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Born's Rule from Record-Formation Constraints: A Thermodynamic-Information Axiomatics

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Abstract

We show the Born rule $P = |\psi|^2$ is the unique probability rule consistent with thermodynamic constraints on record formation, conditional on the regime where Landauer-type bounds apply. The derivation proceeds from five operational postulates: normalization, phase information loss in record formation, interference consistency, tensor product factorization, and continuity. The key physical insight is that creating a classical measurement record requires that phase information is not retained in the record accessible to observers, a not-reversible operation with entropy flow to the bath and Landauer cost $k_B T \ln 2$ per bit. The squared modulus emerges as the unique probability rule that (i) eliminates phase to produce positive probabilities, (ii) preserves interference effects before measurement, and (iii) satisfies standard probability axioms. This thermodynamically motivated derivation complements Gleason's theorem: where Gleason proves the rule is mathematically necessary (dimension ≥ 3), we show it is the unique rule realizable through record formation under these constraints (all dimensions including $d = 2$). The framework provides an answer to "why squared?": the irreversible formation of a classical record, on a Hilbert space whose norm is preserved by unitary evolution, admits no other consistent probability rule.

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1. Introduction

The Born rule, that the probability of obtaining outcome i from a quantum measurement is $P_i = |\alpha_i|^2$ where α_i is the amplitude, is one of the central postulates of quantum mechanics. Despite its empirical success, its foundational status remains debated:

- Is it a separate axiom, or can it be derived?
- Why specifically squared, rather than some other function?
- What is the physical mechanism that produces probabilities from amplitudes?

Previous derivations include:

- **Gleason's theorem** [1]: Shows that any non-contextual probability measure on a Hilbert space of dimension ≥ 3 must take the form $|\langle\phi|\psi\rangle|^2$
- **Deutsch-Wallace** [2]: Derives Born probabilities from rational decision theory in the Everett interpretation
- **Zurek's envariance** [3]: Derives the rule from environment-assisted invariance

We provide a complementary derivation from **constraints motivated by thermodynamic record formation**: given the standard Hilbert space structure, the Born rule is the unique probability assignment compatible with the physical process of creating classical records at finite temperature.

We assume the standard kinematic structure of quantum theory: states form a complex vector space with an inner product fixed by interference and by the unitary time evolution that preserves

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transition amplitudes. This inner product sets normalization independently of the probability rule. The contribution below is conditional on that structure and shows that record formation at finite temperature selects the squared modulus as the unique admissible probability map.

2. Physical Setting

2.1. Quantum Amplitudes and Phase

Consider a quantum system in superposition:

$$|\psi\rangle = \sum_i \alpha_i |i\rangle, \quad \alpha_i = |\alpha_i| e^{i\phi_i} \tag{1}$$

Each amplitude α_i contains:

- **Magnitude** $|\alpha_i|$: The "weight" of pathway i
- **Phase** ϕ_i: Enables interference between pathways

The phase information is responsible for quantum interference:

$$|\alpha_1 e^{i\phi_1} + \alpha_2 e^{i\phi_2}|^2 = |\alpha_1|^2 + |\alpha_2|^2 + 2|\alpha_1||\alpha_2|\cos(\phi_1 - \phi_2) \tag{2}$$

2.2. Classical Records and Phase Information Loss

A classical measurement record, in the record basis, does not exhibit interference between the recorded outcome branches. Therefore, creating such a record requires that relative-phase information is not retained in the record accessible to observers.

Once phase information is not accessible from the record, the map from the quantum state to the record is not reversible. For a thermal environment (subject to the applicability conditions summarized in Sec. III), this implies an entropy cost no smaller than $k_B T \ln 2$ per bit of phase information that is not retained.[4,5,7]

2.3. The Central Question

What probability rule $f : \mathbb{C} \to \mathbb{R}^+$ maps amplitudes to probabilities while respecting both quantum mechanics (before measurement) and thermodynamics (at measurement)?

3. Thermodynamic Constraints

We impose five constraints that any physically realizable probability rule must satisfy:

Definition 1 (Normalization)**.** *For any normalized quantum state $|\psi\rangle = \sum_i \alpha_i |i\rangle$ with $\sum_i |\alpha_i|^2 = 1$, the probabilities must sum to unity. Here the state normalization uses the Hilbert space inner product and is independent of the probability rule:*

$$\sum_i f(\alpha_i) = 1 \tag{3}$$

Outcome-locality note. This definition assumes that the probability of each outcome depends only on its own amplitude, an outcome-locality condition standard for projective measurements. It excludes globally renormalized rules of the form $p_i = g(|\alpha_i|)/\sum_j g(|\alpha_j|)$, which achieve normalization by construction for any power g. The physical motivation is that each record branch is formed independently by local interaction with the apparatus.

Scope note. This work assumes the standard Hilbert space structure, including L^2 normalization and unitary dynamics. The result is conditional on those kinematic assumptions. Given that structure and the record formation constraints stated here, the Born rule is the unique probability assignment compatible with both.

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Definition 2 (Phase Information Loss in Record Formation)**.** *The stabilized classical record, in the record (pointer) basis, does not retain relative-phase information between outcome branches. The probability assigned to each outcome must therefore be independent of phase:*

$$f(\alpha e^{i\phi}) = f(\alpha) \quad \forall \phi \in [0, 2\pi) \tag{4}$$

Formally, this corresponds to the action of a pointer-basis dephasing channel on the record subsystem; see Ref. [8], Sec. III for an explicit CPTP formalization.

Definition 3 (Interference Consistency (Operational))**.** *For two-path preparations with controllable relative phase, the predicted statistics must depend on the phase and exhibit a fringe pattern with nonzero visibility. When the relative phase is randomized (or the paths are fully decohered), the rule must reduce to a classical mixture with no phase dependence. This definition is operational and does not assume a specific functional form for the interference term.*

Definition 4 (Tensor Product Factorization)**.** *For independent systems A and B in product state $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ with amplitudes α_i^A and α_j^B, the joint amplitude is $\alpha_{ij} = \alpha_i^A \alpha_j^B$. The joint probability factorizes:*

$$P(i,j) = f(\alpha_i^A \alpha_j^B) = f(\alpha_i^A) f(\alpha_j^B) \tag{5}$$

This factorization assumes independent preparations yield product states whose records factorize. Whether weaker composition axioms suffice is an open question.

Definition 5 (Continuity)**.** *The function f is continuous in α. Small changes in amplitude produce small changes in probability. This excludes pathological (non-measurable) solutions to the multiplicative functional equation in Step 4; all physically meaningful probability rules satisfy this condition.*

Applicability note. The thermodynamic statements below apply in the operational regime where Landauer-type bounds are valid.[1] Ref. [8] decomposes measurement into three stages: (1) reversible premeasurement (unitary correlation, no Landauer cost), (2) irreversible record formation (boundary event; Landauer cost paid here), and (3) optional reset (separate erasure cost). The Landauer bound applies to Stage 2 under six operational conditions: a thermal bath is present (C1), a classical register is created (C2), the apparatus is cyclic (C3), irreversibility arises from bath coupling (C4), no unaccounted work reservoir is present (C5), and $I(X;Y)$ is the actual mutual information (C6). The derivation should be read as conditional on those assumptions.

4. Derivation of the Born Rule

Theorem 1 (Born Rule from Thermodynamic Constraints)**.** *The unique function $f : \mathbb{C} \to \mathbb{R}^+$ satisfying Definitions 1, 2, 4, and 5 is $f(\alpha) = |\alpha|^2$. Definition 3 (Interference Consistency) is a post-hoc consistency check rather than a derivational constraint.*

Proof. Step 1: By Phase Information Loss (Definition 2), $f(\alpha) = f(|\alpha|)$. Thus f depends only on the modulus. Define $g : \mathbb{R}^+ \to \mathbb{R}^+$ by $g(r) = f(re^{i\phi})$ for any ϕ.

Step 2: By Normalization (Definition 1), when $\sum_i |\alpha_i|^2 = 1$:

$$\sum_i g(|\alpha_i|) = 1 \tag{6}$$

This must hold for all points on the unit sphere in amplitude space.

[1] Norton [15] questions unconditional applications of Landauer's principle; Ladyman and Robertson [16] defend it under specified conditions. Our application is conditional on C1 to C6 (summarized below; see Ref. [8] for experimental diagnostics and falsification criteria).

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Step 3: Consider two specific cases on the unit sphere:

- $|\alpha_1|^2 = 1, |\alpha_2|^2 = 0$: Then $g(1) + g(0) = 1$
- $|\alpha_1|^2 = |\alpha_2|^2 = 1/2$: Then $2g(1/\sqrt{2}) = 1$

For certainty to map to certainty: $g(1) = 1$, hence $g(0) = 0$.
Step 4: By Tensor Product Factorization (Definition 4):

$$g(r_1 \cdot r_2) = g(r_1) \cdot g(r_2) \tag{7}$$

Combined with Continuity (Definition 5), this is Cauchy's multiplicative functional equation,[9] whose unique continuous solution on $\mathbb{R}^+$ is $g(r) = r^n$ for some $n > 0$.
Step 5: The normalization constraint $\sum_i g(|\alpha_i|) = 1$ when $\sum_i |\alpha_i|^2 = 1$ implies

$$\sum_i |\alpha_i|^n = 1 \quad \text{whenever} \quad \sum_i |\alpha_i|^2 = 1. \tag{8}$$

In particular, for an equal-amplitude N-outcome superposition with $|\alpha_i| = 1/\sqrt{N}$, the constraint gives $N(1/\sqrt{N})^n = 1$, i.e. $N^{1-n/2} = 1$ for all N, hence $n = 2$.
Lemma (uniqueness of $n = 2$). The only power n for which the L^2 unit sphere is also an L^n unit sphere is $n = 2$.
Proof sketch. If $n > 2$, take $|\alpha_1| = |\alpha_2| = 1/\sqrt{2}$ and the rest zero. Then $\sum_i |\alpha_i|^2 = 1$ but $\sum_i |\alpha_i|^n = 2^{1-n/2} < 1$, a contradiction. If $n < 2$, take $|\alpha_1| = \sqrt{1-\varepsilon^2}$, $|\alpha_2| = \varepsilon$ (exactly on the L^2 sphere: $|\alpha_1|^2 + |\alpha_2|^2 = 1$). Then $\sum_i |\alpha_i|^n = (1-\varepsilon^2)^{n/2} + \varepsilon^n$. For small $\varepsilon > 0$, $(1-\varepsilon^2)^{n/2} \approx 1 - \frac{n}{2}\varepsilon^2$ while $\varepsilon^n \gg \varepsilon^2$ when $n < 2$, so $\sum_i |\alpha_i|^n > 1$, a contradiction. Hence $n = 2$.
Step 6 (consistency check): The operational interference requirement is met by $f(\alpha) = |\alpha|^2$, which yields the standard phase-dependent fringe visibility in two-path experiments and reduces to a classical mixture under phase randomization. Other power rules generally distort the phase dependence and visibility.
Therefore $f(\alpha) = |\alpha|^2$. □

Remark on scope. Our axioms do not conjure $n = 2$ from nothing: L^2 normalization, which quantum mechanics requires independently for unitarity, already constrains the space of possibilities. The contribution is showing that $n = 2$ is the *unique* rule compatible with all five constraints simultaneously; the Born rule is forced by consistency, not postulated. On an L^p sphere with $p \neq 2$, the same argument would yield $n = p$; the squared modulus is locked in by the L^2 structure that quantum mechanics requires.

5. Physical Interpretation: Why Squaring?

The proof establishes uniqueness, but what is the physical meaning of "squaring"? We provide two complementary interpretations:

5.1. Information-Theoretic Interpretation

The amplitude $\alpha = |\alpha|e^{i\phi}$ contains two pieces of information:

1. The magnitude $|\alpha|$ (positive real)
2. The phase $e^{i\phi}$ (unit complex number)

Classical probability is a **positive real number**. To map from $\mathbb{C}$ to $\mathbb{R}^+$, we must eliminate the phase.
The operation $\alpha \mapsto |\alpha|^2 = \alpha \cdot \alpha^*$ is the **minimal phase-eliminating operation**:

$$|\alpha e^{i\phi}|^2 = (\alpha e^{i\phi})(\alpha e^{i\phi})^* = \alpha\alpha^* e^{i\phi} e^{-i\phi} = |\alpha|^2 \tag{9}$$

The phase cancels through multiplication with the complex conjugate.

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5.2. Thermodynamic Interpretation

Squaring *can be interpreted* as the Landauer cost of creating a classical record, under the operational conditions where Landauer bounds apply.

The phase ϕ carries information that enables interference. When a classical record is created, this information is not retained. The minimum heat dissipation is:

$$Q \geq k_B T \ln 2 \cdot I_{\text{phase}} \tag{10}$$

For a controlled two-phase preparation with $\Delta\phi \in \{0, \pi\}$ and a detector placed at a point of maximal contrast, the phase determines whether interference is constructive or destructive, which encodes 1 bit of distinguishable information.

The Born rule probability $P = |\alpha|^2$ is what remains after this bit has been erased. Squaring is the minimal phase-eliminating operation consistent with Landauer-cost record formation under the stated conditions.

6. Information Content of Phase

Having established $f(\alpha) = |\alpha|^2$ in Sec. IV, we now use this result to analyze the information content of phase and its connection to the Landauer bound. This subsection is illustrative: it uses standard two-path interference statistics implied by the Born rule and does not introduce an additional axiom in the derivation.

6.1. Mutual Information Analysis

Define:

- X: The relative phase $\Delta\phi = \phi_1 - \phi_2$, uniformly distributed on $[0, 2\pi)$
- Y: The measurement outcome (which detector clicks)

Before measurement (with interference):

$$P(Y = 1|\Delta\phi) = \frac{1}{2}(1 + \cos\Delta\phi) \tag{11}$$

The mutual information is:

$$I(X;Y)_{\text{before}} = H(Y) - H(Y|X) \tag{12}$$

For a continuous uniform phase, $I(X;Y)_{\text{before}}$ depends on the detector setting and is less than 1 bit. In a two-phase preparation with $\Delta\phi \in \{0, \pi\}$ and a detector at maximal contrast, $H(Y|X) = 0$ and $I(X;Y)_{\text{before}} = 1$ bit.

After measurement (classical record):

$$P(Y) = |\alpha_1|^2 \quad \text{(independent of phase)} \tag{13}$$

Thus $I(X;Y)_{\text{after}} = 0$.

In this two-phase preparation, the measurement has erased exactly 1 bit of phase information, consistent with the Landauer bound. (For a continuous uniform phase the mutual information is less than 1 bit; for a K-valued discrete phase alphabet, the Landauer cost is $k_B T \ln K$.)

7. Connection to Prior Derivations

7.1. Gleason's Theorem

Gleason's theorem [1] proves that in Hilbert spaces of dimension ≥ 3, any probability measure satisfying non-contextuality must be given by the Born rule.

Our derivation is **complementary**:

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- **Gleason**: Shows the rule is *mathematically necessary* given non-contextuality (dimension ≥ 3)
- **This work**: Shows the rule is the *unique* one realizable under record-formation constraints (all dimensions including $d = 2$)

Our axiom set (factorization (composition) + phase independence (symmetry) + continuity) is structurally closer to the "composition + symmetry" family of derivations (Hardy [13], Masanes and Müller [14]) than to Gleason's measure-theoretic non-contextuality. Phase independence in the record basis replaces Gleason's non-contextuality; this is a different assumption, not a relabeling.

Stacey [17] cautions that many Born-rule derivations merely relocate Gleason's assumptions. Our axiom set differs structurally (phase independence replaces non-contextuality, no $d \geq 3$ restriction), though the concern about assumption-shuffling is legitimate and motivates our explicit scope statement.

7.2. Masanes, Galley, and Müller

Masanes, Galley, and Müller [10] derive the Born rule and measurement postulates from the remaining quantum postulates plus subsystem consistency (a composition axiom related to our Definition 4) without thermodynamic input, using a stronger set of operational postulates. Galley and Masanes [11] classify alternatives to the Born rule in terms of informational properties, providing a complementary perspective on uniqueness.

7.3. Mandhan

Mandhan [12] independently pursues a Landauer-based thermodynamic route to the Born rule via KL-divergence minimization. Our approach differs in using an explicit axiom set (Definitions 1 to 5) rather than an optimization principle, and in providing a direct uniqueness proof via Cauchy's functional equation.

7.4. Envariance

Zurek's envariance derivation [3] uses environment-assisted invariance: probabilities must be invariant under swaps that leave the environment unchanged.

Our framework is *consistent with* envariance and provides a physical mechanism that supports it:

- The environment is the thermal bath receiving the Landauer heat
- Swaps that leave the bath state unchanged cannot change probabilities
- Phase information is not retained in the record accessible to the observer

Envariance is consistent with entropy flow to the bath during record formation.

8. Why Not Linear? Why Not Quartic?

8.1. Linear Rule: $P_i = |\alpha_i|$

A linear rule could be normalized with an L^1 norm, but this breaks the Hilbert-space structure required for unitary evolution and a well-defined inner product. Quantum mechanics uses the L^2 norm precisely because it preserves the inner product and probability under unitary dynamics. With L^2 normalization, $\sum_i |\alpha_i| \neq 1$ generically, and linear probabilities fail to reproduce observed interference.

8.2. Quartic Rule: $P_i = |\alpha_i|^4$

This satisfies phase independence but fails normalization on the unit sphere.

More physically: it would require erasing *more* than the phase information, which is not necessary for creating a classical record.

8.3. General Power Rule: $P_i = |\alpha_i|^n$

Only $n = 2$ satisfies all constraints simultaneously. The Born rule is the unique fixed point of the constraint system.

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9. Experimental Consistency Checks

The following are not predictions unique to this framework, but consistency checks showing the axiom set is compatible with known physics. Calorimetric protocols are developed in Ref. [8].

9.1. Heat-Information Correlation

For measurements creating different amounts of classical information, the heat dissipated should scale with the Shannon entropy of the outcome distribution, with a Landauer prefactor under the applicability conditions:

$$\langle Q \rangle \gtrsim k_B T \ln 2\, I(X;Y) \tag{14}$$

where $I(X;Y) = H(Y) - H(Y|X)$ is the classical mutual information between the prepared state X and the measurement record Y. For ideal projective measurements with known prior, $I(X;Y) = H(P)$ (the Shannon entropy of the outcome distribution); the general bound uses mutual information. [6,8] Calorimetric protocols and sensitivity estimates are provided in Ref. [8].

9.2. Decoherence Interpolation

For weak measurements with partial phase information loss, the unnormalized detection rate (intensity) interpolates between quantum and classical:

$$I_{\text{eff}} = (1-\varepsilon) \cdot |\psi_1 + \psi_2|^2 + \varepsilon \cdot (|\psi_1|^2 + |\psi_2|^2) \tag{15}$$

where ε can be operationally defined from fringe visibility, e.g. $\varepsilon = 1 - V$ with $V = (I_{\max} - I_{\min})/(I_{\max} + I_{\min})$. Note that I_{eff} is not a probability: it can exceed unity for constructive interference. Probabilities are obtained by normalizing I_{eff} over all outcomes.

10. Conclusions

We have derived the Born rule from record-formation constraints motivated by thermodynamics, conditional on the Landauer applicability regime summarized in Sec. III. The key insight is that creating a classical record requires that relative-phase information is not retained in the record, an operation with entropy flow to the bath and Landauer cost. The squared modulus $|\alpha|^2$ is the unique probability rule that:

1. Eliminates relative phase in the record basis (record-formation constraint)
2. Preserves interference before measurement (quantum consistency)
3. Satisfies normalization and factorization (probability axioms)

The Born rule is not an arbitrary postulate. It is the unique bridge between quantum amplitudes and classical probabilities that respects both quantum mechanics and thermodynamics. The proof structure is norm-relative: on an L^p sphere, the same argument yields $n = p$. The Born rule's $n = 2$ is locked in by the L^2 structure that quantum mechanics requires for unitarity. "Why squared?" Because the irreversible formation of a classical record, on a Hilbert space whose norm is preserved by unitary evolution, admits no other consistent probability rule.

A natural next step is to derive phase independence from an explicit CPTP model of the record-formation channel (cf. Ref. [8], Sec. III), which would upgrade the thermodynamic content from motivation to derivation. More broadly, connecting the axiom set to the recent thermodynamic stage decompositions of measurement [18] may clarify the relationship between record formation and the Born rule at a deeper level.

Supplementary code (`simulation.py`) verifies all numerical claims: algebraic identities, uniqueness-lemma sweeps, alternative-rule falsification, information-theoretic quantities, and cross-consistency with Ref. [8]. These checks confirm internal mathematical consistency; they do not validate the physical premises.

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Code Availability: The supplementary simulation code and generated consistency reports are available at https://github.com/MosesRahnama/paper-b-born-rule-supplements.

References

1. A. M. Gleason, "Measures on the closed subspaces of a Hilbert space," J. Math. Mech. **6**, 885 (1957).
2. D. Wallace, *The Emergent Multiverse: Quantum Theory according to the Everett Interpretation* (Oxford University Press, 2012).
3. W. H. Zurek, "Probabilities from entanglement, Born's rule $p_k = |\psi_k|^2$ from envariance," Phys. Rev. A **71**, 052105 (2005).
4. R. Landauer, "Irreversibility and Heat Generation in the Computing Process," IBM J. Res. Dev. **5**, 183 (1961).
5. C. H. Bennett, "Notes on Landauer's principle, reversible computation, and Maxwell's Demon," Stud. Hist. Phil. Mod. Phys. **34**, 501 (2003).
6. T. Sagawa and M. Ueda, "Minimal Energy Cost for Thermodynamic Information Processing," Phys. Rev. Lett. **102**, 250602 (2009).
7. A. Bérut *et al.*, "Experimental verification of Landauer's principle," Nature **483**, 187 (2012).
8. M. Rahnama, "Calorimetric Signature of Quantum Measurement: A Record-Formation Heat Bound and Differential Microcalorimetry Test," (2026), submitted.
9. J. Aczél, *Lectures on Functional Equations and Their Applications* (Academic Press, New York, 1966).
10. L. Masanes, T. D. Galley, and M. P. Müller, "The measurement postulates of quantum mechanics are operationally redundant," Nat. Commun. **10**, 1361 (2019).
11. T. D. Galley and L. Masanes, "Classification of all alternatives to the Born rule in terms of informational properties," Quantum **1**, 15 (2017).
12. S. Mandhan, "Thermodynamic origin of the Born rule," arXiv:2502.06494 (2025).
13. L. Hardy, "Quantum theory from five reasonable axioms," arXiv:quant-ph/0101012 (2001).
14. L. Masanes and M. P. Müller, "A derivation of quantum theory from physical requirements," New J. Phys. **13**, 063001 (2011).
15. J. D. Norton, "Waiting for Landauer," Stud. Hist. Phil. Mod. Phys. **42**, 184 (2011).
16. J. Ladyman and K. Robertson, "Landauer defended: Reply to Norton," Stud. Hist. Phil. Mod. Phys. **48**, 210 (2014).
17. B. C. Stacey, "Is the Born rule a fundamental axiom of quantum theory, or can it be derived?" in *Pandemonia: On the Advantages of a Universal Pandemic* (2021).
18. C. L. Latune and C. Elouard, "A thermodynamically consistent approach to the energy costs of quantum measurements," Quantum **9**, 1614 (2025).