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Integral Forms and Connections of Multi-Parameter Mittag-Leffler Functions with Fox-Wright and Hypergeometric Functions

Pooja Singh

Maharaja Surajmal Institute of Technology; pooja.singh@msit.in

Abstract: In this study, the complex interactions between the well-known Mittag-Leffler functions with two, three, and four parameters are explored. We first study the integral forms of the Mittag-Leffler function $\mathsf{E}^{\rho,\kappa}_{\alpha,\beta}(z)$ with the goal of clarifying its mathematical behaviours and features. Moreover, we create relations between these functions and generalised hypergeometric functions and Fox-Wright functions, expanding the range of their use and comprehension. We explore a number of unique situations by in-depth examination, providing insight into the subtle characteristics of these functions. This investigation advances our theoretical knowledge and opens up possibilities for future applications in a number of scientific fields. As such, this study adds to the current conversation in mathematical analysis and is an important tool for both practitioners and researchers.

Keywords: Mittag-Leffler functions; Wright hypergeometric functions; Fox-Wright function; generalized hypergeometric function

MSC: 33E12; 33B15; 33C20; 33C60; 33C90; 33C99

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1. Introduction and Preliminaries

Despite being a field of study for four centuries, special functions has seen tremendous growth in the last fifty years due to its applications in science and engineering. Bessel functions, generalised hypergeometric functions, generalised Mittag-Leffler functions, Wright hypergeometric functions, H-function, Legendre polynomials, Laguerre polynomials, Hermite polynomials, Srivastava polynomials, and many more special functions are included in the list.

In this paper, we shall be working with the well-known Mittag-Leffler function, which was first presented by Mittag-Leffler [1] in relation to a summation of some divergent series. The Mittag-Leffler function appears in the solution of fractional order differential equations and integral equations, particularly in the study of complex systems, random walks, Levy flights, superdiffusive transport, and the fractional generalisation of the kinetic equation. Numerous works, including Haubold , Mathai and Saxena [2] and Hilfer [3] are available for use as references.

The works of Ankit pal, AK Shukla, JC Prajapati, Hari M Srivastava, Arran Fernandez, F Ghanim, Hiba F Al-Janaby, Omar Bazighifan, Anatoly Kilbas, Anna Koroleva, and Sergei Rogosin [4–9], and many others contain numerous integral formulas connected to the function of Mittag-Leffler. Inspired by their efforts, we provided unified integrals involving the Mittag-Leffler function as defined in (1) in this study, and we take into consideration and comment on several intriguing specific instances as a result.

For our purpose, we begin with the Mittag-Leffler function, $\mathsf{E}_\alpha(z)$, and its two-parameter version, $\mathsf{E}_{\alpha,\beta}(z)$, which are introduced in Mittag-Leffler and Wiman[1,10] in 1903 and 1905 respectively.

$$\mathsf{E}_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \quad \text{and} \quad \mathsf{E}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{1}$$

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where

$$z, \alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0.$$

The Mittag-Leffler functions, $\mathrm{E}_\alpha(z)$ and $\mathrm{E}_{\alpha,\beta}(z)$, are considered to be a generalization of well-known mathematical functions such as the exponential, hyperbolic and trigonometric functions. This can be seen from the following equalities:

$$\mathrm{E}_{2,2}(z^2) = \frac{\sinh z}{z}, \mathrm{E}_2(-z^2) = \cos z, \mathrm{E}_2(z^2) = \cosh z, \mathrm{E}_{1,2}(z) = \frac{e^z - 1}{z} \text{ and } \mathrm{E}_1(z) = e^z.$$

The properties, extensions, and applications of Mittag-Leffler functions $\mathrm{E}_\alpha(z)$ and $\mathrm{E}_{\alpha,\beta}(z)$ have been explored and documented in various works such as Giusti [11], Gorenflo [12], Mainardi and Srivastava [12]. These authors present a range of useful properties with additions to the Mittag-Leffler $\mathrm{E}_\alpha(z)$ function in (1) and its generalized forms, including numerical calculations.

An increase in the number of parameters is associated with the next phase of development within the Mittag-Leffler function theory. Accordingly, Prabhakar [13] presented a three-parametric Mittag-Leffler type function,

$$\mathrm{E}^{\rho}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\rho + k)}{\Gamma(\rho)\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \quad z \in \mathbb{C}; \tag{2}$$

$$\tag{3}$$

and continuing with this function, we get

$$\mathrm{E}^{\rho,\kappa}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\rho + \kappa k) z^k}{\Gamma(\rho)\Gamma(k\alpha + \beta) k!}, \quad z \in \mathbb{C}, \tag{4}$$

as mentioned by Shukla and Prajapati [4] in their work, where $\alpha, \beta, \rho \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\rho) > 0$ and $\kappa \in (0,1) \cup \mathbb{N}$. It is evident that the aforementioned functions exhibit the following relationships:

$$\mathrm{E}^{\rho,1}_{\alpha,\beta}(z) = \mathrm{E}^{\rho}_{\alpha,\beta}(z), \ \mathrm{E}^{1}_{\alpha,\beta}(z) = \mathrm{E}_{\alpha,\beta}(z), \ \mathrm{E}_{\alpha,1}(z) = \mathrm{E}_\alpha(z).$$

For all z, if $\kappa < Re(\alpha) + 1$ the function $\mathrm{E}^{\rho,\kappa}_{\alpha,\beta}(z)$ converges, and for $|z| < 1$, if $\kappa = Re(\alpha) + 1$, the function $\mathrm{E}^{\rho,\kappa}_{\alpha,\beta}(z)$ absolutely converges.

The various multi-parameter extensions and generalizations of the Mittag-Leffler function $\mathrm{E}_\alpha(z)$ mentioned in (1) can all be seen as special cases of the general Fox-Wright function ${}_p\Psi_q$ or its modified version ${}_p\Psi^*_q$, which have p numerator parameters $a_1, \ldots, a_p$ and q denominator parameters $b_1, \ldots, b_q :$ $p, q \in \mathbb{N}_0$ such that

$$a_j \in \mathbb{C} : j = 1, ..., p, \text{and } b_j \in \mathbb{C} \backslash \mathbb{Z}^-_0 \ : j = 1, ..., q.$$

Further details can be found in several references, which include work of Erdelyi, Gorenflo, Mainardi, Srivastava, Pal, Khan, Ahmed, Mathai and Haubold [9,14–16].

For Fox-Wright functions, ${}_p\Psi_q$ and ${}_p\Psi^*_q$, we referred to Srivastava work [17] defined as

$$\begin{aligned} {}_p\Psi^*_q[(a_1, A_1), ..., (a_p, A_p); (b_1, B_1), ..., (b_q, B_q); z] &= \sum_{n=0}^{\infty} \frac{(a_1)_{A_1 n} ... (a_p)_{A_p n}}{(b_1)_{B_1 n} ... (b_q)_{B_q n}} \frac{z^n}{n!} \\ &= \frac{\Gamma(b_1) ... \Gamma(b_q)}{\Gamma(a_1) ... \Gamma(a_p)} {}_p\Psi_q[(a_1, A_1), ..., (a_p, A_p); (b_1, B_1), ..., (b_q, B_q); z] \end{aligned} \tag{5}$$

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where,

$$\Re(A_j) > 0 : j = 1, ..., p; \Re(B_j) > 0\, j = 1, ..., q; \Re\left(\sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j\right) \geq -1,$$

and the notation $(\lambda)_\nu$ used includes Pochhammer symbols, which are shifted factorials, defined as for $\lambda, \nu \in \mathbb{C}$,

$$\begin{aligned}(\lambda)_\nu &= \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} \\ &= \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C}\backslash 0) \\ \lambda(\lambda+1)...(\lambda+n-1) & (\nu = n \in \mathbb{C}). \end{cases} \end{aligned} \tag{6}$$

The convergence of series (5) is subject to the the value of $|z|$ which must be suitably bounded.

$$|z| < \Delta = \left(\prod_{j=1}^{p} A_j^{-A_j}\right)\left(\prod_{j=1}^{q} B_j^{B_j}\right).$$

The special case of the general Fox-Wright function ${}_p\Psi_q$ is the generalized hypergeometric function ${}_pF_q$ with p numerator parameters $a_1, ..., a_p$ and q denominator parameters $b_1, ..., b_q$,where $p, q \in \mathbb{N}_0$, which has been extensively studied and has potential applications. This only happens when

$$A_j = 1\ : j = 1, ..., p \text{ and } B_j = 1\ : j = 1, ..., q.$$

This will provide us with

$$\begin{aligned} {}_pF_q\left[a_1, ..., a_p; b_1, ..., b_q; z\right] &= \sum_{n=0}^{\infty} \frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_p)_n} \frac{z^n}{n!} \\ &= {}_p\Psi_q^*\left[(a_1, 1), ..., (a_p, 1); (b_1, 1), ..., (b_q, 1); z\right] \\ &= \frac{\Gamma(b_1)...\Gamma(b_q)}{\Gamma(a_1)...\Gamma(a_p)}\, {}_p\Psi_q\left[(a_1, 1), ..., (a_p, 1); (b_1, 1), ..., (b_q, 1); z\right]. \end{aligned} \tag{7}$$

To set up our results, we will use a result found in Prudnikov, 2018 [18], which we would like to highlight.

$$\int_r^s \frac{(x-r)^{\gamma-1}(s-x)^{\beta-1}}{[(s-r)+k_1(x-r)+k_2(s-x)]^{\beta+\gamma}}\, dx = \frac{\Gamma(\gamma)\Gamma(\beta)}{\Gamma(\gamma+\beta)} \frac{(1+k_1)^{-\gamma}(1+k_2)^{-\beta}}{(s-r)}, \tag{8}$$

for $r \neq s, \Re(\gamma) > 0, \Re(\beta) > 0$ and we choose k_1, k_2 such that $(1+k_1), (1+k_2)$, $[(s-r)+k_1(x-r)+k_2(s-x)]$ are not zero where $r \leq x \leq s$.

Furthermore, we will look at the Lavoie-Trottier integral formula [19], which is as follows:

$$\int_0^1 y^{\mu-1}(1-y)^{2\nu-1}\left(1-\frac{y}{3}\right)^{2\mu-1}\left(1-\frac{y}{4}\right)^{\nu-1} dy = \left(\frac{2}{3}\right)^{\mu-1} \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}, \tag{9}$$

for $\Re(\mu), \Re(\nu) > 0$.

By using Generalized hypergeometric function representation of $E_{\alpha,\beta}^{\gamma,q}(z)$ as refered in Shukla and Prajapati [4]

$$E^{\beta,\nu}_{\alpha,\gamma}(z) = \frac{1}{\Gamma(\gamma)}{}_{\nu}F_{\alpha}\left[\Delta(\nu;\beta);\Delta(\alpha;\gamma);\frac{\nu^{\nu}z}{\alpha^{\alpha}}\right], \tag{10}$$

where $\Delta(\nu;\beta)$ is a ν-tuple $\frac{\beta}{\nu}, \frac{\beta+1}{\nu}, \ldots, \frac{\beta+\nu-1}{\nu}$; $\Delta(\alpha;\gamma)$ is a α-tuple $\frac{\gamma}{k}, \frac{\gamma+1}{\alpha}, \ldots, \frac{\gamma+\alpha-1}{\alpha}$. Refer to [4] for more information on the convergence criterion.

2. Results

Theorem 1. *If $s \neq r$ and $\beta, \nu, \alpha, \gamma \in \mathbb{C}$ with $\Re(\beta) > 0, \Re(\gamma) > 0, \Re(\nu) > 0, \Re(\alpha) > 0$; there exist k_1, k_2 such that $1+k_1, 1+k_2, (s-r)+k_1(x-r)+k_2(s-x) \neq 0$ for $r \leq x \leq s$ then the integral given below is valid:*

$$\begin{aligned}\int_r^s \frac{(x-r)^{\gamma-1}(s-x)^{\beta-1}}{[(s-r)+k_1(x-r)+k_2(s-x)]^{\beta+\gamma}} E^{\beta,\nu}_{\alpha,\gamma}\left(\frac{(x-r)^{\alpha}}{[(s-r)+k_1(x-r)+k_2(s-x)]^{\alpha}}\right) dx \\ = \Gamma(\beta)\frac{(1+k_1)^{-\gamma}(1+k_2)^{-\beta}}{(s-r)} E^{\beta,\nu}_{\alpha,\gamma+\beta}\big((1+k_1)^{-\alpha}\big).\end{aligned} \tag{11}$$

Proof. Let $\mathcal{I}$ represent the left side of the assertion (11). The integrand of (11) is applied with definition in (4), and the result is obtained by swapping the order of the integral sign and summation, which is confirmed by the involved series' uniform convergence under the specified condition, we get the following

$$\mathcal{I} = \sum_{n=0}^{\infty}\frac{\Gamma(\beta+\nu n)}{\Gamma(\beta)\Gamma(\alpha n+\gamma)n!}\int_r^s \frac{(x-r)^{\gamma+\alpha n-1}(s-x)^{\beta-1}}{[(s-r)+k_1(x-r)+k_2(s-x)]^{\beta+\alpha n+\gamma}}\,dx. \tag{12}$$

In view of the condition given in (8) and since $\Re(\beta) > 0, \Re(\gamma) > 0, \Re(\nu) > 0, \Re(\alpha) > 0$ and $1+k_1, 1+k_2, (s-r)+k_1(x-r)+k_2(s-x) \neq 0$ as given in the conditions of Theorem 1. Thus, we can apply the integral formula in (8) to (12) and obtain the following expression:

$$\mathcal{I} = \sum_{n=0}^{\infty}\frac{\Gamma(\beta+\nu n)}{\Gamma(\beta)\Gamma(\alpha n+\gamma)n!}\frac{\Gamma(\gamma+\alpha n)\Gamma(\beta)}{\Gamma(\gamma+\beta+\alpha n)}\frac{(1+k_1)^{-(\gamma+\alpha n)}(1+k_2)^{-\beta}}{(s-r)}.$$

Now we have

$$\mathcal{I} = \Gamma(\beta)\sum_{n=0}^{\infty}\frac{\Gamma(\beta+\nu n)}{\Gamma(\beta)\Gamma(\gamma+\beta+\alpha n)}\frac{1}{n!}\frac{(1+k_1)^{(\gamma+\alpha n)}(1+k_2)^{-\beta}}{(s-r)}.$$

Therefore we find that

$$\mathcal{I} = \Gamma(\beta)\frac{(1+k_1)^{-\gamma}(1+k_2)^{-\beta}}{(s-r)}E^{\beta,\nu}_{\alpha,\gamma+\beta}(1+k_1)^{-\alpha}. \tag{13}$$

This brings the proof to a close.

We can interpret the Theorem 1 as a specialisation of the generalized hypergeometric function in several variables as defined in (7) and (5), which will lead to a few similar results as stated below.

□

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Corollary 1. *The same parameters as $\alpha = \nu = 1$ in Theorem 1, the integral given below will be valid:*

$$\int_r^s \frac{(x-r)^{\gamma-1}(s-x)^{\beta-1}}{[(s-r)+k_1(x-r)+k_2(s-x)]^{\beta+\gamma}}\, {}_1F_1\left(\beta;\gamma;\frac{(x-r)}{[(s-r)+k_1(x-r)+k_2(s-x)]}\right)dx$$
$$= \frac{\Gamma(\gamma)\Gamma(\beta)}{\Gamma(\gamma+\beta)}\frac{(1+k_1)^{-\gamma}(1+k_2)^{-\beta}}{(s-r)}\, {}_1F_1\left(\beta;\gamma+\beta;(1+k_1)^{-1}\right). \tag{14}$$

We can simply prove the above relation using (10) and the details stated above.

Corollary 2. *The identical conditions as mentioned in Theorem 1, with more general form of corollary 1, the integral given below will be valid:*

$$\int_r^s \frac{(x-r)^{\gamma-1}(s-x)^{\beta-1}}{[(s-r)+k_1(x-r)+k_2(s-x)]^{\beta+\gamma}}$$
$${}_1F_\alpha\left(\beta;\frac{\gamma}{\alpha},\frac{\gamma+1}{\alpha},\ldots,\frac{\gamma+\alpha-1}{\alpha};\frac{(x-r)^\alpha}{\alpha^\alpha[(s-r)+k_1(x-r)+k_2(s-x)]^\alpha}\right)dx$$
$$= \frac{\Gamma(\gamma)\Gamma(\beta)}{\Gamma(\gamma+\beta)}\frac{(1+k_1)^{-\gamma}(1+k_2)^{-\beta}}{(s-r)}\, {}_1F_\alpha\left(\beta;\gamma+\beta;(1+k_1)^{-\alpha}\right). \tag{15}$$

Corollary 3. *The same parameters as $\beta = \nu = 1$ in Theorem 1, the integral given below will be valid:*

$$\int_r^s \frac{(x-r)^{\gamma-1}(s-x)^{\beta-1}}{[(s-r)+k_1(x-r)+k_2(s-x)]^{\gamma}}\, {}_1\Psi_1\left[\begin{array}{c|c} (1,1) & \frac{(x-r)^\alpha}{[(s-r)+k_1(x-r)+k_2(s-x)]^\alpha} \\ (\gamma,\alpha) & \end{array}\right]$$
$$= \Gamma(\beta)\frac{(1+k_1)^{-\gamma}(1+k_2)^{-\beta}}{(s-r)}\, {}_1\Psi_1\left[\begin{array}{c|c} (1,1) & (1+k_1)^{-\alpha} \\ (\gamma,\alpha) & \end{array}\right].$$

Theorem 2. *If $s \neq r$ and $\beta,\nu,\alpha,\gamma \in \mathbb{C}$ with $\Re(\beta) > 0, \Re(\gamma) > 0, \Re(\nu) > 0, \Re(\alpha) > 0$, also there exist k_1, k_2 such that $1+k_1, 1+k_2, (s-r)+k_1(x-r)+k_2(s-x) \neq 0$ for $r \leq x \leq s$ then the integral given below will be valid:*

$$\int_r^s \frac{(x-r)^{\beta-1}(s-x)^{\gamma-1}}{[(s-r)+k_1(x-r)+k_2(s-x)]^{\beta}+\gamma} E^{\beta,\nu}_{\alpha,\gamma}\left(\frac{(s-x)^\alpha}{[(s-r)+k_1(x-r)+k_2(s-x)]^\alpha}\right)dx$$
$$= \Gamma(\beta)\frac{(1+k_1)^{-\beta}(1+k_2)^{-\gamma}}{(s-r)} E^{\beta,\nu}_{\alpha,\gamma+\beta}\left((1+k_2)^{-\alpha}\right). \tag{16}$$

Proof. By applying (4) to the integrand of the aforementioned (16) and then swapping the order of the integral sign and summation yields the following result, which is verifiable by the involved series' uniform convergence under the specified condition, we get

$$\sum_{n=0}^{\infty}\frac{\Gamma(\beta+\nu n)}{\Gamma(\beta)\Gamma(\alpha n+\gamma)n!}\int_r^s \frac{(x-r)^{\beta-1}(s-x)^{\alpha n+\gamma-1}}{[(s-r)+k_1(x-r)+k_2(s-x)]^{\beta+\alpha n+\gamma}}dx$$
$$= \sum_{n=0}^{\infty}\frac{\Gamma(\beta+\nu n)}{\Gamma(\beta)\Gamma(\alpha n+\gamma)n!}\frac{\Gamma(\gamma+\alpha n)\Gamma(\beta)}{\Gamma(\gamma+\beta+\alpha n)}\frac{(1+k_1)^{-\beta}(1+k_2)^{-\alpha n-\gamma}}{(s-r)}.$$

Using integral representation as in (4) will lead to the given result. □

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Corollary 4. *The identical conditions as mentioned in Theorem 2 with $\alpha = \nu = 1$, the integral given below will be valid:*

$$\int_r^s \frac{(x-r)^{\beta-1}(s-x)^{\gamma-1}}{[(s-r)+k_1(x-r)+k_2(s-x)]^{\beta+\gamma}}\, {}_1F_1\left(\beta;\gamma;\frac{(s-x)}{[(s-r)+k_1(x-r)+k_2(s-x)]}\right)dx$$
$$= \frac{\Gamma(\gamma)\Gamma(\beta)}{\Gamma(\gamma+\beta)}\frac{(1+k_1)^{-\beta}(1+k_2)^{-\gamma}}{(s-r)}\, {}_1F_1\left(\beta;\gamma+\beta;(1+k_1)^{-1}\right). \tag{17}$$

Corollary 5. *The identical conditions as mentioned in Theorem 2 with more general form of Corollary 4, the integral given below will be valid:*

$$\int_r^s \frac{(x-r)^{\beta-1}(s-x)^{\gamma-1}}{[(s-r)+k_1(x-r)+k_2(s-x)]^{\beta+\gamma}}$$
$${}_1F_\alpha\left(\beta;\frac{\gamma}{\alpha},\frac{\gamma+1}{\alpha},\ldots,\frac{\gamma+\alpha-1}{\alpha};\left(\frac{(s-x)^alpha}{\alpha^\alpha[(s-r)+k_1(x-r)+k_2(s-x)]}\right)^\alpha\right)dx$$
$$= \frac{\Gamma(\gamma)\Gamma(\beta)}{\Gamma(\gamma+\beta)}\frac{(1+k_1)^{-\beta}(1+k_2)^{-\gamma}}{(s-r)}\, {}_1F_\alpha\left(\beta;\frac{\gamma}{\alpha},\frac{\gamma+1}{\alpha},\ldots,\frac{\gamma+\alpha-1}{\alpha};(\alpha(1+k_1))^{-\alpha}\right).$$

Corollary 6. *The identical conditions as mentioned in Theorem 2, the integral given below will be valid:*

$$\int_r^s \frac{(x-r)^{\beta-1}(s-x)^{\gamma-1}}{[(s-r)+k_1(x-r)+k_2(s-x)]^{\gamma+\beta}}\, {}_1\Psi_1\left[\begin{array}{c|c} (\beta+\nu) & \frac{(s-x)^\alpha}{[(s-r)+k_1(x-r)+k_2(s-x)]^\alpha} \\ (\gamma,\alpha) & \end{array}\right]$$
$$= \Gamma(\beta)\frac{(1+k_1)^{-\beta}(1+k_2)^{-\gamma}}{(s-r)}\, {}_1\Psi_1\left[\begin{array}{c|c} (\beta+\nu) & (1+k_1)^{-\alpha} \\ (\gamma+\beta,\alpha) & \end{array}\right].$$

Theorem 3. *If $s \neq r$ and $\beta,\nu,\alpha,\gamma \in \mathbb{C}$ with $\Re(\beta) > 0, \Re(\gamma) > 0, \Re(\nu) > 0, \Re(\alpha) > 0$, also there exist k_1, k_2 such that $1+k_1, 1+k_2, (s-r)+k_1(x-r)+k_2(s-x) \neq 0$ for $r \leq x \leq s$ then the following integral holds true:*

$$\int_r^s \frac{(x-r)^{\beta-1}(s-x)^{\gamma-1}}{[(s-r)+k_1(x-r)+k_2(s-x)]^{\gamma+\beta}}\, E^{\beta,\nu}_{\alpha,\gamma}\left(\frac{(x-r)^\nu(s-x)^\nu}{[(s-r)+k_1(x-r)+k_2(s-x)]^{2\nu}}\right)$$
$$= \frac{(1+k_1)^{-\beta}(1+k_2)^{\alpha}}{(s-r)}\frac{1}{\Gamma(\beta)}\, {}_3\Psi_2\left[\begin{array}{cc} (\beta,\nu),(\beta,\nu),(\lambda,\nu); & \\ & (1+k_1)^{-\nu}(1+k_2)^{-\nu} \\ (\gamma,\alpha),(\lambda+\beta,2\nu); & \end{array}\right]. \tag{18}$$

Proof. Let $\mathcal{I}$ represent the left side of the statement (18). We will apply (4) to the integrand, and the uniform convergence of the involved series under the specified conditions is used to verify that the order of the integral sign and summation have been changed.

$$\mathcal{I} = \sum_{n=0}^{\infty}\frac{\Gamma(\beta+\nu n)}{\Gamma(\beta)\Gamma(\alpha n+\gamma)}\frac{1}{n!}\int_r^s \frac{(x-r)^{\beta+\nu n-1}(s-x)^{\lambda+\nu n-1}}{[(s-r)+k_1(x-r)+k_2(s-x)]^{\lambda+\beta+2\nu n}}\,dx.$$

In view of conditions given in Theorem 2, we can apply the integral formula of (8) to the above integral and obtain the following expression:

$$\mathcal{I} = \sum_{n=0}^{\infty}\frac{\Gamma(\beta+\nu n)}{\Gamma(\beta)\Gamma(\alpha n+\gamma)}\frac{1}{n!}\frac{\Gamma(\beta+\nu n)\Gamma(\lambda+\nu n)}{\Gamma(\beta+\lambda+2\nu n)}\frac{(1+k_1)^{-(\beta+\nu n)}(1+k_2)^{-(\lambda+\nu n)}}{(s-r)}$$
$$= \frac{(1+k_1)^{-\beta}(1+k_2)^{-\lambda}}{\Gamma(\beta)(s-r)}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\Gamma(\beta+\nu n)\Gamma(\beta+\nu n)\Gamma(\lambda+\nu n)}{\Gamma(\alpha n+\gamma)\Gamma(\beta+\lambda+2\nu n)}[(1+k_1)(1+k_2)]^{-\nu n}.$$

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By applying definition as in (5) we get the following expression:

$$\frac{(1+k_1)^{-\beta}(1+k_2)^{-\lambda}}{\Gamma(\beta)(s-r)}\,{}_3\Psi_2\left[\begin{array}{c}(\beta,\nu),(\beta,\nu),(\lambda,\nu);\\ \\ (\gamma,\alpha),(\lambda+\beta,2\nu);\end{array}\quad(1+k_1)^{-\nu}(1+k_2)^{-\nu}\right]. \tag{19}$$

Finally we interpret the series in (19) as a special case of the Fox-Wright function as defined in (5). We are thus led to the assertion of Theorem 3. □

Corollary 7. *Let the condition of Theorem 3 be satisfied. Then the following integral formula holds true.*

$$\begin{aligned}&\int_r^s \frac{(x-r)^{\beta-1}(s-x)^{\gamma-1}}{[(s-r)+k_1(x-r)+k_2(s-x)]^{\gamma+\beta}}\,{}_1F_1\left(\beta;\gamma;\frac{(x-r)(s-x)}{[(s-r)+k_1(x-r)+k_2(s-x)]^2}\right)\\ &\quad=\frac{(1+k_1)^{-\beta}(1+k_2)^{\alpha}}{(s-r)}\,\mathbf{B}(\beta,\gamma)\,{}_2\Psi_1\left[\begin{array}{c}(\beta,1),(\beta,1),(\lambda,\nu);\\ \\ (\gamma+\beta,2\nu);\end{array}\quad(1+k_1)^{-1}(1+k_2)^{-1}\right].\end{aligned} \tag{20}$$

Theorem 4. *If $\delta,q,\alpha,\beta\in\mathbb{C}$ such that $\Re(\delta)>0,\Re(q)>0,\Re(\alpha)>0,\Re(\beta)>0,\Re(\delta+\beta)w>0$ and $u>0$, the Mittag-Leffler hypergeometric function can be represented by the following integral.*

$$\begin{aligned}&\int_0^1 u^{\beta-1}(1-u)^{2\delta-1}\left(1-\frac{u}{3}\right)^{2\beta-1}\left(1-\frac{u}{4}\right)^{\delta-1}E_{\alpha,\beta}^{\delta,q}\left[w\left(u\left(1-\frac{u}{3}\right)^2\right)^{\nu}\right]\\ &\quad=\left(\frac{2}{3}\right)^{2\beta}\Gamma(\delta)\,{}_1\Psi_1\left[\begin{array}{c}(\beta,\nu);\\ \\ (\delta+\beta,\nu);\end{array}\quad 1\right]E_{\alpha,\beta}^{\delta,q}\left(w\left(\frac{2}{3}\right)^{\nu}\right).\end{aligned} \tag{21}$$

Proof. By using Mittag-Leffler function as in (4) and representing it with $\mathfrak{A}$. After some simplification and interchanging the order of integration and summation which is possible due to uniform convergence of the involved series under the given conditions, we get

$$\begin{aligned}\mathfrak{A}&=\int_0^1 u^{\beta-1}(1-u)^{2\delta-1}\left(1-\frac{u}{3}\right)^{2\beta-1}\left(1-\frac{u}{4}\right)^{\delta-1}\sum_{n=0}^{\infty}\frac{\Gamma(\delta+qn)}{\Gamma(\delta)\Gamma(n\alpha+\beta)}\frac{w^n u^{\nu n}}{n!}\left(1-\frac{u}{3}\right)^{2\nu n}\\ &=\sum_{n=0}^{\infty}\frac{\Gamma(\delta+qn)}{\Gamma(\delta)\Gamma(n\alpha+\beta)}\frac{w^n}{n!}\int_0^1 u^{\beta+\nu n-1}(1-u)^{2\delta-1}\left(1-\frac{u}{3}\right)^{2(\beta+\nu n)-1}\left(1-\frac{u}{4}\right)^{\delta-1}du.\end{aligned}$$

By using (9) integral representation, we get

$$\begin{aligned}\mathfrak{A}&=\sum_{n=0}^{\infty}\frac{\Gamma(\delta+qn)}{\Gamma(\delta)\Gamma(n\alpha+\beta)}\frac{w^n}{n!}\left(\frac{2}{3}\right)^{2(\beta+\nu n)}\frac{\Gamma(\beta+\nu n)\Gamma(\delta)}{\Gamma(\delta+\beta+\nu n)}\\ &=E_{\alpha,\beta}^{\delta,q}\left(w\left(\frac{2}{3}\right)^{\nu}\right)\left(\frac{2}{3}\right)^{2\beta}\sum_{n=0}^{\infty}\frac{\Gamma(\beta+\nu n)\Gamma(\delta)}{\Gamma(\delta+\beta+\nu n)}\\ &=\left(\frac{2}{3}\right)^{2\beta}\Gamma(\delta)\,{}_1\Psi_1\left[\begin{array}{c}(\beta,\nu);\\ \\ (\delta+\beta,\nu);\end{array}\quad 1\right]E_{\alpha,\beta}^{\delta,q}\left(w\left(\frac{2}{3}\right)^{\nu}\right).\end{aligned}$$

□

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Theorem 5. *When $\delta, q, \alpha, \beta \in \mathbb{C}$ are such that $\Re(\delta) > 0, \Re(q) > 0, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\delta+\beta)w > 0$, and $u > 0$, the Mittag-Leffler hypergeometric function may be expressed as the following integral.*

$$\int_0^1 u^{\delta-1}(1-u)^{2\beta-1}\left(1-\frac{u}{3}\right)^{2\delta-1}\left(1-\frac{u}{4}\right)^{\beta-1} E_{\alpha,\beta}^{\delta,q}\left[w\left(u^2(1-u)^{2\beta-1}\right)^{\nu}\right]$$
$$= \left(\frac{2}{3}\right)^{2\beta}\Gamma(\delta)\,{}_1\Psi_1\left[\begin{matrix}(\beta,\nu); \\ \\ (\delta+\beta,\nu);\end{matrix}\quad 1\right] E_{\alpha,\beta}^{\delta,q}\left(w\left(\frac{2}{3}\right)^{\nu}\right). \tag{22}$$

Proof. By using Mittag-Leffler function as in (4) and representing it with $\mathfrak{B}$. After some simplification and interchanging the order of integration and summation which is possible due to uniform convergence of the involved series under the given conditions, we get

$$\mathfrak{B} = \sum_{n=0}^{\infty}\frac{\Gamma(\delta+qn)}{\Gamma(\delta)\Gamma(\alpha n+\beta)}\frac{w^n}{n!}\int_0^1 u^{\delta-1}(1-u)^{2(\beta+\nu n)-1}\left(1-\frac{u}{3}\right)^{2\delta-1}\left(1-\frac{u}{4}\right)^{\beta+\nu n-1} du$$
$$= \sum_{n=0}^{\infty}\frac{\Gamma(\delta+qn)}{\Gamma(\delta)\Gamma(\alpha n+\beta)}\frac{w^n}{n!}\left(\frac{2}{3}\right)^{2\delta}\frac{\Gamma(\delta)\Gamma(\beta+\nu n)}{\Gamma(\delta+\beta+\nu n)}.$$

By applying definition as in (5), we will get the result. □

Corollary 8. *When $\delta, q, \alpha, \beta \in \mathbb{C}$ are such that $\Re(\delta) > 0, \Re(q) > 0, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\delta+\beta)w > 0$, and $u > 0$, the Mittag-Leffler hypergeometric function may be expressed as the following integral.*

$$\int_0^1 u^{\lambda-1}(1-u)^{2(\delta+k)-1}\left(1-\frac{u}{3}\right)^{2\lambda-1}\left(1-\frac{u}{4}\right)^{\delta+k-1} E_{\alpha,\beta}^{\gamma,q}\left(wu(1-u)^2\right)$$
$$= \left(\frac{2}{3}\right)^{2\lambda}\Gamma(\delta+k)\,E_{\alpha,\beta}^{\gamma,q}\left(w\left(\frac{2}{3}\right)^2\right){}_1\Psi_1\left[\begin{matrix}(\lambda,1); \\ \\ (\delta+\lambda+k,1);\end{matrix}\quad 1\right]$$
$$= \left(\frac{2}{3}\right)^{2\lambda}\frac{\Gamma(\delta+k)}{\Gamma(\gamma)}\,{}_2\Psi_2\left[\begin{matrix}(\gamma,q),(\lambda,1); \\ \\ (\beta,\alpha)(\delta+\lambda+k,1);\end{matrix}\quad \left(\tfrac{2}{3}\right)^2 w\right].$$

Corollary 9. *When $\delta, q, \alpha, \beta \in \mathbb{C}$ are such that $\Re(\delta) > 0, \Re(q) > 0, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\delta+\beta)w > 0$, and $u > 0$, the Mittag-Leffler hypergeometric function may be expressed as the following integral.*

$$\int_0^1 u^{\delta+k-1}(1-u)^{2\lambda-1}\left(1-\frac{u}{3}\right)^{2(\delta+k)-1}\left(1-\frac{u}{4}\right)^{\lambda-1} E_{\alpha,\beta}^{\gamma,q}\left(w(1-u)^2\left(1-\frac{u}{4}\right)\right) du$$
$$= \left(\frac{2}{3}\right)^{2\lambda}\Gamma(\delta+k)\,E_{\alpha,\beta}^{\gamma,q}\left(w\left(\frac{2}{3}\right)^2\right){}_1\Psi_1\left[\begin{matrix}(\lambda,1); \\ \\ (\delta+\lambda+k,1);\end{matrix}\quad 1\right]$$
$$= \left(\frac{2}{3}\right)^{2\lambda}\frac{\Gamma(\delta+k)}{\Gamma(\gamma)}\,{}_2\Psi_2\left[\begin{matrix}(\gamma,q),(\lambda,1); \\ \\ (\beta,\alpha)(\delta+\lambda+k,1);\end{matrix}\quad \left(\tfrac{2}{3}\right)^2 w\right].$$

3. Conclusion

Here, we have examined several novel integral formulations that use the Mittag-Leffler function of four parameters. The representation of these formulations is given by suitable special functions. All the obtained results can be expressed in the form of functions like Fox-Wright functions, generalized hypergeometric functions, and others. The results presented in this study can be readily represented

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in terms of a wide variety of special functions by making the necessary parametric replacements. Numerous areas of mathematical physics, probability, and fractional calculus employ the Mittag-Leffler confluent hypergeometric function. The results reported in this paper may have applications in both applied analysis and mathematical physics theory.

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References

1. Mittag-Leffler, G.M. Sur la nouvelle fonction Eα (x). *CR Acad. Sci. Paris* **1903**, *137*, 554–558.
2. Haubold, H.J.; Mathai, A.M.; Saxena, R.K.; others. Mittag-Leffler functions and their applications. *Journal of applied mathematics* **2011**, *2011*.
3. Hilfer, R. *Applications of fractional calculus in physics*; World scientific, 2000.
4. Shukla, A.; Prajapati, J. On a generalization of Mittag-Leffler function and its properties. *Journal of mathematical analysis and applications* **2007**, *336*, 797–811.
5. Srivastava, H.M.; Fernandez, A.; Baleanu, D. Some new fractional-calculus connections between Mittag–Leffler functions. *Mathematics* **2019**, *7*, 485.
6. Ghanim, F.; Al-Janaby, H.F. An analytical study on Mittag-Leffler–confluent hypergeometric functions with fractional integral operator. *Mathematical Methods in the Applied Sciences* **2021**, *44*, 3605–3614.
7. Ghanim, F.; Al-Janaby, H.F.; Bazighifan, O. Some new extensions on fractional differential and integral properties for Mittag-Leffler confluent hypergeometric function. *Fractal and Fractional* **2021**, *5*, 143.
8. Kilbas, A.; Koroleva, A.; Rogosin, S. Multi-parametric Mittag-Leffler functions and their extension. *Fractional Calculus and Applied Analysis* **2013**, *16*, 378–404.
9. Pal, A. Some finite integrals involving Mittag-Leffler confluent hypergeometric function. *Analysis* **2023**.
10. Wiman, A. Über die Nullstellen der FunktionenE a (x). *Acta Mathematica* **1905**, *29*, 217–234.
11. Giusti, A.; Colombaro, I.; Garra, R.; Garrappa, R.; Polito, F.; Popolizio, M.; Mainardi, F. A practical guide to Prabhakar fractional calculus. *Fractional Calculus and Applied Analysis* **2020**, *23*, 9–54.
12. Gorenflo, R.; Mainardi, F.; Srivastava, H.M. Special functions in fractional relaxation-oscillation and fractional diffusion-wave phenomena. Proceedings of the Eighth International Colloquium on Differential Equations, Plovdiv, Bulgaria, 18–23 August, 1997. De Gruyter, 1998, pp. 195–202.
13. Prabhakar, T.R.; others. A singular integral equation with a generalized Mittag-Leffler function in the kernel. *Yokohama math. J* **1971**, *19*, 7–15.
14. Erd*é*lyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F. Higher transcendental functions. vol. II. Melbourne, FL, 1981.
15. Khan, M.A.; Ahmed, S. On some properties of the generalized Mittag-Leffler function. *SpringerPlus* **2013**, *2*, 1–9.
16. Mathai, A.M.; Haubold, H.J. *Special functions for applied scientists*; Vol. 4, Springer, 2008.
17. Srivastava, H.M. A survey of some recent developments on higher transcendental functions of analytic number theory and applied mathematics. *Symmetry* **2021**, *13*, 2294.
18. Prudnikov, A. *Integrals and series*; Routledge, 2018.
19. Lavoie, J.; Trottier, G. On the sum of certain Appell's series. *Ganita* **1969**, *20*, 31–32.