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On a System of Hadamard Fractional Differential Equations with Nonlocal Boundary Conditions on an Unbounded Domain

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Abstract: We study the existence of positive solutions for a system of Hadamard fractional differential equations on an infinite interval with nonnegative nonlinearities, subject to coupled nonlocal boundary conditions which contain Hadamard fractional derivatives and Riemann-Stieltjes integrals. In the proof of the main results, we use the Guo-Krasnosel'skii fixed point theorem and the Leggett-Williams fixed point theorem.

Keywords: Hadamard fractional differential equations; nonlocal boundary conditions; positive solutions

MSC: 34A08; 34B10; 34B15; 34B18

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1. Introduction

We consider the system of nonlinear Hadamard fractional differential equations

$$\begin{cases} {}^H D^{\alpha}_{1+} x(t) + \mathfrak{a}(t)\mathfrak{f}(x(t), y(t)) = 0, \ \ t \in (1, \infty), \\ {}^H D^{\beta}_{1+} y(t) + \mathfrak{b}(t)\mathfrak{g}(x(t), y(t)) = 0, \ \ t \in (1, \infty), \end{cases} \tag{1}$$

supplemented with the nonlocal boundary conditions

$$\begin{cases} x(1) = x'(1) = \cdots = x^{(n-2)}(1) = 0, \\ {}^H D^{\alpha-1}_{1+} x(\infty) = \int_1^\infty x(s)\, d\mathfrak{H}_1(s) + \int_1^\infty y(s)\, d\mathfrak{H}_2(s), \\ y(1) = y'(1) = \cdots = y^{(m-2)}(1) = 0, \\ {}^H D^{\beta-1}_{1+} y(\infty) = \int_1^\infty x(s)\, d\mathfrak{K}_1(s) + \int_1^\infty y(s)\, d\mathfrak{K}_2(s), \end{cases} \tag{2}$$

where $\alpha \in (n-1, n]$, $\beta \in (m-1, m]$, $n, m \in \mathbb{N}$, $n, m \geq 2$, ${}^H D^{\kappa}_{1+}$ denotes the Hadamard fractional derivative of order κ (for $\kappa = \alpha, \beta, \alpha - 1, \beta - 1$), the functions $\mathfrak{a}, \mathfrak{b} : (1, \infty) \to \mathbb{R}_+$ and $\mathfrak{f}, \mathfrak{g} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ verify some assumptions, ($\mathbb{R}_+ = [0, \infty)$), and the integrals from (2) are Riemann-Stieltjes integrals with $\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{K}_1, \mathfrak{K}_2 : [1, \infty) \to \mathbb{R}$ functions of bounded variation.

We will prove the existence of positive solutions of problem (1), (2) under some conditions on the data of problem, by using the Guo-Krasnosel'skii fixed point theorem and the Leggett-Williams fixed point theorem. By a positive solution of (1), (2) we mean a pair of functions $(x(t), y(t))$, $t \in [1, \infty)$ satisfying (1) and (2), with $x(t) \geq 0$, $y(t) \geq 0$ for all $t \in [1, \infty)$, and $x(t) > 0$ for all $t \in (1, \infty)$ or $y(t) > 0$

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for all $t \in (1, \infty)$. This problem is a generalization of the problem investigated in [13]. In the paper [13], the authors studied the system (1) with α, $\beta \in (1, 2]$ $(n = m = 2)$, subject to the boundary conditions

$$\begin{cases} x(1) = 0, \ \ {}^{H}D_{1+}^{\alpha-1}x(\infty) = \sum\limits_{i=1}^{m} \lambda_i \, {}^{H}I_{1+}^{\alpha_i} y(\eta), \\ y(1) = 0, \ \ {}^{H}D_{1+}^{\beta-1}y(\infty) = \sum\limits_{j=1}^{n} \sigma_j \, {}^{H}I_{1+}^{\beta_j} x(\xi), \end{cases} \tag{3}$$

where λ_i, $\sigma_j > 0$ for $i = 1, \ldots, m$, $j = 1, \ldots, n$, $\eta > 1$, $\xi > 1$, and ${}^{H}I_{1+}^{k}$ is the Hadamard fractional integral of order k with lower limit 1 for $k = \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$, defined by

$${}^{H}I_{1+}^{k}h(t) = \frac{1}{\Gamma(k)} \int_1^t \left(\ln \frac{t}{s}\right)^{k-1} \frac{h(s)}{s} \, ds, \ \ t > 0,$$

(for a function $h : [1, \infty) \to \mathbb{R}$), (see [7]). The last conditions for ∞ from (3) are particular cases of our conditions from (2). Indeed we have

$$\sum_{i=1}^{m} \lambda_i \, {}^{H}I_{1+}^{\alpha_i} y(\eta) = \int_1^{\infty} y(s) \, d\mathfrak{H}_2(s), \ \ \text{and} \ \ \sum_{j=1}^{n} \sigma_j \, {}^{H}I_{1+}^{\beta_j} x(\xi) = \int_1^{\infty} x(s) \, d\mathfrak{K}_1(s),$$

with $\mathfrak{H}_2(s) = \sum_{i=1}^{m} \lambda_i \widetilde{H}_i(s)$, ($\mathfrak{H}_1 \equiv 0$), and $\mathfrak{K}_1(s) = \sum_{j=1}^{n} \sigma_j \widetilde{K}_j(s)$, ($\mathfrak{K}_2 \equiv 0$), where

$$\widetilde{H}_i(s) = \begin{cases} \dfrac{1}{\Gamma(\alpha_i + 1)} \left((\ln \eta)^{\alpha_i} - \left(\ln \dfrac{\eta}{s}\right)^{\alpha_i} \right), & \text{if } 0 \le s \le \eta, \\ \dfrac{1}{\Gamma(\alpha_i + 1)} (\ln \eta)^{\alpha_i}, & \text{if } s \ge \eta, \end{cases}$$

and

$$\widetilde{K}_j(s) = \begin{cases} \dfrac{1}{\Gamma(\beta_j + 1)} \left((\ln \xi)^{\beta_j} - \left(\ln \dfrac{\xi}{s}\right)^{\beta_j} \right), & \text{if } 0 \le s \le \xi, \\ \dfrac{1}{\Gamma(\beta_j + 1)} (\ln \xi)^{\beta_j}, & \text{if } s \ge \xi, \end{cases}$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Therefore, in our paper, we consider general orders for the fractional derivatives in the equations of system (1). Besides, in the boundary conditions (2), the fractional derivatives ${}^{H}D_{1+}^{\alpha-1}x$ and ${}^{H}D_{1+}^{\beta-1}y$ for ∞ are dependent on both functions x and y, in comparison with the condition (3) from [13], where the derivative of x is dependent only of y, and the derivative of y is dependent only of x. In addition, the conditions (2) with Riemann-Stieltjes integrals generalize, as we saw before, the conditions (3). These aspects represent the novelties for our problem (1), (2).

In what follows we will present other fractional boundary value problems studied in the last years, and related to problem (1), (2), namely, some Hadamard fractional differential equations subject to various nonlocal boundary conditions. In [14], the authors studied the existence of nonnegative multiple solutions for the Hadamard fractional differential equation with integral boundary conditions

$$\begin{cases} {}^{H}D_{1+}^{\alpha}u(t) + a(t)f(u(t)) = 0, \ \ t \in (0, \infty), \\ u(1) = 0, \ \ {}^{H}D_{1+}^{\alpha-1}u(\infty) = \sum\limits_{i=1}^{m} \lambda_i \, {}^{H}I_{1+}^{\beta_i} u(\eta), \end{cases}$$

where $\alpha \in (1, 2]$, $\eta \in (1, \infty)$, $\beta_i > 0$ and $\lambda_i \ge 0$ for all $i = 1, \ldots, m$. In the proofs of the main results of [14], they applied Leggett-Williams and Guo-Krasnosel'skii fixed point theorems. In [16], by using the Banach fixed point theorem, the authors investigated the existence of the unique solution

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for the Hadamard fractional integro-differential equation subject to Hadamard fractional integral boundary conditions

$$\begin{cases} {}^HD^{\gamma}_{1+}u(t) + f(t, u(t), {}^HI^{q}_{1+}u(t)) = 0, \ \ t \in (1, \infty), \\ u(1) = u'(1) = 0, \ \ {}^HD^{\gamma-1}_{1+}u(\infty) = \sum_{i=1}^{m} \lambda_i \, {}^HI^{\beta_i}_{1+}u(\eta), \end{cases}$$

where $\gamma \in (2, 3)$, $\eta \in (1, \infty)$, $q > 0$, $\beta_i > 0$ and $\lambda_i \geq 0$ for all $i = 1, \ldots, m$. In [15], the authors studied the existence of positive solutions for the nonlinear Hadamard fractional differential equation supplemented with Hadamard integral and multi-point boundary conditions

$$\begin{cases} {}^HD^{q}_{1+}x(t) + \sigma(t)f(t, x(t)) = 0, \ \ t \in (1, \infty), \\ x(1) = x'(1) = 0, \ \ {}^HD^{q-1}_{1+}x(\infty) = a \, {}^HI^{\beta}_{1+}x(\xi) + b\sum_{i=1}^{m-2} \alpha_i x(\eta_i), \end{cases}$$

where $q \in (2, 3]$, $1 < \xi < \eta_1 < \cdots < \eta_{m-2} < \infty$, $a, b \in \mathbb{R}$, and $\alpha_i \geq 0$ for all $i = 1, \ldots, m-2$. In the main theorems they used the monotone iterative method to find two "twin" positive solutions of the problem, and then they presented monotone iterative schemes convergent to a unique positive solution. In [17], the authors investigated the nonlinear Hadamard fractional differential equation with nonlocal boundary conditions

$$\begin{cases} {}^HD^{\alpha}_{1+}x(t) + a(t)f(t, x(t)) = 0, \ \ t \in (1, \infty), \\ x(1) = x'(1) = 0, \ \ {}^HD^{\alpha-1}_{1+}x(\infty) = \sum_{i=1}^{m} \alpha_i \, {}^HI^{\beta_i}_{1+}x(\eta) + b\sum_{j=1}^{n} \sigma_j x(\xi_j), \end{cases} \tag{4}$$

where $\alpha \in (2, 3)$, $\beta_i > 0$ and $\alpha_i \geq 0$ for all $i = 1, \ldots, m$, $\sigma_j \geq 0$ for all $j = 1, \ldots, n$, $1 < \eta < \xi_1 < \ldots < \xi_n < \infty$. They studied the existence, uniqueness and multiplicity of positive solutions for problem (4), by using the Schauder fixed point theorem, the Banach contraction mapping principle, the monotone iterative method, and a fixed point theorem due to Avery and Peterson. In [18], the authors proved a generalization of a fixed point theorem due to Avery and Henderson, and then they applied it to problem (4) and proved the existence of at least three positive solutions of (4). We also mention the monographs [1–4,6–8,10–12,19] for various fractional differential equations and systems of fractional differential equations subject to diverse boundary conditions, and their applications in varied fields.

The paper is organized as follows. In Section 2 we present some preliminary results that will be used in the next section, including the solution for the corresponding linear problem, the Green functions and their properties. In Section 3 we give the main existence theorems for (1), (2), in Section 4 we present an example that illustrates the obtained results, and Section 5 contains the conclusions for our paper.

2. Auxiliary results

In this section we present some preliminary results that we will use in the next section.

Definition 1. *([7]) For a function $z : [a, \infty) \to \mathbb{R}$, ($a \geq 0$), the left-sided Hadamard integral of fractional order $p > 0$ with lower limit a is defined by*

$$({}^HI^{p}_{a+}z)(x) = \frac{1}{\Gamma(p)} \int_a^x \left(\ln\frac{x}{t}\right)^{p-1} \frac{z(t)}{t}\, dt, \ \ x > 0.$$

Definition 2. *([7]) For a function $z : [a, \infty) \to \mathbb{R}$, $(a \geq 0)$, the left-sided Hadamard fractional derivative of order $p \geq 0$ with lower limit a is defined by*

$$ {}^{H}D^{p}_{a+}z(x) = \left(x\frac{d}{dx}\right)^{n} {}^{H}I^{n-p}_{a+}z(x) = \frac{1}{\Gamma(n-p)}\left(x\frac{d}{dx}\right)^{n}\int_a^x \left(\ln\frac{x}{t}\right)^{n-p-1}\frac{z(t)}{t}\,dt, $$

where $n = [p] + 1$.

For $p = m \in \mathbb{N}$, then ${}^{H}D^{m}_{a+}z(x) = (\delta^m z)(x),\ x > a$, where $\delta = x\frac{d}{dx}$ is the δ-derivative.

Lemma 1. *([7]) If α, $\beta > 0$, and $a > 0$, then*

$$ \left({}^{H}I^{\alpha}_{a+}\left(\ln\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\ln\frac{x}{a}\right)^{\beta+\alpha-1}, $$
$$ \left({}^{H}D^{\alpha}_{a+}\left(\ln\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\ln\frac{x}{a}\right)^{\beta-\alpha-1}, $$

and in particular ${}^{H}D^{p}_{a+}(\ln\frac{t}{a})^{p-j}(x) = 0$, for $j = 1, \ldots, [p] + 1$.

Lemma 2. *([7]) Let $p > 0$ and $z, u \in C[1, \infty) \cap L^1(0, \infty)$. Then the Hadamard fractional differential equation ${}^{H}D^{p}_{1+}z(t) = 0$ has the solutions*

$$ z(t) = \sum_{i=1}^{n} c_i(\ln t)^{p-i}, $$

and the following formula holds

$$ {}^{H}I^{p}_{1+}\,{}^{H}D^{p}_{1+}u(t) = u(t) + \sum_{i=1}^{n} d_i(\ln t)^{p-i}, $$

where $c_i, d_i \in \mathbb{R}$, $i = 1, \ldots, n$ and $n = [p] + 1$.

We consider now the system of fractional differential equations

$$ \begin{cases} {}^{H}D^{\alpha}_{1+}x(t) + h(t) = 0, & t \in (1, \infty), \\ {}^{H}D^{\beta}_{1+}y(t) + k(t) = 0, & t \in (1, \infty), \end{cases} \tag{5} $$

where $h, k \in C([1, \infty), \mathbb{R}_+)$, subject to the boundary conditions (2). We denote by

$$ \begin{aligned} & a = \Gamma(\alpha) - \int_1^\infty (\ln s)^{\alpha-1}\,d\mathfrak{H}_1(s),\ \ b = \int_1^\infty (\ln s)^{\beta-1}\,d\mathfrak{H}_2(s), \\ & c = \int_1^\infty (\ln s)^{\alpha-1}\,d\mathfrak{K}_1(s),\ \ d = \Gamma(\beta) - \int_1^\infty (\ln s)^{\beta-1}\,d\mathfrak{K}_2(s), \\ & \Delta = ad - bc. \end{aligned} \tag{6} $$

Lemma 3. *Assume that a, b, c, $d \in \mathbb{R}$ and the functions $h, k \in C([1, \infty), \mathbb{R}_+)$ satisfy the conditions $\int_1^\infty h(s)\,\frac{ds}{s} < \infty$ and $\int_1^\infty k(s)\,\frac{ds}{s} < \infty$. If $\Delta \neq 0$, then the solution of problem (5),(2) is given by*

$$ \begin{aligned} x(t) = & -\frac{1}{\Gamma(\alpha)}\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} h(s)\,\frac{ds}{s} \\ & + \frac{(\ln t)^{\alpha-1}}{\Delta}\left[d\int_1^\infty h(s)\,\frac{ds}{s} - \frac{d}{\Gamma(\alpha)}\int_1^\infty \left(\int_1^s \left(\ln\frac{s}{\tau}\right)^{\alpha-1} h(\tau)\frac{d\tau}{\tau}\right) d\mathfrak{H}_1(s)\right. \end{aligned} $$

$$
\begin{aligned}
&-\frac{d}{\Gamma(\beta)}\int_1^\infty\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\beta-1}k(\tau)\,\frac{d\tau}{\tau}\right)d\mathfrak{H}_2(s)\\
&+b\int_1^\infty k(s)\,\frac{ds}{s}-\frac{b}{\Gamma(\alpha)}\int_1^\infty\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-1}h(\tau)\,\frac{d\tau}{\tau}\right)d\mathfrak{K}_1(s)\\
&-\frac{b}{\Gamma(\beta)}\int_1^\infty\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\beta-1}k(\tau)\,\frac{d\tau}{\tau}\right)d\mathfrak{K}_2(s)\Bigg],\ \ t\in[1,\infty),\\
y(t)=&-\frac{1}{\Gamma(\beta)}\int_1^t\left(\ln\frac{t}{s}\right)^{\beta-1}k(s)\,\frac{ds}{s}\\
&+\frac{(\ln t)^{\beta-1}}{\Delta}\Bigg[a\int_1^\infty k(s)\,\frac{ds}{s}-\frac{a}{\Gamma(\alpha)}\int_1^\infty\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-1}h(\tau)\frac{d\tau}{\tau}\right)d\mathfrak{K}_1(s)\\
&-\frac{a}{\Gamma(\beta)}\int_1^\infty\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\beta-1}k(\tau)\,\frac{d\tau}{\tau}\right)d\mathfrak{K}_2(s)\\
&+c\int_1^\infty h(s)\,\frac{ds}{s}-\frac{c}{\Gamma(\alpha)}\int_1^\infty\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-1}h(\tau)\,\frac{d\tau}{\tau}\right)d\mathfrak{H}_1(s)\\
&-\frac{c}{\Gamma(\beta)}\int_1^\infty\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\beta-1}k(\tau)\,\frac{d\tau}{\tau}\right)d\mathfrak{H}_2(s)\Bigg],\ \ t\in[1,\infty).
\end{aligned}
\tag{7}
$$

Proof. By Lemma 2 the solutions of system (5) are given by

$$
\begin{aligned}
x(t)=&-\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}h(s)\,\frac{ds}{s}\\
&+a_1(\ln t)^{\alpha-1}+a_2(\ln t)^{\alpha-2}+\cdots+a_n(\ln t)^{\alpha-n},\ \ t\in[1,\infty),\\
y(t)=&-\frac{1}{\Gamma(\beta)}\int_1^t\left(\ln\frac{t}{s}\right)^{\beta-1}k(s)\,\frac{ds}{s}\\
&+b_1(\ln t)^{\beta-1}+b_2(\ln t)^{\beta-2}+\cdots+b_m(\ln t)^{\beta-m},\ \ t\in[1,\infty),
\end{aligned}
\tag{8}
$$

with $a_i, b_j\in\mathbb{R}$ for $i=1,\dots,n$, $j=1,\dots,m$. Because $x(1)=x'(1)=\cdots=x^{(n-2)}(1)=0$ and $y(1)=y'(1)=\cdots=y^{(m-2)}(1)=0$ we deduce that $a_2=\cdots=a_n=0$ and $b_2=\cdots=b_m=0$. So, by (8) we find

$$
\begin{aligned}
x(t)&=-\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}h(s)\,\frac{ds}{s}+a_1(\ln t)^{\alpha-1},\ \ t\in[1,\infty),\\
y(t)&=-\frac{1}{\Gamma(\beta)}\int_1^t\left(\ln\frac{t}{s}\right)^{\beta-1}k(s)\,\frac{ds}{s}+b_1(\ln t)^{\beta-1},\ \ t\in[1,\infty).
\end{aligned}
\tag{9}
$$

We have ${}^HD_{1+}^{\alpha-1}x(t)=-\int_1^t h(s)\frac{ds}{s}+a_1\Gamma(\alpha)$ and ${}^HD_{1+}^{\beta-1}y(t)=-\int_1^t k(s)\frac{ds}{s}+b_1\Gamma(\beta)$. Then the last conditions from (2), namely ${}^HD_{1+}^{\alpha-1}x(\infty)=\int_1^\infty x(s)\,d\mathfrak{H}_1(s)+\int_1^\infty y(s)\,d\mathfrak{H}_2(s)$ and ${}^HD_{1+}^{\beta-1}y(\infty)=\int_1^\infty x(s)\,d\mathfrak{K}_1(s)+\int_1^\infty y(s)\,d\mathfrak{K}_2(s)$, for the solution (9), give us

$$
\left\{
\begin{aligned}
&-\int_1^\infty h(s)\,\frac{ds}{s}+a_1\Gamma(\alpha)=\int_1^\infty\left[-\frac{1}{\Gamma(\alpha)}\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-1}h(\tau)\frac{d\tau}{\tau}+a_1(\ln s)^{\alpha-1}\right]d\mathfrak{H}_1(s)\\
&\quad+\int_1^\infty\left[-\frac{1}{\Gamma(\beta)}\int_1^s\left(\ln\frac{s}{\tau}\right)^{\beta-1}k(\tau)\frac{d\tau}{\tau}+b_1(\ln s)^{\beta-1}\right]d\mathfrak{H}_2(s),\\
&-\int_1^\infty k(s)\,\frac{ds}{s}+b_1\Gamma(\beta)=\int_1^\infty\left[-\frac{1}{\Gamma(\alpha)}\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-1}h(\tau)\frac{d\tau}{\tau}+a_1(\ln s)^{\alpha-1}\right]d\mathfrak{K}_1(s)\\
&\quad+\int_1^\infty\left[-\frac{1}{\Gamma(\beta)}\int_1^s\left(\ln\frac{s}{\tau}\right)^{\beta-1}k(\tau)\frac{d\tau}{\tau}+b_1(\ln s)^{\beta-1}\right]d\mathfrak{K}_2(s),
\end{aligned}
\right.
$$

or

$$\begin{cases} a_1\left[\Gamma(\alpha)-\int_1^\infty (\ln s)^{\alpha-1}\,d\mathfrak{H}_1(s)\right]-b_1\int_1^\infty (\ln s)^{\beta-1}\,d\mathfrak{H}_2(s)\\ \quad =\int_1^\infty h(s)\,\frac{ds}{s}-\frac{1}{\Gamma(\alpha)}\int_1^\infty\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-1}h(\tau)\,\frac{d\tau}{\tau}\right)d\mathfrak{H}_1(s)\\ \quad -\frac{1}{\Gamma(\beta)}\int_1^\infty\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\beta-1}k(\tau)\,\frac{d\tau}{\tau}\right)d\mathfrak{H}_2(s),\\ -a_1\int_1^\infty (\ln s)^{\alpha-1}\,d\mathfrak{K}_1(s)+b_1\left[\Gamma(\beta)-\int_1^\infty (\ln s)^{\beta-1}\,d\mathfrak{H}_2(s)\right]\\ \quad =\int_1^\infty k(s)\,\frac{ds}{s}-\frac{1}{\Gamma(\alpha)}\int_1^\infty\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-1}h(\tau)\frac{d\tau}{\tau}\right)d\mathfrak{K}_1(s)\\ \quad -\frac{1}{\Gamma(\beta)}\int_1^\infty\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\beta-1}k(\tau)\,\frac{d\tau}{\tau}\right)d\mathfrak{K}_2(s). \end{cases}$$

The determinant of the above system in the unknowns a_1 and b_1 is Δ, which by the assumption of this lemma is different from zero. So the above system has a unique solution, namely

$$\begin{aligned} a_1=\frac{1}{\Delta}&\left[d\int_1^\infty h(s)\frac{ds}{s}-\frac{d}{\Gamma(\alpha)}\int_1^\infty\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-1}h(\tau)\,\frac{d\tau}{\tau}\right)d\mathfrak{H}_1(s)\right.\\ &-\frac{d}{\Gamma(\beta)}\int_1^\infty\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\beta-1}k(\tau)\,\frac{d\tau}{\tau}\right)d\mathfrak{H}_2(s)\\ &+b\int_1^\infty k(s)\frac{ds}{s}-\frac{b}{\Gamma(\alpha)}\int_1^\infty\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-1}h(\tau)\,\frac{d\tau}{\tau}\right)d\mathfrak{K}_1(s)\\ &\left.-\frac{b}{\Gamma(\beta)}\int_1^\infty\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\beta-1}k(\tau)\,\frac{d\tau}{\tau}\right)d\mathfrak{K}_2(s)\right],\\ b_1=\frac{1}{\Delta}&\left[a\int_1^\infty k(s)\,\frac{ds}{s}-\frac{a}{\Gamma(\alpha)}\int_1^\infty\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-1}h(\tau)\,\frac{d\tau}{\tau}\right)d\mathfrak{K}_1(s)\right.\\ &-\frac{a}{\Gamma(\beta)}\int_1^\infty\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\beta-1}k(\tau)\,\frac{d\tau}{\tau}\right)d\mathfrak{K}_2(s)\\ &+c\int_1^\infty h(s)\,\frac{ds}{s}-\frac{c}{\Gamma(\alpha)}\int_1^\infty\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-1}h(\tau)\,\frac{d\tau}{\tau}\right)d\mathfrak{H}_1(s)\\ &\left.-\frac{c}{\Gamma(\beta)}\int_1^\infty\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\beta-1}k(\tau)\,\frac{d\tau}{\tau}\right)d\mathfrak{H}_2(s)\right]. \end{aligned}$$

By replacing the above formulas for a_1 and b_1 in (9), we obtain the solution of problem (5), (2) given by (7). The converse follows by direct computation. □

Lemma 4. *Under the assumptions of Lemma 3, the solution of problem (5),(2) is*

$$\begin{cases} x(t)=\int_1^\infty \mathcal{G}_1(t,s)h(s)\,\frac{ds}{s}+\int_1^\infty \mathcal{G}_2(t,s)k(s)\,\frac{ds}{s},\ \ t\in[1,\infty),\\ y(t)=\int_1^\infty \mathcal{G}_3(t,s)h(s)\,\frac{ds}{s}+\int_1^\infty \mathcal{G}_4(t,s)k(s)\,\frac{ds}{s},\ \ t\in[1,\infty), \end{cases} \tag{10}$$

where the Green functions $\mathcal{G}_i,\ i=1,\dots,4$ are given by

$$\begin{aligned} \mathcal{G}_1(t,s)&=g_\alpha(t,s)+\frac{(\ln t)^{\alpha-1}}{\Delta}\left(d\int_1^\infty g_\alpha(\tau,s)\,d\mathfrak{H}_1(\tau)+b\int_1^\infty g_\alpha(\tau,s)\,d\mathfrak{K}_1(\tau)\right),\\ \mathcal{G}_2(t,s)&=\frac{(\ln t)^{\alpha-1}}{\Delta}\left(d\int_1^\infty g_\beta(\tau,s)\,d\mathfrak{H}_2(\tau)+b\int_1^\infty g_\beta(\tau,s)\,d\mathfrak{K}_2(\tau)\right),\\ \mathcal{G}_3(t,s)&=\frac{(\ln t)^{\beta-1}}{\Delta}\left(c\int_1^\infty g_\alpha(\tau,s)\,d\mathfrak{H}_1(\tau)+a\int_1^\infty g_\alpha(\tau,s)\,d\mathfrak{K}_1(\tau)\right),\\ \mathcal{G}_4(t,s)&=g_\beta(t,s)+\frac{(\ln t)^{\beta-1}}{\Delta}\left(c\int_1^\infty g_\beta(\tau,s)\,d\mathfrak{H}_2(\tau)+a\int_1^\infty g_\beta(\tau,s)\,d\mathfrak{K}_2(\tau)\right), \end{aligned} \tag{11}$$

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with

$$g_\alpha(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (\ln t)^{\alpha-1} - \left(\ln \frac{t}{s}\right)^{\alpha-1}, & 1 \le s \le t, \\ (\ln t)^{\alpha-1}, \; 1 \le t \le s, \end{cases}$$
$$g_\beta(t,s) = \frac{1}{\Gamma(\beta)} \begin{cases} (\ln t)^{\beta-1} - \left(\ln \frac{t}{s}\right)^{\beta-1}, & 1 \le s \le t, \\ (\ln t)^{\beta-1}, \; 1 \le t \le s. \end{cases} \tag{12}$$

Proof. By (7), we obtain

$$\begin{aligned}
x(t) = & \frac{1}{\Gamma(\alpha)} \int_1^\infty (\ln t)^{\alpha-1} h(s) \frac{ds}{s} - \frac{\Delta}{\Gamma(\alpha)\Delta} \int_1^\infty (\ln t)^{\alpha-1} h(s) \frac{ds}{s} \\
& - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} h(s) \frac{ds}{s} \\
& + \frac{(\ln t)^{\alpha-1}}{\Delta} \left[d \int_1^\infty h(s) \frac{ds}{s} - \frac{d}{\Gamma(\alpha)} \int_1^\infty \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\alpha-1} h(\tau) \frac{d\tau}{\tau} \right) d\mathfrak{H}_1(s) \right. \\
& - \frac{d}{\Gamma(\beta)} \int_1^\infty \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\beta-1} k(\tau) \frac{d\tau}{\tau} \right) d\mathfrak{H}_2(s) \\
& + b \int_1^\infty k(s) \frac{ds}{s} - \frac{b}{\Gamma(\alpha)} \int_1^\infty \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\alpha-1} h(\tau) \frac{d\tau}{\tau} \right) d\mathfrak{K}_1(s) \\
& \left. - \frac{b}{\Gamma(\beta)} \int_1^\infty \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\beta-1} k(\tau) \frac{d\tau}{\tau} \right) d\mathfrak{K}_2(s) \right] \\
= & \underbrace{\frac{1}{\Gamma(\alpha)} \int_1^t \left[(\ln t)^{\alpha-1} - \left(\ln \frac{t}{s}\right)^{\alpha-1} \right] h(s) \frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_t^\infty (\ln t)^{\alpha-1} h(s) \frac{ds}{s}}_{A_0} \\
& + \frac{(\ln t)^{\alpha-1}}{\Delta} \left[- \frac{ad-bc}{\Gamma(\alpha)} \int_1^\infty h(s) \frac{ds}{s} + d \int_1^\infty h(s) \frac{ds}{s} \right. \\
& - \frac{d}{\Gamma(\alpha)} \int_1^\infty \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\alpha-1} h(\tau) \frac{d\tau}{\tau} \right) d\mathfrak{H}_1(s) \\
& - \frac{d}{\Gamma(\beta)} \int_1^\infty \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\beta-1} k(\tau) \frac{d\tau}{\tau} \right) d\mathfrak{H}_2(s) + b \int_1^\infty k(s) \frac{ds}{s} \\
& - \frac{b}{\Gamma(\alpha)} \int_1^\infty \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\alpha-1} h(\tau) \frac{d\tau}{\tau} \right) d\mathfrak{K}_1(s) \\
& \left. - \frac{b}{\Gamma(\beta)} \int_1^\infty \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\beta-1} k(\tau) \frac{d\tau}{\tau} \right) d\mathfrak{K}_2(s) \right] \\
= & A_0 + \frac{(\ln t)^{\alpha-1}}{\Delta} \left\{ - \frac{1}{\Gamma(\alpha)} \left[\Gamma(\alpha)\Gamma(\beta) - \Gamma(\beta) \int_1^\infty (\ln s)^{\alpha-1} d\mathfrak{H}_1(s) \right. \right. \\
& - \Gamma(\alpha) \int_1^\infty (\ln s)^{\beta-1} d\mathfrak{K}_2(s) + \left(\int_1^\infty (\ln s)^{\alpha-1} d\mathfrak{H}_1(s) \right) \left(\int_1^\infty (\ln s)^{\beta-1} d\mathfrak{K}_2(s) \right) \\
& \left. - \left(\int_1^\infty (\ln s)^{\alpha-1} d\mathfrak{K}_1(s) \right) \left(\int_1^\infty (\ln s)^{\beta-1} \mathfrak{H}_2(s) \right) \right] \int_1^\infty h(s) \frac{ds}{s} \\
& + \left[\Gamma(\beta) - \int_1^\infty (\ln s)^{\beta-1} d\mathfrak{K}_2(s) \right] \int_1^\infty h(s) \frac{ds}{s} \\
& - \frac{1}{\Gamma(\alpha)} \left[\Gamma(\beta) - \int_1^\infty (\ln s)^{\beta-1} d\mathfrak{K}_2(s) \right] \int_1^\infty \left(\int_s^\infty \left(\ln \frac{\tau}{s}\right)^{\alpha-1} d\mathfrak{H}_1(\tau) \right) h(s) \frac{ds}{s} \\
& - \frac{1}{\Gamma(\beta)} \left[\Gamma(\beta) - \int_1^\infty (\ln s)^{\beta-1} d\mathfrak{K}_2(s) \right] \int_1^\infty \left(\int_s^\infty \left(\ln \frac{\tau}{s}\right)^{\beta-1} d\mathfrak{H}_2(\tau) \right) k(s) \frac{ds}{s} \\
& + \left(\int_1^\infty (\ln s)^{\beta-1} d\mathfrak{H}_2(s) \right) \int_1^\infty k(s) \frac{ds}{s} \\
& - \frac{1}{\Gamma(\alpha)} \left(\int_1^\infty (\ln s)^{\beta-1} d\mathfrak{H}_2(s) \right) \left(\int_1^\infty \left(\int_s^\infty \left(\ln \frac{\tau}{s}\right)^{\alpha-1} d\mathfrak{K}_1(\tau) \right) h(s) \frac{ds}{s} \right) \\
& \left. - \frac{1}{\Gamma(\beta)} \left(\int_1^\infty (\ln s)^{\beta-1} d\mathfrak{H}_2(s) \right) \left(\int_1^\infty \left(\int_s^\infty \left(\ln \frac{\tau}{s}\right)^{\beta-1} d\mathfrak{K}_2(\tau) \right) k(s) \frac{ds}{s} \right) \right\}
\end{aligned}$$

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$$
\begin{aligned}
&= A_0 + \frac{(\ln t)^{\alpha-1}}{\Delta} \Big\{ \int_1^\infty \Big[\frac{\Gamma(\beta)}{\Gamma(\alpha)} \int_1^\infty (\ln \tau)^{\alpha-1}\, d\mathfrak{H}_1(\tau) \\
&\quad - \frac{1}{\Gamma(\alpha)} \Big(\int_1^\infty (\ln \tau)^{\alpha-1}\, d\mathfrak{H}_1(\tau) \Big) \Big(\int_1^\infty (\ln \tau)^{\beta-1}\, d\mathfrak{K}_2(\tau) \Big) \\
&\quad + \frac{1}{\Gamma(\alpha)} \Big(\int_1^\infty (\ln \tau)^{\alpha-1}\, d\mathfrak{K}_1(\tau) \Big) \Big(\int_1^\infty (\ln \tau)^{\beta-1}\, d\mathfrak{H}_2(\tau) \Big) \\
&\quad - \frac{\Gamma(\beta)}{\Gamma(\alpha)} \int_s^\infty \Big(\ln \frac{\tau}{s} \Big)^{\alpha-1}\, d\mathfrak{H}_1(\tau) \\
&\quad + \frac{1}{\Gamma(\alpha)} \Big(\int_1^\infty (\ln \tau)^{\beta-1}\, d\mathfrak{K}_2(\tau) \Big) \int_s^\infty \Big(\ln \frac{\tau}{s} \Big)^{\alpha-1}\, d\mathfrak{H}_1(\tau) \\
&\quad - \frac{1}{\Gamma(\alpha)} \Big(\int_1^\infty (\ln \tau)^{\beta-1}\, d\mathfrak{H}_2(\tau) \Big) \Big(\int_s^\infty \Big(\ln \frac{\tau}{s} \Big)^{\alpha-1}\, d\mathfrak{K}_1(\tau) \Big) \Big] h(s)\, \frac{ds}{s} \\
&\quad + \int_1^\infty \Big[- \int_s^\infty \Big(\ln \frac{\tau}{s} \Big)^{\beta-1}\, d\mathfrak{H}_2(\tau) \\
&\quad + \frac{1}{\Gamma(\beta)} \Big(\int_1^\infty (\ln \tau)^{\beta-1}\, d\mathfrak{K}_2(\tau) \Big) \int_s^\infty \Big(\ln \frac{\tau}{s} \Big)^{\beta-1}\, d\mathfrak{H}_2(\tau) \\
&\quad + \Big(\int_1^\infty (\ln \tau)^{\beta-1}\, d\mathfrak{H}_2(\tau) \Big) \\
&\quad - \frac{1}{\Gamma(\beta)} \Big(\int_1^\infty (\ln \tau)^{\beta-1}\, d\mathfrak{H}_2(\tau) \Big) \int_s^\infty \Big(\ln \frac{\tau}{s} \Big)^{\beta-1}\, d\mathfrak{K}_2(\tau) \Big] k(s)\, \frac{ds}{s} \Big\} \\
&= A_0 + \frac{(\ln t)^{\alpha-1}}{\Delta} \Big\{ \int_1^\infty \Big\{ \frac{\Gamma(\beta)}{\Gamma(\alpha)} \Big[\int_1^\infty (\ln \tau)^{\alpha-1}\, d\mathfrak{H}_1(\tau) - \int_s^\infty \Big(\ln \frac{\tau}{s} \Big)^{\alpha-1}\, d\mathfrak{H}_1(\tau) \Big] \\
&\quad - \frac{1}{\Gamma(\alpha)} \Big(\int_1^\infty (\ln \tau)^{\beta-1}\, d\mathfrak{K}_2(\tau) \Big) \Big[\int_1^\infty (\ln \tau)^{\alpha-1}\, d\mathfrak{H}_1(\tau) - \int_s^\infty \Big(\ln \frac{\tau}{s} \Big)^{\alpha-1}\, d\mathfrak{H}_1(\tau) \Big] \\
&\quad + \frac{1}{\Gamma(\alpha)} \Big(\int_1^\infty (\ln \tau)^{\beta-1}\, d\mathfrak{H}_2(\tau) \Big) \\
&\quad \times \Big[\int_1^\infty (\ln \tau)^{\alpha-1}\, d\mathfrak{K}_1(\tau) - \int_s^\infty \Big(\ln \frac{\tau}{s} \Big)^{\alpha-1}\, d\mathfrak{K}_1(\tau) \Big] \Big\} h(s)\, \frac{ds}{s} \\
&\quad + \int_1^\infty \Big\{ \Big[\int_1^\infty (\ln \tau)^{\beta-1}\, d\mathfrak{H}_2(\tau) - \int_s^\infty \Big(\ln \frac{\tau}{s} \Big)^{\beta-1}\, d\mathfrak{H}_2(\tau) \Big] \\
&\quad + \frac{1}{\Gamma(\beta)} \Big[\Big(\int_1^\infty (\ln \tau)^{\beta-1}\, d\mathfrak{K}_2(\tau) \Big) \Big(\int_s^\infty \Big(\ln \frac{\tau}{s} \Big)^{\beta-1}\, d\mathfrak{H}_2(\tau) \Big) \\
&\quad - \Big(\int_1^\infty (\ln \tau)^{\beta-1}\, d\mathfrak{K}_2(\tau) \Big) \Big(\int_1^\infty (\ln \tau)^{\beta-1}\, d\mathfrak{H}_2(\tau) \Big) \\
&\quad + \Big(\int_1^\infty (\ln \tau)^{\beta-1}\, d\mathfrak{H}_2(\tau) \Big) \Big(\int_1^\infty (\ln \tau)^{\beta-1}\, d\mathfrak{K}_2(\tau) \Big) \\
&\quad - \Big(\int_1^\infty (\ln \tau)^{\beta-1}\, d\mathfrak{H}_2(\tau) \Big) \Big(\int_s^\infty \Big(\ln \frac{\tau}{s} \Big)^{\beta-1}\, d\mathfrak{K}_2(\tau) \Big) \Big] \Big\} k(s)\, \frac{ds}{s} \Big\} \\
&= A_0 + \frac{(\ln t)^{\alpha-1}}{\Delta} \Big\{ \int_1^\infty \Big\{ \frac{d}{\Gamma(\alpha)} \Big[\int_1^\infty (\ln \tau)^{\alpha-1}\, d\mathfrak{H}_1(\tau) - \int_s^\infty \Big(\ln \frac{\tau}{s} \Big)^{\alpha-1}\, d\mathfrak{H}_1(\tau) \Big] \\
&\quad + \frac{b}{\Gamma(\alpha)} \Big[\int_1^\infty (\ln \tau)^{\alpha-1}\, d\mathfrak{K}_1(\tau) - \int_s^\infty \Big(\ln \frac{\tau}{s} \Big)^{\alpha-1}\, d\mathfrak{K}_1(\tau) \Big] \Big\} h(s)\, \frac{ds}{s} \\
&\quad + \int_1^\infty \Big\{ \frac{d}{\Gamma(\beta)} \Big[\int_1^\infty (\ln \tau)^{\beta-1}\, d\mathfrak{H}_2(\tau) - \int_s^\infty \Big(\ln \frac{\tau}{s} \Big)^{\beta-1}\, d\mathfrak{H}_2(\tau) \Big] \\
&\quad + \frac{b}{\Gamma(\beta)} \Big[\int_1^\infty (\ln \tau)^{\beta-1}\, d\mathfrak{K}_2(\tau) - \int_s^\infty \Big(\ln \frac{\tau}{s} \Big)^{\beta-1}\, d\mathfrak{K}_2(\tau) \Big] \Big\} k(s)\, \frac{ds}{s} \Big\} \\
&= \int_1^\infty g_\alpha(t,s) h(s)\, \frac{ds}{s} + \frac{(\ln t)^{\alpha-1}}{\Delta} \Big[d \int_1^\infty \Big(\int_1^\infty g_\alpha(\tau,s)\, d\mathfrak{H}_1(\tau) \Big) h(s)\, \frac{ds}{s} \\
&\quad + b \int_1^\infty \Big(\int_1^\infty g_\alpha(\tau,s)\, d\mathfrak{K}_1(\tau) \Big) h(s)\, \frac{ds}{s} \\
&\quad + d \int_1^\infty \Big(\int_1^\infty g_\beta(\tau,s)\, d\mathfrak{H}_2(\tau) \Big) k(s)\, \frac{ds}{s} \\
&\quad + b \int_1^\infty \Big(\int_1^\infty g_\beta(\tau,s)\, d\mathfrak{K}_2(\tau) \Big) k(s)\, \frac{ds}{s} \Big] \\
&= \int_1^\infty \mathcal{G}_1(t,s) h(s)\, \frac{ds}{s} + \int_1^\infty \mathcal{G}_2(t,s) k(s)\, \frac{ds}{s},
\end{aligned}
$$

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where g_α and g_β are given by (12), and $\mathcal{G}_1$ and $\mathcal{G}_2$ are given by (11). In the last relations above we used the equality

$$\begin{aligned}
&\int_1^\infty (\ln\tau)^{\alpha-1}\,d\mathfrak{H}_1(\tau) - \int_s^\infty \left(\ln\frac{\tau}{s}\right)^{\alpha-1} d\mathfrak{H}_1(\tau)\\
&\quad = \int_1^s (\ln\tau)^{\alpha-1}\,d\mathfrak{H}_1(\tau) + \int_s^\infty \left[(\ln\tau)^{\alpha-1} - \left(\ln\frac{\tau}{s}\right)^{\alpha-1}\right] d\mathfrak{H}_1(\tau)\\
&\quad = \Gamma(\alpha)\int_1^\infty g_\alpha(\tau,s)\,d\mathfrak{H}_1(\tau),
\end{aligned}$$

and similar equalities for $\mathfrak{H}_2(\tau)$, $\mathfrak{K}_1(\tau)$ and $\mathfrak{K}_2(\tau)$.

In a similar manner, by (7) we find

$$\begin{aligned}
y(t) &= \frac{1}{\Gamma(\beta)}\int_1^\infty (\ln t)^{\beta-1}k(s)\,\frac{ds}{s} - \frac{\Delta}{\Gamma(\beta)\Delta}\int_1^\infty (\ln t)^{\beta-1}k(s)\,\frac{ds}{s}\\
&\quad -\frac{1}{\Gamma(\beta)}\int_1^t \left(\ln\frac{t}{s}\right)^{\beta-1} k(s)\,\frac{ds}{s}\\
&\quad +\frac{(\ln t)^{\beta-1}}{\Delta}\left[a\int_1^\infty k(s)\,\frac{ds}{s} - \frac{a}{\Gamma(\alpha)}\int_1^\infty\left(\int_1^s \left(\ln\frac{s}{\tau}\right)^{\alpha-1} h(\tau)\frac{d\tau}{\tau}\right) d\mathfrak{K}_1(s)\right.\\
&\quad -\frac{a}{\Gamma(\beta)}\int_1^\infty\left(\int_1^s \left(\ln\frac{s}{\tau}\right)^{\beta-1} k(\tau)\,\frac{d\tau}{\tau}\right) d\mathfrak{K}_2(s)\\
&\quad +c\int_1^\infty h(s)\,\frac{ds}{s} - \frac{c}{\Gamma(\alpha)}\int_1^\infty\left(\int_1^s \left(\ln\frac{s}{\tau}\right)^{\alpha-1} h(\tau)\,\frac{d\tau}{\tau}\right) d\mathfrak{H}_1(s)\\
&\quad \left.-\frac{c}{\Gamma(\beta)}\int_1^\infty\left(\int_1^s \left(\ln\frac{s}{\tau}\right)^{\beta-1} k(\tau)\,\frac{d\tau}{\tau}\right) d\mathfrak{H}_2(s)\right]\\
&= \underbrace{\frac{1}{\Gamma(\beta)}\int_1^t \left[(\ln t)^{\beta-1} - \left(\ln\frac{t}{s}\right)^{\beta-1}\right] k(s)\,\frac{ds}{s} + \frac{1}{\Gamma(\beta)}\int_t^\infty (\ln t)^{\beta-1}k(s)\,\frac{ds}{s}}_{B_0}\\
&\quad +\frac{(\ln t)^{\beta-1}}{\Delta}\left[-\frac{ad-bc}{\Gamma(\beta)}\int_1^\infty k(s)\,\frac{ds}{s} + a\int_1^\infty k(s)\,\frac{ds}{s}\right.\\
&\quad -\frac{a}{\Gamma(\alpha)}\int_1^\infty\left(\int_1^s \left(\ln\frac{s}{\tau}\right)^{\alpha-1} h(\tau)\,\frac{d\tau}{\tau}\right) d\mathfrak{K}_1(s)\\
&\quad -\frac{a}{\Gamma(\beta)}\int_1^\infty\left(\int_1^s \left(\ln\frac{s}{\tau}\right)^{\beta-1} k(\tau)\,\frac{d\tau}{\tau}\right) d\mathfrak{K}_2(s)\\
&\quad +c\int_1^\infty h(s)\,\frac{ds}{s} - \frac{c}{\Gamma(\alpha)}\int_1^\infty\left(\int_1^s \left(\ln\frac{s}{\tau}\right)^{\alpha-1} h(\tau)\,\frac{d\tau}{\tau}\right) d\mathfrak{H}_1(s)\\
&\quad \left.-\frac{c}{\Gamma(\beta)}\int_1^\infty\left(\int_1^s \left(\ln\frac{s}{\tau}\right)^{\beta-1} k(\tau)\,\frac{d\tau}{\tau}\right) d\mathfrak{H}_2(s)\right]\\
&= B_0 + \frac{(\ln t)^{\beta-1}}{\Delta}\left\{-\frac{1}{\Gamma(\beta)}\left[\Gamma(\alpha)\Gamma(\beta) - \Gamma(\beta)\int_1^\infty (\ln s)^{\alpha-1}\,d\mathfrak{H}_1(s)\right.\right.\\
&\quad -\Gamma(\alpha)\int_1^\infty (\ln s)^{\beta-1}\,d\mathfrak{K}_2(s) + \left(\int_1^\infty (\ln s)^{\alpha-1}\,d\mathfrak{H}_1(s)\right)\left(\int_1^\infty (\ln s)^{\beta-1}\,d\mathfrak{K}_2(s)\right)\\
&\quad \left.-\left(\int_1^\infty (\ln s)^{\alpha-1}\,d\mathfrak{K}_1(s)\right)\left(\int_1^\infty (\ln s)^{\beta-1}\,d\mathfrak{H}_2(s)\right)\right]\int_1^\infty k(s)\,\frac{ds}{s}\\
&\quad +\left[\Gamma(\alpha) - \int_1^\infty (\ln s)^{\alpha-1}\,d\mathfrak{H}_1(s)\right]\int_1^\infty k(s)\,\frac{ds}{s}\\
&\quad -\frac{1}{\Gamma(\alpha)}\left[\Gamma(\alpha) - \int_1^\infty (\ln s)^{\alpha-1}\,d\mathfrak{H}_1(s)\right]\int_1^\infty\left(\int_1^s \left(\ln\frac{s}{\tau}\right)^{\alpha-1} h(\tau)\frac{d\tau}{\tau}\right) d\mathfrak{K}_1(s)\\
&\quad -\frac{1}{\Gamma(\beta)}\left[\Gamma(\alpha) - \int_1^\infty (\ln s)^{\alpha-1}\,d\mathfrak{H}_1(s)\right]\int_1^\infty\left(\int_1^s \left(\ln\frac{s}{\tau}\right)^{\beta-1} k(\tau)\,\frac{d\tau}{\tau}\right) d\mathfrak{K}_2(s)\\
&\quad +\left(\int_1^\infty (\ln s)^{\alpha-1}\,d\mathfrak{K}_1(s)\right)\int_1^\infty h(s)\,\frac{ds}{s}\\
&\quad -\frac{1}{\Gamma(\alpha)}\left(\int_1^\infty (\ln s)^{\alpha-1}\,d\mathfrak{K}_1(s)\right)\left(\int_1^\infty\left(\int_1^s \left(\ln\frac{s}{\tau}\right)^{\alpha-1} h(\tau)\,\frac{d\tau}{\tau}\right) d\mathfrak{H}_1(s)\right)\\
&\quad \left.-\frac{1}{\Gamma(\beta)}\left(\int_1^\infty (\ln s)^{\alpha-1}\,d\mathfrak{K}_1(s)\right)\left(\int_1^\infty\left(\int_1^s \left(\ln\frac{s}{\tau}\right)^{\beta-1} k(\tau)\,\frac{d\tau}{\tau}\right) d\mathfrak{H}_2(s)\right)\right\}
\end{aligned}$$

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$$
\begin{aligned}
&= B_0 + \frac{(\ln t)^{\beta-1}}{\Delta}\left\{\int_1^\infty \left[-\int_s^\infty \left(\ln\frac{\tau}{s}\right)^{\alpha-1} d\mathfrak{K}_1(\tau)\right.\right.\\
&\quad + \frac{1}{\Gamma(\alpha)}\left(\int_1^\infty (\ln\tau)^{\alpha-1}\, d\mathfrak{H}_1(\tau)\right)\int_s^\infty \left(\ln\frac{\tau}{s}\right)^{\alpha-1} d\mathfrak{K}_1(\tau)\\
&\quad + \int_1^\infty (\ln\tau)^{\alpha-1}\, d\mathfrak{K}_1(\tau)\\
&\quad - \left.\frac{1}{\Gamma(\alpha)}\left(\int_1^\infty (\ln\tau)^{\alpha-1}\, d\mathfrak{K}_1(\tau)\right)\int_s^\infty \left(\ln\frac{\tau}{s}\right)^{\alpha-1} d\mathfrak{H}_1(\tau)\right] h(s)\,\frac{ds}{s}\\
&\quad + \int_1^\infty \left[\frac{\Gamma(\alpha)}{\Gamma(\beta)}\int_1^\infty (\ln\tau)^{\beta-1}\, d\mathfrak{K}_2(\tau)\right.\\
&\quad - \frac{1}{\Gamma(\beta)}\left(\int_1^\infty (\ln\tau)^{\alpha-1}\, d\mathfrak{H}_1(\tau)\right)\left(\int_1^\infty (\ln\tau)^{\beta-1}\, d\mathfrak{K}_2(\tau)\right)\\
&\quad + \frac{1}{\Gamma(\beta)}\left(\int_1^\infty (\ln\tau)^{\alpha-1}\, d\mathfrak{K}_1(\tau)\right)\left(\int_1^\infty (\ln\tau)^{\beta-1}\, d\mathfrak{H}_2(\tau)\right)\\
&\quad - \frac{\Gamma(\alpha)}{\Gamma(\beta)}\int_s^\infty \left(\ln\frac{\tau}{s}\right)^{\beta-1} d\mathfrak{K}_2(\tau)\\
&\quad + \frac{1}{\Gamma(\beta)}\left(\int_1^\infty (\ln\tau)^{\alpha-1}\, d\mathfrak{H}_1(\tau)\right)\int_s^\infty \left(\ln\frac{\tau}{s}\right)^{\beta-1} d\mathfrak{K}_2(\tau)\\
&\quad - \left.\left.\frac{1}{\Gamma(\beta)}\left(\int_1^\infty (\ln\tau)^{\alpha-1}\, d\mathfrak{K}_1(\tau)\right)\int_s^\infty \left(\ln\frac{\tau}{s}\right)^{\beta-1} d\mathfrak{H}_2(\tau)\right] k(s)\,\frac{ds}{s}\right\}\\
&= B_0 + \frac{(\ln t)^{\beta-1}}{\Delta}\left\{\int_1^\infty \left\{\left[\int_1^\infty (\ln\tau)^{\alpha-1}\, d\mathfrak{K}_1(\tau) - \int_s^\infty \left(\ln\frac{\tau}{s}\right)^{\alpha-1} d\mathfrak{K}_1(\tau)\right]\right.\right.\\
&\quad + \frac{1}{\Gamma(\alpha)}\left(\int_1^\infty (\ln\tau)^{\alpha-1}\, d\mathfrak{H}_1(\tau)\right)\int_s^\infty \left(\ln\frac{\tau}{s}\right)^{\alpha-1} d\mathfrak{K}_1(\tau)\\
&\quad - \frac{1}{\Gamma(\alpha)}\left(\int_1^\infty (\ln\tau)^{\alpha-1}\, d\mathfrak{H}_1(\tau)\right)\left(\int_1^\infty (\ln\tau)^{\alpha-1}\, d\mathfrak{K}_1(\tau)\right)\\
&\quad + \frac{1}{\Gamma(\alpha)}\left(\int_1^\infty (\ln\tau)^{\alpha-1}\, d\mathfrak{H}_1(\tau)\right)\left(\int_1^\infty (\ln\tau)^{\alpha-1}\, d\mathfrak{K}_1(\tau)\right)\\
&\quad - \left.\frac{1}{\Gamma(\alpha)}\left(\int_1^\infty (\ln\tau)^{\alpha-1}\, d\mathfrak{K}_1(\tau)\right)\left(\int_s^\infty \left(\ln\frac{\tau}{s}\right)^{\alpha-1} d\mathfrak{H}_1(\tau)\right)\right\} h(s)\,\frac{ds}{s}\\
&\quad + \int_1^\infty \left\{\frac{\Gamma(\alpha)}{\Gamma(\beta)}\left[\int_1^\infty (\ln\tau)^{\beta-1}\, d\mathfrak{K}_2(\tau) - \int_s^\infty \left(\ln\frac{\tau}{s}\right)^{\beta-1} d\mathfrak{K}_2(\tau)\right]\right.\\
&\quad - \frac{1}{\Gamma(\beta)}\left(\int_1^\infty (\ln\tau)^{\alpha-1}\, d\mathfrak{H}_1(\tau)\right)\left[\int_1^\infty (\ln\tau)^{\beta-1}\, d\mathfrak{K}_2(\tau) - \int_s^\infty \left(\ln\frac{\tau}{s}\right)^{\beta-1} d\mathfrak{K}_2(\tau)\right]\\
&\quad + \frac{1}{\Gamma(\beta)}\left(\int_1^\infty (\ln\tau)^{\alpha-1}\, d\mathfrak{K}_1(\tau)\right)\left[\int_1^\infty (\ln\tau)^{\beta-1}\, d\mathfrak{H}_2(\tau)\right.\\
&\quad - \left.\left.\left.\int_s^\infty \left(\ln\frac{\tau}{s}\right)^{\beta-1} d\mathfrak{H}_2(\tau)\right]\right\} k(s)\,\frac{ds}{s}\right\}\\
&= B_0 + \frac{(\ln t)^{\beta-1}}{\Delta}\left\{\int_1^\infty \left\{\frac{a}{\Gamma(\alpha)}\left[\int_1^\infty (\ln\tau)^{\alpha-1}\, d\mathfrak{K}_1(\tau) - \int_s^\infty \left(\ln\frac{\tau}{s}\right)^{\alpha-1} d\mathfrak{K}_1(\tau)\right]\right.\right.\\
&\quad + \left.\frac{c}{\Gamma(\alpha)}\left[\int_1^\infty (\ln\tau)^{\alpha-1}\, d\mathfrak{H}_1(\tau) - \int_s^\infty \left(\ln\frac{\tau}{s}\right)^{\alpha-1} d\mathfrak{H}_1(\tau)\right]\right\} h(s)\,\frac{ds}{s}\\
&\quad + \int_1^\infty \left\{\frac{a}{\Gamma(\beta)}\left[\int_1^\infty (\ln\tau)^{\beta-1}\, d\mathfrak{K}_2(\tau) - \int_s^\infty \left(\ln\frac{\tau}{s}\right)^{\beta-1} d\mathfrak{K}_2(\tau)\right]\right.\\
&\quad + \left.\left.\frac{c}{\Gamma(\beta)}\left[\int_1^\infty (\ln\tau)^{\beta-1}\, d\mathfrak{H}_2(\tau) - \int_s^\infty \left(\ln\frac{\tau}{s}\right)^{\beta-1} d\mathfrak{H}_2(\tau)\right]\right\} k(s)\,\frac{ds}{s}\right\}\\
&= \int_1^\infty g_\beta(t,s)k(s)\,\frac{ds}{s} + \frac{(\ln t)^{\beta-1}}{\Delta}\left[a\int_1^\infty \left(\int_1^\infty g_\alpha(\tau,s)\, d\mathfrak{K}_1(\tau)\right) h(s)\,\frac{ds}{s}\right.\\
&\quad + c\int_1^\infty \left(\int_1^\infty g_\alpha(\tau,s)\, d\mathfrak{H}_1(\tau)\right) h(s)\,\frac{ds}{s}\\
&\quad + a\int_1^\infty \left(\int_1^\infty g_\beta(\tau,s)\, d\mathfrak{K}_2(\tau)\right) k(s)\,\frac{ds}{s}\\
&\quad + \left. c\int_1^\infty \left(\int_1^\infty g_\beta(\tau,s)\, d\mathfrak{H}_2(\tau)\right) k(s)\,\frac{ds}{s}\right]\\
&= \int_1^\infty \mathcal{G}_3(t,s)h(s)\,\frac{ds}{s} + \int_1^\infty \mathcal{G}_4(t,s)k(s)\,\frac{ds}{s},
\end{aligned}
$$

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where $\mathcal{G}_3$ and $\mathcal{G}_4$ are given by (11). □

Based on the definitions of the functions g_α, g_β, $\mathcal{G}_i$, $i = 1, \ldots, 4$, we obtain easily the next lemma.

Lemma 5. *We assume that the functions $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathfrak{K}_1$, $\mathfrak{K}_2$ are nondecreasing functions, a, $d \in \mathbb{R}_+$, b, $c \in \mathbb{R}$, and $\Delta > 0$, and let $\theta > 1$. Then the functions g_α, g_β, and $\mathcal{G}_i$, $i = 1, \ldots, 4$ have the following properties:*

a) The functions g_α and g_β are continuous on $[1, \infty) \times [1, \infty)$;

b) $0 \le g_\alpha(t,s) \le \dfrac{1}{\Gamma(\alpha)}(\ln t)^{\alpha-1}, \ \ 0 \le g_\beta(t,s) \le \dfrac{1}{\Gamma(\beta)}(\ln t)^{\beta-1}, \ \ \forall\, t,s \in [1,\infty);$

c) $0 \le \dfrac{g_\alpha(t,s)}{1+(\ln t)^{\alpha-1}} \le \dfrac{1}{\Gamma(\alpha)}, \ \ 0 \le \dfrac{g_\beta(t,s)}{1+(\ln t)^{\beta-1}} \le \dfrac{1}{\Gamma(\beta)}, \ \ \forall\, t,s \in [1,\infty);$

d) The functions $\mathcal{G}_i$, $i = 1, \ldots, 4$ are continuous on $[1, \infty) \times [1, \infty)$;

e) $\mathcal{G}_i(t,s) \ge 0$ for all $(t,s) \in [1,\infty) \times [1,\infty)$ and $i = 1, \ldots, 4$;

f) $$\frac{\mathcal{G}_1(t,s)}{1+(\ln t)^{\alpha-1}} \le \frac{1}{\Gamma(\alpha)} + \frac{1}{\Delta\Gamma(\alpha)}\left(d\int_1^\infty (\ln\tau)^{\alpha-1}\,d\mathfrak{H}_1(\tau) + b\int_1^\infty (\ln\tau)^{\alpha-1}\,d\mathfrak{K}_1(\tau)\right)$$
$$= \frac{d}{\Delta} =: \Lambda_1 > 0, \ \ \forall\, t,s \in [1,\infty);$$

g) $$\frac{\mathcal{G}_2(t,s)}{1+(\ln t)^{\alpha-1}} \le \frac{1}{\Delta\Gamma(\beta)}\left(d\int_1^\infty (\ln\tau)^{\beta-1}\,d\mathfrak{H}_2(\tau) + b\int_1^\infty (\ln\tau)^{\beta-1}\,d\mathfrak{K}_2(\tau)\right)$$
$$= \frac{b}{\Delta} =: \Lambda_2 \ge 0, \ \ \forall\, t,s \in [1,\infty);$$

h) $$\frac{\mathcal{G}_3(t,s)}{1+(\ln t)^{\beta-1}} \le \frac{1}{\Delta\Gamma(\alpha)}\left(c\int_1^\infty (\ln\tau)^{\alpha-1}\,d\mathfrak{H}_1(\tau) + a\int_1^\infty (\ln\tau)^{\alpha-1}\,d\mathfrak{K}_1(\tau)\right)$$
$$= \frac{c}{\Delta} =: \Lambda_3 \ge 0, \ \ \forall\, t,s \in [1,\infty);$$

i) $$\frac{\mathcal{G}_4(t,s)}{1+(\ln t)^{\beta-1}} \le \frac{1}{\Gamma(\beta)} + \frac{1}{\Delta\Gamma(\beta)}\left(c\int_1^\infty (\ln\tau)^{\beta-1}\,d\mathfrak{H}_2(\tau) + a\int_1^\infty (\ln\tau)^{\beta-1}\,d\mathfrak{K}_2(\tau)\right)$$
$$= \frac{a}{\Delta} =: \Lambda_4 > 0, \ \ \forall\, t,s \in [1,\infty);$$

j) $$\min_{t\in[\theta,\infty)} \frac{\mathcal{G}_1(t,s)}{1+(\ln t)^{\alpha-1}} \ge \frac{(\ln\theta)^{\alpha-1}}{\Delta(1+(\ln\theta)^{\alpha-1})}$$
$$\times\left(d\int_1^\infty g_\alpha(\tau,s)\,d\mathfrak{H}_1(\tau) + b\int_1^\infty g_\alpha(\tau,s)\,d\mathfrak{K}_1(\tau)\right), \forall s \in [1,\infty);$$

k) $$\min_{t\in[\theta,\infty)} \frac{\mathcal{G}_2(t,s)}{1+(\ln t)^{\alpha-1}} = \frac{(\ln\theta)^{\alpha-1}}{\Delta(1+(\ln\theta)^{\alpha-1})}$$
$$\times\left(d\int_1^\infty g_\beta(\tau,s)\,d\mathfrak{H}_2(\tau) + b\int_1^\infty g_\beta(\tau,s)\,d\mathfrak{K}_2(\tau)\right), \forall s \in [1,\infty);$$

l) $$\min_{t\in[\theta,\infty)} \frac{\mathcal{G}_3(t,s)}{1+(\ln t)^{\beta-1}} = \frac{(\ln\theta)^{\beta-1}}{\Delta(1+(\ln\theta)^{\beta-1})}$$
$$\times\left(c\int_1^\infty g_\alpha(\tau,s)\,d\mathfrak{H}_1(\tau) + a\int_1^\infty g_\alpha(\tau,s)\,d\mathfrak{K}_1(\tau)\right), \forall s \in [1,\infty);$$

m) $$\min_{t\in[\theta,\infty)} \frac{\mathcal{G}_4(t,s)}{1+(\ln t)^{\beta-1}} \ge \frac{(\ln\theta)^{\beta-1}}{\Delta(1+(\ln\theta)^{\beta-1})}$$
$$\times\left(c\int_1^\infty g_\beta(\tau,s)\,d\mathfrak{H}_2(\tau) + a\int_1^\infty g_\beta(\tau,s)\,d\mathfrak{K}_2(\tau)\right), \forall s \in [1,\infty).$$

Remark 1. *Under assumptions of Lemma 5 we find that a, $d > 0$ and b, $c \ge 0$, so Λ_1, $\Lambda_4 > 0$ and Λ_2, $\Lambda_3 \ge 0$.*

We present now the fixed point theorems that we will use in the next section.

Let E be a real Banach space with the norm $\|\cdot\|$.

Definition 3. *A nonempty convex closed set $K \subset E$ is a cone if it satisfies the conditions:*

a) if $u \in K$ and $\alpha \ge 0$, then $\alpha u \in K$;

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b) if $u \in K$ and $-u \in K$, then $u = 0$,
where 0 is the zero element of E.

A cone $K \subset E$ defines a partial ordering in E given by $x \leq y$ if and only if $y - x \in K$.

Definition 4. *An operator $A : D(A) \subset E \to E$ is compact if it maps bounded sets into relatively compact sets. An operator $A : D(A) \subset E \to E$ is completely continuous if it is continuous and compact.*

Theorem 1. *(Guo-Krasnosels'kii fixed point theorem - the fixed point theorem of cone expansion and compression of norm type, [5]). Let E be a real Banach space with the norm $\|\cdot\|$, and let $K \subset X$ be a cone in E. Assume Ω_1 and Ω_2 are bounded open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let $\mathcal{A} : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that, either*
i) $\|\mathcal{A}u\| \leq \|u\|, \ \forall u \in K \cap \partial\Omega_1$, and $\|\mathcal{A}u\| \geq \|u\|, \ \forall u \in K \cap \partial\Omega_2$, or
ii) $\|\mathcal{A}u\| \geq \|u\|, \ \forall u \in K \cap \partial\Omega_1$, and $\|\mathcal{A}u\| \leq \|u\|, \ \forall u \in K \cap \partial\Omega_2$.
Then $\mathcal{A}$ has at least one fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

We consider Θ a nonnegative continuous concave functional on the cone K, and $0 < c < d$. We define the convex sets K_c, $\overline{K}_c$, $K(\Theta, c, d)$ by
$K_c = \{u \mid u \in K, \ \|u\| < c\}$, $\overline{K}_c = \{u \mid u \in K, \ \|u\| \leq c\}$, and
$K(\Theta, c, d) = \{u \mid u \in K, \ \Theta(u) \geq c, \ \|u\| \leq d\}$.

Theorem 2. *(Leggett-Williams fixed point theorem, [9]) Let K be a cone in a real Banach space E and $\Theta(u)$ a nonnegative continuous concave functional on K satisfying $\Theta(u) \leq \|u\|$ for all $u \in \overline{K}_l$, $(l > 0)$. Suppose that $A : \overline{K}_l \to \overline{K}_l$ is a completely continuous operator and there exist positive numbers $0 < m < c < d < l$ such that*
i) $\{u \mid u \in K(\Theta, c, d), \ \Theta(u) > c\} \neq \emptyset$ and $\Theta(Au) > c$ for $u \in K(\Theta, c, d)$;
ii) $\|Au\| < m$ for $u \in \overline{K}_m$;
iii) $\Theta(Au) > c$ for $u \in K(\Theta, c, l)$ and $\|Au\| > d$.
Then A has at least three fixed points u_1, u_2 and u_3 satisfying

$$\|u_1\| < m, \ \ \Theta(u_2) > c, \ \ \|u_3\| > m \ \ and \ \ \Theta(u_3) < c.$$

3. Main results

We introduce the space

$$X_1 = \left\{ x \in C(I, \mathbb{R}), \ \sup_{t \in I} \frac{|x(t)|}{1 + (\ln t)^{\alpha - 1}} < \infty \right\},$$

where $I = [1, \infty)$, with the norm $\|x\|_1 = \sup_{t \in I} \frac{|x(t)|}{1 + (\ln t)^{\alpha - 1}}$, the space

$$X_2 = \left\{ y \in C(I, \mathbb{R}), \ \sup_{t \in I} \frac{|y(t)|}{1 + (\ln t)^{\beta - 1}} < \infty \right\},$$

with the norm $\|y\|_2 = \sup_{t \in I} \frac{|y(t)|}{1 + (\ln t)^{\beta - 1}}$, and the space $X = X_1 \times X_2$ with the norm $\|(x, y)\| = \|x\|_1 + \|y\|_2$. The spaces $(X_1, \|\cdot\|_1)$, $(X_2, \|\cdot\|_2)$ and $(X, \|(\cdot, \cdot)\|)$ are Banach spaces (see [14], Lemma 2.7).

Lemma 6. *([14], Lemma 2.8) Let $\Omega \subset X_1$ be a bounded set, which satisfy the following conditions:*
(i) The functions $\dfrac{x(t)}{1 + (\ln t)^{\alpha - 1}}$, $x \in \Omega$ are equicontinuous on any compact interval of I;

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(ii) For any $\epsilon > 0$, there exists a constant $T = T(\epsilon) > 0$ such that

$$\left| \frac{x(t_1)}{1 + (\ln t_1)^{\alpha-1}} - \frac{x(t_2)}{1 + (\ln t_2)^{\alpha-1}} \right| < \epsilon, \ \ \forall\, t_1, t_2 \geq T, \ \ \forall\, x \in \Omega.$$

Then Ω is relatively compact in X_1.

We define now the positive cone $\mathcal{P} \subset X$ by

$$\mathcal{P} = \{(x,y) \in X, \ \ x(t) \geq 0, \ y(t) \geq 0, \ \forall\, t \in [1,\infty)\},$$

and the operator $\mathcal{A} : \mathcal{P} \to X$ by $\mathcal{A}(x,y) = (\mathcal{A}_1(x,y), \mathcal{A}_2(x,y))$, $(x,y) \in \mathcal{P}$, where the operators $\mathcal{A}_1 : \mathcal{P} \to X_1$ and $\mathcal{A}_2 : \mathcal{P} \to X_2$ are defined by

$$\begin{aligned} \mathcal{A}_1(x,y)(t) &= \int_1^\infty \mathcal{G}_1(t,s)\mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s} + \int_1^\infty \mathcal{G}_2(t,s)\mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s}, \ \ t \in I, \\ \mathcal{A}_2(x,y)(t) &= \int_1^\infty \mathcal{G}_3(t,s)\mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s} + \int_1^\infty \mathcal{G}_4(t,s)\mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s}, \ \ t \in I, \end{aligned}$$

for $(x,y) \in \mathcal{P}$.

In what follows we present the basic assumptions that we will use in our main results.

- $(H1)$ $\alpha \in (n-1,n]$, $\beta \in (m-1,m]$, $n,m \in \mathbb{N}$, $n,m \geq 2$, $\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{K}_1, \mathfrak{K}_2 : [1,\infty) \to \mathbb{R}$ are nondecreasing functions, $a,d \in \mathbb{R}_+$, $b,c \in \mathbb{R}$, and $\Delta > 0$.
- $(H2)$ The functions $\mathfrak{f}$, $\mathfrak{g} \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, $\mathfrak{f}$, $\mathfrak{g} \not\equiv 0$ on any subinterval of $(0,\infty) \times (0,\infty)$, and $\mathfrak{f}$, $\mathfrak{g}$ are bounded on $\mathbb{R}_+ \times \mathbb{R}_+$.
- $(H3)$ The functions $\mathfrak{a}$, $\mathfrak{b} : [1,\infty) \to \mathbb{R}_+$ are not identical zero on any subinterval of $[1,\infty)$, and $0 < \int_1^\infty \frac{\mathfrak{a}(s)}{s} ds < \infty$, $0 < \int_1^\infty \frac{\mathfrak{b}(s)}{s} ds < \infty$.

Lemma 7. *If $(H1) - (H3)$ hold, then the operator $\mathcal{A} : \mathcal{P} \to \mathcal{P}$ is completely continuous.*

Proof. Under the assumptions of this lemma, we have $\mathcal{A}(x,y) \in \mathcal{P}$ for all $(x,y) \in \mathcal{P}$, that is $\mathcal{A} : \mathcal{P} \to \mathcal{P}$. We will prove this lemma in four steps.

(I) We show firstly that the operator $\mathcal{A}$ is uniformly bounded on $\mathcal{P}$. Let $\mathcal{S}$ be a bounded set of $\mathcal{P}$. Then there exists $r > 0$ such that $\|(x,y)\| \leq r$, and so $\|x\|_1 \leq r$ and $\|y\|_2 \leq r$ for all $(x,y) \in \mathcal{S}$. By $(H2)$, there exist $M_1 > 0$ and $M_2 > 0$ such that $\mathfrak{f}(x,y) \leq M_1$ and $\mathfrak{g}(x,y) \leq M_2$ for all x, $y \in \mathbb{R}_+$. Then by $(H3)$, for any $(x,y) \in \mathcal{S}$, we obtain

$$\begin{aligned} \|\mathcal{A}_1(x,y)\|_1 &= \sup_{t\in I} \frac{|\mathcal{A}_1(x,y)(t)|}{1+(\ln t)^{\alpha-1}} \\ &= \sup_{t\in I} \frac{1}{1+(\ln t)^{\alpha-1}} \left(\int_1^\infty \mathcal{G}_1(t,s)\mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s} \right. \\ &\quad \left. + \int_1^\infty \mathcal{G}_2(t,s)\mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s} \right) \\ &\leq M_1\Lambda_1 \int_1^\infty \mathfrak{a}(s)\,\frac{ds}{s} + M_2\Lambda_2 \int_1^\infty \mathfrak{b}(s)\,\frac{ds}{s} =: M_3 < \infty, \\ \|\mathcal{A}_2(x,y)\|_2 &= \sup_{t\in I} \frac{|\mathcal{A}_2(x,y)(t)|}{1+(\ln t)^{\beta-1}} \\ &= \sup_{t\in I} \frac{1}{1+(\ln t)^{\beta-1}} \left(\int_1^\infty \mathcal{G}_3(t,s)\mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s} \right. \\ &\quad \left. + \int_1^\infty \mathcal{G}_4(t,s)\mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s} \right) \\ &\leq M_1\Lambda_3 \int_1^\infty \mathfrak{a}(s)\,\frac{ds}{s} + M_2\Lambda_4 \int_1^\infty \mathfrak{b}(s)\,\frac{ds}{s} =: M_4 < \infty. \end{aligned}$$

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So

$$\|\mathcal{A}(x,y)\| = \|\mathcal{A}_1(x,y)\|_1 + \|\mathcal{A}_2(x,y)\|_2 \leq M_3 + M_4 < \infty, \ \forall (x,y) \in \mathcal{S},$$

that is, operator $\mathcal{A}$ is uniformly bounded.

(II) Next we will prove that $\mathcal{A}$ is equicontinuous on any compact interval of I. We consider the interval $[1,T]$, where $T > 1$. Then for any $t_1, t_2 \in [1,T]$, with $t_1 < t_2$ and $(x,y) \in \mathcal{S}$, we have

$$\begin{aligned}
\Xi_1 =& \left| \frac{\mathcal{A}_1(x,y)(t_2)}{1+(\ln t_2)^{\alpha-1}} - \frac{\mathcal{A}_1(x,y)(t_1)}{1+(\ln t_1)^{\alpha-1}} \right| \\
=& \left| \int_1^\infty \frac{\mathcal{G}_1(t_2,s)}{1+(\ln t_2)^{\alpha-1}} \mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s} + \int_1^\infty \frac{\mathcal{G}_2(t_2,s)}{1+(\ln t_2)^{\alpha-1}} \mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s} \right. \\
& \left. - \int_1^\infty \frac{\mathcal{G}_1(t_1,s)}{1+(\ln t_1)^{\alpha-1}} \mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s} - \int_1^\infty \frac{\mathcal{G}_2(t_1,s)}{1+(\ln t_1)^{\alpha-1}} \mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s} \right| \\
\leq& \left| \int_1^\infty \left(\frac{\mathcal{G}_1(t_2,s)}{1+(\ln t_2)^{\alpha-1}} - \frac{\mathcal{G}_1(t_1,s)}{1+(\ln t_1)^{\alpha-1}} \right) \mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s} \right| \\
& + \left| \int_1^\infty \left(\frac{\mathcal{G}_2(t_2,s)}{1+(\ln t_2)^{\alpha-1}} - \frac{\mathcal{G}_2(t_1,s)}{1+(\ln t_1)^{\alpha-1}} \right) \mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s} \right| \\
\leq& \int_1^\infty \left| \frac{\mathcal{G}_1(t_2,s)}{1+(\ln t_2)^{\alpha-1}} - \frac{\mathcal{G}_1(t_1,s)}{1+(\ln t_1)^{\alpha-1}} \right| \mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s} \\
& + \int_1^\infty \left| \frac{\mathcal{G}_2(t_2,s)}{1+(\ln t_2)^{\alpha-1}} - \frac{\mathcal{G}_2(t_1,s)}{1+(\ln t_1)^{\alpha-1}} \right| \mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s} \\
\leq& \int_1^\infty \left[\left| \frac{g_\alpha(t_2,s)}{1+(\ln t_2)^{\alpha-1}} - \frac{g_\alpha(t_1,s)}{1+(\ln t_1)^{\alpha-1}} \right| \right. \\
& + \frac{1}{\Delta} \left(d \int_1^\infty g_\alpha(\tau,s)\,d\mathfrak{H}_1(\tau) + b \int_1^\infty g_\alpha(\tau,s)\,d\mathfrak{K}_1(\tau) \right) \\
& \times \left. \left(\frac{(\ln t_2)^{\alpha-1}}{1+(\ln t_2)^{\alpha-1}} - \frac{(\ln t_1)^{\alpha-1}}{1+(\ln t_1)^{\alpha-1}} \right) \right] \mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s} \\
& + \int_1^\infty \frac{1}{\Delta} \left(d \int_1^\infty g_\beta(\tau,s)\,d\mathfrak{H}_2(\tau) + b \int_1^\infty g_\beta(\tau,s)\,d\mathfrak{K}_2(\tau) \right) \\
& \times \left(\frac{(\ln t_2)^{\alpha-1}}{1+(\ln t_2)^{\alpha-1}} - \frac{(\ln t_1)^{\alpha-1}}{1+(\ln t_1)^{\alpha-1}} \right) \mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s} \\
\leq& M_1 \underbrace{\int_1^\infty \left| \frac{g_\alpha(t_2,s)}{1+(\ln t_2)^{\alpha-1}} - \frac{g_\alpha(t_1,s)}{1+(\ln t_1)^{\alpha-1}} \right| \mathfrak{a}(s)\,\frac{ds}{s}}_{\Xi_0} \\
& + \frac{M_1}{\Delta\Gamma(\alpha)} \left(d \int_1^\infty (\ln \tau)^{\alpha-1}\,d\mathfrak{H}_1(\tau) + b \int_1^\infty (\ln \tau)^{\alpha-1}\,d\mathfrak{K}_1(\tau) \right) \\
& \times \left(\frac{(\ln t_2)^{\alpha-1}}{1+(\ln t_2)^{\alpha-1}} - \frac{(\ln t_1)^{\alpha-1}}{1+(\ln t_1)^{\alpha-1}} \right) \int_1^\infty \mathfrak{a}(s)\,\frac{ds}{s} \\
& + \frac{M_2}{\Delta\Gamma(\beta)} \left(d \int_1^\infty (\ln \tau)^{\beta-1}\,d\mathfrak{H}_2(\tau) + b \int_1^\infty (\ln \tau)^{\beta-1}\,d\mathfrak{K}_2(\tau) \right) \\
& \times \left(\frac{(\ln t_2)^{\alpha-1}}{1+(\ln t_2)^{\alpha-1}} - \frac{(\ln t_1)^{\alpha-1}}{1+(\ln t_1)^{\alpha-1}} \right) \int_1^\infty \mathfrak{b}(s)\,\frac{ds}{s} \\
=& M_1\Xi_0 + \frac{M_1(d\Gamma(\alpha)-\Delta)}{\Delta\Gamma(\alpha)} \left(\frac{(\ln t_2)^{\alpha-1}}{1+(\ln t_2)^{\alpha-1}} - \frac{(\ln t_1)^{\alpha-1}}{1+(\ln t_1)^{\alpha-1}} \right) \int_1^\infty \mathfrak{a}(s)\,\frac{ds}{s} \\
& + \frac{M_2 b}{\Delta} \left(\frac{(\ln t_2)^{\alpha-1}}{1+(\ln t_2)^{\alpha-1}} - \frac{(\ln t_1)^{\alpha-1}}{1+(\ln t_1)^{\alpha-1}} \right) \int_1^\infty \mathfrak{b}(s)\,\frac{ds}{s}.
\end{aligned}$$

For Ξ_0 we deduce

$$\begin{aligned}
\Xi_0 =& \int_1^{t_1} \left| \frac{g_\alpha(t_2,s)}{1+(\ln t_2)^{\alpha-1}} - \frac{g_\alpha(t_1,s)}{1+(\ln t_1)^{\alpha-1}} \right| \mathfrak{a}(s)\,\frac{ds}{s} \\
& + \int_{t_1}^{t_2} \left| \frac{g_\alpha(t_2,s)}{1+(\ln t_2)^{\alpha-1}} - \frac{g_\alpha(t_1,s)}{1+(\ln t_1)^{\alpha-1}} \right| \mathfrak{a}(s)\,\frac{ds}{s} \\
& + \int_{t_2}^\infty \left| \frac{g_\alpha(t_2,s)}{1+(\ln t_2)^{\alpha-1}} - \frac{g_\alpha(t_1,s)}{1+(\ln t_1)^{\alpha-1}} \right| \mathfrak{a}(s)\,\frac{ds}{s}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left| \frac{(\ln t_2)^{\alpha-1} - \left(\ln \frac{t_2}{s}\right)^{\alpha-1}}{1+(\ln t_2)^{\alpha-1}} - \frac{(\ln t_1)^{\alpha-1} - \left(\ln \frac{t_1}{s}\right)^{\alpha-1}}{1+(\ln t_1)^{\alpha-1}} \right| \mathfrak{a}(s)\, \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left| \frac{(\ln t_2)^{\alpha-1} - \left(\ln \frac{t_2}{s}\right)^{\alpha-1}}{1+(\ln t_2)^{\alpha-1}} - \frac{(\ln t_1)^{\alpha-1}}{1+(\ln t_1)^{\alpha-1}} \right| \mathfrak{a}(s)\, \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_2}^{\infty} \left(\frac{(\ln t_2)^{\alpha-1}}{1+(\ln t_2)^{\alpha-1}} - \frac{(\ln t_1)^{\alpha-1}}{1+(\ln t_1)^{\alpha-1}} \right) \mathfrak{a}(s)\, \frac{ds}{s} \\
&\le \frac{1}{\Gamma(\alpha)} \left(\frac{(\ln t_2)^{\alpha-1}}{1+(\ln t_2)^{\alpha-1}} - \frac{(\ln t_1)^{\alpha-1}}{1+(\ln t_1)^{\alpha-1}} \right) \int_1^{t_1} \mathfrak{a}(s)\, \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left(\frac{\left(\ln \frac{t_2}{s}\right)^{\alpha-1}}{1+(\ln t_2)^{\alpha-1}} - \frac{\left(\ln \frac{t_1}{s}\right)^{\alpha-1}}{1+(\ln t_1)^{\alpha-1}} \right) \mathfrak{a}(s)\, \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \left(\frac{(\ln t_2)^{\alpha-1}}{1+(\ln t_2)^{\alpha-1}} - \frac{(\ln t_1)^{\alpha-1}}{1+(\ln t_1)^{\alpha-1}} \right) \int_{t_1}^{t_2} \mathfrak{a}(s)\, \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{\left(\ln \frac{t_2}{s}\right)^{\alpha-1}}{1+(\ln t_2)^{\alpha-1}} \mathfrak{a}(s)\, \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \left(\frac{(\ln t_2)^{\alpha-1}}{1+(\ln t_2)^{\alpha-1}} - \frac{(\ln t_1)^{\alpha-1}}{1+(\ln t_1)^{\alpha-1}} \right) \int_{t_2}^{\infty} \mathfrak{a}(s)\, \frac{ds}{s}.
\end{aligned}
$$

Because the functions $\frac{(\ln t)^{\alpha-1}}{1+(\ln t)^{\alpha-1}}$ and $\frac{(\ln(t/s))^{\alpha-1}}{1+(\ln t)^{\alpha-1}}$ are uniformly continuous on $[1,T]$, respectively on $[1,T]\times[1,T]$, and the integrals $\int_0^\infty \mathfrak{a}(s)\,\frac{ds}{s}$, $\int_0^\infty \mathfrak{b}(s)\,\frac{ds}{s}$ are convergent, we conclude that $\Xi_0 \to 0$ and $\Xi_1 \to 0$ as $t_2 \to t_1$ uniformly with respect to $(x,y) \in \mathcal{S}$.

In a similar manner we obtain

$$
\left| \frac{\mathcal{A}_2(x,y)(t_2)}{1+(\ln t_2)^{\beta-1}} - \frac{\mathcal{A}_2(x,y)(t_1)}{1+(\ln t_1)^{\beta-1}} \right| \to 0, \ \text{ as } \ t_2 \to t_1,
$$

uniformly with respect to $(x,y) \in \mathcal{S}$. Then $\mathcal{A}(\mathcal{S})$ is equicontinuous on $[1,T]$.

(III) In what follows we will show that $\mathcal{A}$ is equiconvergent at ∞. We will prove firstly that $\mathcal{A}_1$ is equiconvergent at ∞, that is, for any $\epsilon > 0$ there exists $T_0 > 0$ such that for all $t_1, t_2 \ge T_0$, and $(x,y) \in \mathcal{S}$ we have

$$
\left| \frac{\mathcal{A}_1(x,y)(t_2)}{1+(\ln t_2)^{\alpha-1}} - \frac{\mathcal{A}_1(x,y)(t_1)}{1+(\ln t_1)^{\alpha-1}} \right| < \Lambda_0 \epsilon, \ \ (\Lambda_0 > 0).
$$

For this, let $\epsilon > 0$. Then there exists $\delta_1 > 1$ such that $\int_{\delta_1}^\infty \mathfrak{a}(s)\,\frac{ds}{s} < \epsilon$ and $\int_{\delta_1}^\infty \mathfrak{b}(s)\frac{ds}{s} < \epsilon$. Because $\lim_{t\to\infty} \frac{(\ln t)^{\alpha-1}}{1+(\ln t)^{\alpha-1}} = 1$ and $\lim_{t\to\infty} \frac{g_\alpha(t,\delta_1)}{1+(\ln t)^{\alpha-1}} = 0$, we deduce that there exist $\delta_2 > 0$ and $\delta_3 > \delta_1$ such that for any $t_1, t_2 > \delta_2$ we have

$$
\left| \frac{(\ln t_2)^{\alpha-1}}{1+(\ln t_2)^{\alpha-1}} - \frac{(\ln t_1)^{\alpha-1}}{1+(\ln t_1)^{\alpha-1}} \right| < \epsilon,
$$

and for any $t_1, t_2 > \delta_3$ and $1 \le s \le \delta_1$ we have

$$
\begin{aligned}
\left| \frac{g_\alpha(t_2,s)}{1+(\ln t_2)^{\alpha-1}} - \frac{g_\alpha(t_1,s)}{1+(\ln t_1)^{\alpha-1}} \right| &\le \left| \frac{g_\alpha(t_2,s)}{1+(\ln t_2)^{\alpha-1}} \right| + \left| \frac{g_\alpha(t_1,s)}{1+(\ln t_1)^{\alpha-1}} \right| \\
&\le \left| \frac{g_\alpha(t_2,\delta_1)}{1+(\ln t_2)^{\alpha-1}} \right| + \left| \frac{g_\alpha(t_1,\delta_1)}{1+(\ln t_1)^{\alpha-1}} \right| < \epsilon.
\end{aligned}
$$

Then we choose $T_0 > \max\{\delta_2, \delta_3\}$, and then for any $(x,y) \in \mathcal{S}$ and $t_1, t_2 \geq T_0$ we obtain

$$\begin{aligned}
&\left| \frac{\mathcal{A}_1(x,y)(t_2)}{1+(\ln t_2)^{\alpha-1}} - \frac{\mathcal{A}_1(x,y)(t_1)}{1+(\ln t_1)^{\alpha-1}} \right| \\
&\leq \int_1^\infty \left| \frac{\mathcal{G}_1(t_2,s)}{1+(\ln t_2)^{\alpha-1}} - \frac{\mathcal{G}_1(t_1,s)}{1+(\ln t_1)^{\alpha-1}} \right| \mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s} \\
&\quad + \int_1^\infty \left| \frac{\mathcal{G}_2(t_2,s)}{1+(\ln t_2)^{\alpha-1}} - \frac{\mathcal{G}_2(t_1,s)}{1+(\ln t_1)^{\alpha-1}} \right| \mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s} \\
&\leq \int_1^{\delta_1} \left| \frac{\mathcal{G}_1(t_2,s)}{1+(\ln t_2)^{\alpha-1}} - \frac{\mathcal{G}_1(t_1,s)}{1+(\ln t_1)^{\alpha-1}} \right| \mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s} \\
&\quad + \int_{\delta_1}^\infty \left| \frac{\mathcal{G}_1(t_2,s)}{1+(\ln t_2)^{\alpha-1}} - \frac{\mathcal{G}_1(t_1,s)}{1+(\ln t_1)^{\alpha-1}} \right| \mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s} \\
&\quad + \int_1^{\delta_1} \left| \frac{\mathcal{G}_2(t_2,s)}{1+(\ln t_2)^{\alpha-1}} - \frac{\mathcal{G}_2(t_1,s)}{1+(\ln t_1)^{\alpha-1}} \right| \mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s} \\
&\quad + \int_{\delta_1}^\infty \left| \frac{\mathcal{G}_2(t_2,s)}{1+(\ln t_2)^{\alpha-1}} - \frac{\mathcal{G}_2(t_1,s)}{1+(\ln t_1)^{\alpha-1}} \right| \mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s} \\
&\leq M_1 \int_1^{\delta_1} \left| \frac{g_\alpha(t_2,s)}{1+(\ln t_2)^{\alpha-1}} - \frac{g_\alpha(t_1,s)}{1+(\ln t_1)^{\alpha-1}} \right| \mathfrak{a}(s)\,\frac{ds}{s} \\
&\quad + M_1 \int_{\delta_1}^\infty \left| \frac{g_\alpha(t_2,s)}{1+(\ln t_2)^{\alpha-1}} - \frac{g_\alpha(t_1,s)}{1+(\ln t_1)^{\alpha-1}} \right| \mathfrak{a}(s)\,\frac{ds}{s} \\
&\quad + \frac{M_1}{\Delta} \left| \frac{(\ln t_2)^{\alpha-1}}{1+(\ln t_2)^{\alpha-1}} - \frac{(\ln t_1)^{\alpha-1}}{1+(\ln t_1)^{\alpha-1}} \right| \\
&\quad \times \left(d \int_1^\infty \frac{1}{\Gamma(\alpha)} \tau^{\alpha-1}\, d\mathfrak{H}_1(\tau) + b \int_1^\infty \frac{1}{\Gamma(\alpha)} \tau^{\alpha-1}\, d\mathfrak{K}_1(\tau) \right) \int_1^\infty \mathfrak{a}(s)\,\frac{ds}{s} \\
&\quad + \frac{M_2}{\Delta} \left| \frac{(\ln t_2)^{\alpha-1}}{1+(\ln t_2)^{\alpha-1}} - \frac{(\ln t_1)^{\alpha-1}}{1+(\ln t_1)^{\alpha-1}} \right| \\
&\quad \times \left(d \int_1^\infty \frac{1}{\Gamma(\beta)} \tau^{\beta-1}\, d\mathfrak{H}_2(\tau) + b \int_1^\infty \frac{1}{\Gamma(\beta)} \tau^{\beta-1}\, d\mathfrak{K}_2(\tau) \right) \int_1^\infty \mathfrak{b}(s)\,\frac{ds}{s} \\
&\leq \epsilon M_1 \int_1^{\delta_1} \mathfrak{a}(s)\,\frac{ds}{s} + \frac{2\epsilon M_1}{\Gamma(\alpha)} + \frac{\epsilon M_1 (d\Gamma(\alpha) - \Delta)}{\Delta\Gamma(\alpha)} \int_1^\infty \mathfrak{a}(s)\,\frac{ds}{s} \\
&\quad + \frac{\epsilon M_2 b}{\Delta} \int_1^\infty \mathfrak{b}(s)\,\frac{ds}{s} = \Lambda_0 \epsilon, \quad (\Lambda_0 > 0).
\end{aligned}$$

So $\mathcal{A}_1$ is equiconvergent at ∞. In a similar manner we show that $\mathcal{A}_2$ is equiconvergent at ∞, and then we deduce that $\mathcal{A}$ is equiconvergent at ∞.

(IV) In the last part of the proof we will prove that the operator $\mathcal{A}$ is continuous. Let $((x_n, y_n))_{n\in\mathbb{N}}$, $(x,y) \in X$, $(x_n, y_n) \to (x,y)$ in $X = X_1 \times X_2$, for $n \to \infty$, that is

$$\sup_{t\in I} \frac{|x_n(t) - x(t)|}{1+(\ln t)^{\alpha-1}} \to 0, \text{ and } \sup_{t\in I} \frac{|y_n(t) - y(t)|}{1+(\ln t)^{\beta-1}} \to 0, \text{ as } n \to \infty.$$

Then we deduce that for any $s \in I$, $x_n(s) - x(s) \to 0$ and $y_n(s) - y(s) \to 0$, as $n \to \infty$.

Because

$$\begin{aligned}
|\mathfrak{f}(x_n(s), y_n(s)) - \mathfrak{f}(x(s), y(s))| \leq 2M_1, \quad \forall s \in I, \quad n \in \mathbb{N}, \\
|\mathfrak{g}(x_n(s), y_n(s)) - \mathfrak{g}(x(s), y(s))| \leq 2M_2, \quad \forall s \in I, \quad n \in \mathbb{N},
\end{aligned}$$

we obtain by the Lebesgue convergence theorem that

$$\begin{aligned}
\int_1^\infty \mathfrak{a}(s) |\mathfrak{f}(x_n(s), y_n(s)) - \mathfrak{f}(x(s), y(s))|\,\frac{ds}{s} \to 0, \text{ as } n \to \infty, \\
\int_1^\infty \mathfrak{b}(s) |\mathfrak{g}(x_n(s), y_n(s)) - \mathfrak{g}(x(s), y(s))|\,\frac{ds}{s} \to 0, \text{ as } n \to \infty.
\end{aligned} \tag{13}$$

By using Lemma 5 and (13), we find

$$
\begin{aligned}
\|\mathcal{A}_1(x_n,y_n)-\mathcal{A}_1(x,y)\|_1 &= \sup_{t\in I}\frac{|\mathcal{A}_1(x_n,y_n)(t)-\mathcal{A}_1(x,y)(t)|}{1+(\ln t)^{\alpha-1}}\\
&= \sup_{t\in I}\Bigg|\int_1^\infty \frac{\mathcal{G}_1(t,s)}{1+(\ln t)^{\alpha-1}}\mathfrak{a}(s)\mathfrak{f}(x_n(s),y_n(s))\,\frac{ds}{s}\\
&+\int_1^\infty \frac{\mathcal{G}_2(t,s)}{1+(\ln t)^{\alpha-1}}\mathfrak{b}(s)\mathfrak{g}(x_n(s),y_n(s))\,\frac{ds}{s}\\
&-\int_1^\infty \frac{\mathcal{G}_1(t,s)}{1+(\ln t)^{\alpha-1}}\mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s}-\int_1^\infty \frac{\mathcal{G}_2(t,s)}{1+(\ln t)^{\alpha-1}}\mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s}\Bigg|\\
&\le \sup_{t\in I}\left(\int_1^\infty \frac{\mathcal{G}_1(t,s)}{1+(\ln t)^{\alpha-1}}\mathfrak{a}(s)|\mathfrak{f}(x_n(s),y_n(s))-\mathfrak{f}(x(s),y(s))|\,\frac{ds}{s}\right.\\
&\left.+\int_1^\infty \frac{\mathcal{G}_2(t,s)}{1+(\ln t)^{\alpha-1}}\mathfrak{b}(s)|\mathfrak{g}(x_n(s),y_n(s))-\mathfrak{g}(x(s),y(s))|\,\frac{ds}{s}\right)\\
&\le \Lambda_1\int_1^\infty \mathfrak{a}(s)|\mathfrak{f}(x_n(s),y_n(s))-\mathfrak{f}(x(s),y(s))|\,\frac{ds}{s}\\
&+\Lambda_2\int_1^\infty \mathfrak{b}(s)|\mathfrak{g}(x_n(s),y_n(s))-\mathfrak{g}(x(s),y(s))|\,\frac{ds}{s}\to 0,\ \text{as } n\to\infty.
\end{aligned}
$$

Then we deduce that $\|\mathcal{A}_1(x_n,y_n)-\mathcal{A}_1(x,y)\|_1\to 0$, as $n\to\infty$. In a similar manner we can prove that $\|\mathcal{A}_2(x_n,y_n)-\mathcal{A}_2(x,y)\|_2\to 0$, as $n\to\infty$. Therefore we obtain $\|\mathcal{A}(x_n,y_n)-\mathcal{A}(x,y)\|\to 0$ as $n\to\infty$. So the operator $\mathcal{A}$ is continuous.

From the above steps and Lemma 6, we conclude that the operator $\mathcal{A}$ is completely continuous. □

Now for $\theta>1$ we introduce the following constants

$$
\begin{aligned}
&Q_1=\int_1^\infty \mathfrak{a}(s)\,\frac{ds}{s},\ \ Q_2=\int_1^\infty \mathfrak{b}(s)\,\frac{ds}{s},\ \ Q_3=\int_\theta^\infty \mathfrak{a}(s)\,\frac{ds}{s},\ \ Q_4=\int_\theta^\infty \mathfrak{b}(s)\,\frac{ds}{s},\\
&\widetilde{L}_1=\frac{d}{\Gamma(\alpha)}\int_1^\theta(\ln\tau)^{\alpha-1}\,d\mathfrak{H}_1(\tau)+\frac{b}{\Gamma(\alpha)}\int_1^\theta(\ln\tau)^{\alpha-1}\,d\mathfrak{K}_1(\tau),\\
&\widetilde{L}_2=\frac{d}{\Gamma(\beta)}\int_1^\theta(\ln\tau)^{\beta-1}\,d\mathfrak{H}_2(\tau)+\frac{b}{\Gamma(\beta)}\int_1^\theta(\ln\tau)^{\beta-1}\,d\mathfrak{K}_2(\tau),\\
&\widetilde{L}_3=\frac{c}{\Gamma(\alpha)}\int_1^\theta(\ln\tau)^{\alpha-1}\,d\mathfrak{H}_1(\tau)+\frac{a}{\Gamma(\alpha)}\int_1^\theta(\ln\tau)^{\alpha-1}\,d\mathfrak{K}_1(\tau),\\
&\widetilde{L}_4=\frac{c}{\Gamma(\beta)}\int_1^\theta(\ln\tau)^{\beta-1}\,d\mathfrak{H}_2(\tau)+\frac{a}{\Gamma(\beta)}\int_1^\theta(\ln\tau)^{\beta-1}\,d\mathfrak{K}_2(\tau),\\
&L_1=\frac{\widetilde{L}_1(\ln\theta)^{\alpha-1}}{\Delta(1+(\ln\theta)^{\alpha-1})},\ \ L_2=\frac{\widetilde{L}_2(\ln\theta)^{\alpha-1}}{\Delta(1+(\ln\theta)^{\alpha-1})},\\
&L_3=\frac{\widetilde{L}_3(\ln\theta)^{\beta-1}}{\Delta(1+(\ln\theta)^{\beta-1})},\ \ L_4=\frac{\widetilde{L}_4(\ln\theta)^{\beta-1}}{\Delta(1+(\ln\theta)^{\beta-1})},\\
&\Upsilon_1=Q_3(L_1+L_3),\ \ \Upsilon_2=Q_4(L_2+L_4),\ \ \Upsilon_3=Q_1(\Lambda_1+\Lambda_3),\ \ \Upsilon_4=Q_2(\Lambda_2+\Lambda_4).
\end{aligned}
\tag{14}
$$

In our first main theorem we will prove the existence of at least one positive solution for problem (1), (2) by using the Guo-Krasnosel'skii fixed point theorem (Theorem 1).

Theorem 3. *We assume that $(H1)-(H3)$ hold, and there exists $\theta>1$ such that $Q_j>0$, $j=3,4$, and $\widetilde{L}_i>0$, $i=1,\dots,4$. In addition, we suppose that there exist positive constants r_1,r_2 with $r_1<r_2$, and $\sigma_1\in[\Upsilon_1^{-1},\infty)$, $\sigma_2\in[\Upsilon_2^{-1},\infty)$, $\sigma_3\in(0,\Upsilon_3^{-1}]$ and $\sigma_4\in(0,\Upsilon_4^{-1}]$ such that*

$(H4)$ $\mathfrak{f}((1+(\ln t)^{\alpha-1})x,(1+(\ln t)^{\beta-1})y)\ge\frac{\sigma_1 r_1}{2},\ \forall t\in[\theta,\infty),\ (x,y)\in[0,r_1]\times[0,r_1],$
$\mathfrak{g}((1+(\ln t)^{\alpha-1})x,(1+(\ln t)^{\beta-1})y)\ge\frac{\sigma_2 r_1}{2},\ \forall t\in[\theta,\infty),\ (x,y)\in[0,r_1]\times[0,r_1];$

$(H5)$ $\mathfrak{f}((1+(\ln t)^{\alpha-1})x,(1+(\ln t)^{\beta-1})y)\le\frac{\sigma_3 r_2}{2},\ \forall t\in I,\ (x,y)\in[0,r_2]\times[0,r_2],$
$\mathfrak{g}((1+(\ln t)^{\alpha-1})x,(1+(\ln t)^{\beta-1})y)\le\frac{\sigma_4 r_2}{2},\ \forall t\in I,\ (x,y)\in[0,r_2]\times[0,r_2].$

Then problem (1), (2) has at least one positive solution (x,y) such that $r_1\le\|(x,y)\|\le r_2$.

Proof. By Lemma 7, the operator $\mathcal{A}$ is completely continuous. We introduce the set $\Omega_1 = \{(x,y) \in X,\ \|(x,y)\| < r_1\}$. Then for $(x,y) \in \mathcal{P} \cap \partial\Omega_1$, we have $\|x\|_1 + \|y\|_2 = r_1$, so $\|x\|_1 \leq r_1$ and $\|y\|_2 \leq r_1$, that is $0 \leq \frac{x(t)}{1+(\ln t)^{\alpha-1}} \leq r_1$ and $0 \leq \frac{y(t)}{1+(\ln t)^{\beta-1}} \leq r_1$ for all $t \in I$.

Therefore by $(H4)$ and Lemma 5, we obtain

$$\begin{aligned}
\|\mathcal{A}_1(x,y)\|_1 &= \sup_{t\in I} \frac{1}{1+(\ln t)^{\alpha-1}} \left(\int_1^\infty \mathcal{G}_1(t,s)\mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\, \frac{ds}{s} \right.\\
&\quad \left. + \int_1^\infty \mathcal{G}_2(t,s)\mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\, \frac{ds}{s} \right)\\
&\geq \inf_{t\in[\theta,\infty)} \frac{1}{1+(\ln t)^{\alpha-1}} \left(\int_1^\infty \mathcal{G}_1(t,s)\mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\, \frac{ds}{s} \right.\\
&\quad \left. + \int_1^\infty \mathcal{G}_2(t,s)\mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\, \frac{ds}{s} \right)\\
&\geq \inf_{t\in[\theta,\infty)} \frac{1}{1+(\ln t)^{\alpha-1}} \int_1^\infty \mathcal{G}_1(t,s)\mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\, \frac{ds}{s}\\
&\quad + \inf_{t\in[\theta,\infty)} \frac{1}{1+(\ln t)^{\alpha-1}} \int_1^\infty \mathcal{G}_2(t,s)\mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\, \frac{ds}{s}\\
&\geq \int_1^\infty \inf_{t\in[\theta,\infty)} \frac{\mathcal{G}_1(t,s)}{1+(\ln t)^{\alpha-1}} \mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\, \frac{ds}{s}\\
&\quad + \int_1^\infty \inf_{t\in[\theta,\infty)} \frac{\mathcal{G}_2(t,s)}{1+(\ln t)^{\alpha-1}} \mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\, \frac{ds}{s}\\
&\geq \frac{(\ln\theta)^{\alpha-1}}{\Delta(1+(\ln\theta)^{\alpha-1})} \int_\theta^\infty \left(d\int_1^\infty g_\alpha(\tau,s)\, d\mathfrak{H}_1(\tau) \right.\\
&\quad \left. + b\int_1^\infty g_\alpha(\tau,s)\, d\mathfrak{K}_1(\tau) \right) \mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\, \frac{ds}{s}\\
&\quad + \frac{(\ln\theta)^{\alpha-1}}{\Delta(1+(\ln\theta)^{\alpha-1})} \int_\theta^\infty \left(d\int_1^\infty g_\beta(\tau,s)\, d\mathfrak{H}_2(\tau) \right.\\
&\quad \left. + b\int_1^\infty g_\beta(\tau,s)\, d\mathfrak{K}_2(\tau) \right) \mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\, \frac{ds}{s}\\
&\geq \frac{(\ln\theta)^{\alpha-1}}{\Delta(1+(\ln\theta)^{\alpha-1})} \int_\theta^\infty \left(d\int_1^\theta g_\alpha(\tau,s)\, d\mathfrak{H}_1(\tau) \right.\\
&\quad \left. + b\int_1^\theta g_\alpha(\tau,s)\, d\mathfrak{K}_1(\tau) \right) \mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\, \frac{ds}{s}\\
&\quad + \frac{(\ln\theta)^{\alpha-1}}{\Delta(1+(\ln\theta)^{\alpha-1})} \int_\theta^\infty \left(d\int_1^\theta g_\beta(\tau,s)\, d\mathfrak{H}_2(\tau) \right.\\
&\quad \left. + b\int_1^\theta g_\beta(\tau,s)\, d\mathfrak{K}_2(\tau) \right) \mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\, \frac{ds}{s}\\
&= \frac{(\ln\theta)^{\alpha-1}}{\Delta(1+(\ln\theta)^{\alpha-1})} \int_\theta^\infty \mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\, \frac{ds}{s}\\
&\quad \times \left(\frac{d}{\Gamma(\alpha)} \int_1^\theta (\ln\tau)^{\alpha-1}\, d\mathfrak{H}_1(\tau) + \frac{b}{\Gamma(\alpha)} \int_1^\theta (\ln\tau)^{\alpha-1}\, d\mathfrak{K}_1(\tau) \right)\\
&\quad + \frac{(\ln\theta)^{\alpha-1}}{\Delta(1+(\ln\theta)^{\alpha-1})} \int_\theta^\infty \mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\, \frac{ds}{s}\\
&\quad \times \left(\frac{d}{\Gamma(\beta)} \int_1^\theta (\ln\tau)^{\beta-1}\, d\mathfrak{H}_2(\tau) + \frac{b}{\Gamma(\beta)} \int_1^\theta (\ln\tau)^{\beta-1}\, d\mathfrak{K}_2(\tau) \right)\\
&= \frac{(\ln\theta)^{\alpha-1}\widetilde{L}_1}{\Delta(1+(\ln\theta)^{\alpha-1})} \frac{\sigma_1 r_1}{2} \int_\theta^\infty \mathfrak{a}(s)\, \frac{ds}{s} + \frac{(\ln\theta)^{\alpha-1}\widetilde{L}_2}{\Delta(1+(\ln\theta)^{\alpha-1})} \frac{\sigma_2 r_1}{2} \int_\theta^\infty \mathfrak{b}(s)\, \frac{ds}{s}\\
&= \frac{\sigma_1 r_1 L_1 Q_3}{2} + \frac{\sigma_2 r_1 L_2 Q_4}{2} = \left(\frac{\sigma_1 L_1 Q_3}{2} + \frac{\sigma_2 L_2 Q_4}{2} \right) r_1.
\end{aligned}$$

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In a similar manner we find

$$\begin{aligned}\|\mathcal{A}_2(x,y)\|_2 &\geq \frac{(\ln\theta)^{\beta-1}\widetilde{L}_3}{\Delta(1+(\ln\theta)^{\beta-1})}\frac{\sigma_1 r_1}{2}\int_\theta^\infty \mathfrak{a}(s)\,\frac{ds}{s} + \frac{(\ln\theta)^{\beta-1}\widetilde{L}_4}{\Delta(1+(\ln\theta)^{\beta-1})}\frac{\sigma_2 r_1}{2}\int_\theta^\infty \mathfrak{b}(s)\,\frac{ds}{s}\\ &= \frac{\sigma_1 r_1 L_3 Q_3}{2} + \frac{\sigma_2 r_1 L_4 Q_4}{2} = \left(\frac{\sigma_1 L_3 Q_3}{2} + \frac{\sigma_2 L_4 Q_4}{2}\right) r_1.\end{aligned}$$

Therefore we deduce

$$\begin{aligned}\|\mathcal{A}(x,y)\| &= \|\mathcal{A}_1(x,y)\|_1 + \|\mathcal{A}_2(x,y)\|_2\\ &\geq \left[\frac{(L_1+L_3)\sigma_1 Q_3}{2} + \frac{(L_2+L_4)\sigma_2 Q_4}{2}\right] r_1 = \left(\frac{\sigma_1 \Upsilon_1}{2} + \frac{\sigma_2 \Upsilon_2}{2}\right) r_1 \geq r_1,\end{aligned}$$

which gives us

$$\|\mathcal{A}(x,y)\| \geq \|(x,y)\|, \ \forall\,(x,y) \in \mathcal{P}\cap\partial\Omega_1. \tag{15}$$

Now we introduce the set $\Omega_2 = \{(x,y) \in X,\ \|(x,y)\| < r_2\}$. Then for any $(x,y) \in \mathcal{P}\cap\partial\Omega_2$, we have $\|x\|_1 + \|y\|_2 = r_2$, and then $\|x\|_1 \leq r_2$ and $\|y\|_2 \leq r_2$, that is $0 \leq \frac{x(t)}{1+(\ln t)^{\alpha-1}} \leq r_2$ and $0 \leq \frac{y(t)}{1+(\ln t)^{\beta-1}} \leq r_2$ for all $t \in I$.

Then by $(H5)$ and Lemma 5, we obtain

$$\begin{aligned}\|\mathcal{A}_1(x,y)\|_1 &= \sup_{t\in I}\frac{\mathcal{A}_1(x,y)}{1+(\ln t)^{\alpha-1}}\\ &\leq \sup_{t\in I}\frac{1}{1+(\ln t)^{\alpha-1}}\int_1^\infty \mathcal{G}_1(t,s)\mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s}\\ &\quad + \sup_{t\in I}\frac{1}{1+(\ln t)^{\alpha-1}}\int_1^\infty \mathcal{G}_2(t,s)\mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s}\\ &\leq \int_1^\infty \sup_{t\in I}\frac{\mathcal{G}_1(t,s)}{1+(\ln t)^{\alpha-1}}\mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s}\\ &\quad + \int_1^\infty \sup_{t\in I}\frac{\mathcal{G}_2(t,s)}{1+(\ln t)^{\alpha-1}}\mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s}\\ &\leq \Lambda_1\int_1^\infty \mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s} + \Lambda_2\int_1^\infty \mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s}\\ &\leq \frac{\sigma_3 r_2 \Lambda_1}{2}\int_1^\infty \mathfrak{a}(s)\,\frac{ds}{s} + \frac{\sigma_4 r_2 \Lambda_2}{2}\int_1^\infty \mathfrak{b}(s)\,\frac{ds}{s}\\ &= \left(\frac{\sigma_3 Q_1 \Lambda_1}{2} + \frac{\sigma_4 Q_2 \Lambda_2}{2}\right) r_2,\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{A}_2(x,y)\|_2 &= \sup_{t\in I}\frac{\mathcal{A}_2(x,y)}{1+(\ln t)^{\beta-1}}\\ &\leq \sup_{t\in I}\frac{1}{1+(\ln t)^{\beta-1}}\int_1^\infty \mathcal{G}_3(t,s)\mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s}\\ &\quad + \sup_{t\in I}\frac{1}{1+(\ln t)^{\beta-1}}\int_1^\infty \mathcal{G}_4(t,s)\mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s}\\ &\leq \int_1^\infty \sup_{t\in I}\frac{\mathcal{G}_3(t,s)}{1+(\ln t)^{\beta-1}}\mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s}\\ &\quad + \int_1^\infty \sup_{t\in I}\frac{\mathcal{G}_4(t,s)}{1+(\ln t)^{\beta-1}}\mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s}\\ &\leq \Lambda_3\int_1^\infty \mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s} + \Lambda_4\int_1^\infty \mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s}\\ &\leq \frac{\sigma_3 r_2 \Lambda_3}{2}\int_1^\infty \mathfrak{a}(s)\,\frac{ds}{s} + \frac{\sigma_4 r_2 \Lambda_4}{2}\int_1^\infty \mathfrak{b}(s)\,\frac{ds}{s}\\ &= \left(\frac{\sigma_3 Q_1 \Lambda_3}{2} + \frac{\sigma_4 Q_2 \Lambda_4}{2}\right) r_2.\end{aligned}$$

Then we deduce

$$\|\mathcal{A}(x,y)\| \le \left[\frac{(\Lambda_1+\Lambda_3)\sigma_3 Q_1}{2} + \frac{(\Lambda_2+\Lambda_4)\sigma_4 Q_2}{2}\right] r_2 = \left(\frac{\sigma_3 \Upsilon_3}{2} + \frac{\sigma_4 \Upsilon_4}{2}\right) r_2 \le r_2,$$

that is

$$\|\mathcal{A}(x,y)\| \le \|(x,y)\|, \ \forall (x,y) \in \mathcal{P} \cap \partial\Omega_2. \tag{16}$$

Therefore by (15), (16) and Theorem 1 ii), we conclude that the operator $\mathcal{A}$ has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$, with $r_1 \le \|(x,y)\| \le r_2$. So $x(t) \ge 0$ and $y(t) \ge 0$ for all $t \in I$, and by $(H2)$,$(H3)$, we obtain that $x(t) > 0$ for all $t \in (1,\infty)$ or $y(t) > 0$ for all $t \in (1,\infty)$, that is (x,y) is a positive solution of problem (1), (2). □

In a similar manner as we proved Theorem 3, we can obtain the following theorem.

Theorem 4. *We assume that $(H1)-(H3)$ hold, and there exists $\theta > 1$ such that $Q_j > 0$, $j = 3,4$, and $\widetilde{L}_i > 0$, $i = 1,\ldots,4$. In addition, we suppose that there exist positive constants r_1, r_2 with $r_1 < r_2$, and $\sigma_1 \in [\Upsilon_1^{-1},\infty)$, $\sigma_2 \in [\Upsilon_2^{-1},\infty)$, $\sigma_3 \in (0,\Upsilon_3^{-1}]$ and $\sigma_4 \in (0,\Upsilon_4^{-1}]$ such that*

$(H6)$ $\mathfrak{f}((1+(\ln t)^{\alpha-1})x,(1+(\ln t)^{\beta-1})y) \le \frac{\sigma_3 r_1}{2}, \ \forall t \in I, \ (x,y) \in [0,r_1]\times[0,r_1]$,
$\mathfrak{g}((1+(\ln t)^{\alpha-1})x,(1+(\ln t)^{\beta-1})y) \le \frac{\sigma_4 r_1}{2}, \ \forall t \in I, \ (x,y) \in [0,r_1]\times[0,r_1]$;

$(H7)$ $\mathfrak{f}((1+(\ln t)^{\alpha-1})x,(1+(\ln t)^{\beta-1})y) \ge \frac{\sigma_1 r_2}{2}, \ \forall t \in [\theta,\infty), \ (x,y) \in [0,r_2]\times[0,r_2]$,
$\mathfrak{g}((1+(\ln t)^{\alpha-1})x,(1+(\ln t)^{\beta-1})y) \ge \frac{\sigma_2 r_2}{2}, \ \forall t \in [\theta,\infty), \ (x,y) \in [0,r_2]\times[0,r_2]$.

Then problem (1), (2) has at least one positive solution (x,y) such that $r_1 \le \|(x,y)\| \le r_2$.

In what follows, we will prove the existence of at least three positive solutions for problem (1), (2) by applying the Leggett-Williams fixed point theorem (Theorem 2).

Theorem 5. *We assume that $(H1)-(H3)$ hold, and there exists $\theta > 1$ such that $Q_j > 0$, $j = 3,4$, and $\widetilde{L}_i > 0$, $i = 1,\ldots,4$. In addition, we suppose that there exist positive constants $a_0 < b_0 < c_0$ such that*

$(H8)$ $\mathfrak{f}((1+(\ln t)^{\alpha-1})x,(1+(\ln t)^{\beta-1})y) < \frac{a_0}{2\Upsilon_3}, \ \forall t \in I, \ (x,y) \in [0,a_0]\times[0,a_0]$,
$\mathfrak{g}((1+(\ln t)^{\alpha-1})x,(1+(\ln t)^{\beta-1})y) < \frac{a_0}{2\Upsilon_4}, \ \forall t \in I, \ (x,y) \in [0,a_0]\times[0,a_0]$;

$(H9)$ $\mathfrak{f}((1+(\ln t)^{\alpha-1})x,(1+(\ln t)^{\beta-1})y) > \frac{b_0}{2\Upsilon_1}, \ \forall t \in [\theta,\infty), \ x,y \ge 0, \ b_0 \le x+y \le c_0$,
$\mathfrak{g}((1+(\ln t)^{\alpha-1})x,(1+(\ln t)^{\beta-1})y) > \frac{b_0}{2\Upsilon_2}, \ \forall t \in [\theta,\infty), \ x,y \ge 0, \ b_0 \le x+y \le c_0$;

$(H10)$ $\mathfrak{f}((1+(\ln t)^{\alpha-1})x,(1+(\ln t)^{\beta-1})y) \le \frac{c_0}{2\Upsilon_3}, \ \forall t \in I, \ (x,y) \in [0,c_0]\times[0,c_0]$,
$\mathfrak{g}((1+(\ln t)^{\alpha-1})x,(1+(\ln t)^{\beta-1})y) \le \frac{c_0}{2\Upsilon_4}, \ \forall t \in I, \ (x,y) \in [0,c_0]\times[0,c_0]$.

Then problem (1),(2) has at least three positive solutions (x_1,y_1), (x_2,y_2) and (x_3,y_3) such that $\|(x_1,y_1)\| < a_0$, $\|(x_3,y_3)\| > a_0$, and

$$\inf_{t\in[\theta,\infty)}\left(\frac{x_2(t)}{1+(\ln t)^{\alpha-1}} + \frac{y_2(t)}{1+(\ln t)^{\beta-1}}\right) > b_0, \ \inf_{t\in[\theta,\infty)}\left(\frac{x_3(t)}{1+(\ln t)^{\alpha-1}} + \frac{y_3(t)}{1+(\ln t)^{\beta-1}}\right) < b_0.$$

Proof. We show firstly that operator $\mathcal{A} : \overline{\mathcal{P}}_{c_0} \to \overline{\mathcal{P}}_{c_0}$. For any $(x,y) \in \overline{\mathcal{P}}_{c_0}$, we have $\|(x,y)\| \le c_0$, and so $\|x\|_1 \le c_0$, $\|y\|_2 \le c_0$. Using the assumption $(H10)$ and Lemma 5, we find

$$\|\mathcal{A}_1(x,y)\|_1 \le \sup_{t\in I}\frac{1}{1+(\ln t)^{\alpha-1}}\int_1^\infty \mathcal{G}_1(t,s)\mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s} + \sup_{t\in I}\frac{1}{1+(\ln t)^{\alpha-1}}\int_1^\infty \mathcal{G}_2(t,s)\mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s}$$

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$$
\begin{aligned}
&\le \int_1^\infty \sup_{t\in I} \frac{\mathcal{G}_1(t,s)}{1+(\ln t)^{\alpha-1}}\mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s} \\
&\quad + \int_1^\infty \sup_{t\in I} \frac{\mathcal{G}_2(t,s)}{1+(\ln t)^{\alpha-1}}\mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s} \\
&\le \Lambda_1 \int_1^\infty \mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s} + \Lambda_2 \int_1^\infty \mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s} \\
&\le \Lambda_1 \frac{c_0}{2\Upsilon_3}\int_1^\infty \mathfrak{a}(s)\,\frac{ds}{s} + \Lambda_2 \frac{c_0}{2\Upsilon_4}\int_1^\infty \mathfrak{b}(s)\,\frac{ds}{s} \\
&= \left(\frac{\Lambda_1 Q_1}{\Upsilon_3} + \frac{\Lambda_2 Q_2}{\Upsilon_4}\right)\frac{c_0}{2},
\end{aligned}
$$

and

$$
\begin{aligned}
\|\mathcal{A}_2(x,y)\|_2 &\le \sup_{t\in I} \frac{1}{1+(\ln t)^{\beta-1}} \int_1^\infty \mathcal{G}_3(t,s)\mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s} \\
&\quad + \sup_{t\in I} \frac{1}{1+(\ln t)^{\beta-1}} \int_1^\infty \mathcal{G}_4(t,s)\mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s} \\
&\le \int_1^\infty \sup_{t\in I} \frac{\mathcal{G}_3(t,s)}{1+(\ln t)^{\beta-1}}\mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s} \\
&\quad + \int_1^\infty \sup_{t\in I} \frac{\mathcal{G}_4(t,s)}{1+(\ln t)^{\beta-1}}\mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s} \\
&\le \Lambda_3 \int_1^\infty \mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s} + \Lambda_4 \int_1^\infty \mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s} \\
&\le \Lambda_3 \frac{c_0}{2\Upsilon_3}\int_1^\infty \mathfrak{a}(s)\,\frac{ds}{s} + \Lambda_4 \frac{c_0}{2\Upsilon_4}\int_1^\infty \mathfrak{b}(s)\,\frac{ds}{s} \\
&= \left(\frac{\Lambda_3 Q_1}{\Upsilon_3} + \frac{\Lambda_4 Q_2}{\Upsilon_4}\right)\frac{c_0}{2}.
\end{aligned}
$$

Then we obtain

$$
\|\mathcal{A}(x,y)\| \le \left[\frac{(\Lambda_1+\Lambda_3)Q_1}{\Upsilon_3} + \frac{(\Lambda_2+\Lambda_4)Q_2}{\Upsilon_4}\right]\frac{c_0}{2} = c_0,
$$

so $\mathcal{A} : \overline{\mathcal{P}}_{c_0} \to \overline{\mathcal{P}}_{c_0}$.

We consider $d_0 \in (b_0, c_0)$, and we define the concave nonnegative continuous functional w on $\mathcal{P}$ by

$$
w(x,y) = \inf_{t\in[\theta,\infty)} \left(\frac{x(t)}{1+(\ln t)^{\alpha-1}} + \frac{y(t)}{1+(\ln t)^{\beta-1}}\right), \ (x,y) \in \mathcal{P}.
$$

We see easily that $w(x,y) \le \|(x,y)\|$ for all $(x,y) \in \overline{\mathcal{P}}_{c_0}$.

Next we will verify the conditions of Theorem 2, with $E = X$, $K = \mathcal{P}$, $A = \mathcal{A}$, $m = a_0$, $c = b_0$, $l = c_0$, $d = d_0$, $\Theta = w$. We verify first condition ii). For $(x,y) \in \overline{\mathcal{P}}_{a_0}$ we will show that $\|\mathcal{A}(x,y)\| < a_0$. For this, let $(x,y) \in \overline{\mathcal{P}}_{a_0}$. We obtain as in the above inequalities that

$$
\|\mathcal{A}_1(x,y)\|_1 < \left(\frac{\Lambda_1 Q_1}{\Upsilon_3} + \frac{\Lambda_2 Q_2}{\Upsilon_4}\right)\frac{a_0}{2}, \ \|\mathcal{A}_2(x,y)\|_2 < \left(\frac{\Lambda_3 Q_1}{\Upsilon_3} + \frac{\Lambda_4 Q_2}{\Upsilon_4}\right)\frac{a_0}{2},
$$

and so $\|\mathcal{A}(x,y)\| < a_0$. Then we have assumption ii) of Theorem 2.

We verify condition i) from Theorem 2. We choose the element

$$
(x_0(t), y_0(t)) = \left(\frac{b_0+d_0}{4}\left(1+(\ln t)^{\alpha-1}\right), \frac{b_0+d_0}{4}(1+(\ln t)^{\beta-1})\right), \ t \in I.
$$

Because $x_0(t) \ge 0$, $y_0(t) \ge 0$ for all $t \in I$, $\|(x_0,y_0)\| = \frac{b_0+d_0}{2} < d_0$ and $w(x_0,y_0) = \frac{b_0+d_0}{2} > b_0$, we deduce that $(x_0,y_0) \in \{(x,y) \mid (x,y) \in \mathcal{P}(w,b_0,d_0),\ w(x,y) > b_0\}$. Now, let $(x,y) \in \mathcal{P}(w,b_0,d_0)$, that is $(x,y) \in \mathcal{P}$, $w(x,y) \ge b_0$ and $\|(x,y)\| \le d_0$. So we have $\frac{x(t)}{1+(\ln t)^{\alpha-1}} + \frac{y(t)}{1+(\ln t)^{\beta-1}} \le d_0$ for all $t \in I$, and

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$\inf_{t\in[\theta,\infty)}\left(\frac{x(t)}{1+(\ln t)^{\alpha-1}}+\frac{y(t)}{1+(\ln t)^{\beta-1}}\right)\geq b_0$. Then by $(H9)$ and Lemma 5, and similar computations as those from the first part of the proof of Theorem 3, we find

$$\begin{aligned}
w(\mathcal{A}(x,y)) &= \inf_{t\in[\theta,\infty)}\left(\frac{\mathcal{A}_1(x,y)}{1+(\ln t)^{\alpha-1}}+\frac{\mathcal{A}_2(x,y)}{1+(\ln t)^{\beta-1}}\right)\\
&\geq L_1\int_\theta^\infty \mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s}+L_2\int_\theta^\infty \mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s}\\
&\quad +L_3\int_\theta^\infty \mathfrak{a}(s)\mathfrak{f}(x(s),y(s))\,\frac{ds}{s}+L_4\int_\theta^\infty \mathfrak{b}(s)\mathfrak{g}(x(s),y(s))\,\frac{ds}{s}\\
&> L_1\frac{b_0Q_3}{2\Upsilon_1}+L_2\frac{b_0Q_4}{2\Upsilon_2}+L_3\frac{b_0Q_3}{2\Upsilon_1}+L_4\frac{b_0Q_4}{2\Upsilon_2}\\
&= (L_1+L_3)\frac{b_0Q_3}{2\Upsilon_1}+(L_2+L_4)\frac{b_0Q_4}{2\Upsilon_2}=b_0.
\end{aligned}$$

Therefore $w(\mathcal{A}(x,y))>b_0$, and we have assumption i) of Theorem 2.

Now we verify condition iii) of Theorem 2, namely $w(\mathcal{A}(x,y))>b_0$ for $(x,y)\in\mathcal{P}(w,b_0,c_0)$ and $\|\mathcal{A}(x,y)\|>d_0$. So let $(x,y)\in\mathcal{P}(w,b_0,c_0)$ and $\|\mathcal{A}(x,y)\|>d_0$. By similar arguments used before we deduce that $w(\mathcal{A}(x,y))>b_0$, that is assumption iii) is satisfied.

Then by Theorem 2 and $(H2)$,$(H3)$ we deduce that problem (1), (2) has at least three positive solutions (x_1,y_1), (x_2,y_2) and (x_3,y_3) with $\|(x_1,y_1)\|<a_0$, $\|(x_3,y_3)\|>a_0$, $w(x_2,y_2)>b_0$ and $w(x_3,y_3)<b_0$. $\square$

4. An example

Let $\alpha=\frac{5}{2}$, $\beta=\frac{10}{3}$, $n=3$, $m=4$, $\mathfrak{a}(t)=\frac{1}{(t-1/2)^{1/3}}$, $t\in[1,\infty)$, $\mathfrak{b}(t)=\frac{t+2}{(t-1/3)^{5/4}}$, $t\in[1,\infty)$, $\mathfrak{f}(x,y)=6+e^{-|x-y|}\sin^2(xy)$, $x,y\geq 0$, $\mathfrak{g}(x,y)=\frac{2}{3}+95e^{-xy}\cos^4(x+y)$, $x,y\geq 0$, $\mathfrak{H}_1(s)=\{\frac{19}{110},\ s\in[1,3);\ \frac{s}{11}-\frac{1}{10},\ s\in[3,7);\ \frac{59}{110},\ s\in[7,10);\ \frac{116}{165},\ s\in[10,\infty)\}$, $\mathfrak{H}_2(s)=\{\frac{5}{136}(s-1)^{17/5},\ s\in[1,2);\ \frac{5}{136},\ s\in(2,\infty)\}$, $\mathfrak{K}_1(s)=\{1,\ s\in[1,\frac{5}{3});\ \frac{14}{13},\ s\in[\frac{5}{3},2);\ \frac{7}{290}(s-2)^{29/7}+\frac{14}{13},\ s\in[2,\frac{15}{6});\ \frac{7}{290}(\frac{1}{2})^{29/7}+\frac{14}{13},\ s\in[\frac{15}{6},\infty)\}$, $\mathfrak{K}_2(s)=\{3,\ s\in[1,4);\ \frac{286}{95},\ s\in[4,9);\ \frac{s^2}{146}+\frac{34061}{13870},\ s\in[9,11);\ \frac{45556}{13870},\ s\in[11,\infty)\}$.

We consider the system of fractional differential equations

$$\begin{cases}
{}^HD^{5/2}_{1+}x(t)+\dfrac{1}{(t-1/2)^{1/3}}\left(6+e^{-|x(t)-y(t)|}\sin^2(x(t)y(t))\right)=0,\ \ t\in(1,\infty),\\
{}^HD^{10/3}_{1+}y(t)+\dfrac{t+2}{(t-1/3)^{5/4}}\left(\dfrac{2}{3}+95e^{-x(t)y(t)}\cos^4(x(t)+y(t))\right)=0,\ \ t\in(1,\infty),
\end{cases}\tag{17}$$

subject to the boundary conditions

$$\begin{cases}
x(1)=x'(1)=0,\ \ y(1)=y'(1)=y''(1)=0,\\
{}^HD^{3/2}_{1+}x(\infty)=\dfrac{1}{11}\displaystyle\int_3^7 x(s)\,ds+\dfrac{1}{6}x(10)+\dfrac{1}{8}\int_1^2(s-1)^{12/5}y(s)\,ds,\\
{}^HD^{7/3}_{1+}y(\infty)=\dfrac{1}{13}x\left(\dfrac{5}{3}\right)+\dfrac{1}{10}\displaystyle\int_2^{15/6}(s-2)^{22/7}x(s)\,ds+\dfrac{1}{95}y(4)+\dfrac{1}{73}\int_9^{11}sy(s)\,ds.
\end{cases}\tag{18}$$

By using Mathematica program, we obtain $a\approx 0.01753901$, $b\approx 0.01031779$, $c\approx 0.02920547$, $d\approx 0.83235873$, $\Delta\approx 0.01429742>0$, $\int_1^\infty \mathfrak{a}(s)\,\frac{ds}{s}\approx 3.15855472$, $\int_1^\infty \mathfrak{b}(s)\,\frac{ds}{s}\approx 6.53061854$. So assumptions $(H1)-(H3)$ are satisfied. In addition, we take $\theta=2$, and we deduce $\Lambda_1=\frac{d}{\Delta}\approx 58.21742336$, $\Lambda_2=\frac{b}{\Delta}\approx 0.72165469$, $\Lambda_3=\frac{c}{\Delta}\approx 2.04270988$, $\Lambda_4=\frac{a}{\Delta}\approx 1.22672621$, $\widetilde{L}_1\approx 0.00021797$, $\widetilde{L}_2\approx 0.00309129$, $\widetilde{L}_3\approx 0.00037053$, $\widetilde{L}_4\approx 0.00010846$, $L_1\approx 0.00557882$, $L_2\approx 0.07911642$, $L_3\approx 0.00773204$, $L_4\approx 0.00226336$, $Q_1\approx 3.15855472$, $Q_2\approx 6.53061854$, $Q_3\approx 2.43620073$, $Q_4\approx 4.28276191$, $\Upsilon_1\approx 0.03242794$, $\Upsilon_2\approx 0.34853023$, $\Upsilon_3\approx 190.33492843$, $\Upsilon_4\approx 12.72413242$.

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We also consider $r_1 = \frac{1}{3}$, $r_2 = 2800$, $\sigma_1 = 31 > \Upsilon_1^{-1}$, $\sigma_2 = 3 > \Upsilon_2^{-1}$, $\sigma_3 = 0.005 < \Upsilon_3^{-1}$ and $\sigma_4 = 0.07 < \Upsilon_4^{-1}$. Then we obtain the inequalities

$$\begin{aligned}
&\mathfrak{f}((1+(\ln t)^{3/2})x, (1+(\ln t)^{7/3})y) \geq 6 > \tfrac{\sigma_1 r_1}{2} = \tfrac{31}{6}, \ \ \forall t \in [2,\infty), \ x, y \in [0, \tfrac{1}{3}],\\
&\mathfrak{g}((1+(\ln t)^{3/2})x, (1+(\ln t)^{7/3})y) \geq \tfrac{2}{3} > \tfrac{\sigma_2 r_1}{2} = \tfrac{1}{2}, \ \ \forall t \in [2,\infty), \ x, y \in [0, \tfrac{1}{3}],
\end{aligned}$$

so assumption $(H4)$ is satisfied. In addition we have

$$\begin{aligned}
&\mathfrak{f}((1+(\ln t)^{3/2})x, (1+(\ln t)^{7/3})y) \leq 7 = \tfrac{\sigma_3 r_2}{2}, \ \ \forall t \in [1,\infty), \ x, y \in [0, 2800],\\
&\mathfrak{g}((1+(\ln t)^{3/2})x, (1+(\ln t)^{7/3})y) \leq \tfrac{287}{3} < \tfrac{\sigma_4 r_2}{2} = 98, \ \ \forall t \in [1,\infty), \ x, y \in [0, 2800],
\end{aligned}$$

hence assumption $(H5)$ is also satisfied. Therefore by Theorem 3 we conclude that problem (17), (18) has at least one positive solution $(x(t), y(t))$, $t \in [1,\infty)$, with $\frac{1}{3} \leq \sup_{t\in[1,\infty)} \frac{x(t)}{1+(\ln t)^{3/2}} + \sup_{t\in[1,\infty)} \frac{y(t)}{1+(\ln t)^{7/3}} \leq 2800$.

5. Conclusions

In this paper we investigate the system of nonlinear fractional differential equations (1) with Hadamard derivatives of various orders $\alpha \in (n-1, n]$ and $\beta \in (m-1, m]$, respectively, and nonnegative nonlinearities on the infinite interval $(1,\infty)$. The system (1) is supplemented with general nonlocal boundary conditions (2), where the unknown functions x and y in the point 1 and their derivatives until orders $n-2$ and $m-2$, respectively, are all 0, and the Hadamard derivatives of x and y of order $n-1$ and $m-1$ at ∞ are dependent on both Riemann-Liouville integrals of x and y. Our problem generalizes the problem studied in [13], by considering here different orders for the fractional derivatives in the equations of system (1), and also a general form of the boundary conditions from (2) at ∞. Under some assumptions on the data of this problem, we give firstly the solution of the associated linear boundary value problem, and the corresponding Green functions with their properties. Then in the main section of paper we prove the existence of positive solutions of (1), (2) by applying the Guo-Krasnosl'skii fixed point theorem and the Leggett-Williams fixed point theorem. We also present finally an example for illustrating our results.

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