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Development of a Four-Parameter Statistical Neutrosophic Nadarajah-Haghighi Distribution: Estimation, Simulation, and Application to Neutrosophic Monthly Temperature Data

Rihab Ahmed Abed [1], Wafaa A. Ashour [1] and Nooruldeen A. Noori [2,*]

[1] Statistics Departments, College Management and Economics, University of Basra, Iraq

[2] Anbar Education Directorate, Anbar, Iraq

* Correspondence: nooruldeen.a.noori35508@st.tu.edu.iq

Abstract

This paper proposes a new four-parameter statistical distribution based on the neutrosophic Gompertz (NGo-G) family and the extended Nadarajah-Haghighi distribution in neutrosophic logic called neutrosophic Gompertz Nadarajah-Haghighi (NGoNH) distribution to deal with uncertain and indeterminate data, known as neutrosophic data. The basic distribution functions and some properties are derived and the parameters of the proposed distribution are estimated using three different methods. To compare the performance of the different estimation methods, numerical simulations are performed using the evaluation criteria: MSE, RMSE, and bias. NGoNH distribution is applied to real data set representing the monthly minimum and maximum temperatures in Lahore, Pakistan, for the period 2016-2020. To verify the consistency of the data with the proposed distribution, the properties of the neutrosophic data are tested and the three components (True, Indeterminate, False) are plotted. In addition, the performance of the proposed distribution is compared with six other neutrosophic distributions using some information criteria and some goodness-of-fit tests. The extent to which the data fit the proposed distribution is plotted to demonstrate its effectiveness. The results indicate that the proposed distribution provides a better fit to neutrosophic data than other distributions, which enhances its effectiveness in analyzing data with uncertainty.

Keywords: neutrosophic Gompertz family; Nadarajah-Haghighi; neutrosophic data; estimation methods; goodness-of-fit tests

1. Introduction

Statistical distributions are essential tools in data analysis and modeling random phenomena, as they provide a mathematical representation of the relationships between different variables. From this principle, quite a few continuous statistical distributions have been proposed, as well as a large group of families of continuous distributions, the most famous of which is the T-X method [1]. This method paved the way for generating continuous distribution. Example of these families are: Kw-TG [2], OEHL-G [3], EOF-G [4], GOIE-G [5], MAPW-X [6], LOG [7], NOGEE−G [8], NGLog-X [9], TIHL [10], and HOE-Φ [11].

However, uncertain and incomplete data collected in fields such as the environment, meteorology, and social sciences require models capable of dealing with these types of ambiguity. The neutrosophic logic, introduced by Smarandache 1999 [12], provides a powerful framework for dealing with uncertain data by dividing it into three components: True membership, Indeterminate membership, and False membership. Many statistical distributions have been developed and incorporated into the neutrosophic framework, examples of which appear in [13–17,18]. The NGoNH distribution is based on NGo-G family, whose neutrosophic cumulative probability function (NCDF) and neutrosophic probability density function (NPDF) are given by the equations below, respectively:

M_NGo (x_N,r_N,u_N)=1-e^(r_N/u_N (1-e^(u_N Φ(x_N,)/(1-Φ(x_N)))))   (1)

m_NGo (x_N,r_N,u_N)=(r_N φ(x_N) e^(u_N Φ(x_N,)/(1-Φ(x_N))))/(1-Φ(x_N))^2 e^(r_N/u_N (1-e^(u_N Φ(x_N,)/(1-Φ(x_N)))))(2)

where Φ(x_N), and φ(x_N) are NCDF and NPDF respectively for any baseline distribution with shape neutrosophic parameters r_N and u_N.

Most current statistical distributions do not adequately address the neutrosophic nature of environmental data, as climate data often include uncertainty arising from sudden weather changes or inaccurate measurements. Using a new, more flexible distribution that combines the NGo-G family and neutrosophic Nadarajah-Haghighi (NNH) distribution will provide a more efficient tool for modeling these data. Although previous studies have addressed neutrosophic distributions, they have often focused on transforming the basic distribution into neutrosophic distribution. These studies have not addressed the composition of distributions or its extension and adding parameters to the basic distribution while integrating it into neutrosophic logic, to obtain a more flexible distribution, capable of adapting to different data formats. The specific research gap in this study is represented by:

The lack of previous studies that addressed the development of a four-parameter statistical distribution that combines the NGo-G family and NNH distribution.

The absence of the use of the T-X method in neutrosophic distribution composition, as previous studies were limited to one simple distribution instead of combining two distributions to create a more complex model.

Lack of studies that compared the efficiency of parameter estimation methods (maximum likelihood, least squares, weighted least squares) in neutrosophic distributions.

Failure to evaluate neutrosophic distributions using advanced goodness-of-fit criteria when applied to real environmental data.

This paper aims to develop a new four-parameter statistical distribution that combines the NGo-G family and NNH distribution, and estimating the neutrosophic parameters of new distribution using three estimation method in addition to conducting a numerical simulation to evaluate the performance of estimation methods using MSE, RMSE, and bias. Then. we apply the NGoNH distribution to monthly temperature data and compare the performance of NGoNH with six other neutrosophic distributions using information criteria, and goodness-of-fit tests.

2. Neutrosophic Gompertz Nadarajah-Haghighi (NGoNH) Distribution

Let x represent any random variable, then the CDF and PDF of the Nadarajah-Haghighi distribution with shape parameters a, b are, respectively [19,20]:

$$\Phi(x) = 1 - e^{[1-(1+bx)^{a}]}, \quad x \geq 0, \quad a, b > 0 \tag{1}$$

$$\varphi(x) = ab(1 + bx)^{a-1} e^{[1-(1+bx)^{a}]} \tag{2}$$

Now, we will find the neutrosophic Nadarajah-Haghighi distribution, by updating the random variable and parameters to neutrosophic random variable and neutrosophic parameters, as follows:

Definition: Let $X_N = d + tI$, $tI \in [X_L, X_U]$, where X_L, X_U are the bottom and upper values of neutrosophic Nadarajah-Haghighi (NNH) random variable, which has an indeterminate portion tI, $tI \in [I_L, I_U]$ and definite part d. When $X_L = X_U$ the NNH will simplify classical Nadarajah-Haghighi. The neutrosophic shape parameters are $a_N \in [a_L, a_U]$, $b_N \in [b_L, b_U]$, and the NCDF, and NPDF of NNH have the following form:

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$$\Phi(x) = 1 - e^{[1-(1+b_N x_N)^{a_N}]}, \quad x_N \geq 0, \quad a_N, b_N > 0 \tag{3}$$

$$\varphi(x) = a_N b_N (1 + b_N x_N)^{a_N - 1} e^{[1-(1+b_N x_N)^{a_N}]} \tag{4}$$

To get the NCDF for NGoNH substituting the equation (5) in equation (1) to get the form:

$$M(x_N) = 1 - e^{\frac{r_N}{u_N}\left(1 - e^{u_N \frac{1-e^{[1-(1+b_N x_N)^{a_N}]}}{e^{[1-(1+b_N x_N)^{a_N}]}}}\right)}, x_N \geq 0, \quad r_N, u_N, a_N, b_N > 0 \tag{5}$$

Figure 1 shows the NCDF curve of a random variable following the NGoNH distribution. The function is plotted using different intervals of parameters to illustrate their effect on the shape of the function, the curve reflects how the probability of values occurring up to a certain threshold change according to the interval parameters of the distribution. Figure 2 shows a 3D plot of NCDF of NGoNH with varying parameters. It helps to understand how different values of the distribution parameters affect the cumulative function and provides a visual representation of the variation in probability as the input values change.

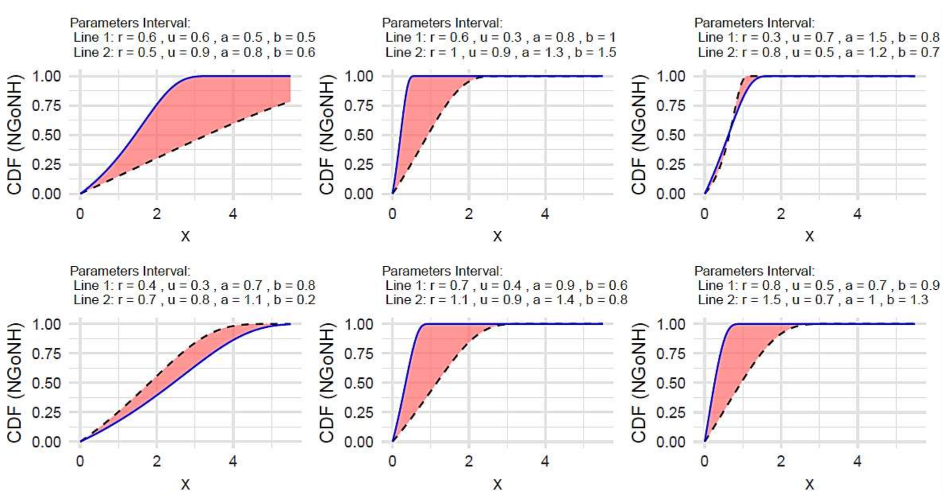

Figure 1. Plot the NCDF for NGoNH with different interval of parameters.

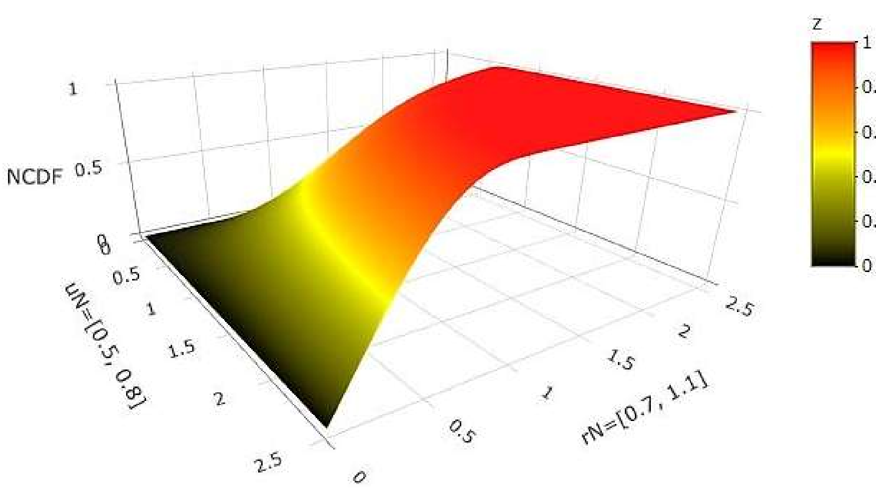

Figure 2. 3D-plot the NCDF for NGoNH with $a_N = 0.7, b_N = 0.9$.

To find the NPDF function for NGoNH distribution, substitute equation 5 and 6 into equation 2 to get:

$$m(x_N)=\frac{r_N a_N b_N(1+b_N x_N)^{a_N-1} e^{u_N\frac{1-e^{[1-(1+b_N x_N)^{a_N}]}}{e^{[1-(1+b_N x_N)^{a_N}]}}}}{e^{[1-(1+b_N x_N)^{a_N}]}}\, e^{\frac{r_N}{u_N}\left(1-e^{u_N\frac{1-e^{[1-(1+b_N x_N)^{a_N}]}}{e^{[1-(1+b_N x_N)^{a_N}]}}}\right)} \qquad (6)$$

Figure 3 shows how random variables are distributed according to the Npdf, the curve represent the effect of varying parameters on the shape of the distribution, it shows how the probabilities of different values of a given random variable change. Figure 4 provides a 3D view of the Npdf, and it shows how the probability density values change based on different intervals of parameters. Moreover, it allows us to see the effect of each parameter on the distribution of possible data.

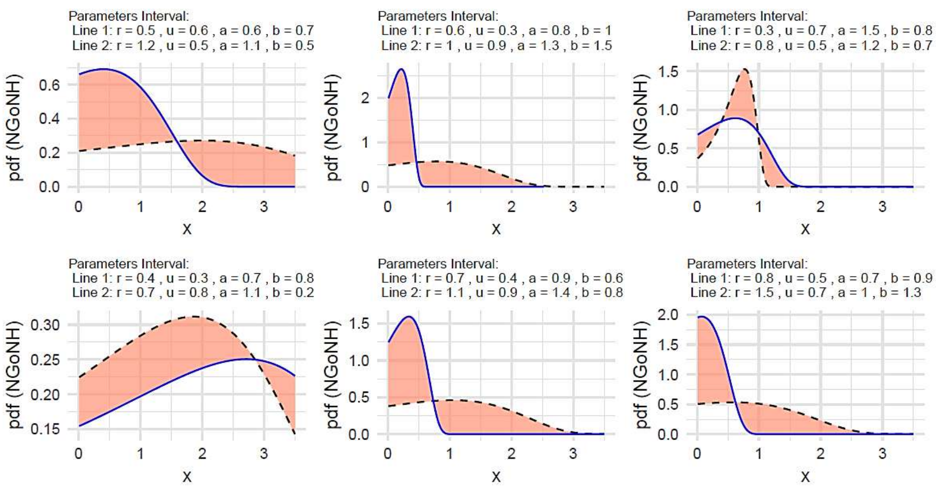

Figure 3. plot of the NPDF for NGoNH with different interval of parameters.

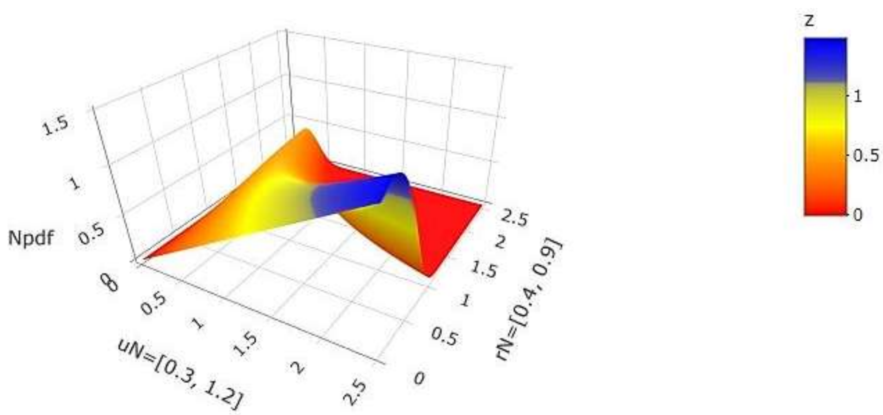

Figure 4. 3D-plot of the NPDF for NGoNH with $a_N = 1.1, b_N = 0.5$.

The survival function is given by [21]:

$$S(x) = 1 - M(x) \qquad (7)$$

Then using the equation above and substituting equation (7), we get the survival function for NGoNH by the form:

$$S(x_N) = e^{\frac{r_N}{u_N}\left(1-e^{u_N\frac{1-e^{[1-(1+b_N x_N)^{a_N}]}}{e^{[1-(1+b_N x_N)^{a_N}]}}}\right)} \tag{8}$$

Figure 5 shows the probability that a random variable will remain above a certain value. The survival function is widely used in systems reliability studies and life data analysis, it shows how survival rates vary with different intervals parameters.

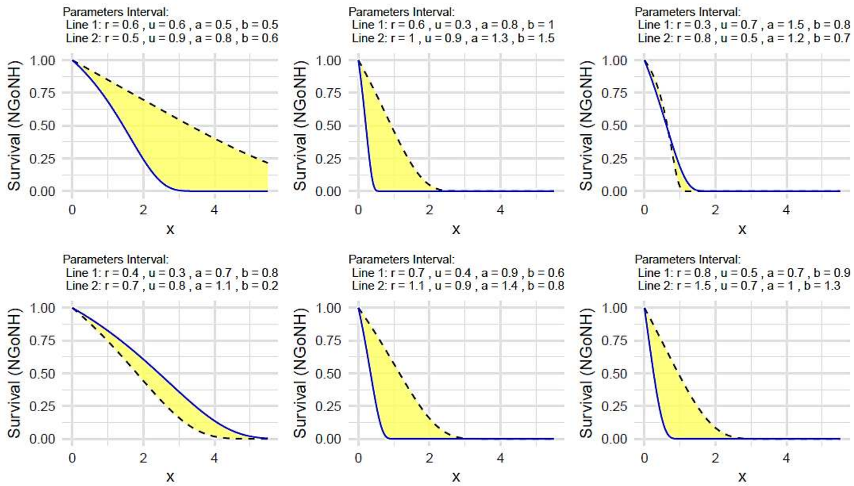

Figure 5. plot the survival for NGoNH with different interval of parameters.

The hazard function takes the form [22]:

$$h(x) = \frac{m(x)}{s(x)} \tag{9}$$

Then using the equation above and substitute equation (7) and (8) we get the hazard function for NGoNH by the form:

$$h(x_N) = \frac{r_N a_N b_N (1+b_N x_N)^{a_N-1} e^{u_N\frac{1-e^{[1-(1+b_N x_N)^{a_N}]}}{e^{[1-(1+b_N x_N)^{a_N}]}}}}{e^{[1-(1+b_N x_N)^{a_N}]}} \tag{10}$$

The Materials and Methods should be described with sufficient details to allow others to replicate and build on the published results. Please note that the publication of your manuscript implicates that you must make all materials, data, computer code, and protocols associated with the publication available to readers. Please disclose at the submission stage any restrictions on the availability of materials or information. New methods and protocols should be described in detail while well-established methods can be briefly described and appropriately cited.

Research manuscripts reporting large datasets that are deposited in a publicly available database should specify where the data have been deposited and provide the relevant accession numbers. If the accession numbers have not yet been obtained at the time of submission, please state that they will be provided during review. They must be provided prior to publication.

Interventionary studies involving animals or humans, and other studies that require ethical approval, must list the authority that provided approval and the corresponding ethical approval code.

In this section, where applicable, authors are required to disclose details of how generative artificial intelligence (GenAI) has been used in this paper (e.g., to generate text, data, or graphics, or to assist in study design, data collection, analysis, or interpretation). The use of GenAI for superficial text editing (e.g., grammar, spelling, punctuation, and formatting) does not need to be declared.

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3. Statistical Properties of NGoNH Distribution

3.1. NCD and Npdf Expansion

Due to the difficulty of the functions for the proposed distribution, the basic distribution functions are expanded to use these expansions to derive the properties of the proposed distribution. These expansions are done using the exponential function expansion and the binomial expansion to obtain the NCDF function in the form:

$$M(x_N) = 1 - \Psi e^{l_N[1-(1+b_N x_N)^{a_N}]} \tag{11}$$

where $\Psi = \sum_{i_N=j_N=k_N=s_N=l_N=0}^{\infty} \frac{(-1)^{i_N+j_N+k_N+l_N}\Gamma(k_N+s_N)}{i_N!k_N!s_N!z_N!\Gamma(k_N)} \binom{i_N}{j_N}\binom{k_N+s_N}{l_N} b_N^{c_N} r_N^{i_N} u_N^{k_N-i_N} j_N^{k_N}$

While the $M^{\delta_N}(x_N)$ which has the form:

$$M^{\delta_N}(x_N) = \left(1 - e^{\frac{r_N}{u_N}\left(1-e^{u_N \frac{1-e^{[1-(1+b_N x_N)^{a_N}]}}{e^{[1-(1+b_N x_N)^{a_N}]}}}\right)}\right)^{\delta_N} \tag{12}$$

And the finally expansion for above function has a form:

$$M^{\delta_N}(x_N) = T e^{t_N[1-(1+b_N x_N)^{a_N}]} \tag{13}$$

where $T = \sum_{l_N=w_N=r_N=z_N=p_N=t_N=0}^{\infty} \frac{(-1)^{l_N+w_N+r_N+z_N+t_N}\Gamma(z_N+p_N)}{w_N!z_N!p_N!\Gamma(z_N)} \binom{\delta_N}{l_N}\binom{w_N}{r_N}\binom{z_N+p_N}{t_N} r_N^{w_N} l_N^{w_N} u_N^{z_N-w_N} j_N^{z_N}$

Using the same way, we can expand the Npdf to get:

$$m(x_N) = \Upsilon(1 + b_N x_N)^{a_N-1} e^{(k_N+1)[1-(1+b_N x_N)^{a_N}]} \tag{14}$$

where $\Upsilon = \sum_{i_N=j_N=t_N=v_N=k_N=0}^{\infty} \frac{(-1)^{i_N+j_N+t_N+k_N}\Gamma(t_N+2+v_N)}{i_N!b_N^{i_N}t_N!v_N!\Gamma(t_N+2)} \binom{i_N}{j_N}\binom{t_N+v_N}{k_N} r_N^{i_N+1} u_N^{t_N} (j_N+1)^{t_N} a_N b_N$

Also, due to the need for an expansion of the function m^{β_N} in deriving some properties, the function is expanded in the same way as the functions above to obtain the from:

$$m^{\beta_N}(x_N) = \mathrm{E}(1 + b_N x_N)^{\beta_N(a_N-1)} e^{(\beta_N+\gamma_N)[1-(1+b_N x_N)^{a_N}]} \tag{15}$$

where $\mathrm{E} = \sum_{d_N=q_N=m_N=\varepsilon_N=\gamma_N=0}^{\infty} \frac{(-1)^{d_N+q_N+m_N+\gamma_N}}{d_N!\, b_N^{d_N-m_N}} \binom{d_N}{q_N} r_N^{d_N+\beta_N} \beta_N^{d_N} u_N^{m_N} (q_N+\beta_N)^{m_N} a_N^{\beta_N} b_N^{\beta_N}$

3.2. N.Quantile Function

The Neutrosophic Quantile (N.Quantile) function is the inverse of NCDF [22–24]:

$$F^{-1}(x_N) = y_N$$

Thus its obtained using the N.Quantile function of NGoNH in the form:

$$x_N = \frac{\left(1-\ln\left\{1-\frac{ln\left[1-\frac{u_N}{r_N}ln(1-y_N)\right]}{u_N+ln\left[1-\frac{u_N}{r_N}ln(1-y_N)\right]}\right\}\right)^{\frac{1}{a_N}} - 1}{b_N} \tag{16}$$

To show the method of distributing the values over the quartilies with different intervals of parameters. The table below showing that:

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Table 1. N.Quantile function values of NGoNH distribution for different intervals.

y_N	(r_N, u_N, a_N, b_N)				
	[0.4, 1.4],[0.9, 1.9], [0.4, 1.4],[0.7, 1.7]	[0.6, 1.6],[0.4, 1.4], [0.5, 1.5],[0.8, 1.8]	[0.5, 1.5],[0.3, 1.3], [0.5, 1.5],[0.6, 1.6]	[0.7, 1.7],[0.5, 1.5] [0.8, 1.8],[0.7, 1.7]	[0.6, 1.6],[0.9, 1.9], [0.7, 1.7],[1, 2]
0.1	[0.028291,0.88254]	[0.022418,0.42260]	[0.026837,0.67735]	[0.018603,0.24594]	[0.017491,0.22254]
0.2	[0.053782,1.72686]	[0.043547,0.85493]	[0.052051,1.37010]	[0.036108,0.47681]	[0.033515,0.42004]
0.3	[0.077338,2.53170]	[0.063897,1.29692]	[0.076159,2.07634]	[0.052867,0.69704]	[0.048507,0.59977]
0.4	[0.099522,3.30867]	[0.083669,1.75189]	[0.099620,2.80065]	[0.069180,0.91078]	[0.062787,0.76743]
0.5	[0.120994,4.07428]	[0.103349,2.22638]	[0.122845,3.55326]	[0.085367,1.12252]	[0.076720,0.92800]
0.6	[0.142297,4.85002]	[0.123397,2.73179]	[0.146455,4.35195]	[0.101816,1.33765]	[0.090676,1.08650]
0.7	[0.164360,5.66782]	[0.144564,3.28861]	[0.171284,5.22876]	[0.119148,1.56433]	[0.105160,1.24938]
0.8	[0.188588,6.58739]	[0.168293,3.93978]	[0.199047,6.25040]	[0.138527,1.81788]	[0.121200,1.42768]
0.9	[0.218907,7.77210]	[0.198471,4.80898]	[0.234227,7.60857]	[0.163117,2.14021]	[0.141309,1.65004]

The N.Quantile function rises as y_N increases, indicating an increasingly positive distribution. A range with higher upper limits give large values, reflecting the effect of expanding the boundaries of the distribution. Analyzing these values helps in understanding the behavior of the distribution and predicting future values based on the model used.

3.3. N.Moments Function

A strong mathematical and statistical tool for comprehending and describing the fundamental characteristics of data distribution is the neutrosophic moments (N.Moment) which is given by [8,25,26]:

$$\mu_{N_m} = E(x_N^m) = \int_{-\infty}^{\infty} x_N^m m(x_N) dx_N \tag{17}$$

Using the above expression and the Npdf for NGoNH distribution from equation (17) we get:

$$\mu_{N_m} = E(x_N^m) = \int_0^{\infty} x_N^m (1 + b_N x_N)^{\beta_N(a_N-1)} e^{(\beta_N+\gamma_N)[1-(1+b_N x_N)^{a_N}]} dx_N$$

let $\mathrm{t} = (1 + \mathrm{b_N x_N})^{\mathrm{a_N}} \Longrightarrow \mathrm{dt} = \mathrm{a_N b_N}(1 + \mathrm{b_N x_N})^{\mathrm{a_N}-1}\mathrm{dx} \Longrightarrow \mathrm{dx} = \frac{\mathrm{dt}}{\mathrm{a_N b_N}(1+\mathrm{b_N x_N})^{\mathrm{a_N}-1}}$

when $\mathrm{x_N} = 0 \Longrightarrow \mathrm{t} = 1$, and as $\mathrm{x_N} \longrightarrow \infty \Longrightarrow \mathrm{t} \longrightarrow \infty$

then we get $\mathrm{x_N} = \frac{\mathrm{t}^{\mathrm{a_N}-1}}{\mathrm{b_N}}$

$$\mu_{N_m} = \frac{1}{a_N b_N^{m+1}} \int_1^{\infty} (t^{a_N} - 1)^m t^{\beta_N(a_N-1) - \frac{a_N-1}{a_N}} e^{(\beta_N+\gamma_N)[1-t]} dt$$

Using binomial expansion for $(\mathrm{t}^{\mathrm{a_N}} - 1)^{\mathrm{m}}$ to get:

$$\mu_{N_m} = \frac{1}{a_N b_N^{m+1}} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \int_1^{\infty} t^{\alpha} e^{(\beta_N+\gamma_N)[1-t]} dt, \qquad \alpha = \beta_N(a_N - 1) - \frac{a_N - 1}{a_N} + \frac{k}{a_N}$$

$$\mu_{N_m} = \frac{1}{a_N b_N^{m+1}} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} e^{\beta_N+\gamma_N} \int_1^{\infty} t^{\alpha} e^{-(\beta_N+\gamma_N)t} dt$$

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$$\mu_{N_m} = \frac{\Upsilon e^{\beta_N+\gamma_N}}{a_N b_N^{m+1}(\beta_N+\gamma_N)^{\alpha+1}} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k}\, \Gamma(\alpha+1, \beta_N+\gamma_N) \tag{18}$$

where $\alpha = \beta_N(a_N - 1) - \frac{a_N - 1}{a_N} + \frac{k}{a_N}$

Table 2 shows the values of the first four moments of NGoNH distribution as well as the values of neutrosophic skewness and neutrosophic Kurtosis. The results indicate that the larger the intervals of the mean of the distribution. The results of neutrosophic Variance σ_N^2, neutrosophic skewness S_N and neutrosophic Kurtosis K_N change significantly, reflecting the effect of the intervals on the shape of the distribution. Small values such as [0.4, 1.4] give a shape and centered distribution, while large values such as [0.9, 1.9] give a more dispersed and flat distribution. At [0.4, 1.4] the K_N is very high, indicating that the values are clustered around the mean, while at [0.9, 1.9] it is more moderate, indicating a more even distribution.

Table 2. some intervals of N.moments for NGoNH distribution.

r_N	u_N	q_N	p_N	$\grave{\mu}_{1_N}$	$\grave{\mu}_{2_N}$	$\grave{\mu}_{3_N}$	$\grave{\mu}_{4_N}$	σ_N^2	S_N	K_N
[0.4,1.4]	[0.6,1.6]	[0.3,1.3]	[0.1,1.1]	[0.00587, 0.21612]	[0.00390, 0.06271]	[0.00292, 0.02095]	[0.00233, 0.00765]	[0.00387, 0.01600]	[1.33438,11 .9842]	[1.94536,15 3.278]
			[0.2,1.2]	[0.01151, 0.19811]	[0.00763, 0.05270]	[0.00571, 0.01614]	[0.00456, 0.00540]	[0.00750, 0.01345]	[1.33326,8. 5562]	[2.04766,78 .16698]
		[0.5,1.5]	[0.3,1.3	[0.03071, 0.15551]	[0.02053, 0.03229]	[0.00770,0. 01543]	[0.00200,0. 01236]	[0.00811,0. 01959]	[1.32683,5. 24347]	[2.9579,29. 30875]
			[0.4,1.4]	[0.04131, 0.14440]	[0.02766, 0.02784]	[0.00616,0. 02079]	[0.00148,0. 01666]	[0.00699,0. 02595]	[1.32323,4. 52152]	[1.92041,21 .78839]
	[0.9,1.9]	[0.7,1.7]	[0.5,1.5]	[0.09010, 0.11228]	[0.01661,0. 06187]	[0.00280,0. 04722]	[0.00051,0. 03823]	[0.00400,0. 05375]	[1.305903.0 6888]	[1.94335,9. 98682]
			[0.6,1.6]	[0.10526,0. 11362]	[0.01460,0. 07842]	[0.00231,0. 06004]	[0.00039,0. 09895]	[0.00352,0. 06551]	[1.403502.7 3416]	[1.86435,16 .09007]
		[0.9,1.9]	[0.7,1.7]	[0.08769,0. 22536]	[0.01010,0. 15967]	[0.00132,0. 12412]	[0.00018,0. 10166]	[0.00241,0. 10888]	[1.30590,1. 94536]	[1.85137,3. 987623]
			[0.8,1.8]	[00.08909,. 20711]	[0.01047,0. 14532]	[0.00140,0. 11231]	[0.00020,0. 09164]	[0.00253,0. 10243]	[1.31132,2. 02742]	[1.86904,4. 339501]

3.4. N.Moment Generating Function

The form of [26] (N.mgf) neutrosophic Moment generating function (NMGF) of NGoNH is derived from equation N.moment (20), to get:

$$M'_{x_N}(y_N) = \sum_{s=0}^{\infty} \frac{y_N^s}{s!} \left[\frac{\Upsilon e^{\beta_N+\gamma_N}}{a_N b_N^{m+1}(\beta_N+\gamma_N)^{\alpha+1}} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k}\, \Gamma(\alpha+1, \beta_N+\gamma_N) \right] \tag{19}$$

3.5. Incomplete Moments

The following formula provides the neutrosophic incomplete moments (N.I.M) of a random variable X_N for NGoNH distribution:

$$\grave{\mu}_n(y_N) = \int_0^{y_N} x_N^n \Upsilon(1+b_N x_N)^{a_N-1} e^{(k_N+1)[1-(1+b_N x_N)^{a_N}]} dx_N$$

By substituting equation Npdf (12) and performing the integration, we get:

$$\grave{\mu}_n(y_N) = \frac{\Upsilon e^{k_N+1}}{a_N b_N^{n+1}(k_N+1)^{\xi+1}} \sum_{s=0}^{n} \binom{n}{k} (-1)^{n-s} \{\Gamma(\xi+1,(k_N+1)) - \Gamma(\xi+1,(k_N+1))(1-(1+b_N x_N)^{a_N})\} \tag{20}$$

where $\xi = \frac{(a_N-1)^2+k}{a_N}$

3.6. Neutrosophic Entropy

3.6.1. Neutrosophic Rényi Entropy

The Neutrosophic Rényi's Entropy of a random variable x_N is defined as [16,19,21]:

$$\grave{I}_R(\beta_N)_N = \frac{1}{1-\beta_N} \log \int_0^{\infty} m^{\beta_N}(x_N) dx_N$$

From the equation above and from equation (17) we get:

$$\grave{I}_R(\beta_N)_N = \frac{1}{1-\beta_N} \log\left[\mathrm{E}\int_0^{\infty} (1+b_N x_N)^{\beta_N(a_N-1)} e^{(\beta_N+\gamma_N)[1-(1+b_N x_N)^{a_N}]} dx_N\right]$$

Then the final form of above integral is:

$$\grave{I}_R(\lambda_N)_N = \frac{1}{1-\lambda_N} \log\left[\frac{\mathrm{E}e^{\beta_N+\gamma_N}\Gamma(\pi+1,\beta_N+\gamma_N)}{a_N b_N^{m+1}(\beta_N+\gamma_N)^{\pi+1}}\right], \pi = \frac{(a_N-1)(a_N\beta_N-1)}{a_N} \tag{21}$$

3.6.2. Neutrosophic Arimoto Entropy

This entropy can be calculated using the equation:

$$\grave{A}(\beta_N)_N = \frac{\beta_N}{1-\beta_N}\left(\left[\int_0^{\infty} m^{\beta_N}(x_N) dx_N\right]^{\frac{1}{\beta_N}} - 1\right)$$

By carrying out the integral we obtain:

$$\grave{A}(\beta_N)_N = \frac{\beta_N}{1-\beta_N}\left[\left\{\frac{\mathrm{E}e^{\beta_N+\gamma_N}\Gamma(\pi+1,\beta_N+\gamma_N)}{a_N b_N^{m+1}(\beta_N+\gamma_N)^{\pi+1}}\right\}^{\frac{1}{\beta_N}} - 1\right] \tag{22}$$

3.6.3. Neutrosophic Havrda and Charvat Entropy

This entropy can be calculated using the equation:

$$\grave{HC}(\beta_N)_N = \frac{1}{2^{1-\beta_N}-1}\left(\left[\int_0^{\infty} m^{\beta_N}(x_N) dx_N\right]^{\frac{1}{\beta_N}} - 1\right)$$

By integrating the above we get:

$$\grave{HC}(\beta_N)_N = \frac{1}{2^{1-\beta_N}-1}\left[\left\{\frac{\mathrm{E}e^{\beta_N+\gamma_N}\Gamma(\pi+1,\beta_N+\gamma_N)}{a_N b_N^{m+1}(\beta_N+\gamma_N)^{\pi+1}}\right\}^{\frac{1}{\beta_N}} - 1\right] \tag{23}$$

4. Estimation

4.1. Maximum Likelihood Estimation

The NGoNH distribution parameters are computed using the maximum likelihood estimation approach. For $x_{N_1}, x_{N_2}, \dots, x_{N_m}$ which is a random sample [27–29] the NGoNH distribution Npdf is:

$$L(\theta_N, x_{N_i}) = \prod_{i=1}^{m} \frac{r_N a_N b_N (1 + b_N x_{N_i})^{a_N - 1} e^{u_N \frac{1 - e^{[1-(1+b_N x_{N_i})^{a_N}]}}{e^{[1-(1+b_N x_{N_i})^{a_N}]}}}}{e^{[1-(1+b_N x_{N_i})^{a_N}]}} e^{\frac{r_N}{u_N}\left(e^{u_N \frac{1 - e^{[1-(1+b_N x_{N_i})^{a_N}]}}{e^{[1-(1+b_N x_{N_i})^{a_N}]}}} \right)}$$

we find the log- likelihood as:

$$\begin{aligned} L = {} & mlog(r_N) + mlog(a_N) + mlog(b_N) + (a_N - 1)\sum_{i=1}^{m} \log(1 + b_n x_{N_i}) \\ & - \sum_{i=1}^{m} \left\{1 - (1 + b_n x_{N_i})^{a_N}\right\} + u_N \sum_{i=1}^{m} \frac{1 - e^{\left\{1-(1+b_n x_{N_i})^{a_N}\right\}}}{e^{\left\{1-(1+b_n x_{N_i})^{a_N}\right\}}} \\ & - \frac{r_N}{u_N} \sum_{i=1}^{m} e^{u_N \frac{1 - e^{[1-(1+b_N x_{N_i})^{a_N}]}}{e^{[1-(1+b_N x_{N_i})^{a_N}]}}} \end{aligned} \tag{24}$$

4.2. Least Square Estimation

The following formula can be used to estimate the parameters using the least square estimation (LSE) method [30]:

$$\varphi(\theta_N) = \sum_{i=1}^{m} \left[1 - e^{\frac{r_N}{u_N}\left(1 - e^{u_N \frac{1 - e^{[1-(1+b_N x_{N_i})^{a_N}]}}{e^{[1-(1+b_N x_{N_i})^{a_N}]}}} \right)} - \frac{1}{n+1} \right]^2 \tag{25}$$

4.3. Weighted Least Square Estimation

The following formula can be used to estimate the parameters using the weighted least square estimation (WLSE) method [30]:

$$\varphi(\theta_N) = \sum_{i=1}^{m} \frac{(n+1)^2 (n+2)}{i(n-i+1)} \left[1 - e^{\frac{r_N}{u_N}\left(1 - e^{u_N \frac{1 - e^{[1-(1+b_N x_{N_i})^{a_N}]}}{e^{[1-(1+b_N x_{N_i})^{a_N}]}}} \right)} - \frac{i}{n+1} \right]^2 \tag{26}$$

Estimation of the parameters for the three previously described methods may be obtained by finding the partial derivative of the four parameters and setting their equations equal to zero. Computer software such as R language are used since it is difficult to find these values using analytical solutions.

5. Simulation

A Monte Carlo simulation is carried out for the three methods discussed in the fourth section to show the effectiveness of the estimation of the NGoNH distribution. The sizes of the generated samples were based on n=50, 100, 150, 200, 300, 400 and 500 up to 1000. The mean square error (MSE) and its root (RMSE) values were calculated, as well as the bias which have respectively the forms [31,32]:

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$$MSE = \frac{1}{n}\sum_{t=1}^{n}(x_t - \hat{x}_t)^2 \tag{27}$$

$$RMSE = \sqrt{\frac{1}{n}\sum_{t=1}^{n}(x_t - \hat{x}_t)^2} \tag{28}$$

$$bias = (\hat{\gamma}) = \frac{\sum_{i=1}^{N}\hat{\gamma}_i}{N} - \gamma \tag{29}$$

The above formulas are used to measure the efficiency of parameter estimation for the three methods. Tables 3, 4, 5, and 6 displays respectively the Mean value intervals, MSE value intervals, RMSE value intervals, and Bias value intervals of the Monte Carlo simulations conducted for the NGoNH distribution.

Table 3. Mean value intervals of Monte Carlo simulations conducted for the NGoNH distribution.

$\boldsymbol{r_N}$ = [0.3,1.3], $\boldsymbol{u_N}$ = [0.6,1.6], $\boldsymbol{b_N}$ = [0.7,1.7], $\boldsymbol{c_N}$ = [0.8,1.8]				
N	**Ess. Par.**	**MLE**	**LSE**	**WLSE**
25	$\widehat{r_N}$	[0.278725, 1.208506]	[0.330885, 1.77051]	[0.325109, 1.712059]
	$\widehat{u_N}$	[0.545109, 1.512762]	[0.529641, 0.982358]	[0.574482, 1.159052]
	$\widehat{a_N}$	[0.82538, 2.26148]	[0.727016, 1.90030]	[0.6985924, 1.917410]
	$\widehat{b_N}$	[0.836688, 1.871466]	[0.827649, 1.802932]	[0.884834, 1.761696]
50	$\widehat{r_N}$	[0.288682, 1.175954]	[0.323403, 1.455807]	[0.320551, 1.441650]
	$\widehat{u_N}$	[0.733040, 1.582329]	[0.488049, 1.111214]	[0.552887, 1.403803]
	$\widehat{a_N}$	[0.772505, 2.09839]	[0.727572, 2.005198]	[0.726416, 1.913894]
	$\widehat{b_N}$	[0.847302, 1.921580]	[0.816181, 1.710567]	[0.836340, 1.808019]
100	$\widehat{r_N}$	[0.2827085, 1.208919]	[0.3142788, 1.326015]	[0.3091922, 1.309548]
	$\widehat{u_N}$	[0.798041, 1.644623]	[0.508863, 1.408090]	[0.551583, 1.407206]
	$\widehat{a_N}$	[0.735851, 1.99728]	[0.722367, 1.892478]	[0.720947, 1.881493]
	$\widehat{b_N}$	[0.848726, 1.90989]	[0.8144451, 1.777535]	[0.826489, 1.8062446]
150	$\widehat{r_N}$	[0.2964969, 1.220680]	[0.3071746, 1.272246]	[0.3100239, 1.289856]
	$\widehat{u_N}$	[0.86246, 1.612384]	[0.543385, 1.30409]	[0.624793, 1.528528]
	$\widehat{a_N}$	[0.717331, 1.99700]	[0.715691, 1.90893]	[0.6926641, 1.835397]
	$\widehat{b_N}$	[0.836636, 1.842685]	[0.820484, 1.792420]	[0.850076, 1.801155]
200	$\widehat{r_N}$	[0.2920368, 1.285749]	[0.2990094, 1.278224]	[0.2989902, 1.284404]
	$\widehat{u_N}$	[0.856185, 1.90406]	[0.5572717, 1.23905]	[0.5965633, 1.48762]
	$\widehat{a_N}$	[0.702390, 1.875348]	[0.7133606, 1.89546]	[0.7085301, 1.81871]
	$\widehat{b_N}$	[0.856278, 1.862480]	[0.826951, 1.812227]	[0.837854, 1.831246]
300	$\widehat{r_N}$	[0.3040763, 1.282362]	[0.3016217, 1.265423]	[0.3045414, 1.298720]
	$\widehat{u_N}$	[0.893900, 1.819303]	[0.5513485, 1.461314]	[0.6917411, 1.634536]
	$\widehat{a_N}$	[0.700084, 1.84783]	[0.719860, 1.80666]	[0.6958297, 1.770661]
	$\widehat{b_N}$	[0.820012, 1.845923]	[0.814926, 1.841863]	[0.8229515, 1.828610]
400	$\widehat{r_N}$	[0.304597,1.311825]	[0.2933161, 1.254802]	[0.2961900, 1.2908484]

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	$\widehat{u_N}$	[0.905151,1.857734]	[0.588903, 1.421155]	[0.630648, 1.532744]
	$\widehat{a_N}$	[0.695743, 1.81612]	[0.7052426, 1.781923]	[0.7031947, 1.795610]
	$\widehat{b_N}$	[0.814393, 1.857870]	[0.836268, 1.862511]	[0.827265, 1.8332795]
500	$\widehat{r_N}$	[0.3111381, 1.316864]	[0.2947219, 1.282619]	[0.2985773, 1.298005]
	$\widehat{u_N}$	[0.927286, 1.888777]	[0.589510, 1.467209]	[0.679843, 1.598864]
	$\widehat{a_N}$	[0.683244, 1.789918]	[0.708064, 1.818735]	[0.687633, 1.771254]
	$\widehat{b_N}$	[0.814162, 1.853697]	[0.825335, 1.808836]	[0.837494, 1.824339]

Table 4. MSE value intervals of Monte Carlo simulations conducted for the NGoNH distribution.

$\boldsymbol{r_N}$ = [0.3,1.3], $\boldsymbol{u_N}$ = [0.6,1.6], $\boldsymbol{b_N}$ = [0.7,1.7], $\boldsymbol{c_N}$ = [0.8,1.8]				
N	**Ess. Par.**	**MLE**	**LSE**	**WLSE**
25	$\widehat{r_N}$	[0.023135, 0.706801]	[0.032444, 1.62494]	[0.030712, 1.31117]
	$\widehat{u_N}$	[4.14523,0.497087]	[0.080344, 1.063434]	[0.12249, 0.9756436]
	$\widehat{a_N}$	[0.059763, 1.11365]	[0.017808, 0.37968]	[0.015584, 0.4181343]
	$\widehat{b_N}$	[0.119858, 0.503096]	[0.031222, 0.2345520]	[0.063167, 0.213283]
50	$\widehat{r_N}$	[0.012067, 0.346669]	[0.0160610.717349]	[0.015734, 0.87509]
	$\widehat{u_N}$	[0.734766, 3.679494]	[0.049355, 0.57713]	[0.274055, 1.17055]
	$\widehat{a_N}$	[0.058392, 0.794207]	[0.010602, 0.34445]	[0.017254, 0.31337]
	$\widehat{b_N}$	[0.109587, 0.433128]	[0.027279, 0.128267]	[0.034144, 0.229846]
100	$\widehat{r_N}$	[0.0079689, 0.253872]	[0.009457, 0.198970]	[0.008064, 0.130369]
	$\widehat{u_N}$	[0.83836, 2.824828]	[0.047707, 0.61153]	[0.086030, 0.645480]
	$\widehat{a_N}$	[0.038626, 0.682035]	[0.006328, 0.281991]	[0.010779, 0.210703]
	$\widehat{b_N}$	[0.079480, 0.269563]	[0.019532, 0.115940]	[0.033326, 0.121073]
150	$\widehat{r_N}$	[0.006884, 0.191622]	[0.0055073, 0.144637]	[0.006099, 0.140694]
	$\widehat{u_N}$	[0.86152, 2.381664]	[0.057725, 0.62285]	[0.121972, 0.768219]
	$\widehat{a_N}$	[0.035500, 0.561778]	[0.006314, 0.20831]	[0.008967, 0.16158]
	$\widehat{b_N}$	[0.065723, 0.221268]	[0.014683, 0.088061]	[0.024392, 0.087131]
200	$\widehat{r_N}$	[0.0062275, 0.218678]	[0.004615, 0.097623]	[0.004289, 0.113431]
	$\widehat{u_N}$	[0.753041, 3.14446]	[0.0587917, 0.60065]	[0.1047137, 0.62307]
	$\widehat{a_N}$	[0.030672, 0.49517]	[0.0061114, 0.13265]	[0.009599, 0.154766]
	$\widehat{b_N}$	[0.061232, 0.214991]	[0.011810, 0.072328]	[0.020838, 0.078229]
300	$\widehat{r_N}$	[0.0046038, 0.148718]	[0.003744, 0.073007]	[0.003550, 0.082893]
	$\widehat{u_N}$	[0.72283, 2.106184]	[0.071511, 0.616261]	[0.172864, 0.725281]
	$\widehat{a_N}$	[0.029830, 0.412794]	[0.006769, 0.117654]	[0.012307, 0.136178]
	$\widehat{b_N}$	[0.044213, 0.161359]	[0.014176, 0.065852]	[0.017359, 0.065906]
400	$\widehat{r_N}$	[0.0044547, 0.147368]	[0.002093, 0.066562]	[0.002366, 0.0648016]
	$\widehat{u_N}$	[0.780242, 2.137328]	[0.086523, 0.404274]	[0.086714, 0.465084]

	$\widehat{a_N}$	[0.026265, 0.394238]	[0.0077952, 0.072529]	[0.010696, 0.180237]
	$\widehat{b_N}$	[0.041377, 0.163266]	[0.014921, 0.053476]	[0.01818, 0.073968]
500	$\widehat{r_N}$	[0.0045347, 0.134610]	[0.001561, 0.058319]	[0.002277, 0.0628932]
	$\widehat{u_N}$	[0.64707, 2.246025]	[0.073237, 0.59519]	[0.128656, 0.6032856]
	$\widehat{a_N}$	[0.024108, 0.340620]	[0.0072589, 0.130788]	[0.009566, 0.1360325]
	$\widehat{b_N}$	[0.037528, 0.150351]	[0.009878, 0.0543079]	[0.015243, 0.062339]

Table 5. RMSE value intervals of Monte Carlo simulations conducted for the NGoNH distribution.

$r_N = [0.3,1.3]$, $u_N = [0.6,1.6]$, $b_N = [0.7,1.7]$, $c_N = [0.8,1.8]$				
N	**Ess. Par.**	**MLE**	**LSE**	**WLSE**
25	$\widehat{r_N}$	[0.1521022, 0.840714]	[0.180122, 1.27473]	[0.175249, 1.145065]
	$\widehat{u_N}$	[0.705044, 2.0359839]	[0.283450, 1.031229]	[0.349994, 0.987746]
	$\widehat{a_N}$	[0.244465, 1.055298]	[0.133447, 0.616186]	[0.1248374, 0.646633]
	$\widehat{b_N}$	[0.346205, 0.709292]	[0.176698, 0.4843057]	[0.251330, 0.4618259]
50	$\widehat{r_N}$	[0.109851, 0.588786]	[0.1267339, 0.8469651]	[0.125436, 0.935467]
	$\widehat{u_N}$	[0.85718, 1.918200]	[0.222160, 0.759692]	[0.523503, 1.08192]
	$\widehat{a_N}$	[0.241644, 0.891183]	[0.102970, 0.58689]	[0.131355, 0.559796]
	$\widehat{b_N}$	[0.3310399, 0.658124]	[0.165165, 0.358144]	[0.184782, 0.47942306]
100	$\widehat{r_N}$	[0.0892688, 0.503857]	[0.0972493, 0.446061]	[0.089801, 0.3610666]
	$\widehat{u_N}$	[0.915622, 1.680722]	[0.218419, 0.782004]	[0.293309, 0.803417]
	$\widehat{a_N}$	[0.1965366, 0.825854]	[0.079553, 0.531028]	[0.103825, 0.459024]
	$\widehat{b_N}$	[0.281921, 0.519195]	[0.139759, 0.340500]	[0.182554, 0.3479566]
150	$\widehat{r_N}$	[0.0829748, 0.437747]	[0.074211, 0.3803117]	[0.0780961, 0.375092]
	$\widehat{u_N}$	[0.92818, 1.543264]	[0.2402606, 0.789210]	[0.349244, 0.876481]
	$\widehat{a_N}$	[0.1884161, 0.749518]	[0.0794657, 0.456420]	[0.0946980, 0.4019703]
	$\widehat{b_N}$	[0.256366, 0.470392]	[0.121176, 0.296751]	[0.1561804, 0.295180]
200	$\widehat{r_N}$	[0.0789145, 0.4676308]	[0.067934, 0.3124475]	[0.065494, 0.336795]
	$\widehat{u_N}$	[0.86777, 1.77326]	[0.24247, 0.775017]	[0.3235949, 0.789352]
	$\widehat{a_N}$	[0.1751355, 0.703688]	[0.078175, 0.36422]	[0.097977, 0.3934033]
	$\widehat{b_N}$	[0.247451, 0.463671]	[0.108677, 0.26893]	[0.144355, 0.279695]
300	$\widehat{r_N}$	[0.0678518, 0.385640]	[0.061196, 0.270198]	[0.0595834, 0.287911]
	$\widehat{u_N}$	[0.850196, 1.451269]	[0.2674172, 0.785023]	[0.415769, 0.851634]
	$\widehat{a_N}$	[0.1727160, 0.64249]	[0.0822794, 0.343008]	[0.1109371, 0.369024]
	$\widehat{b_N}$	[0.210270, 0.401695]	[0.1190656, 0.2566176]	[0.1317556, 0.256722]
400	$\widehat{r_N}$	[0.0667440, 0.383886]	[0.0457507, 0.257997]	[0.0486476, 0.25456166]
	$\widehat{u_N}$	[0.883313, 1.461960]	[0.2941494, 0.635825]	[0.294472, 0.681971]
	$\widehat{a_N}$	[0.1620668, 0.627883]	[0.08829061, 0.2693129]	[0.103425, 0.424543]

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	$\widehat{b_N}$	[0.2034155, 0.404062]	[0.122153, 0.231250]	[0.1348423, 0.271970]
500	$\widehat{r_N}$	[0.0673402, 0.366892]	[0.0395125, 0.2414946]	[0.0477265, 0.2507853]
	$\widehat{u_N}$	[0.804408, 1.498674]	[0.2706251, 0.771488]	[0.3586869, 0.7767146]
	$\widehat{a_N}$	[0.1552686, 0.583626]	[0.0851996, 0.361647]	[0.0978079, 0.3688259]
	$\widehat{b_N}$	[0.1937220, 0.387751]	[0.0993905, 0.2330407]	[0.1234648, 0.2496794]

Table 6. Bias value intervals of Monte Carlo simulations conducted for the NGoNH distribution.

r_N = [0.3,1.3], u_N = [0.6,1.6], b_N = [0.7,1.7], c_N = [0.8,1.8]				
N	**Ess. Par.**	**MLE**	**LSE**	**WLSE**
25	$\widehat{r_N}$	[0.0212749, 0.0914936]	[0.030885, 0.470514]	[0.025109, 0.412059]
	$\widehat{u_N}$	[0.0548905, 0.0872377]	[0.070358, 0.617641]	[0.025517, 0.440947]
	$\widehat{a_N}$	[0.1253840, 0.561482]	[0.027016, 0.200302]	[0.0014075, 0.2174105]
	$\widehat{b_N}$	[0.036688, 0.0714662]	[0.027649, 0.0029324]	[0.0848340, 0.0383038]
50	$\widehat{r_N}$	[0.011317, 0.124046]	[0.023403, 0.155807]	[0.020551, 0.141650]
	$\widehat{u_N}$	[0.133040, 0.0176707]	[0.111950, 0.488785]	[0.047112, 0.196197]
	$\widehat{a_N}$	[0.072505, 0.398390]	[0.027572, 0.305198]	[0.026416, 0.213894]
	$\widehat{b_N}$	[0.047302, 0.121580]	[0.016181, 0.089432]	[0.0080191,0.036340]
100	$\widehat{r_N}$	[0.0172914, 0.091080]	[0.0142788, 0.026015]	[0.0091922, 0.192793]
	$\widehat{u_N}$	[0.0446233,0.198041]	[0.091136, 0.191909]	[0.048416, 0.181493]
	$\widehat{a_N}$	[0.035851, 0.297286]	[0.022367, 0.192478]	[0.0062446,0.020947]
	$\widehat{b_N}$	[0.048726, 0.109893]	[0.014445, 0.022464]	[0.026489, 0.079319]
150	$\widehat{r_N}$	[0.003503, 0.012384]	[0.007174, 0.027753]	[0.010023, 0.010143]
	$\widehat{u_N}$	[0.262469, 0.297001]	[0.056614, 0.295905]	[0.02479, 0.0714713]
	$\widehat{a_N}$	[0.017331, 0.042685]	[0.0156912, 0.208932]	[0.0073358, 0.135397]
	$\widehat{b_N}$	[0.027753,0.036636]	[0.007579,0.020484]	[0.0500766, 0.0011552]
200	$\widehat{r_N}$	[0.0079631, 0.0142501]	[0.0009905, 0.021775]	[0.001009, 0.015595]
	$\widehat{u_N}$	[0.256185, 0.304060]	[0.0427282, 0.360942]	[0.0034366, 0.112378]
	$\widehat{a_N}$	[0.002390, 0.175348]	[0.0133606, 0.195463]	[0.0085301, 0.1187108]
	$\widehat{b_N}$	[0.056278, 0.062480]	[0.026951, 0.012227]	[0.0378548, 0.0312468]
300	$\widehat{r_N}$	[0.0040763, 0.017637]	[0.00162177, 0.034576]	[0.001279,0.004541]
	$\widehat{u_N}$	[0.219303,0.293900]	[0.0486514, 0.138685]	[0.034536,0.0917411]
	$\widehat{a_N}$	[0.0000843, 0.147830]	[0.0198602, 0.106663]	[0.0041702, 0.0706614]
	$\widehat{b_N}$	[0.0200124, 0.045923]	[0.014926, 0.0418630]	[0.0229515, 0.0286102]
400	$\widehat{r_N}$	[0.0045970, 0.0118250]	[0.006683, 0.045197]	[0.0038099, 0.0091515]
	$\widehat{u_N}$	[0.257734,0.305151]	[0.0110961, 0.178844]	[0.030648, 0.067255]
	$\widehat{a_N}$	[0.0042561, 0.116129]	[0.0052426, 0.0819237]	[0.0031947, 0.0956105]
	$\widehat{b_N}$	[0.0143930, 0.057870]	[0.036268, 0.062511]	[0.0272654, 0.0332795]

500	$\widehat{r_N}$	[0.0111381, 0.016864]	[0.0052780, 0.017380]	[0.0014226, 0.001994]
	$\widehat{u_N}$	[0.288777,0.327286]	[0.010489, 0.13279]	[0.001135,0.079843]
	$\widehat{a_N}$	[0.016755, 0.089918]	[0.0080641, 0.118735]	[0.012366, 0.071254]
	$\widehat{b_N}$	[0.014162, 0.053697]	[0.0088362,0.0253352]	[0.037494, 0.024339]

Table 3 shows the mean values of the three estimation methods (MLE, LSE, WLSE) at different sample sizes. The table shows the stability of the estimation of the basic parameters (r_N, u_N, a_N, b_N) as the sample sizes increases. At a small sample (N=25, 50), the confidence intervals of the parameters are wide, indicating high variability in the estimate. For the samples (N=100 to 500), the intervals become more compressed, indicating improved estimation accuracy and stability of the values. The MLE method shows ranges closer to the true values than LSE and WLSE, indicating that it is more accurate at large sample sizes. We note that the value u_N, which is one of the basic parameters, tends to stabilize at large samples, but the LSE method shows greater variability compared to MLE and WLSE.

Table 4 shows that at small samples (N=25, 50), the MSE is high, indicating that the parameter estimation is unstable and contains larger error. As the sample size increases, the MSE values decrease significantly, indicating that the performance of the estimation methods improves. The MLE method shows the lowest MSE values compared to LSE and WLSE, confirming its high effectiveness, especially for large samples. The WLSE method shows similar performance to MLE in some cases, but is less stable for small samples. As the sample increases (N=500), the minimum MSE values decrease to their lowest levels, indicating that the estimates converge towards the true values of the parameters.

Table 5 shows that the high values at samples (N=25, 50) mean that the initial estimates are inaccurate, but the values decrease significantly at N=100 and above. The MLE method shows the lowest RMSE at all samples, indicating its stability and robustness. LSE shows a higher RMSE compared to MLE, indicating that this method is less efficient in estimating the parameters. WLSE shows an average performance between the other two methods, but becomes more stable at $N \geq 200$.

Table 6 shows that at N=25, the bias is high for all methods, indicating that the initial estimates are inaccurate. As the sample sizes increases, Bias values gradually decrease, indicating that the estimates converge towards the true values. The MLE shows the lowest Bias values, meaning that it is the most accurate at all sizes. LSE shows a larger deviation from the true values, making it less reliable, especially at small samples. At N=500, Bias becomes very low for all methods, meaning that all methods become more accurate with larger samples.

From the above results, it is clear that MLE is the most efficient method, showing the lowest values for MSE, RMSE, and Bias, especially at large samples, while WLSE shows an average performance between MLE and LSE, but becomes more accurate at large samples. LSE shows lower performance than the other two methods, with Bias and MSE being higher at all sizes. Increasing the sample size improves the accuracy of all methods, but MLE gives the best results even with medium samples.

6. Application

The application side serves as a foundation for illustrating the breadth of the NGoNH distribution's quality, effectiveness, and adaptability in real-world applications. In this section, genuine neutrosophic data represented by temperatures in Lahore, Pakistan, for the period 2016-2020 [33]. The data utilized is checked to accomplish the neutrosophic qualities represent by achieving the condition $T + F + I = 1$ before the analytic process begins. This condition is similar to the probability condition, but it is different in that the values must lie within interval [0,1]. The following is what sets apart the neutrosophic logic in Table 7.

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Table 7. Data used, Truth, False, indeterminacy values and verification condition.

No.	intervals	Truth (T)	Falsity (F)	Indeterminacy (I)	Sum (T+F+I)	Satisfy neutrosophic condition or not
1	[46,72]	59.0	-26	-32.0	1	yes
2	[49,80]	64.5	-31	-32.5	1	yes
3	[60,87]	73.5	-27	-45.5	1	yes
4	[71,98]	84.5	-27	-56.5	1	yes
5	[84,107]	95.5	-23	-71.5	1	yes
6	[91,110]	100.5	-19	-80.5	1	yes
7	[88,104]	96.0	-16	-79.0	1	yes
8	[84,102]	93.0	-18	-74.0	1	yes
9	[79,103]	91.0	-24	-66.0	1	yes
10	[69,97]	83.0	-28	-54.0	1	yes
11	[61,86]	73.5	-25	-47.5	1	yes
12	[53,79]	66.0	-26	-39.0	1	yes
13	[47,69]	58.0	-22	-35.0	1	yes
14	[50,79]	64.5	-29	-34.5	1	yes
15	[56,87]	71.5	-31	-39.5	1	yes
16	[72,102]	87.0	-30	-56.0	1	yes
17	[83,107]	95.0	-24	-70.0	1	yes
18	[80,102]	91.0	-22	-68.0	1	yes
19	[87,108]	97.5	-21	-75.5	1	yes
20	[87,107]	97.0	-20	-76.0	1	yes
21	[88,104]	96.0	-16	-79.0	1	yes
22	[86,104]	95.0	-18	-76.0	1	yes
23	[72,96]	84.0	-24	-59.0	1	yes
24	[63,83]	73.0	-20	-52.0	1	yes
25	[56,75]	65.5	-19	-45.5	1	yes
26	[49,73]	61.0	-24	-36.0	1	yes
27	[54,78]	66.0	-24	-41.0	1	yes
28	[62,89]	75.5	-27	-47.5	1	yes
29	[72,98]	85.0	-26	-58.0	1	yes
30	[85,106]	95.5	-21	-73.5	1	yes
31	[92,108]	100.0	-16	-83.0	1	yes
32	[89,102]	95.5	-13	-81.5	1	yes
33	[86,102]	94.0	-16	-77.0	1	yes
34	[80,100]	90.0	-20	-69.0	1	yes
35	[82,98]	90.0	-16	-73.0	1	yes
36	[71,86]	78.5	-15	-62.5	1	yes
37	[60,76]	68.0	-16	-51.0	1	yes
38	[54,69]	61.5	-15	-45.5	1	yes
39	[55,71]	63.0	-16	-46.0	1	yes
40	[61,81]	71.0	-20	-50.0	1	yes
41	[79,101]	90.0	-22	-67.0	1	yes
42	[94,114]	104.0	-20	-83.0	1	yes
43	[90,106]	98.0	-16	-81.0	1	yes
44	[85,103]	94.0	-18	-75.0	1	yes
45	[82,101]	91.5	-19	-71.5	1	yes
46	[77,97]	87.0	-20	-66.0	1	yes
47	[67,82]	74.5	-15	-58.5	1	yes
48	[56,72]	64.0	-16	-47.0	1	yes
49	[43,64]	53.5	-21	-31.5	1	yes
50	[50,72]	61.0	-22	-38.0	1	yes
51	[58,81]	69.5	-23	-45.5	1	yes
52	[69,94]	81.5	-25	-55.5	1	yes
53	[78,103]	90.5	-25	-64.5	1	yes
54	[80,101]	90.5	-21	-68.5	1	yes
55	[80,95]	87.5	-15	-71.5	1	yes
56	[80,94]	87.0	-14	-72.0	1	yes
57	[77,94]	85.5	-17	-67.5	1	yes
58	[69,91]	80.0	-22	-57.0	1	yes
59	[54,78]	66.0	-24	-41.0	1	yes
60	[45,69]	57.0	-24	-32.0	1	yes

Mean	81.03333	-21.1666	-58.86667	
Sd-values	13.86003	4.588736	15.849361	
Max-values	104.0	-13.0	-31.5	
Min-values	53.5	-31.0	-83.0	

The temperatures in Lahore, Pakistan, for the period 2016-2020 is represented by actual data in Table 7. The table demonstrates that every value satisfies the requirement, demonstrating the data's conformance to the neutrosophic logic's characteristics. Additionally, it is seen that the T values gradually decline with time, indicating a shift in real values. The table uses the neutrosophic logic to accurately depict actual data. The values show how adaptably T, F, and I may be represented by the system. The findings support the applicability of using neutrosophic logic to the analysis of complicated and ambiguous data.

The data utilized, the type of analysis, and the components of the analysis are depicted in the following figures.

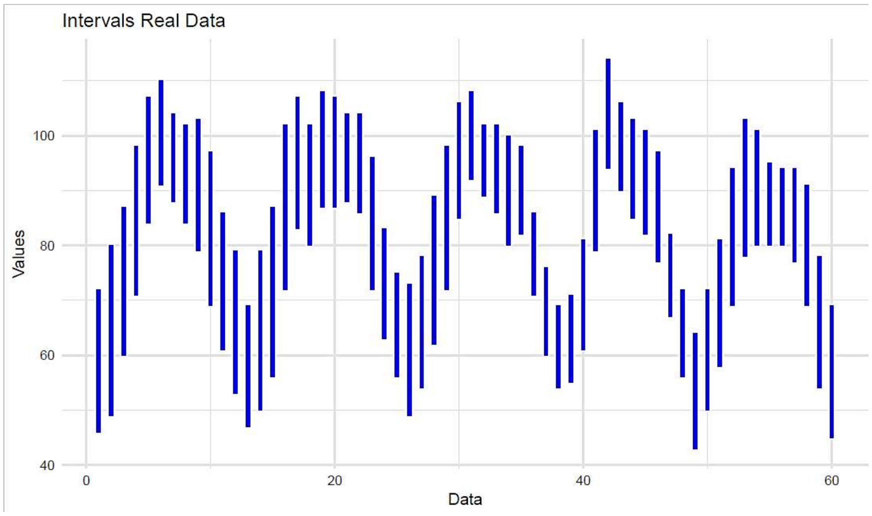

Figure 5. Plot of the intervals for the data used.

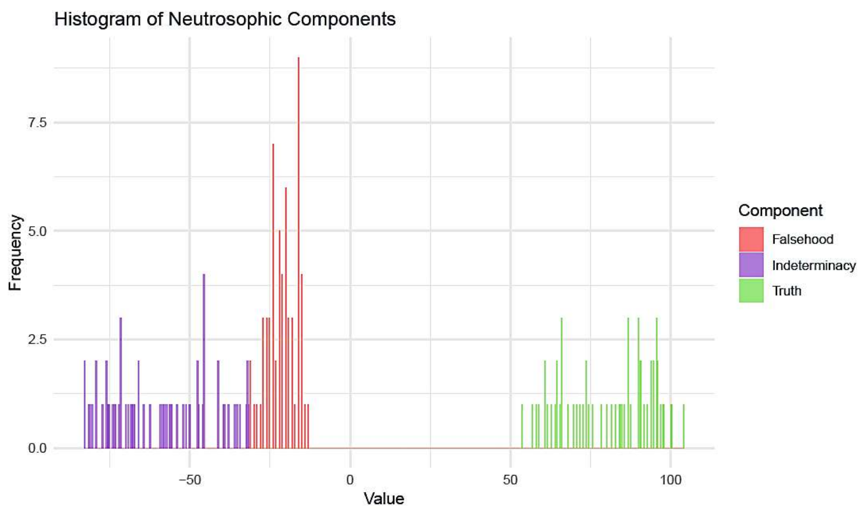

Figure 6. Histogram of Neutrosophic Components.

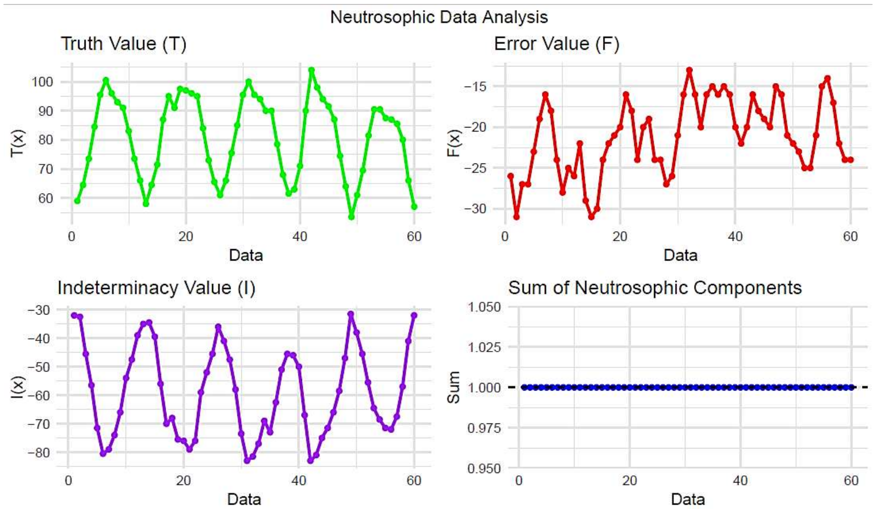

Figure 7. Neutrosophic parts and sum of Neutrosophic Components.

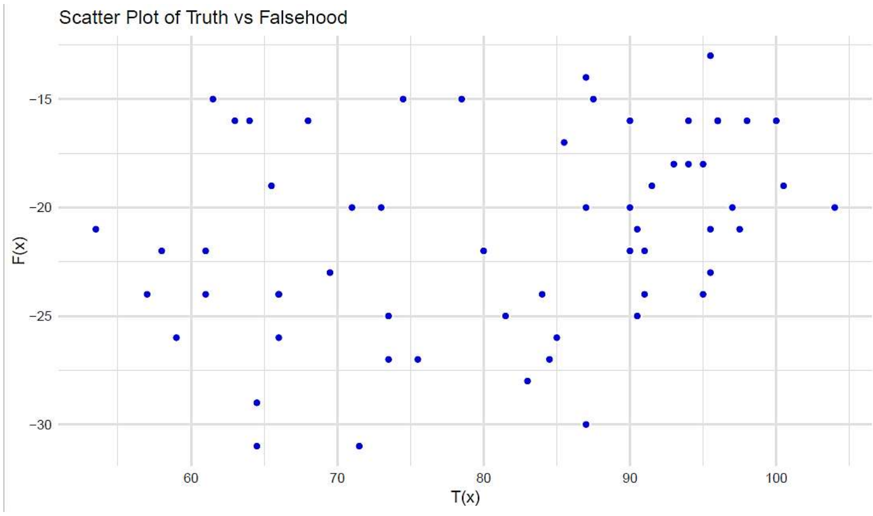

Figure 8. Scatter plot of Truth vs Falshood.

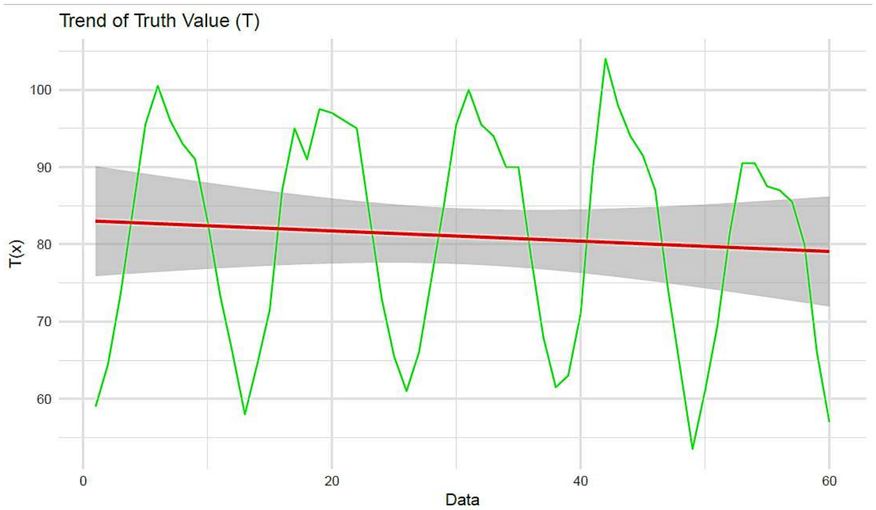

Figure 9. Trend plot of Truth value.

The NGoNH distribution's results are compared to six alternative neutrosophic distributions, which are:

Neutrosophic Wiebull Nadarajah-Haghighi (NWeNH)

Neutrosophic Kumaraswamy Nadarajah-Haghighi (NKuNH)

Neutrosophic Exponeted Generalized Exponential Nadarajah-Haghighi (NEGNH)

Neutrosophic [0,1] Truncated Exponeted Exponential Nadarajah-Haghighi (NTEENH)

Neutrosophic Beta Nadarajah-Haghighi (NBeNH)

Neutrosophic Nadarajah-Haghighi (NNH)

Four information criteria AIC [33], CAIC [34], HQIC [35], and BIC [36] as well as the statistical measures such as Kolmogorov-Smirnov (KS), Anderson- Darling (A), Cramér-von Mises (W), and p-value [16,21,37] were used for this comparison. The results of the distribution's criterion are displayed in Table 8, the values of the statistical measures are displayed in Table 9, and the estimated parameters in MLE are displayed in Table 10.

Table 8. Results of the criteria for the distributions.

Dist.	-2L	AIC	CAIC	BIC	HQIC
NGoNH	[238.5133,244.946]	[485.026,497.892]	[485.753,498.619]	[493.40,506.2695]	[488.303,501.168]
NWeNH	[239.031,251.0866]	[486.062,510.173]	[486.789,510.900]	[494.439,518.550]	[489.3392, 513.45]
NKuNH	[240.336,247.0285]	[488.672,502.057]	[489.400,502.784]	[497.050,510.434]	[491.949,505.333]
NEGNH	[245.750,248.0053]	[499.501,504.010]	[500.228,504.737]	[507.878,512.387]	[502.777,507.287]
NTEENH	[255.7479,272.326]	[519.495,552.653]	[520.223,553.380]	[527.873,561.030]	[522.772,555.930]
NBeNH	[242.1966,247.903]	[492.393, 503.806]	[493.120,504.533]	[500.770,512.183]	[495.67, 507.0829]
NNH	[299.898,315.7357]	[603.795,635.471]	[604.006,635.681]	[607.9846,639.66]	[605.434,637.109]

Table 9. Values of the statistical measures.

Dist.	W	A	K-S	p-value
NGoNH	[0.1631278, 0.1742447]	[1.033034, 1.076228]	[0.104354,0.109531]	[0.4677146,0.5306625]
NWeNH	[0.2179362,0.2454571]	[1.296429,1.415124]	[0.127022,0.144822]	[0.1613447,0.2876714]
NKuNH	[0.2586521,0.3033063]	[1.526916,1.73937]	[0.143952,0.152958]	[0.1662749, 0.1206781]
NEGNH	[0.2847026,0.3772915]	[1.671658,2.156582]	[0.155954,0.162163]	[0.08521509,0.1079987]
NTEENH	[0.3286531,0.3724759]	[1.920758,2.134018]	[0.194677,0.324559]	[0.021178,6.475093e-06]
NBeNH	[0.2819412,0.3405313]	[1.655773,1.950182]	[0.151771,0.158347]	[0.09868349,0.1517713]
NNH	[0.2379204,0.3124073]	[1.40662,1.789007]	[0.47210,0.5410004]	[1.11022e-15,4.8459e-12]

Table 10. Estimator value interval for the parameters using the MLE method.

Dist.	$\hat{r}_N$	$\hat{u}_N$	$\hat{q}_N$	$\hat{p}_N$
NGoNH	[0.006457,0.00810040]	[0.5863147,3.7445267]	[0.37624873,0.694243]	[0.043232,0.0654481
NWeNH	[4.2912505,9.7916074]	[0.0379921,0.1424287]	[0.0552390,0.3434586]	[0.006357,0.009962]
NKuNH	[17.614578,31.743331]	[3.319084,11.2087923]	[1.1577578,1.4533959]	[0.018980,0.021025]
NEGNH	[0.164599,0.28248229]	[18.732066,54.537074]	[1.440879,2.00402975]	[0.037163,0.099447]
NTEENH	[19.613499,20.100419]	[26.350845,66.978545]	[0.4644331,0.5031479]	[0.005143,0.008562]

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NBeNH	[0.126028,23.6324741]	[1.301384,41.1872412]	[2.006751,6.47165680]	[0.015163,1.305300]
NNH	---	---	[4.5300068,5.3838006]	[0.002039,0.002223]

Table 8 shows the values of the different evaluation criteria for the mentioned distributions. These criteria are used to compare the quality of the statistical models. -2L represents twice the negative value of the log-likelihood function. Here, we notice that the NGoNH distribution has the lowest values, which indicates that it is the best among the distributions.

Table 9 shows the statistical values which are used to evaluate the quality of distributions, the value of W was for the NGoNH, it has the lowest values, which indicates that it is the best. The A test is used to test the distribution's fit to the data. The smaller value is the best distribution. this is confirmed by NGoNH values. The K-S values compares the experimental distribution with theoretical distribution. NGoNH has lowest values for K-S. NGoNH has the highest values for p-value, which indicates that it is the best.

Table 10 shows the estimated parameters for the NGoNH distribution. It shows reasonable and well-defined values of parameters for the NWeNH distribution with greater variability, which may indicate that the distribution is less stable. The NKuNH parameters also show large variability, which may indicate difficulty in estimating them accurately. The NEGNH shows moderate variability, while NTEENH, NBeNH and NNH shows large variability, especially for the parameter u_N.

Figure 10 shows the theoretical probability density (Npdf) of NGoNH compared to actual data used in study. Ff the curve fits well with the bar distribution, it indicates that NGoNH describes the data with high accuracy. NGoNH provides a good fit with the real data, as the curve approaches the peaks of the bars. This indicates that the distribution is effective in describing the real temperature values. Figure 11 shows the NCDF of NGoNH compared to the actual data distribution. The good fit between the real data and the distribution curve indicates that the NGoNH distribution can be an accurate model for predicting climate data.

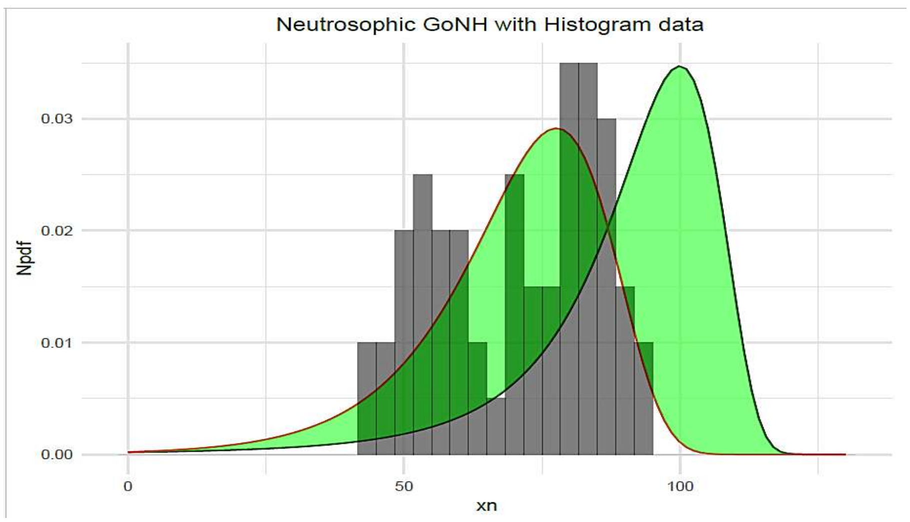

Figure 10. The fitted Npdf with histogram of the data used.

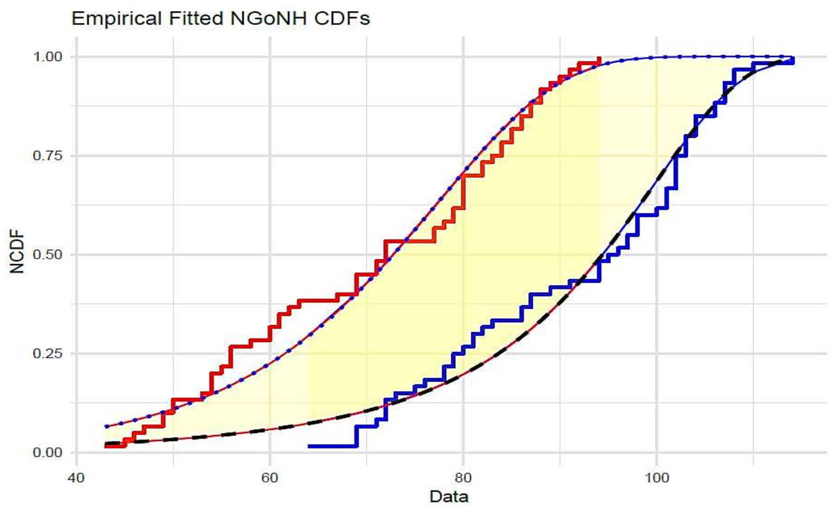

Figure 11. The empirical fitted NGoNH NCDF.

7. Conclusion

A new four-parameter distribution is developed that combines NGo-G family and Nadarajah-Haghighi distribution within the framework of neutrosophic logic. This expansion contributes to providing more accurate models for representing ambiguous and uncertain data, enhancing the capabilities of statistical modeling. The underling function of many properties of the distribution are derived, providing a solid mathematical foundation for its use in various applications. The results shows that the NGoNH provides the best fit to the data compared to other distributions. According to the information criteria's, statistical values confirm that NGoNH has lowest error, making it the most accurate model. Analysis of neutrosophic values: true values reflect well the data pattern, with a general trend that can be explained based on environmental or temporal factors. The indeterminate values and false values contribute to understanding the extent of uncertainty in the data, which supports the use of neutrosophic logic in analyzing complex phenomena. The estimation method using MLE gave more stable and accurate results compared to other methods, especially for large samples. As the sample size increased, the values of MSE, RMSE, and Bias decreased, indicating an improvement in the accuracy of the estimating with more data. The study confirms that the proposed distribution can be applied to complex climate data, which opens the door to its use in other applications such as climate forecasting, market analysis, and biological systems. The study suggests improving the model by adding supplementary variables that may help explain fluctuations in neutrosophic values, such as the effects of climate change or economic factors, in addition to using large data to verify the stability of the parameters and ensure that the model works efficiently in different contexts. The model can also be compared with other models such as neural networks or complex time series models to see how well it excels in analyzing ambiguous data. The logical limits of NGoNH distribution can be summarized as follows:

Type of limit	Explanation	Possible solution
Mathematical constraints	The distribution must be non-negative, and its coefficients must be positive	Checking that mathematical conditions are met
Statistical constraints	Sensitivity of the estimate to sample size and initial values	Using large samples and techniques to improve the estimate
Application limitations	Limitations of its application in non-neutrosophic data	Verification of its suitability using goodness-of-fit tests
Numerical limitations	Need for approximate solutions for some properties	Use of mathematical expansions and numerical analysis

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These boundaries define the scope of distribution and guide its development to improvement of future applications.

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Conflicts of Interest: The authors declare no conflicts of interest.

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