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Leonid Shaikhet *

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About Unsolved Problems in Stabilization of Two Coupled Controlled Inverted Pendulums under Stochastic Perturbations

Leonid Shaikhet

Department of Mathematics, Ariel University, Ariel 40700, Israel; leonid.shaikhet@usa.net

Abstract

To readers attention two known theorems on the stabilization of a controlled inverted pendulum under stochastic perturbations in the form of a combination of white noise and Poisson's jumps are presented. As unsolved problems, a generalization of these theorems is proposed for a mathematical model, described two coupled controlled inverted pendulums.

Keywords: Inverted pendulums; zero and nonzero equilibria; stochastic perturbations; white noise; Poisson's jumps; stability in probability

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1. Introduction

The proposed paper continues a series of papers devoted to unsolved problems in the theory of stability and optimal control for stochastic systems (see [1–5]). Here the problem of the controlled inverted pendulum is considered, which appears many years ago [6] and has a long history (see [7] and the references therein). The mathematical model of the controlled inverted pendulum has the form of nonlinear differential equation of the second order

$$\ddot{x}(t) - a\sin x(t) = u(t), \qquad a > 0, \qquad t \geq 0, \tag{1}$$

where $x(t)$ measures the angle between the rod and the upward vertical (Figure 1).

The classical way of stabilization for the equation (1) uses [6] the control $u(t)$, which is a linear combination of the state and velocity of the pendulum, i.e.,

$$u(t) = -b_1 x(t) - b_2\dot{x}(t), \qquad b_1 > a, \qquad b_2 > 0.$$

But this type of control, which represents instantaneous feedback, is quite difficult to realize because usually it is necessary to have some finite time to make measurements of the coordinates and velocities, to treat the results of the measurements and to implement them in the control action.

Unlike of the classical way of stabilization another way of stabilization is considered in [8]. It is supposed that only the trajectory of the pendulum is observed and the control $u(t)$ does not depend on the velocity, but depends on the all previous values of the trajectory $x(s)$, $s \leq t$, and is given in the form

$$u(t) = \int_0^\infty dK(\tau)x(t-\tau), \tag{2}$$

where the kernel $K(\tau)$ is a continuous from the right function of bounded variation on $[0,\infty)$ and the integral is understood in the Stieltjes sense. It means, in particular, that both distributed and discrete delays can be used depending on the concrete choice of the kernel $K(\tau)$.

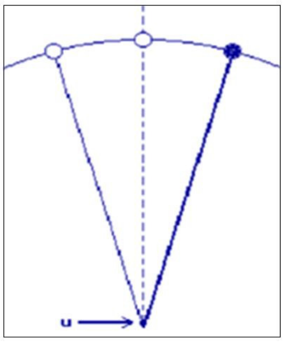

Figure 1. Controlled inverted pendulum

2. Zero and Nonzero Equilibria

It is clear that (1), (2) define the delay differential equation

$$\ddot{x}(t) - a\sin x(t) = \int_0^\infty dK(\tau)x(t-\tau), \qquad a > 0, \qquad t \geq 0, \tag{3}$$

which has the zero solution. However, this equation has not the zero solution only. Put

$$k_i = \int_0^\infty \tau^i dK(\tau), \quad i = 0,1, \qquad k_2 = \int_0^\infty \tau^2 |dK(\tau)|, \tag{4}$$

and note that the nonzero equilibrium $x(t) = \hat{x}$ of the equation (3) is a root of the equation

$$a\sin\hat{x} + k_0\hat{x} = 0 \tag{5}$$

or $S(\hat{x}) = 0$, where

$$S(x) = \frac{\sin x}{x} + \frac{k_0}{a}.$$

Remark 1. *As it is noted in [7] for all* $x \neq 0$

$$-\alpha \leq \frac{\sin x}{x} < 1,$$

where $0.217233 < \alpha < 0.217234$. *Therefore, if*

$$-\alpha \leq -\frac{k_0}{a} < 1$$

or

$$0 < a + k_0 \leq (1+\alpha)a$$

then there exists at least one nonzero root of the equation (5).

3. Two Types of Stochastic Perturbations

In addition, it is supposed that the pendulum is under influence of stochastic perturbations, so, the considered stabilization problem is a problem of the theory of stochastic functional differential equations [8–10].

Let $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ be a complete probability space, $\{\mathfrak{F}_t, t \geq 0\}$ be a nondecreasing family of sub-σ-algebras of $\mathfrak{F}$, i.e., $\mathfrak{F}_s \subset \mathfrak{F}_t$ for $s < t$, $\mathbf{E}$ be the mathematical expectation with respect to the probability $\mathbf{P}$. Let $w(t)$ and $\nu(t)$ be respectively $\mathfrak{F}_t$-measurable the Wiener and the Poisson processes, $\mathbf{E}\nu(t) = \lambda t$, $\lambda > 0$, $\tilde{\nu}(t) = \nu(t) - \lambda t$ [9,10],

$$\xi(t) = \sigma w(t) + \gamma\tilde{\nu}(t). \tag{6}$$

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Supposing for the beginning that the parameter a in (1) is influenced by additive stochastic perturbations, i.e., $a \to a + \dot{\xi}(t)$, where $\xi(t)$ is defined in (6), and using the control (2), rewrite the equation (1) as

$$\ddot{x}(t) - (a + \dot{\xi}(t)) \sin x(t) = \int_0^\infty dK(\tau)x(t-\tau), \qquad a > 0, \qquad t \geq 0, \tag{7}$$

or, using $x_1(t) = x(t)$, $x_2 = \dot{x}(t)$ and (6), in the form of the system of stochastic differential equations [9,10] with the zero solution

$$\begin{aligned} dx_1(t) &= x_2(t)dt, \\ dx_2(t) &= \left(a \sin x_1(t) + \int_0^\infty dK(\tau)x_1(t-\tau) \right) dt \\ &\quad + \sin x_1(t)(\sigma dw(t) + \gamma d\tilde{v}(t)). \end{aligned} \tag{8}$$

From the other hand let us suppose that the equation (3) is influenced by additive stochastic perturbations of the form $(x_1(t) - \hat{x})\dot{\xi}(t)$, where $\hat{x}$ is a nonzero root of the equation (5) and $\xi(t)$ is defined in (6). Then similarly to (8) we obtain the system of stochastic differential equations with the solution $x_1(t) = \hat{x}$, $x_2(t) = 0$:

$$\begin{aligned} dx_1(t) &= x_2(t)dt, \\ dx_2(t) &= \left(a \sin x_1(t) + \int_0^\infty dK(\tau)x_1(t-\tau) \right) dt \\ &\quad + (x_1(t) - \hat{x})(\sigma dw(t) + \gamma d\tilde{v}(t)). \end{aligned} \tag{9}$$

Using the general method of Lyapunov functionals construction [8], in [7] the following two theorems about stability in probability (see definition in [7,8]) have been proven

Theorem 1. *Let be*

$$\begin{aligned} &\alpha_1 = -(a + k_0) > 0, \qquad k_1 > 0, \qquad k_2 < 2, \\ &\sigma^2 + \lambda\gamma^2 < 2\alpha_1 \left(k_1 - k_2 \sqrt{\frac{\alpha_1}{2(2-k_2)}} \right), \end{aligned} \tag{10}$$

where k_i, $i = 0, 1, 2$, are defined in (4). Then the zero solution of the equation (7) (or the system (8)) is stable in probability.

Theorem 2. *Let be*

$$\begin{aligned} &\alpha_2 = -(a \cos(\hat{x}) + k_0) > 0, \qquad k_1 > 0, \qquad k_2 < 2, \\ &\sigma^2 + \lambda\gamma^2 < 2\alpha_2 \left(k_1 - k_2 \sqrt{\frac{\alpha_2}{2(2-k_2)}} \right), \end{aligned} \tag{11}$$

where k_i, $i = 0, 1, 2$, are defined in (4). Then the nonzero equilibrium $\hat{x}$ of the system (9) is stable in probability.

4. Unsolved Problems

Let us consider the mathematical model of two coupled controlled inverted pendulums in the form of the system of two nonlinear differential equations of the second order:

$$\begin{aligned} &\ddot{x}_1(t) - a_1 \sin x_1(t) - b_1 x_2(t) = u_1(t), \\ &\ddot{x}_2(t) - a_2 \sin x_2(t) - b_2 x_1(t) = u_2(t), \\ &a_1 > 0, \qquad a_2 > 0, \qquad t \geq 0. \end{aligned} \tag{12}$$

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It is assumed that only the trajectories of both pendulums are observed, the controls $u_1(t)$ and $u_2(t)$ do not depend on the velocity, but depend respectively on all previous values of the trajectories $x_1(s)$ and $x_2(s)$, $s \le t$, and are given in the form

$$u_1(t) = \int_0^\infty dK_1(\tau)x_1(t-\tau), \quad u_2(t) = \int_0^\infty dK_2(\tau)x_2(t-\tau). \tag{13}$$

Similarly to (2) here it is supposed that $K_1(\tau)$ and $K_2(\tau)$ are continuous from the right functions of bounded variation on $[0,\infty)$ and the integrals are understood in the Stieltjes sense.

If $b_1 = b_2 = 0$ then the system (12), (13) splits into two independent equations, for each from which the statements of both Theorems 1 and 2 can be formulated.

Putting

$$x_{11}(t) = x_1(t), \quad x_{12}(t) = \dot{x}_1(t), \quad x_{21}(t) = x_2(t), \quad x_{22}(t) = \dot{x}_2(t),$$

we obtain the following two problems.

4.1. The Problem 1

Supposing that the parameters a_i, $i = 1,2$, in (12) are influenced by stochastic perturbations

$$a_i \to a_i + \dot{\xi}_i(t), \quad \xi_i(t) = \sigma_i w_i(t) + \gamma_i \tilde{\nu}_i(t), \quad i = 1,2,$$

where $\tilde{\nu}_i(t) = \nu_i(t) - \lambda_i t$, $w_i(t)$ and $\nu_i(t)$ are mutually independent the Wiener and the Poisson processes, we obtain

$$\begin{aligned}
\dot{x}_{11}(t) &= x_{12}(t), \\
\dot{x}_{12}(t) &= (a_1 + \dot{\xi}_1(t))\sin x_{11}(t) + b_1 x_{21}(t) + \int_0^\infty dK_1(\tau)x_{11}(t-\tau), \\
\dot{x}_{21}(t) &= x_{22}(t), \\
\dot{x}_{22}(t) &= (a_2 + \dot{\xi}_2(t))\sin x_{21}(t) + b_2 x_{11}(t) + \int_0^\infty dK_2(\tau)x_{21}(t-\tau).
\end{aligned}$$

or

$$\begin{aligned}
dx_{11}(t) &= x_{12}(t)dt, \\
dx_{12}(t) &= \left(a_1 \sin x_{11}(t) + b_1 x_{21}(t) + \int_0^\infty dK_1(\tau)x_{11}(t-\tau)\right)dt \\
&\quad + \sin x_{11}(t)(\sigma_1 dw_1(t) + \gamma_1 d\tilde{\nu}_1(t)), \\
dx_{21}(t) &= x_{22}(t)dt, \\
dx_{22}(t) &= \left(a_2 \sin x_{21}(t) + b_2 x_{11}(t) + \int_0^\infty dK_2(\tau)x_{21}(t-\tau)\right)dt \\
&\quad + \sin x_{21}(t)(\sigma_2 dw_2(t) + \gamma_2 d\tilde{\nu}_2(t)).
\end{aligned} \tag{14}$$

Problem 1: Formulate and prove for the system (14) an analogue of the Theorem 1.

4.2. The Problem 2

The nonzero equilibrium $(\hat{x}_1, \hat{x}_2)$ of the system (12), (13) is a solution of the system of algebraic equations

$$\begin{aligned}
a_1 \sin \hat{x}_1 + b_1 \hat{x}_2 + k_{01}\hat{x}_1 &= 0, \\
a_2 \sin \hat{x}_2 + b_2 \hat{x}_1 + k_{02}\hat{x}_2 &= 0,
\end{aligned} \tag{15}$$

where $k_{0i} = \int_0^\infty dK_i(\tau)$, $i = 1, 2$, and is also the equilibrium of the system

$$
\begin{aligned}
dx_{11}(t) &= x_{12}(t)dt, \\
dx_{12}(t) &= \left(a_1 \sin x_{11}(t) + b_1 x_{21}(t) + \int_0^\infty dK_1(\tau) x_{11}(t-\tau) \right) dt + (x_{11}(t) - \hat{x}_1) d\xi_1(t), \\
dx_{21}(t) &= x_{22}(t)dt, \\
dx_{22}(t) &= \left(a_2 \sin x_{21}(t) + b_2 x_{11}(t) + \int_0^\infty dK_2(\tau) x_{21}(t-\tau) \right) dt + (x_{21}(t) - \hat{x}_2) d\xi_2(t).
\end{aligned}
\tag{16}
$$

Problem 2: Formulate and prove for the system (16) an analogue of the Theorem 2.

5. Conclusions

The list of the author's unsolved problems continues to increase. But the author continues to hope that his unsolved problems will be solved someday, and invites interested readers to help him with this.

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