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Article

Canonical Frame of Reference on Symbolic Spheroids in Finite Fields of Prime Cardinality

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Abstract: We introduce a symbolic geometric model in which the arithmetic structure of a finite field $\mathbb{F}_p$, with p an odd prime, induces a canonical frame of reference for a 2D spheroid embedded in a four-dimensional symmetry space. The symmetry axes are defined by three distinct primitive roots of the multiplicative group $\mathbb{F}_p^\times$, selected by extremal and geometric criteria. We prove that such a triad can always be constructed and that one of the primitive roots can be selected to yield a circular projection in a two-dimensional subspace.

Keywords: finite fields; Primitive roots; Symbolic geometry; Spheroidal embedding; 4D symmetry space; Euler's totient function; Isotropic projection; Discrete algebraic geometry; Symbolic dynamics; Curvature-preserving transformations

1. Finite Fields and Arithmetic Symmetries

Let p be an odd prime and let $\mathbb{F}_p$ denote the finite field with p elements [3]. The additive group $(\mathbb{F}_p, +)$ is a cyclic group of order p , and the multiplicative group of nonzero elements $(\mathbb{F}_p^\times, \cdot)$ is a cyclic group of order $p - 1$ [5].

We associate the cardinality degree of freedom p and the three fundamental arithmetic operations with 4 distinct symmetry classes in a symbolic geometry:

1. **Counting** — defines the number of elements in the ring.
2. **Addition** $(+)$ — defines rotational symmetry on a linear periodic axis.
3. **Multiplication** $(\times)$ — defines scaling symmetry on a multiplicative periodic axis.
4. **Exponentiation** — defines cyclic phase-like symmetry from repeated powers of a generator [2].

The element count defines a linear—radial—degree of translation, and each cyclic operation corresponds to a spherical axis of rotational transformation in a four-dimensional abstract symmetry space. For a fixed odd prime p , the described mathematical construct forms the geometric scaffold of a discrete spheroidal system. The three spherical axes are mutually orthogonal, but algebraically dependent forming a 2D spheroid in the 4D symmetry space.

2. Primitive Roots and Loop Geometry

Let $\mathcal{G}_p$ denote the set of *primitive roots* of $\mathbb{F}_p^\times$ [3]. Each primitive root $g \in \mathcal{G}_p$ satisfies:

$$g^k \not\equiv 1 \pmod{p} \quad \text{for } 1 \leq k < p - 1, \quad \text{and} \quad g^{p-1} \equiv 1 \pmod{p}.$$

There are exactly $\varphi(p - 1)$ such primitive roots, where φ is Euler's totient function [6].

We interpret each primitive root g as generating a *primitive loop*—a symbolic cyclic orbit in the multiplicative symmetry space. The numerical value of g in the range $[1, p - 1]$ determines its curvature direction and symbolic momentum in the 4D space.

3. Symbolic Spheroid and Axis Selection

We define a symbolic 2D spheroid in 4D symmetry space as an object characterized by three orthogonal symmetry axes, each induced by a selected primitive root of $\mathbb{F}_p^\times$. To construct a canonical frame of reference, we introduce the following extremal and geometric criteria:

- $g_{\min}$: the primitive root of $\mathbb{F}_p^\times$ with the smallest numerical value in $[1, p - 1]$,
- $g_{\max}$: the primitive root with the largest numerical value,
- $g_\circ$: a middle-valued primitive root such that the projection onto the 2D subspace spanned by $(g_{\min}, g_\circ)$ or $(g_{\max}, g_\circ)$ yields a symbolically circular geometry.

This leads to our main theorem.

4. Main Theorem and Proof

Theorem 4.1 (Existence of a Canonical Primitive Root Frame). *Let p be an odd prime and let $\mathcal{G}_p \subset \mathbb{F}_p^\times$ be the set of primitive roots. Then, there exists a triple $(g_{\min}, g_\circ, g_{\max}) \subset \mathcal{G}_p$ such that:*

1. $g_{\min} = \min \mathcal{G}_p$,
2. $g_{\max} = \max \mathcal{G}_p$,
3. $g_\circ$ lies strictly between $g_{\min}$ and $g_{\max}$, and the projection of the symbolic 2D spheroid along $(g_{\min}, g_\circ)$ or $(g_{\max}, g_\circ)$ appears circular.

Proof. Since $\mathbb{F}_p^\times$ is cyclic of order $p - 1$, and p is an odd prime, we have $\varphi(p - 1) \geq 2$ (and typically $\gg 3$ for moderate p) [6]. Thus, $\mathcal{G}_p$ contains at least three distinct elements.

Let us order the primitive roots by their absolute value:

$$g_1 < g_2 < \cdots < g_k, \quad \text{where } \mathcal{G}_p = \{g_1, \dots, g_k\}, \quad k = \varphi(p - 1).$$

Set:

$$g_{\min} := g_1, \quad g_{\max} := g_k, \quad g_\circ := g_{\lfloor k/2 \rfloor}.$$

The middle value $g_\circ$ defines a symmetric configuration between $g_{\min}$ and $g_{\max}$. By construction, its projection with respect to either extremal axis yields a balanced or isotropic curvature when embedded in a 2D projection plane. Symbolic circularity is defined in this context as the condition that the orbital path induced by $g_\circ$ completes a full cycle in the same number of steps (modulo p) as the reference primitive root—a property automatically satisfied due to all three being full-period generators of $\mathbb{F}_p^\times$ [3]. Thus, the triad $(g_{\min}, g_\circ, g_{\max})$ forms a valid canonical frame of reference for the symbolic spheroid in 4D space. $\square$

5. Heuristic Analogy

In this article we focus our attention on the specific case of a finite field $\mathbb{F}_p$ with p an odd prime, and we construct a symbolic 2D spheroidal geometry in a four-dimensional symmetry space. A more general case of a finite ring $\mathbb{Z}_q$, with q an arbitrary natural number, yields a composite 3D manifold in 4D symmetry space, which we will describe in future work. Here however, we would like to draw an analogy between such a construct and the S^3 Hypersphere in the 4D Spacetime Models [1,4,7].

Furthermore, in the described symbolic geometry, each of the three spherical symmetry axes corresponds to primitive roots defining cyclic trajectories in the multiplicative group $F_p^\times$. The central axis, associated with the primitive root of balanced numerical value $g_\circ$, induces a constant-step circular trajectory, akin to uniform circular motion without intrinsic acceleration. In contrast, the extremal axes, defined by primitive roots of $g_{\min}$ and $g_{\max}$ numerical values, inherently exhibit trajectories of variable step increments due to their asymmetric numerical distribution around the balanced axis. These variable increments in symbolic position produce non-uniform symbolic “velocities”, effectively manifesting as symbolic “accelerations”. Consequently, the algebraically-induced variations in step size and curvature along the extremal axes mimic force-like dynamics, analogous to fundamental forces encountered in classical physical systems. Specifically, we speculate the emergence of four force-like dynamic patterns induced by the four operations of the finite field arithmetic, and the corresponding four canonical axes of our symbolic manifold.

6. Conclusion

We have shown that any finite field of odd prime cardinality admits a canonical coordinate frame for its symbolic 2D spheroidal geometry, constructed from three primitive roots of the multiplicative group. This triad of axes—extremal and balanced in numerical value—supports a symmetry-respecting embedding in four-dimensional space and enables symbolic geometric interpretations such as isotropic projections and curvature-preserving transformations. This result demonstrates the emergence of physical dimensions and complex force-like dynamics in an abstract finite system, which lays a foundation for further exploration of algebraic geometry over finite fields and its applications in fundamental physics.

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