frac{\omega(n)}{\gamma(n)}\right). \tag{13}$$

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Proof. From the first inequality of (11) one has

$$1+\frac{n}{\varphi(n)}\geq 2\left(1+\frac{\omega(n)}{\gamma(n)}\right),$$

so, the second inequality of (13) follows.

The first inequality of (13) is due to Ch. R. Wall, but without a proof; it has been proved in [3] in the form

$$\frac{2\psi(n)}{n}\geq 1+\frac{n}{\varphi(n)}. \quad \square$$

Theorem 5. *For $n>1$*

$$\frac{n\omega(n)}{\gamma(n)}\leq n-\varphi(n)\leq\psi(n)-n. \tag{14}$$

Proof. The first inequality of (14) is exactly Corollary 2 (see (9)). The second inequality, written in the form

$$\varphi(n)+\psi(n)\geq 2n$$

is due to Ch. R. Wall [4]. $\square$

We must mention that the relation (14) refines the first inequality of (6).

Theorem 6. *For $n>1$ one has*

$$\psi(n)\geq n+2^{\omega(n)+\Omega(n)-2}\geq n+1. \tag{15}$$

Proof. By using of (2) and the definitions of $\omega(n)$ and $\Omega(n)$, one has that

$$\psi(n)-n=\prod_{i=1}^{r}p_i^{\alpha_i-1}\left(\prod_{i=1}^{r}(p_i+1)-\prod_{i=1}^{r}p_i\right).$$

Since

$$\prod_{i=1}^{r}p_i^{\alpha_i-1}\geq 2^{\sum\limits_{i=1}^{r}\alpha_i-r}=2^{\Omega(n)-\omega(n)},$$

then in order to prove (15), we have to prove that

$$\prod_{i=1}^{r}(p_i+1)-\prod_{i=1}^{r}p_i\geq 2^{2r-2}. \tag{16}$$

For $r=1$ in (16) there is an equality, while for $r=2$ one has

$$\begin{aligned}(p_1+1)(p_2+1)-p_1p_2 &= p_1+p_2+1\\ &\geq\ 2+3+1\\ &=6>4\\ &=\ 2^2.\end{aligned}$$

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We can prove that there is strict inequality in (16) for $r \geq 2$. Having in mind that $p_1 \geq 2, p_2 \geq 3$ for each i $(3 \leq i \leq r)$: $p_i \geq 5$, we obtain

$$\begin{aligned}\prod_{i=1}^{r}(p_i+1) - \prod_{i=1}^{r} p_i - 2^{2r-2} &\geq 12\prod_{i=3}^{r}(p_i+1) - 6\prod_{i=3}^{r} p_i - 2^{2r-2} \\ &> 6\prod_{i=3}^{r}(p_i+1) - 2^{2r-2} \\ &\geq 6.6^{r-2} - 2^{2r-2} \\ &= 6^{r-1} - 4^{r-1} \\ &> 0,\end{aligned}$$

which proves (16) and therefore (15). □

From the above proof it follows that there is equality in (15) only for $n = 2^{\alpha}$ for $\alpha = 1, 2, \ldots$ or for n being a prime number.

The second inequality of (15) follows from $\Omega(n) + \omega(n) \geq 2$.

Theorem 7. *Let*

$$\lambda = \frac{\ln 2}{\ln 3}.$$

Then

$$\varphi(n) \geq n.\begin{cases} \frac{1}{(\gamma(n))^{1-\lambda}}, & \text{if } n > 1 \text{ is odd} \\ \frac{1}{2^{\lambda}(\gamma(n))^{1-\lambda}}, & \text{if } n > 1 \text{ is even} \end{cases}. \tag{17}$$

Proof. Let us consider the application

$$f_1(x) = \frac{x-1}{x^{\lambda}}$$

for $x > 1$. Then an easy computation gives

$$f_1'(x) = \frac{1}{x^{\lambda+1}}((1-\lambda)x + \lambda) > 0$$

for $0 < \lambda < 1$. Thus, the function f_1 is strictly increasing. Particularly, for $p \geq 3$

$$f(p) \geq f(3) = \frac{2}{3^{\lambda}} = 1$$

for $3^{\lambda} = 2$, i.e., it is valid only for

$$\lambda = \frac{\ln 2}{\ln 3} = 0.63092\ldots .$$

Thus, we have the inequality

$$p - 1 \geq p^{\lambda} \tag{18}$$

for $p \geq 3$ with equality only for $p = 3$.

Now, it is well known that

$$\frac{\varphi(n)}{n} = \prod_{p|n}\left(1 - \frac{1}{p}\right),$$

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where p runs through the prime divisors of n. Now, for $n > 1$ odd, by (18) we obtain

$$\prod_{p|n}\left(1-\frac{1}{p}\right) = \frac{\prod_{p|n}(p-1)}{\prod_{p|n} p} \geq \frac{\prod_{p|n} p^{\lambda}}{\prod_{p|n} p}.$$

Thus, the first inequality of (17) follows. When n is even, then let $p_1 = 2$ be the least prime divisor of n. Then by (18)

$$\frac{\varphi(n)}{n} = \frac{1}{2}\prod_{p|n,p\geq 3}\left(1-\frac{1}{p}\right) \geq \frac{1}{\prod_{p|n} p}\left(\frac{\prod_{p|n} p}{2}\right)^{\lambda}.$$

Thus, the second inequality of (17) follows as well. □

Remark 1. *As $\lambda > \frac{1}{2}$ and $\gamma(n) \geq 2$, by (17) we get the weaker inequality*

$$\varphi(n) \geq n.\begin{cases} \frac{1}{\sqrt{\gamma(n)}}, & \text{if } n > 1 \text{ is odd,} \\ \frac{1}{\sqrt{2\gamma(n)}}, & \text{if } n > 1 \text{ is even,} \end{cases}$$

proved in [5].

Remark 2. *The number $\lambda = \frac{\ln 2}{\ln 3}$ is an irrational number. Indeed, as $0 < \lambda < 1$, it cannot be an integer. If it would be rational, i.e.,*

$$\frac{\ln 2}{\ln 3} = \frac{a}{b}$$

for some integers $a, b > 1$, then we would obtain $2^b = 3^a$, that is impossible, as the left side is even and the right side is odd. But λ is even a transcendental number, according for the famous theorem of Gelfond–Schneider [6]. If a and b are algebraic numbers with $a \notin \{0,1\}$ and b not a rational, then a^b is transcendental. In our case, $3^{\lambda} = 2$ and since λ is irrational, by the above theorem, if λ would be algebraic, we would obtain a contradiction.

Theorem 8. *Let*

$$\mu = \frac{\ln 4}{\ln 3}.$$

Then

$$\psi(n) \leq n.\begin{cases} (\gamma(n))^{\mu-1}, & \text{if } n > 1 \text{ is odd} \\ \frac{3}{2^{\mu}}(\gamma(n))^{\mu-1}), & \text{if } n > 1 \text{ is even} \end{cases}. \quad (19)$$

Proof. Let us define

$$f_2(x) = \frac{x^{\mu}}{x+1}$$

for $x > 1$. For the derivative of this function, one has

$$f_2'(x) = \frac{x^{\mu-1}}{(x+1)^2}(x(\mu-1)+\mu) > 0,$$

as $\mu > 1$. This f_2 is strictly increasing and implying

$$f_2(p) \geq f_2(3) = \frac{3^{\mu}}{4} = 1$$

for $p \geq 3$. This implies the inequality

$$p + 1 \leq p^{\mu} \quad (20)$$

for $p \geq 3$ with μ satisfying $\frac{3^{\mu}}{4}$, i.e., $\mu = \frac{\ln 4}{\ln 3} = 1.26185\ldots$.

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Now, the proof of (19) follows by applying (20) in the same manner, as in the proof of Theorem 7. □

Remark 3. *As $\mu - 1 = 0.26185\ldots < \frac{1}{2}$, it is easy to see from (19) we get the weaker relation*

$$\psi(n) \leq n.\begin{cases} \sqrt{\gamma(n)}, & \text{if } n > 1 \text{ is odd,} \\ \sqrt{2\gamma(n)}, & \text{if } n > 1 \text{ is even.} \end{cases}$$

Remark 4. *From the proof of Theorem 7, it follows that there is an equality in the first part of (17) only for $n = 3^k$, where $k \geq 1$ is integer, and for $n = 2^k$ in the second part. Similarly, for the relation (19).*

Remark 5. *As $\mu = \frac{2\ln 2}{\ln 3}$, from Remark 2 we get that μ is also a transcendental number.*

Theorem 9. *For each $n > 1$*

$$\varphi(n)\psi(n) \leq n^2 - \omega(n)\left(\frac{n}{\gamma(n)}\right)^2. \tag{21}$$

Proof. When n is prime, (21) is obviously true. Let us assume that (21) is valid for some $n > 1$ with $\omega(n) \geq 2$ and let $p \geq 2$ is not a divisor of n. Then

$$\begin{aligned}
&(np)^2 - \omega(np)\left(\frac{np}{\gamma(np)}\right)^2 - \varphi(np)\psi(np) \\
&\quad = n^2p^2 - (\omega(n)+1)\left(\frac{n}{\gamma(n)}\right)^2 - \varphi(n)\psi(n)(p^2-1) \\
&\quad \geq n^2p^2 - (\omega(n)+1)\left(\frac{n}{\gamma(n)}\right)^2 - \left(n^2 - \omega(n)\left(\frac{n}{\gamma(n)}\right)^2\right)(p^2-1) \\
&\quad = n^2p^2 - \omega(n)\left(\frac{n}{\gamma(n)}\right)^2 - \left(\frac{n}{\gamma(n)}\right)^2 \\
&\qquad - n^2p^2 + \omega(n)\left(\frac{n}{\gamma(n)}\right)^2 p^2 + n^2 - \omega(n)\left(\frac{n}{\gamma(n)}\right)^2 \\
&\quad = \omega(n)\left(\frac{n}{\gamma(n)}\right)^2 (p^2-2) + n^2 - \left(\frac{n}{\gamma(n)}\right)^2 \\
&\quad > 0.
\end{aligned}$$

Let $p \geq 2$ be a divisor of n. Then

$$\begin{aligned}
&(np)^2 - \omega(np)\left(\frac{np}{\gamma(np)}\right)^2 - \varphi(np)\psi(np) \\
&\quad = n^2p^2 - \omega(n)\left(\frac{n}{\gamma(n)}\right)^2 p^2 - \varphi(n)\psi(n)p^2 \\
&\quad \geq n^2p^2 - \omega(n)\left(\frac{n}{\gamma(n)}\right)^2 p^2 - \left(n^2 - \omega(n)\left(\frac{n}{\gamma(n)}\right)^2\right)p^2 \\
&\quad = 0,
\end{aligned}$$

which proves the theorem. □

Remark 6. *Let $\sigma(n)$ denote the sum of the divisors of n. By the known inequality for $n > 1$*

$$\varphi(n)\sigma(n) < n^2,$$

we get from (17) *the following relation for* $\sigma(n)$:

$$\sigma(n) < n.\begin{cases} (\gamma(n))^{1-\lambda}, & \text{if } n > 1 \text{ is odd,} \\ \frac{1}{2^{\lambda}}(\gamma(n))^{1-\lambda}), & \text{if } n > 1 \text{ is even.} \end{cases}$$

As $\gamma(n) \le n$ *and* $1 - \lambda < \frac{1}{2}$, *this improves the inequaity*

$$\sigma(n) < n\sqrt{n}$$

by C. C. Lindner (see, e.g., [1]).

Remark 7. *By using the known inequality for* $n > 1$ *(see, e.g., [7])*

$$\sigma(n) < \frac{\pi^2}{6}\psi(n) \tag{22}$$

and combining with relation (19), *one can obtain another upper bound for* $\sigma(n)$. *For example, when* $n > 1$ *is odd, we get from* (22)

$$\sigma(n) < \frac{\pi^2}{6}n(\gamma(n))^{\mu-1}. \tag{23}$$

Since $\gamma(n) \le n$, *a simple computation shows that the right side of* (23) *is less than* $\frac{6}{\pi^2}n\sqrt{n}$ *for* $n \ge 77$, *so we get an improvement of the inequality for* $n \ge 9$

$$\sigma(n) < \frac{6}{\pi^2}n\sqrt{n}$$

due to U. Annapurna (see, [8]).

3. Conclusion

In the authors' book [9], a lot of inequalities related to the arithmetic functions φ and ψ were given. For a brief survey of some inequalities for arithmetic functions, see paper [10].

In the present paper some new inequalities with these functions were formulated and their validity was proven.

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