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(m,n)-Closed Submodules of Modules Over Commutative Rings

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Abstract: Let R be a commutative ring and m, n be positive integers. We define a proper submodule N of an R-module M to be (m,n)-closed if for $r \in R$ and $b \in M$, $r^m b \in N$ implies $r^n \in (N :_R M)$ or $b \in N$. This class of submodules lies properly between the classes of prime and primary submodules. Many characterizations, properties and supporting examples concerning this class of submodules are provided. The notion of (m,n)-modules is introduced and characterized. Furhermore, the (m,n)-closed avoidance theorem is proved. Finally, the (m,n)-closed submodules in amalgamated modules are studied.

Keywords: (m,n)-closed ideals; (m,n)-prime ideals; (m,n)-closed submodules; n-absorbing submodules

1. Introduction

Throughout, we assume that all rings are commutative with identity and all modules are unital. Let R be a ring, M an R-module, N a proper submodule of M and m,n,k be positive integers. By $M\text{-}rad(N)$ and $(N :_R M)$, we denote the radical of N (the intersection of all prime submodules containing N) and the ideal of all elements r of R for which $rM \subseteq N$, respectively. Moreover, by $Nil(M)$ we mean the set $\{r \in R : r^k M = 0_M$ for some positive integer $k\}$ of all nilpotent elements in M. An R-module M is called multiplication if every submodule N of M has the form IM for some ideal I of R. Moreover, M is called faithful if it has a zero annihilator in R, that is $ann_R(M) = (0 :_R M) = 0$.

The concepts of prime and primary submodules, which are important subjects of module theory have been widely studied by various authors. Recall that a proper submodule N of an R-module M is a prime (resp. primary) submodule if for $r \in R$ and $b \in M$, $rb \in N$ implies $r \in (N :_R M)$ (resp. $r \in \sqrt{(N :_R M)}$) or $b \in N$. The concepts of prime and primary submodules have been generalized in several ways (see, for example, [2,5,6,10,11,20,24].)

Following [11], N is called an n-absorbing submodule (resp. semi-k-absorbing submodule [23]) if whenever $r_1 r_2 \cdots r_n b \in N$ for $r_1, r_2, \ldots, r_n \in R$ and $b \in M$ (resp. $r^k b \in N$ for $r \in R$ and $b \in M$), then either $r_1 r_2 \cdots r_n \in (N :_R M)$ or there are $n-1$ of r_i's whose product with b is in N. (resp. $r^k \in (N :_R M)$ or $r^{k-1} b \in N$). In particular, as a subclass of 2-absorbing primary submodules defined in [18], the notion of 1-absorbing primary (resp. 1-absorbing prime) submodules are initiated in [24] (resp. in [4]). N is said to be a 1-absorbing primary (resp. 1-absorbing prime) submodule if whenever non-unit elements $r, s \in R$ and $b \in M$ with $rsb \in N$, then either $rs \in (N :_R M)$ or $b \in M\text{-}rad(N)$ (resp. $b \in N$). The n-absorbing $(n \geq 1)$ structures have received a significant amount of attention; see for instance [2,4,5,7,24].

As one of the generalizations of prime submodules, in [20], the concept of semiprime submodules is introduced. N is called a semiprime (resp. an (m,n)-semiprime [19]) submodule if for $r \in R$ and $b \in M$, $r^m b \in N$ implies $rb \in N$ (resp. $r^n b \in N$). In 2017, Anderson and Badawi introduced the class of (m,n)-closed ideals. According to [3], a proper ideal I of R is called an (m,n)-closed ideal if whenever $r^m \in I$ for some $r \in R$, then $r^n \in I$. In a recent work [13], the class of (m,n)-prime ideals which is a subclass of the previous structure is defined and studied. A proper ideal I of R is called (m,n)-prime

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if for $r, s \in R$, $r^m s \in I$ implies either $r^n \in I$ or $s \in I$. Note that this subclass is proper in the class of (m,n)-closed ideals. For example, the ideal $I = 16\mathbb{Z}$ is $(3,2)$-closed in the ring of integers which is not $(3,2)$-prime as $2^3 \cdot 2 \in I$ but $2^2, 2 \notin I$. For more classes of ideals related to (m,n)-closed ideals, one can see [14] and [15].

The main objective of this work is to describe the structure of (m,n)-closed submodules as a submodule synonym of (m,n)-prime ideals. A proper submodule N of an R-module M is called an (m,n)-closed submodule if for $r \in R$ and $b \in M$, $r^m b \in N$ implies either $r^n \in (N :_R M)$ or $b \in N$. In this case, $P = \{r \in R : r^n \in (N :_R M)\}$ is a prime ideal of R and we sometimes refer to N as P-(m,n)-closed. We clarify that this class of submodules lies properly between the classes of prime and primary submodules. The paper is organized as follows. In Section 2, first, we determine the exact place of this class of submodules in the literature. Many supporting examples and counterexamples are presented. Various characterizations of this class of submodules are provided. (see Theorems 4, 10 and 18, Corollary 6) We obtain a characterization for faithful multiplication modules for which every proper submodule is P-(m,n)-closed. (see Corollary 17) Furthermore, we give a characterization for P-(m,n)-closed modules that is an R-module whose zero submodule is P-(m,n)-closed. (see Theorem 30). For any class of submodules, it is natural to investigate its behavior under homomorphisms, localizations, direct sums and extensions. (see Propositions 20-28, Theorems 26 and 27). Moreover, as generalization of ZPI-rings, we introduce and characterize (m,n)-modules M for which every proper submodule N is either an (m,n)-closed submodule or $N = I_1 I_2 \cdots I_k K$ where I_i's are P_i-(m,n)-prime ideals of R and K is a Q-(m,n)-closed submodule, (see Theorem 30) We also justify (see Proposition 28) the relationship between the (m,n)-closed submodules in an R-module M and the (m,n)-prime ideals of the idealization ring $R(+)M$. At the end of this section, the (m,n)-closed avoidance theorem is verified (see Theorem 33).

Let $f : R_1 \to R_2$ be a ring homomorphism, J be an ideal of R_2, M_1 be an R_1-module, M_2 be an R_2-module (which is an R_1-module induced naturally by f) and $\varphi : M_1 \to M_2$ be an R_1-module homomorphism. We conclude section 3 by investigating some kinds of (m,n)-closed submodules in the the amalgamation $(R_1 \bowtie^f J)$-module $M_1 \bowtie^\varphi JM_2$ of M_1 and M_2 along J with respect to φ (see Theorems 34 and 35).

2. (m,n)-closed Submodules

In this section, we introduce the class of (m,n)-closed submodules and clarify its relationship with some other classes of submodules. We determine some of properties and characterizations for (m,n)-closed submodules. Moreover, we investigate the behavior of this structure under module homomorphisms, localizations, quotient modules, Cartesian products and idealizations.

Definition 1. *Let N be a proper submodule of an R-module M and m,n be positive integers. Then N is called an (m,n)-closed submodule if for $r \in R$ and $b \in M$, $r^m b \in N$ implies either $r^n \in (N :_R M)$ or $b \in N$.*

It is clear that the (m,n)-prime ideals of a ring R coincides with the (m,n)-closed submodules of the R-module R. If N is an (m,n)-closed submodule of an R-module M, then clearly N is a primary submodule of M and so $P = \sqrt{(N :_R M)}$ is a prime ideal of R. In this case, we call N a P-(m,n)-closed submodule of M. Moreover, we note by [13, Lemma 1] that $P = \{r \in R : r^n \in (N :_R M)\}$.

In the following remark, we clarify the relationship between (m,n)-closed submodules and some other kinds of submodules.

Remark 2. *Let N be a proper submodule of an R-module M and n,m be positive integers.*

1. If N is a prime submodule of M, then clearly it is (m,n)-closed for all $m,n \in \mathbb{N}$. Moreover, N is a prime submodule of M if and only if N is $(1,1)$-closed.
2. If N is a 1-absorbing prime submodule of M, then N is an (m,n)-closed submodule for $n \geq 2$. Indeed, let $r^m b \in N$ and $b \notin N$ for $r \in R$ and $b \in M$. If r is a unit, then $b \in N$, a contradiction.

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Hence, assume that r is nonunit. Then by assumption, $r^{m-1}r \in (N :_R M)$ and so either $r \in (N :_R M)$ or $r^{m-2}r \in (N :_R M)$ as $(N :_R M)$ is 1-absorbing prime in R. By continuing this process, we get $r^2 \in (N :_R M)$ and so $r^n \in (N :_R M)$, as needed. The converse is also true if $(N :_R M)$ a radical ideal of R.

3. In general, N can be a 2-absorbing submodule of M that is not an (m,n)-closed submodule for all integers m and n. For example, the ideal $I = \langle 4, 2x, 2y, xy, xz, x^2 \rangle$ is a 2-absorbing submodule of the R-module $R = \mathbb{Z}[x,y,z]$ which is not a primary submodule, see [5, Example 2.12]. Thus, I is not an (m,n)-closed submodule of R for all integers m and n. For another example, the submodule $\langle \bar{0} \rangle$ of the $\mathbb{Z}$-module $\mathbb{Z}_6$ is a 2-absorbing submodule as it is an intersection of two distinct prime submodules, [10, Theorem 2.3]. However, for all integers m and n, $\langle \bar{0} \rangle$ is not (m,n)-closed as also it is not primary.
4. If N is (m,n)-closed in M, then N is a semi n-absorbing submodule of M. Indeed, let $r \in R$ and $b \in M$ such that $r^n b \in N$ and $r^{n-1}b \notin N$ (so that $b \notin N$). If $m \geq n$, then $r^m b \in N$ implies $r^n \in (N :_R M)$ as N is (m,n)-closed. If $m \lneqq n$, then $r^m r^{n-m} b \in N$ and $r^{n-m} b \notin N$ as $n-1 \geq n-m$. Thus, again $r^n \in (N :_R M)$ and N is a semi n-absorbing submodule of M.
5. If N is (m,n)-closed in M, then it is (m',n')-closed for $n \leq n'$ and $m' \leq m$.

We illustrate the place of the class of (m,n)-closed submodules for all positive integers n and m by the following diagram:

$$\begin{array}{ccccc}
\text{semiprime} & \longrightarrow & \text{semi } n\text{-absorbing} & \longleftarrow & n\text{-absorbing} \\
\uparrow & & \uparrow & & \\
\text{prime} & \longrightarrow & (m,n)\text{-closed} & \longrightarrow & \text{primary} \\
\downarrow & \nearrow & \downarrow & & \\
\text{1-absorbing prime} & & (m,n)\text{-semiprime} & &
\end{array}$$

However, the implications in the above diagram are proper as we can see in the following example.

Example 3.

1. The submodule $N = \bar{0}$ of the $\mathbb{Z}$-module $M = \mathbb{Z}_4$ is not prime in M. On the other hand, N is $(m,2)$-closed in M for all $m \in \mathbb{N}$. Indeed, let $r \in \mathbb{Z}$ and $b \in \mathbb{Z}_4$ such that $r^m b \in N$ and $b \notin N$. Then clearly, $r^m \in 2\mathbb{Z}$ and so $r \in 2\mathbb{Z}$. Thus, $r^2 \in 4\mathbb{Z} = (N :_R M)$ and N is $(m,2)$-closed in M.
2. Semiprime submodules and (m,n)-closed submodules are not comparable in general. The submodule $N = \bar{0}$ of the $\mathbb{Z}$-module $M = \mathbb{Z}_4$ is also not semiprime in M as $2^2 \cdot \bar{1} \in N$ but $2 \cdot \bar{1} \notin N$. Moreover, any non primary radical ideal of any ring R is an example of a semiprime submodule of the R-module R that is not (m,n)-closed for any $n, m \in \mathbb{N}$.
3. The submodule $N = \bar{8}\mathbb{Z}_{16}$ of the $\mathbb{Z}$-module $M = \mathbb{Z}_{16}$ is clearly a primary submodule. However, N is not $(m,2)$-closed in M for all $m \in \mathbb{N}$. For example, $2^m.\bar{4} \in N$ for all $m \in \mathbb{N}$ but $\bar{4} \notin N$ and $2^2 \notin 8\mathbb{Z} = (N :_{\mathbb{Z}} M)$.
4. The ideal $I = 6\mathbb{Z}$ is a semi-2-absorbing submodule of the $\mathbb{Z}$-module $\mathbb{Z}$ since for $r, b \in \mathbb{Z}$, $r^2 b \in 6\mathbb{Z}$ implies $rb \in 2\mathbb{Z} \cap 3\mathbb{Z} = 6\mathbb{Z}$. On the other hand, for all $m, n \in \mathbb{N}$, I is not an (m,n)-closed submodule of $\mathbb{Z}$ since it is not primary.
5. The submodule $I = 8\mathbb{Z}$ is $(m,3)$-closed in the $\mathbb{Z}$-module $\mathbb{Z}$ for all $m \in \mathbb{N}$, [13, Theorem 3]. However, $8\mathbb{Z}$ is not 2-absorbing (and so not 1-absorbing prime) in $\mathbb{Z}$ since $2 \cdot 2 \cdot 2 \in I$ but $2 \cdot 2 \notin I$.
6. One can easily verify that the submodule $\bar{6}\mathbb{Z}_{12}$ is $(2,1)$-semiprime in the $\mathbb{Z}$-module $\mathbb{Z}_{12}$. However, $\bar{6}\mathbb{Z}_{12}$ is not primary and so not $(2,1)$-closed in $\mathbb{Z}_{12}$.

Let N be a proper submodule of an R-module M and I be an ideal of R. The residual of N by I is the set $(N :_M I) = \{m \in M : Im \subseteq N\}$. It is clear that $(N :_M I)$ is a submodule of M containing N. More generally, for any subset $A \subseteq R$, $(N :_M A)$ is a submodule of M containing N. We next give one characterization of (m,n)-closed submodules of an R-module.

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Theorem 4. *Let N be a proper submodule of an R-module M and n, m be positive integers. The following are equivalent.*

1. N is an (m, n)-closed submodule of M.
2. For all $r \in R$ with $r^n \notin (N :_R M)$, $(N :_M r^m) = N$.
3. Whenever $r \in R$ and K is a submodule of M with $r^m K \subseteq N$, then $r^n \in (N :_R M)$ or $K \subseteq N$.

Proof. (1)⇒(2) Let $b \in (N :_M r^m)$ so that $r^m b \in N$. Since $r^n \notin (N :_R M)$, we have by assumption $b \in N$ and so $(N :_M r^m) \subseteq N$. The other containment is clear.

(2)⇒(3) Let $r \in R$ and K be a submodule of M with $r^m K \subseteq N$. If $r^n \notin (N :_R M)$, then by assumption we conclude $K \subseteq (N :_M r^m) = N$.

(3)⇒(1) Suppose that $r^m b \in N$ for some $r \in R$ and $b \in M$. The result follows by putting $K = Rb$ in (3). □

Corollary 5. *Let N be an (m, n)-closed submodule of an R-module M. Then*

1. $(N :_R M)$ is a (m, n)-prime ideal of R.
2. For any subset $A \subseteq R$, either $N = AM$ or $(N :_M A)$ is an (m, n)-closed submodule of M.

Proof. (1) Clearly, $(N :_R M)$ is proper in R. Let $a, b \in R$ such that $a^m b \in (N :_R M)$. Then $a^m(bM) \subseteq N$ and so by Theorem 4, $a^n \in (N :_R M)$ or $bM \subseteq N$. Thus, $a^n \in (N :_R M)$ or $b \in (N :_R M)$, as required.

(2) Suppose that $N \neq AM$. Then $(N :_M A)$ is proper in M. Let $r \in R$ and $b \in M$ such that $r^m b \in (N :_M A)$. Then $r^m(Ab) \subseteq N$ and so by Theorem 4, we have $r^n \in (N :_R M) \subseteq ((N :_M A) :_R M)$ or $Ab \subseteq N$, that is $b \in (N :_M A)$. □

Note that the converse of (2) of Corollary 5 need not be true. For example, $N = \bar{8}\mathbb{Z}_{16}$ is not $(m, 2)$-closed in the $\mathbb{Z}$-module $M = \mathbb{Z}_{16}$ for all $m \in \mathbb{N}$ (see Example 3(3)). But, for $I = 4\mathbb{Z}$, $(N :_M I) = \bar{2}\mathbb{Z}_{16}$ is clearly $(m, 2)$-closed in M.

Corollary 6. *Let N be a submodule of a multiplication R-module M and n, m be positive integers. Then*

1. N is (m, n)-closed in M if and only if $(N :_R M)$ is a (m, n)-prime ideal of R.
2. If N is (m, n)-closed, then M-$rad(N)$ is a prime submodule of M.
3. If $P = \{r \in R : r^n \in (N :_R M)\}$ is a maximal ideal of R, then N is P-(m, n)-closed in M.

Proof. (1) ⇒) Follows by (1) of Corollary 5.

⇐) Let $r \in R$ and K be a submodule of M with $r^m K \subseteq N$. Since M is multiplication, there is a presentation ideal I in R such that $K = IM$. Hence, $r^m I \subseteq (N :_R M)$ and [13, Corollary 2(3)] yields either $r^n \in (N :_R M)$ or $I \subseteq (N :_R M)$. Thus, $r^n \in (N :_R M)$ or $K = IM \subseteq (N :_R M)M = N$.

(2) Since by (1), $(N :_R M)$ is a (m, n)-prime ideal of R, then $\sqrt{(N :_R M)}$ is clearly a prime ideal of R. It is verified in [21, Theorem 2.12] that if M is a multiplication R-module, then M-$rad(N) = \sqrt{(N :_R M)}M$. Thus, M-$rad(N) = \sqrt{(N :_R M)}M$ is a prime submodule of M.

(3) If $P = \{r \in R : r^n \in (N :_R M)\}$ is maximal in R, then $(N :_R M)$ is (m, n)-prime in R by [13, Proposition 1]. Now, the result follows by (1). □

Unless M is multiplication, being $(N :_R M)$ a (m, n)-prime ideal does not imply that N is an (m, n)-closed submodule. For example, the submodule $N = 0 \times \bar{4}\mathbb{Z}_8$ of the $\mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z}_8$ is clearly not prime (and so not $(1, 1)$-closed). On the other hand, $(N :_R M) = 0$ is a prime (and so $(1, 1)$-prime) ideal of $\mathbb{Z}$.

Following [22], a proper submodule N of an R-module M is called an n-submodule if $rb \in N$ for $r \in R$ and $b \in M$ implies $r \in Nil(M)$ or $b \in N$. Next, we determine the relationship between n-submodules and (m, n)-closed submodules.

Proposition 7. *Let N be a submodule of an R-module M and n, m be positive integers.*

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1. Let N be a P-(m,n)-closed submodule of M. Then N is an n-submodule if and only if $P = Nil(M)$.
2. If N is an n-submodule of M and $P = Nil(M)$, then N is a P-(m,n)-closed submodule of M.

Proof. (1) Suppose N is an n-submodule of M and let $r \in P$. Then $r^n \in (N :_R M)$ and so $r^n b \in N$ for some $b \in M \backslash N$ as N is proper. Since N is n-submodule, we have $r^n \in Nil(M)$ and so $r \in Nil(M)$. Thus, $P = Nil(M)$ as the reverse inclusion always holds. Conversely, suppose $P = Nil(M)$ and let $r \in R$, $b \in M$ such that $rb \in N$. Since $r^m b \in N$ and N is P-(m,n)-closed in M, we conclude that $r \in P$ or $b \in N$. Thus, N is an n-submodule as $P = Nil(M)$.

(2) Suppose that $r^m b \in N$ where $r \in R$ and $b \in M$. Then by assumption, $r^m \in Nil(M)$ or $b \in N$ and so clearly, $r \in P$ or $b \in N$ as $P = Nil(M)$. Thus, N is an (m,n)-closed submodule of M. □

As it is shown in Example 3(1), there are (m,n)-closed submodules which are not prime. In the following, we conclude a condition for an (m,n)-closed submodule to be prime.

Theorem 8. *Let N be a maximal (m,n)-closed submodule with respect to inclusion. Then N is a prime submodule of M.*

Proof. Let $r \in R$, $b \in M$ such that $rb \in N$ and $r \notin (N :_R M)$. Then $rM \nsubseteq N$ and so $(N :_M r)$ is proper in M. Moreover, $(N :_M r)$ is an (m,n)-closed submodule of M by Corollary 5(2). Since $N \subseteq (N :_M r)$ and by the maximality of N, we conclude that $N = (N :_M r)$. Thus, $b \in N$ and N is a prime submodule of M. □

Corollary 9. *Let M be an R-module. If M has an (m,n)-closed submodule, then it has a prime submodule.*

Proof. Let N be an (m,n)-closed submodule and set $\Omega = \{K : K$ is an (m,n)-closed submodule containing $N\}$. Then Ω is non-empty as $N \in \Omega$. Let $K_1 \subseteq K_2 \subseteq \cdots$ be a chain in Ω. We show that $\cup K_i$ is (m,n)-closed in M. Suppose that $r^m b \in \cup K_i$ for some $r \in R$, $b \in M$ and $b \notin \cup K_i$. Then $r^m b \in K_i$ and $b \notin K_i$ for some i which imply that $r^n \in (K_i :_R M) \subseteq (\cup K_i :_R M)$. Hence, $\cup K_i$ is (m,n)-closed and an upper bound of the chain. Now, Zorn's Lemma yields that Ω has a maximal element, say $L \in \Omega$. Thus, L is a prime submodule of M by Theorem 8. □

Recall that the product of two submodules $N = IM$ and $K = JM$ of a multiplication module M is defined as $NK = IJM$. In particular, for $b_1, b_2 \in M$ by $b_1 b_2$, we mean the product of the submodules Rb_1 and Rb_2. Let M be a multiplication R-module where R is a principal ideal domain. In this case, we conclude further characterizations for (m,n)-closed submodules.

Theorem 10. *Let R be a principal ideal domain, N be a proper submodule of a multiplication R-module M and n, m be positive integers. The following statements are equivalent.*

1. N is an (m,n)-closed submodule of M.
2. If I is an ideal of R and $b \in M \backslash N$ with $I^m b \subseteq N$, then $I^n \subseteq (N :_R M)$.
3. For an ideal I of R and a submodule K of M, $I^m K \subseteq N$ implies either $I^n \subseteq (N :_R M)$ or $K \subseteq N$.
4. For an ideal I of R, $I^n \nsubseteq (N :_R M)$ implies $(N :_M I^m) = N$.
5. For $b_1, b_2 \in M$ such that $b_1^m b_2 \subseteq N$, we have either $b_1^n \subseteq N$ or $b_2 \in N$.

Proof. (1)⇒(2) Suppose that $I^m b \subseteq N$ for an ideal I of R and $b \in M \backslash N$. Put $I = \langle r \rangle$ for some $r \in R$. Then $r^m b \in N$ which implies $r^n \in (N :_R M)$ and so $I^n \subseteq (N :_R M)$.

(2)⇒(3) Suppose that $I^m K \subseteq N$ but $K \nsubseteq N$. Then $I^n b \subseteq N$ for some $b \in K \backslash N$. Thus, by (2), we conclude that $I^n \subseteq (N :_R M)$.

(3)⇒(4) Straightforward.

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(4)⇒(5) Let $b_1, b_2 \in M$ such that $b_1^m b_2 \subseteq N$ and $b_1^n \nsubseteq N$. If I and J are the presentation ideals of Rb_1 and Rb_2, respectively, then $I^n \nsubseteq (N :_R M)$ and so by (4), we conclude that $b_2 \in Rb_2 = JM \subseteq (N :_M I^m) = N$.

(5)⇒(1) Let $r^m b \in N$ for some $r \in R$ and $b \in M \backslash N$. Let $b' = \langle r \rangle M$ and I be the presentation ideal of Rb. Then $b'^m b \subseteq N$ with $b \notin N$ and so by assumption, $\langle r \rangle^n M = b'^n \subseteq N$. Therefore, $r^n \in (N :_R M)$ and the result follows. □

Corollary 11. *Let I be an ideal of a ring R and N be an $(m, 2)$-closed submodule of an R-module M. If $I^m \subseteq (N :_R M)$, then, $2I^2 \subseteq (N :_R M)$. Additionally, if 2 is unit in R, then $I^2 \subseteq (N :_R M)$.*

Proof. Suppose that $I^m \subseteq (N :_R M)$. Then for all $r_1, r_2 \in I$ and $b \in M$, we have $r_1^m b, r_2^m b, (r_1 + r_2)^m b \in N$. Since N is $(m, 2)$-closed, we conclude that either $r_1^2 \in (N :_R M)$ (resp. $r_2^2 \in (N :_R M)$, $(r_1 + r_2)^2 \in (N :_R M)$) or $b \in N$. Thus, $r_1^2 b, r_2^2 b, (r_1 + r_2)^2 b \in N$ and so $2r_1 r_2 b = ((r_1 + r_2)^2 - r_1^2 - r_2^2)b \in N$. Therefore, $2I^2 \subseteq (N :_R M)$. In particular, if 2 is unit, then $I^2 \subseteq (N :_R M)$. □

Corollary 12. *Let I be an ideal of a ring R, N_1, N_2 be submodules of an R-module M and n, m be positive integers.*

1. If for $i = 1, 2$, N_i is a P_i-(m, n) closed submodule and $I \nsubseteq P_1 \cup P_2$, then $IN_1 = IN_2$ implies $N_1 = N_2$.
2. If IN_1 is a P-(m, n)-closed submodule of M with $I \nsubseteq P$, then $IN_1 = N_1$.

Proof. (1) Suppose that $I \nsubseteq P_1 \cup P_2$ and $IN_1 = IN_2$. Then $I \nsubseteq P_1$ and so there is an element $r \in I$ such that $r^n \notin (N_1 :_R M)$. Now, since $r^m N_2 \subseteq IN_1 \subseteq N_1$ and N_1 is (m, n)-closed, we have $N_2 \subseteq N_1$ by Theorem 4. Similarly, $I \nsubseteq P_2$ implies $N_1 \subseteq N_2$ and the required equality holds.

(2) Let IN_1 be an (m, n)-closed submodule of M with $I \nsubseteq P$. Then there exists $r \in I$ such that $r^n \notin (IN_1 :_R M)$. Since $r^m N_1 \subseteq IN_1$ and $r^n \notin (IN_1 :_R M)$, we have $N_1 \subseteq IN_1$ and so $IN_1 = N_1$, as needed. □

We recall that a module M is torsion (resp. torsion-free) if $T(M) = M$ (resp. $T(M) = \{0\}$) where $T(M) = \{m \in M :$ there exists $0 \neq r \in R$ such that $rm = 0\}$. A submodule N of an R-module M is called a pure submodule if for any $r \in R$, $(rM) \cap N = rN$.

Proposition 13. *Let N be a submodule of a torsion free R-module M such that $P = \{r \in R : r^n \in (N :_R M)\} = 0$. Then for all $m \in \mathbb{N}$, N is a pure submodule of M if and only if N is (m, n)-closed.*

Proof. Suppose that N is a pure submodule of M and note that N is proper in M since otherwise $R = P = 0$. Let $r^m b \in N$ for some $r \in R$, $b \in M$. Then $r^m b \in (r^m M) \cap N = r^m N$ and so $r^m b = r^m t$ for some $t \in N$ which yields, $r^m (b - t) = 0$. If $r = 0$, then $r^n \in (N :_R M)$. If $r \neq 0$, then M is torsion free and $r^m (b - t) = 0$ imply $r^{m-1}(b - t) = 0$. Continue in this process to get $b = t \in N$.

Conversely, suppose that N is an (m, n)-closed submodule of M and note that $rN \subseteq rM \cap N$ for any $r \in R$. If $r = 0$, then the containment $rM \cap N \subseteq rN$ is clear. So, assume that r is nonzero. Let $rb \in rM \cap N$ for some $b \in M$ and assume on contrary that $b \notin N$. Then clearly $r^m b \in N$ and since N is (m, n)-closed, we conclude $r \in P = 0$, a contradiction. Thus $b \in N$ and $rM \cap N \subseteq rN$, as required. □

Lemma 14. *[1] For an ideal I of a ring R and a submodule N of a finitely generated faithful multiplication R-module M, the following hold.*

1. *$(IN :_R M) = I(N :_R M)$.*
2. *If I is finitely generated faithful multiplication, then*

   (a) $(IN :_M I) = N$.

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(b) *Whenever* $N \subseteq IM$, *then* $(JN :_M I) = J(N :_M I)$ *for any ideal* J *of* R.

Proposition 15. *Let* M *be a finitely generated faithful multiplication* R*-module,* N *a submodule of* M *and* I *a finitely generated faithful multiplication ideal of* R*. Then for* $n, m \in \mathbb{N}$*, we have*

1. N is (m,n)-closed in IM if and only if $(N :_M I)$ is (m,n)-closed in M.
2. If IN is (m,n)-closed in M, then either I is (m,n)-closed in R or N is (m,n)-closed in M.

Proof. (1) Suppose N is an (m,n)-closed submodule of IM. If $(N :_M I) = M$, then by Lemma 14, $N = (IN :_M I) = I(N :_M I) = IM$, a contradiction. Thus, $(N :_M I)$ is proper in M. Let $r^m b \in (N :_M I)$ for $r \in R$ and $b \in M$ such that $r^n \notin ((N :_M I) :_R M)$. If $r^n \in (N :_R IM)$, then $r^n M = r^n(IM :_M I) \subseteq (r^n IM :_M I) \subseteq (N :_M I)$, a contradiction. Thus, $r^m Ib \subseteq N$ and $r^n \notin (N :_R IM)$. By Theorem 4, we have $Ib \subseteq N$ and so $b \in (N :_M I)$. Conversely, suppose $(N :_M I)$ is an (m,n)-closed submodule of M. Then N is proper in IM since otherwise Lemma 14 implies $(N :_M I) = (IM : I) = M$, a contradiction. Let $r \in R$ and $ab \in IM$ ($a \in I$) such that $r^m ab \in N$ and $r^n \notin (N :_R IM)$. Then $r^m(\langle ab\rangle :_M I) \subseteq (\langle r^m ab\rangle :_M I) \subseteq (N :_M I)$. Moreover, since clearly $r^n \notin ((N :_M I) :_R M)$, then by Theorem 4, $(\langle ab\rangle :_M I) \subseteq (N :_M I)$. Thus, by Lemma 14, $ab \in (\langle ab\rangle I :_M I) = I(\langle ab\rangle :_M I) \subseteq I(N :_M I) = (IN :_M I) = N$. Therefore, N is (m,n)-closed in IM.

(2) Suppose IN is (m,n)-closed in M. If $N = M$, then $I = I(N :_R M) = (IN :_R M)$ is an (m,n)-closed ideal of R by Corollary 6. Suppose N is proper in M. By Lemma 14, $N = (IN :_M I)$ and so $(N :_R M) = ((IN :_M I) :_R M) = (I(N :_R M) :_M I)$. Now, let $r \in R$, $b \in M$ such that $r^m b \in N$ and $r^n \notin (N :_R M)$. Then $r^m Ib \subseteq IN$ and clearly $r^n \notin (IN :_R M)$. By assumption, $Ib \subseteq IN$ and so again by Lemma 14, $b \in (IN :_M I) = N$. Therefore, N is an (m,n)-closed submodule of M. □

As we can see in [13, Remark 2], if I is an (m,n)-closed ideal of R and N is an (m,n)-closed submodule of M, then IN need not be (m,n)-closed in M.

Proposition 16. *Let* M *be a faithful multiplication* R*-module,* I *be an ideal of* R *and* n, m *be positive integers. Then* I *is a* P*-*(m,n)*-prime ideal of* R *if and only if* IM *is a* P*-*(m,n)*-closed submodule of* M*.*

Proof. $\Rightarrow$) If $IM = M$, then $I = (IM :_R M) = R$, a contradiction. Let $r \in R$, $b \in M$ with $r^m b \in IM$ and $b \notin IM$. Since I is (m,n)-prime, it is primary which implies that IM is a primary submodule of M. Thus, clearly we have $r \in \sqrt{(IM :_R M)} = \sqrt{I} = P$. By [13, Lemma 1], we have $P = \{r \in R : r^n \in I\}$, and so $r^n \in I = (IM :_R M)$. Therefore, IM is a P-(m,n)-closed submodule of M.

$\Leftarrow$) As $I = (IM :_R M)$, then clearly I is proper. Now, let $r, s \in R$ such that $r^m s \in I$ and $r^n \notin I = (IM :_R M)$. By Theorem 4, $r^m sM \subseteq IM$ implies $sM \subseteq IM$ and so $s \in (IM :_R M) = I$, as needed. □

Corollary 17. *Let* M *be a faithful multiplication* R*-module and* n, m *be positive integers. The following are equivalent.*

1. Every proper submodule of M is P-(m,n)-closed.
2. Every proper ideal of R is P-(m,n)-prime.
3. R has no non-trivial idempotents, $\dim(R) = 0$ and $r^n = 0$ for all $r \in Nil(R)$.

Proof. (1) $\Leftrightarrow$ (2): Follows directly by Proposition 16.

(2) $\Leftrightarrow$ (3): [13, Theorem 2]. □

Now, we are ready to present a general characterization for P-(m,n)-closed submodules in multiplication modules.

Theorem 18. *Let* M *be a multiplication* R*-module,* N *be a submodule of* M *and* n, m *be positive integers. The following statements are equivalent.*

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1. N is a P-(m,n)-closed submodule of M.
2. $(N :_R M)$ is a P-(m,n)-prime ideal of R.
3. $N = IM$ for some P-(m,n)-prime ideal I of R including $ann_R(M)$.

Proof. (1)⇒(2) Corollary 6.

(2)⇒(3) We choose $I = (N :_R M)$.

(3)⇒(1) Suppose that $N = IM$ for some P-(m,n)-prime ideal I of R including $ann_R(M)$. It is well-known that M is faithful as an $R/ann_R(M)$-module. Hence, $(I/ann_R(M))M = IM = N$ is a P-(m,n)-closed submodule of the $R/ann_R(M)$-module M by Proposition 16. We show that N is a P-(m,n)-closed submodule of the R-module M. Suppose $r^m b \in N$ for some $r \in R$ and $b \in M$. Then $(r + ann_R(M))^m b \in N$ which implies that $(r + ann_R(M))^n \in (N :_{R/ann_R(M)} M)$ or $b \in N$. In the case $(r + ann_R(M))^n \in (N :_{R/ann_R(M)} M)$, we have $r^n M = (r^n + ann_R(M))M \subseteq N$, that is, $r^n \in (N :_R M)$. Since also, $\sqrt{(N :_R M)} = \sqrt{(IM :_R M)} = \sqrt{(I/ann_R(M))M :_R M)} = \sqrt{I} = P$, then N is a P-(m,n)-closed submodule of M. □

Recall that for an R-module M, $Z_R(M)$ denotes the set of all zero divisors on M. Following [16], a submodule N of an R-module M is called an r-submodule if whenever $rb \in N$ for $r \in R$ and $b \in M$ with $ann_M(r) = 0$, then $b \in N$. We call an R-module M a P-(m,n)-closed module if the submodule $\{0\}$ is P-(m,n)-closed. Next, we give a characterization for (m,n)-closed modules.

Theorem 19. *Let M be an R-module and $n, m \in \mathbb{N}$. If $P = \{r \in R : r^n \in (0 :_R M)\}$, then the following are equivalent.*

1. M is a P-(m,n)-closed module.
2. $P = Z_R(M)$.
3. Every r-submodule of M is (m,n)-closed.

Proof. (1)⇒(2) Let $r \in P$ so that $r^n M = 0$. Let k be the least positive integer such that $r^k M = 0$. Then $r(r^{k-1}M) = 0$ and $r^{k-1}M \neq 0$. Thus, $r \in Z_R(M)$ and $P \subseteq Z_R(M)$. Conversely, let $r \in Z_R(M)$. Then $rb = 0$ for some $0 \neq b \in M$. Since $r^m b = 0$ and $b \notin \{0\}$, we have $r^n \in (0 :_R M)$, and so $r \in P$. Thus, $Z_R(M) \subseteq P$ and the equality holds.

(2)⇒(3) Suppose $P = Z_R(M)$ and N is an r-submodule of M. Let $r \in R$ and $b \in M$ such that $r^m b \in N$ and $r^n \notin (N :_R M)$. Since $r \notin P = Z_R(M)$, then clearly $r^m \notin Z_R(M)$ and so $ann_M(r^m) = 0$. It follows that that $b \in N$ as N is an r-submodule. Therefore, N is an (m,n)-closed submodule of M.

(3)⇒(1) Note that the submodule $\{0\}$ is always an r-submodule. Hence, the claim follows from (3). □

Let I be an ideal of a ring R and N be a submodule of an R-module M. By $Z_I(R)$ and $Z_N(M)$, we denote the sets $\{r \in R : ra \in I$ for some $a \in R\backslash I\}$ and $\{r \in R : rm \in N$ for some $m \in M\backslash N\}$.

Proposition 20. *Let N be a proper submodule of an R-module M and S be a multiplicatively closed subset of R such that $(N :_R M) \cap S = \emptyset$. Then for $n, m \in \mathbb{N}$, we have*

1. If N is an (m,n)-closed submodule of M, then $S^{-1}N$ is an (m,n)-closed submodule of $S^{-1}M$.
2. If $S^{-1}N$ is an (m,n)-closed submodule of $S^{-1}M$ and $Z_{(N:_RM)}(R) \cap S = Z_N(M) = \phi$, then N is an (m,n)-closed submodule of M.

Proof. (1) Let $\left(\frac{r}{s_1}\right)^m \left(\frac{b}{s_2}\right) \in S^{-1}N$ for $\frac{r}{s_1} \in S^{-1}R$ and $\frac{b}{s_2} \in S^{-1}M$. Then $(ur)^m b \in N$ for some $u \in S$. Since N is (m,n)-closed, then either $(ur)^n \in (N :_R M)$ or $b \in N$. Thus, $\left(\frac{r}{s_1}\right)^n = \frac{u^n r^n}{u^n s_1^n} \in S^{-1}(N :_R M) = (S^{-1}N :_{S^{-1}R} S^{-1}M)$ or $\left(\frac{b}{s_2}\right) \in S^{-1}N$.

(2) Suppose that $r^m b \in N$ for some $r \in R$ and $b \in M$. Then $\left(\frac{r}{1}\right)^m \left(\frac{b}{1}\right) \in S^{-1}N$ which implies either $\left(\frac{r}{1}\right)^n \in (S^{-1}N :_{S^{-1}R} S^{-1}M) = S^{-1}(N :_R M)$ or $\left(\frac{b}{1}\right) \in S^{-1}N$. Thus, $ur^n \in (N :_R M)$ or $vb \in N$

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for some $u, v \in S$. Since $Z_{(N:_R M)}(R) \cap S = Z_N(M) = \emptyset$, we have $r^n \in (N :_R M)$ or $b \in N$ and we are done. □

Proposition 21. *Let M be a P-(m,n)-closed R-module, S be a multiplicatively closed subset of R and $\varphi : M \to S^{-1}M$ be a natural homomorphism defined by $m \mapsto \frac{m}{1}$ for all $m \in M$. Then φ is a monomorphism or $S^{-1}M = 0$.*

Proof. Suppose that φ is not a monomorphism. Choose $0 \neq b \in Ker(\varphi)$. Since $\varphi(b) = \frac{b}{1} = 0$, there exists some $s \in S$ such that $sb = 0$. Since $s^m b = 0$ and 0 is (m,n)-closed, we have $s^n \in (0 :_R M)$. Thus, $s^n m = 0$ for all $m \in M$ and so $S^{-1}M = 0$. □

Proposition 22. *Let M_1 and M_2 be R-modules, $f : M_1 \to M_2$ be an R-module homomorphism and n, m be positive integers.*

1. If N_2 is an (m,n)-closed submodule of M_2, then $f^{-1}(N_2)$ is an (m,n)-closed submodule of M_1.
2. If f is onto and N_1 is an (m,n)-closed submodule of M_1 containing $Ker(f)$, then $f(N_1)$ is an (m,n)-closed submodule of M_2.

Proof. (1) Let $r^m b \in f^{-1}(N_2)$ for some $r \in R$ and $b \in M_1$. Then $r^m f(b) \in N_2$ implies either $r^n \in (N_2 :_R M_2)$ or $f(b) \in N_2$. Suppose $r^n \in (N_2 :_R M_2)$ so that $r^n f(M_1) \subseteq N_2$. Then $r^n M_1 \subseteq f^{-1}(N_2)$ and so $r^n \in (f^{-1}(N_2) :_R M_1)$. If $f(b) \in N_2$, then $b \in f^{-1}(N_2)$, as required.

(2) Suppose that $r^m b_2 \in f(N_1)$ for some $r \in R$ and $b_2 \in M_2$. Let $b_2 := f(b_1)$ for some $b_1 \in M_1$. Then $r^m b_1 \in N_1$ as $Ker(f) \subseteq N_1$ and so either $r^n \in (N_1 :_R M_1)$ or $b_1 \in N_1$. Therefore, $r^n M_2 = r^n f(M_1) = f(r^n M_1) \subseteq f(N_1)$ or $b_2 \in f(N_1)$ and $f(N_1)$ is an (m,n)-closed submodule of M_2. □

As a direct consequence of Proposition 22, we have the following.

Corollary 23. *Let R be a ring, M_1 and M_2 be R-modules and n, m be positive integers.*

1. If $M_1 \subseteq M_2$ and N is an (m,n)-closed submodule of M_2, then $N \cap M_1$ is an (m,n)-closed submodule of M_1.
2. Let $K \subseteq N$ be submodules of M_1. Then N/K is an (m,n)-closed submodule of M_1/K if and only if N is an (m,n)-closed submodule of M_1.

Proposition 24. *Let $\{m_i, n_i\}_{i=1}^k$ be positive integers and M be an R-module.*

1. If $\{N_i\}_{i=1}^k$ are P-(m_i,n_i)-closed submodules of an R-module M, then $\bigcap_{i=1}^k N_i$ is a P-(m,n)-closed submodule of M for all $m \leq \min\{m_1, m_2, \cdots, m_k\}$ and $n \geq \max\{n_1, n_2, \cdots, n_k\}$.
2. If $\{P_i\}_{i\in I}$ is a family of prime submodules of M and $\bigcap_{i\in I} P_i$ is an (m,n)-closed submodule of M (for $n, m \in \mathbb{N}$), then $\bigcap_{i\in I} P_i$ is a prime submodule of M.

Proof. (1) Suppose N_i is P-(m_i,n_i)-closed in M for all $i \in \{1,2,\cdots,k\}$. Let $m \leq \min\{m_1, m_2, \cdots, m_k\}$ such that $r^m b \in \bigcap_{i=1}^k N_i$ and $b \notin \bigcap_{i=1}^k N_i$ for $r \in R$ and $b \in M$. Then $b \notin N_j$ for some $j \in \{1,2,\cdots,k\}$. Since $r^{m_j} b \in N_j$, then by assumption $r^{n_j} \in (N_j :_R M)$ and so $r \in \sqrt{(N_j :_R M)} = P$ as $n \geq n_j$. By [13, Lemma 1], we have for all $i \in \{1,2,\cdots,k\}$, $P = \{r \in R : r^{n_i} \in (N_i :_R M)\}$. Thus, $r^n \in \bigcap_{i=1}^k (N_i :_R M) = (\bigcap_{i=1}^k N_i :_R M)$ as $n \geq \max\{n_1, n_2, \cdots, n_k\}$. Since also $\sqrt{\bigcap_{i=1}^k (N_i :_R M)} = \bigcap_{i=1}^k \sqrt{(N_i :_R M)} = P$, then $\bigcap_{i=1}^k N_i$ is a P-(m,n)-closed submodule of M.

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(2) Suppose that $rb \in \bigcap_{i\in I} P_i$ and $r \notin \left(\bigcap_{i\in I} P_i :_R M\right) = \bigcap_{i\in I}(P_i :_R M)$. Then $r^m b \in \bigcap_{i\in I} P_i$ and $r \notin (P_j : M)$ for some $j \in I$. Since $(P_j : M)$ is a prime ideal of R, then $r^n \notin (P_j : M)$ and so $r^n \notin \left(\bigcap_{i\in I} P_i :_R M\right)$. Since $\bigcap_{i\in I} P_i$ is (m,n)-closed, we have $b \in \bigcap_{i\in I} P_i$ and we are done. □

Corollary 25. *Let N be a submodule of an R-module M and n, m be positive integers. Then M-rad(N) is an (m,n)-closed submodule of M if and only if M-rad(N) is prime.*

In general, if N_1 and N_2 are two (m,n)-closed submodules of an R-module M with $\sqrt{(N_1 :_R M)} \neq \sqrt{(N_2 :_R M)}$, then $N_1 \cap N_2$ need not be (m,n)-closed, see [13], Remark 2.

Next, we discuss the (m,n)-closed submodules in a direct sum of modules.

Theorem 26. *Let M_1 and M_2 be R-modules, N_1 and N_2 be submodules of M_1 and M_2, respectively and n, m be positive integers.*

1. If $N_1 \oplus N_2$ is an (m,n)-closed submodule of $M_1 \oplus M_2$, then N_i is an (m,n)-closed submodule of M_i for all $i = 1,2$ such that $N_i \neq M_i$.
2. If N_1 and N_2 are P-(m,n)-closed submodules of M_1 and M_2, respectively, then $N = N_1 \oplus N_2$ is a P-(m,n)-closed submodule of $M = M_1 \oplus M_2$.
3. Suppose $\sqrt{(N_1 :_R M_1)} \neq \sqrt{(N_2 :_R M_2)}$. Then $N = N_1 \oplus N_2$ is an (m,n)-closed submodule of $M = M_1 \oplus M_2$ if and only if one of the following statements holds:
   (a) $N = N_1 \oplus M_2$ where N_1 is an (m,n)-closed submodule of M_1.
   (b) $N = M_1 \oplus N_2$ where N_2 is an (m,n)-closed submodule of M_2.

Proof. (1) Suppose $N_1 \oplus N_2$ is an (m,n)-closed submodule of $M_1 \oplus M_2$ and suppose, say, $N_1 \neq M_1$. Let $r^m b \in N_1$ such that $b \notin N_1$. Then $r^m(b,0) \in N_1 \oplus N_2$ with $(b,0) \notin N_1 \oplus N_2$. Therefore, $r^n \in (N_1 \oplus N_2 :_R M_1 \oplus M_2)$ and then clearly, $r^n \in (N_1 :_R M_1)$, as needed.

(2) Suppose that N_1 and N_2 are P-(m,n)-closed submodules of M_1 and M_2, respectively. First, we show that $N_1 \oplus M_2$ is a P-(m,n)-closed submodule of M. Let $r \in R$ and $(b_1, b_2) \in M$ with $r^m(b_1, b_2) \in N_1 \oplus M_2$. Then $r^m b_1 \in N_1$ which implies $r^n \in (N_1 :_R M_1)$ or $b_1 \in N_1$. Thus, $r^n \in (N_1 \oplus M_2 :_R M)$ or $(b_1, b_2) \in N_1 \oplus M_2$ and $N_1 \oplus M_2$ is an (m,n)-closed submodule of M. By using the same manner, it can be verified that $M_1 \oplus N_2$ is a P-(m,n)-closed submodule of M. Since $N = N_1 \oplus N_2 = (N_1 \oplus M_2) \cap (M_1 \oplus N_2)$, we are done by Proposition 24.

(3) Suppose $N = N_1 \oplus N_2$ is an (m,n)-closed submodule of M. Suppose that N_1 is proper in M_1 so that $(N_1 :_R M_1) \neq R$. Choose $r \in \sqrt{(N_2 :_R M_2)} \backslash \sqrt{(N_1 :_R M_1)}$ and let $b \in M_2$. Then $r^k \in (N_2 :_R M_2)$ for some positive integer k and hence $(r^k)^m(0,b) \in N$. It follows that $r^{kn} \in (N :_R M)$ or $(0,b) \in N$. In the first case, we get $r^{kn} \in (N_1 :_R M_1)$ which is a contradiction. Thus, $b \in N_2$ and so $N_2 = M_2$. Similarly, if N_2 is proper in M_2, then we must have $N_1 = M_1$. Without loss of generality, we may assume that $N_2 = M_2$ and show that N_1 is an (m,n)-closed submodule of M_1. Let $r^m b \in N_1$ such that $b \notin N_1$. Then $r^m(b,0) \in N$ with $(b,0) \notin N$. This yields that $r^n \in (N :_R M)$ and then clearly, $r^n \in (N_1 :_R M_1)$, as needed. The converse part follows by the proof of (2). □

The direct sum of two (m,n)-closed submodules need not to be an (m,n)-closed submodule in general. Let p and q be distinct prime integers. Then the submodules $N_1 = p^2\mathbb{Z}$ and $N_2 = q^2\mathbb{Z}$ are $(m,2)$-closed in the $\mathbb{Z}$-module $M = \mathbb{Z}$ for all $m \in \mathbb{N}$, [13, Theorem 3]. However, $p^2\mathbb{Z} \oplus q^2\mathbb{Z}$ is not an $(m,2)$-closed submodule of $\mathbb{Z} \oplus \mathbb{Z}$ as $q^m(p^2, 1) \in p^2\mathbb{Z} \oplus q^2\mathbb{Z}$ for all $m \geq 2$, but neither $q^2 \in (p^2\mathbb{Z} \oplus q^2\mathbb{Z} :_{\mathbb{Z}} \mathbb{Z} \oplus \mathbb{Z}) = p^2q^2\mathbb{Z}$ nor $(p^2, 1) \in p^2\mathbb{Z} \oplus q^2\mathbb{Z}$. Note that $\sqrt{(N_1 :_{\mathbb{Z}} M)} \neq \sqrt{(N_2 :_{\mathbb{Z}} M)}$.

We next generalize Theorem 26 to a finite direct sum of submodules.

Theorem 27. *Let $M_1, M_2, \cdots, M_k$ be R-modules, $N_1, N_2, \cdots, N_k$ be submodules of $M_1, M_2, \cdots, M_k$, respectively and n, m be positive integers.*

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1. If $N_1 \oplus N_2 \oplus \cdots \oplus N_k$ is an (m,n)-closed submodule of $M_1 \oplus M_2 \oplus \cdots \oplus M_k$, then N_i is an (m,n)-closed submodule of M_i for all i such that $N_i \neq M_i$.
2. If $N_1, ..., N_k$ are P-(m,n)-closed submodules of $M_1, M_2, ..., M_k$, respectively, then $N = N_1 \oplus N_2 \oplus \cdots \oplus N_k$ is a P-(m,n)-closed submodule of $M = M_1 \oplus M_2 \oplus \cdots \oplus M_k$.
3. Suppose $\bigcap_{i \neq j} \sqrt{(N_i :_R M_i)} \nsubseteq \sqrt{(N_j :_R M_j)}$ for some $j \in \{1, 2, ..., k\}$. Then $N = N_1 \oplus N_2 \oplus \cdots \oplus N_k$ is an (m,n)-closed submodule of $M = M_1 \oplus M_2 \oplus \cdots \oplus M_k$ if and only if $N = M_1 \oplus \cdots \oplus N_j \oplus \cdots \oplus M_k$ where N_j is an (m,n)-closed submodule of M_j $(j = 1, 2, \cdots, k)$.

Proof. The proofs of (1) and (2) are similar to those of (1) and (2) in Theorem 26.

(3) With no loss of generality, we may assume that $j = 1$. We use mathematical induction on k. Clearly, the result is true for $k = 2$ by (1) of Theorem 26. Suppose the result is true for k and let $N = N_1 \oplus N_2 \oplus \cdots \oplus N_k \oplus N_{k+1} = N' \oplus N_{k+1}$ where $\bigcap_{i=2}^{k+1} \sqrt{(N_i :_R M_i)} \nsubseteq \sqrt{(N_1 :_R M_1)}$. Suppose N is an (m,n)-closed submodule of $M = M_1 \oplus M_2 \oplus \cdots \oplus M_k \oplus M_{k+1} = M' \oplus M_{k+1}$. Now, clearly $\sqrt{(N' :_R M')} \neq \sqrt{(N_{k+1} :_R M_{k+1})}$. If $N' = M'$, then $(N_1 :_R M_1) = R$ which is impossible. Thus, again by Theorem 26, we have $N_{k+1} = M_{k+1}$ and N' is (m,n)-closed in M'. By induction hypothesis, we get $N' = N_1 \oplus M_2 \oplus \cdots \oplus M_k$ (and so $N = N_1 \oplus M_2 \oplus \cdots \oplus M_k \oplus M_{k+1}$) where N_1 is (m,n)-closed in M_1. Conversely, suppose $N = N_1 \oplus M_2 \oplus \cdots \oplus M_k \oplus M_{k+1}$ where N_1 is (m,n)-closed in M_1. Then by induction hypothesis, $L = N_1 \oplus M_2 \oplus \cdots \oplus M_k$ is (m,n)-closed in $M_1 \oplus M_2 \oplus \cdots \oplus M_k$. Thus, $N = L \oplus M_{k+1}$ is (m,n)-closed in $M_1 \oplus M_2 \oplus \cdots \oplus M_k \oplus M_{k+1}$ by Theorem 26(2). □

Note that the sum of two (m,n)-closed submodules is not an (m,n)-closed submodule in general. Consider the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}$ and the submodules $N = p\mathbb{Z} \oplus \mathbb{Z}$ and $N = \mathbb{Z} \oplus q\mathbb{Z}$ where p and q are prime integers. By Theorem 26, both N and K are (m,n)-closed submodules of $\mathbb{Z} \oplus \mathbb{Z}$. However, $N + K = \mathbb{Z} \oplus \mathbb{Z}$ is not (m,n)-closed.

Recall that the idealization ring of an R-module M is the set $R(+)M = R \oplus M = \{(r,m) : r \in R, m \in M\}$ with coordinate-wise addition and multiplication defined as $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$. If I is an ideal of R and N a submodule of M, then $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$. Next, we justify the relationship between the (m,n)-closed submodules of M and the (m,n)-prime ideals of the idealization ring $R(+)M$.

Proposition 28. *Let I be an ideal of a ring R, N be a proper submodule of an R-module M and n, m be positive integers.*

1. *If $I(+)N$ is an (m,n)-closed ideal of $R(+)M$, then N is an (m,n)-closed submodule of M and I is a (m,n)-prime ideal of R.*
2. *If $I = (N :_R M)$ is a (m,n)-prime ideal of R and N is an (m,n)-closed submodule of M, then $I(+)N$ is an $(m, n+1)$-closed ideal of $R(+)M$.*

Proof. (1) Suppose $I(+)N$ is an (m,n)-closed ideal of $R(+)M$. Then I is a (m,n)-prime ideal of R by [13, Proposition 6]. Now, let $r \in R$ and $b \in M$ such that $r^m b \in N$ and $r^n \notin (N :_R M)$. Then $(r,0)^m (0,b) = (0, r^m b) \in I(+)N$ and $(r,0)^n \notin I(+)N$ since otherwise $r^n M \subseteq IM \subseteq N$, a contradiction. Therefore, $(0,b) \in I(+)N$ and so $b \in N$, as required.

(2) Suppose $I = (N :_R M)$. Let $(r,x), (s,y) \in R(+)M$ such that $(r,x)^m (s,y) = (r^m s, r^m y + m r^{m-1} s x) \in I(+)N$. Then $r^m s \in I$ and $r^m y + m r^{m-1} s x \in N$. We have two cases:

Case I: $r^n \in I$. In this case, we have $(r,x)^{n+1} = (r^{n+1}, (n+1) r^n x) \in I(+)N$ as $IM \subseteq N$.

Case II: $r^n \notin I = (N :_R M)$. Since I is (m,n)-prime in R, then $s \in I$ and so $m r^{m-1} s x \in IM \subseteq N$. Therefore, $r^m y \in N$. Since N is (m,n)-closed in M and $r^n \notin (N :_R M)$, then $y \in N$ and so $(s,y) \in I(+)N$. Therefore, $I(+)N$ is an $(m, n+1)$-closed ideal of $R(+)M$. □

We note that if N is an (m,n)-closed submodule of M and I is a (m,n)-prime ideal of R, then $I(+)N$ need not be (m,n)-closed in $R(+)M$. For example, while $2\mathbb{Z}$ is a $(2,1)$-prime ideal and $\bar{2}\mathbb{Z}_4$

is a $(2,1)$-closed submodule of the $\mathbb{Z}$-module $\mathbb{Z}_4$, the ideal $2\mathbb{Z}(+)\bar{2}\mathbb{Z}_4$ is not $(2,1)$-prime in $\mathbb{Z}(+)\mathbb{Z}_4$. Indeed, $(2,\bar{1})^2 = (4,\bar{0}) \in 2\mathbb{Z}(+)\bar{2}\mathbb{Z}_4$ but $(2,\bar{1}) \notin 2\mathbb{Z}(+)\bar{2}\mathbb{Z}_4$.

Recall that a ring (resp. a domain) R is called a ZPI-ring (resp. a Dedekind domain) if every proper ideal of R is a product of prime ideals. As a generalization of this structure, we define (m,n)-rings and (m,n)-modules.

Definition 29. *Let R be a ring, M be an R-module and n,m be positive integers.*

1. We call R an (m,n)-ring if every proper ideal I of R can be written as a finite product of P_i-(m,n)-prime ideals I_i of R, that is, $I = I_1 I_2 \cdots I_k$ for some positive integer k.
2. We call M an (m,n)-module if every proper submodule N of M is either an (m,n)-closed submodule or $N = I_1 I_2 \cdots I_k K$ where I_i's are P_i-(m,n)-prime ideals of R and K is a Q-(m,n)-closed submodule.

Theorem 30. *Let n,m be positive integers. Then every faithful multiplication module over an (m,n)-ring R is an (m,n)-module. In particular, a faithful multiplication module over a ZPI-ring is an (m,n)-module.*

Proof. Let N be a proper submodule of M. Since M is multiplication, $N = IM$ for some ideal I of R. Now, $I = I_1 I_2 \cdots I_k$ where I_i is a P_i-(m,n)-prime ideal of R for $i = 1,2,\cdots,k$. Hence, $N = I_1 I_2 \cdots I_k M$ where $I_i M \neq M$ for some $i \in \{1,...,k\}$ as N is proper. These proper submodules $I_i M$ are (m,n)-closed by Proposition 16. Thus, N is either an (m,n)-closed submodule or it has an (m,n)-factorization. The particular part is clear as a ZPI-ring is an (m,n)-ring. □

In the following, we show that the converse of Theorem 30 also holds if M is finitely generated.

Theorem 31. *Let M be a finitely generated faithful multiplication R-module and n,m be positive integers. If M is an (m,n)-module, then R is an (m,n)-ring.*

Proof. Let I be a proper ideal of R. Then IM is a proper submodule of M since otherwise, $I = (IM :_R M) = R$ by Lemma 14. By assumption, $IM = I_1 I_2 \cdots I_k K$ where I_i's are P_i-(m,n)-prime ideals of R and K is a Q-(m,n)-closed submodule. Since M is multiplication, $K = (K :_R M)M$ and $(K :_R M)$ is a (m,n)-prime ideal by Corollary 5. Thus, $IM = I_1 I_2 \cdots I_k (K :_R M)M$ and $I = (IM : M) = N_1 N_2 \cdots N_k (K :_R M)$ again by Lemma 14. Therefore, R is an (m,n)-ring. □

We end up this section by the (m,n)-closed avoidance theorem. We assume in the rest of this section that M is a finitely generated multiplication R-module and $N, N_1, ..., N_k$ are submodules of M. We recall that a covering $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$ (resp. $N = N_1 \cup N_2 \cup \cdots \cup N_n$) is said to be an efficient covering (resp. efficient union) if no N_k is superfluous (resp. excluded), [17]. A covering (union) of a submodule by two submodules is never efficient.

Theorem 32. *Let $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_k$ be an efficient covering of submodules $N_1, N_2, ..., N_k$ of an R-module M where $k \gneqq 2$. Suppose that $\sqrt{(N_i :_R M)} \nsubseteq \sqrt{(N_j : b)}$ for all $b \in M \backslash M$-rad(N_i) whenever $i \neq j$. Then no N_i $(1 \leq i \leq k)$ is an (m,n)-closed submodule of M for all positive integers n,m such that $n \leq m$.*

Proof. Suppose on contrary that N_j is an (m,n)-closed submodule of M for some $1 \leq j \leq k$. It can be easily observed that $N \subseteq \bigcup_{i=1}^{k} (N_i \cap N)$ is an efficient covering as $N \subseteq \bigcup_{i=1}^{k} N_i$ is efficient. From [17, Lemma 2.1], we conclude the following inclusion

$$\left(\bigcap_{i \neq j} N_i \right) \cap N = \left(\bigcap_{i=1}^{k} N_i \right) \cap N \subseteq N_j \cap N$$

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Choose $r_i \in \sqrt{(N_i : M)} \backslash \sqrt{(N_j : b)}$ where $i \neq j$ and $b \in M \backslash M\text{-}rad(N_j)$. Then, there exists the least positive integer m_i such that $r_i^{m_i} \in (N_i :_R M)$ for each $i \neq j$. Set $r := r_1 r_2 \cdots r_{j-1}$, $s := r_{j+1} r_{j+2} \cdots r_k$ and $m = \max\{m_1, m_2, ..., m_{j-1}, m_{j+1}, ..., m_n\}$. Then $r^m s^m b \in \left(\bigcap_{i \neq j} N_i \right) \cap N$. We show that $r^m s^m b \notin (N_j \cap N)$. For this purpose, assume on the opposite that $r^m s^m b \in N_j \cap N$. Then $r^m s^m \in (N_j :_R b) \subseteq \sqrt{(N_j :_R b)}$. Since $\sqrt{(N_j : b)}$ is a prime ideal and $r_1^m r_2^m \cdots r_{j-1}^m r_{j+1}^m r_{j+2}^m \cdots r_k^m \in \sqrt{(N_j : b)}$, we conclude $r_i \in \sqrt{(N_j :_R b)}$ for some $i \neq j$, a contradiction. Consequently, $r^m s^m b \notin (N_j \cap N)$, and so $r^m s^m b \in \left(\left(\bigcap_{i \neq j} N_i \right) \cap N \right) \setminus (N_j \cap N)$ which contradicts the inclusion $\left(\bigcap_{i \neq j} N_i \right) \cap N \subseteq N_j \cap N$. Therefore, no N_i is an (m, n)-closed submodule. □

Theorem 33. *((m,n)-closed avoidance Theorem) Let n, m be positive integers and let N, $N_1, N_2, ..., N_k$ $(k \geq 2)$ be submodules of an R-module M such that at most two of $N_1, N_2, ..., N_k$ are not (m, n)-closed and $\sqrt{(N_i :_R M)} \nsubseteq \sqrt{(N_j : b)}$ for all $b \in M \backslash M\text{-}rad(N_i)$ whenever $i \neq j$. If $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_k$, then $N \subseteq N_j$ for some $1 \leq j \leq k$.*

Proof. Since a covering of an ideal by two ideals is never efficient, assume that $k \geq 3$. As any covering can be reduced to an efficient covering by omitting any unnecessary terms, we may suppose that $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_k$ is an efficient covering of ideals of R. From Theorem 32, no N_j is an (m, n)-closed submodule of M. However, our assumption states that at most two of N_i's are not (m, n)-closed. Consequently, $N \subseteq N_j$ for some $1 \leq j \leq k$. □

3. (m, n)-closed Submodules of Amalgamation Modules

Let R be a ring, J an ideal of R and M an R-module. The amalgamated duplication of R along J is defined as

$$R \bowtie J = \{(r, r + j) : r \in R, j \in J\}$$

which is a subring of $R \times R$, see [9]. The duplication of the R-module M along the ideal J denoted by $M \bowtie J$ is defined recently in [8] as

$$M \bowtie J = \{(b, b') \in M \times M : b - b' \in JM\}$$

which is an $(R \bowtie J)$-module with scalar multiplication defined by $(r, r + j)(b, b') = (rb, (r + j)b')$ for $r \in R$, $j \in J$ and $(b, b') \in M \bowtie J$. Many properties and results concerning this kind of modules can be found in [8].

Let N be a submodule of an R-module M and J be an ideal of R. Then clearly

$$N \bowtie J = \{(a, b) \in N \times M : a - b \in JM\}$$

and

$$\bar{N} = \{(b, a) \in M \times N : b - a \in JM\}$$

are submodules of $M \bowtie J$.

In general, let $f : R_1 \to R_2$ be a ring homomorphism, J be an ideal of R_2, M_1 be an R_1-module, M_2 be an R_2-module (which is an R_1-module induced naturally by f) and $\varphi : M_1 \to M_2$ be an R_1-module homomorphism. The subring

$$R_1 \bowtie^f J = \{(r, f(r) + j) : r \in R_1, j \in J\}$$

of $R_1 \times R_2$ is called the amalgamation of R_1 and R_2 along J with respect to f. In [12], the amalgamation of M_1 and M_2 along J with respect to φ is defined as

$$M_1 \bowtie^{\varphi} JM_2 = \{(b_1, \varphi(b_1) + b_2) : b_1 \in M_1 \text{ and } b_2 \in JM_2\}$$

which is an $(R_1 \bowtie^f J)$-module with the scalar product defined as

$$(r, f(r) + j)(b_1, \varphi(b_1) + b_2) = (rb_1, \varphi(rb_1) + f(r)b_2 + j\varphi(b_1) + jb_2)$$

For submodules N_1 and N_2 of M_1 and M_2, respectively, clearly the sets

$$N_1 \bowtie^{\varphi} JM_2 = \{(b_1, \varphi(b_1) + b_2) \in M_1 \bowtie^{\varphi} JM_2 : b_1 \in N_1\}$$

and

$$\overline{N_2}^{\varphi} = \{(b_1, \varphi(b_1) + b_2) \in M_1 \bowtie^{\varphi} JM_2 : \ \varphi(b_1) + b_2 \in N_2\}$$

are submodules of $M_1 \bowtie^{\varphi} JM_2$.

In the following two theorems, we justify conditions under which $N_1 \bowtie^{\varphi} JM_2$ and $\overline{N_2}^{\varphi}$ are (m, n)-closed in $M_1 \bowtie^{\varphi} JM_2$.

Theorem 34. *Consider the $(R_1 \bowtie^f J)$-module $M_1 \bowtie^{\varphi} JM_2$ defined as above. Let N_1 be a submodule of M_1 and n, m be positive integers. Then $N_1 \bowtie^{\varphi} JM_2$ is an (m, n)-closed submodule of $M_1 \bowtie^{\varphi} JM_2$ if and only if N_1 is an (m, n)-closed submodule of M_1.*

Proof. First, we note that N_1 is a proper submodule of M_1 if and only if $N_1 \bowtie^{\varphi} JM_2$ is a proper submodule of $M_1 \bowtie^{\varphi} JM_2$.

Suppose $N_1 \bowtie^{\varphi} JM_2$ is (m, n)-closed in $M_1 \bowtie^{\varphi} JM_2$ and let $r_1^m b_1 \in N_1$ for $r_1 \in R_1$ and $b_1 \in M_1$. Then $(r_1, f(r_1)) \in R_1 \bowtie^f J$ and $(b_1, \varphi(b_1)) \in M_1 \bowtie^{\varphi} JM_2$ with $(r_1, f(r_1))^m (b_1, \varphi(b_1)) = (r_1^m b_1, \varphi(r_1^m b_1)) \in N_1 \bowtie^{\varphi} JM_2$. Thus, either $(r_1, f(r_1))^n \in (N_1 \bowtie^{\varphi} JM_2 :_{R_1 \bowtie^f J} M_1 \bowtie^{\varphi} JM_2)$ or $(b_1, \varphi(b_1)) \in N_1 \bowtie^{\varphi} JM_2$. In the first case, for all $b \in M_1$, $(r_1, f(r_1))^n (b, \varphi(b)) \in N_1 \bowtie^{\varphi} JM_2$ and so $r_1^n M_1 \subseteq N_1$. In the second case, $b_1 \in N_1$ and so N_1 is an (m, n)-closed submodule of M_1. Conversely, let $(r_1, f(r_1) + j) \in R_1 \bowtie^f J$ and $(b_1, \varphi(b_1) + b_2) \in M_1 \bowtie^{\varphi} JM_2$ such that $(r_1, f(r_1) + j)^m (b_1, \varphi(b_1) + b_2) \in N_1 \bowtie^{\varphi} JM_2$. Then $r_1^m b_1 \in N_1$ and hence either $r_1^n \in (N_1 :_{R_1} M_1)$ or $b_1 \in N_1$. If $r_1^n \in (N_1 :_{R_1} M_1)$, then clearly $(r_1, f(r_1) + j)^n \in (N_1 \bowtie^{\varphi} JM_2 :_{R_1 \bowtie^f J} M_1 \bowtie^{\varphi} JM_2)$ and if $b_1 \in N_1$, then $(b_1, \varphi(b_1) + b_2) \in N_1 \bowtie^{\varphi} JM_2$. Therefore, $N_1 \bowtie^{\varphi} JM_2$ is an (m, n)-closed submodule of $M_1 \bowtie^{\varphi} JM_2$. □

Theorem 35. *Consider the $(R_1 \bowtie^f J)$-module $M_1 \bowtie^{\varphi} JM_2$ defined as in Theorem 34 where f and φ are epimorphisms. Let N_2 be a submodule of M_2 and n, m be positive integers. Then*

1. N_2 is an (m, n)-closed submodule of M_2 if and only if $\overline{N_2}^{\varphi}$ is an (m, n)-closed submodule of $M_1 \bowtie^{\varphi} JM_2$.
2. If $\overline{N_2}^{\varphi}$ is an (m, n)-closed submodule of $M_1 \bowtie^{\varphi} JM_2$ and $J \nsubseteq (N_2 :_{R_2} M_2)$, then $(N_2 :_{M_2} J)$ is an (m, n)-closed submodule of M_2.

Proof. (1) Suppose N_2 is an (m, n)-closed submodule of M_2. Suppose $\overline{N_2}^{\varphi} = M_1 \bowtie^{\varphi} JM_2$ and let $b_2 = \varphi(b_1) \in M_2$ for some $b_1 \in M_1$. Then $(b_1, b_2) \in M_1 \bowtie^{\varphi} JM_2 = \overline{N_2}^{\varphi}$ and so $b_2 \in N_2$. Thus, $N_2 = M_2$ which is a contradiction. Therefore, $\overline{N_2}^{\varphi}$ is proper in $M_1 \bowtie^{\varphi} JM_2$. Let $(r_1, f(r_1) + j) \in R_1 \bowtie^f J$ and $(b_1, \varphi(b_1) + b_2) \in M_1 \bowtie JM_2$ such that $(r_1, f(r_1) + j)^m (b_1, \varphi(b_1) + b_2) \in \overline{N_2}^{\varphi}$. Then $(f(r_1) + j)^m (\varphi(b_1) + b_2) \in N_2$ and so $(f(r_1) + j)^n \in (N_2 :_{R_2} M_2)$ or $(\varphi(b_1) + b_2) \in N_2$. If $(f(r_1) + j)^n \in (N_2 :_{R_2} M_2)$, then for all $(b_1, \varphi(b_1) + b_2) \in M_1 \bowtie^{\varphi} JM_2$, clearly $(r_1, f(r_1) + j)^n (b_1, \varphi(b_1) + b_2) \in \overline{N_2}^{\varphi}$ and so $(r_1, f(r_1) + j)^n \in (\overline{N_2}^{\varphi} :_{R_1 \bowtie^f J} M_1 \bowtie^{\varphi} JM_2)$. If $(\varphi(b_1) + b_2) \in N_2$, then $(b_1, \varphi(b_1) + b_2) \in \overline{N_2}^{\varphi}$ and the result follows.

Conversely, suppose $\overline{N_2}^{\varphi}$ is an (m,n)-closed submodule of $M_1 \bowtie^{\varphi} JM_2$. Clearly, N_2 is proper in M_2. Let $r_2 = f(r_1) \in R_2$ and $b_2 = \varphi(b_1) \in M_2$ such that $r_2^m b_2 \in N_2$. Then $(r_1, r_2) \in R_1 \bowtie^f J$ and $(b_1, b_2) \in M_1 \bowtie^{\varphi} JM_2$ with $(r_1, r_2)^m (b_1, b_2) \in \overline{N_2}^{\varphi}$. Thus, $(r_1, r_2)^n \in (\overline{N_2}^{\varphi} :_{R_1 \bowtie^f J} M_1 \bowtie^{\varphi} JM_2)$ or $(b_1, b_2) \in \overline{N_2}^{\varphi}$. If $(r_1, r_2)^n (M_1 \bowtie^{\varphi} JM_2) \subseteq \overline{N_2}^{\varphi}$, then for all $b = \varphi(b') \in M_2$, we have $(r_1, r_2)^n (b', b) \in \overline{N_2}^{\varphi}$ and so $r_2^n \in (N_2 :_{R_2} M_2)$. If $(b_1, b_2) \in \overline{N_2}^{\varphi}$, then $b_2 \in N_2$ and we are done.

(2) Since $J \nsubseteq (N_2 :_{R_2} M_2)$, $(N_2 :_{M_2} J)$ is proper in M_2. Suppose $\overline{N_2}^{\varphi}$ is an (m,n)-closed submodule of $M_1 \bowtie^{\varphi} JM_2$. Let $r_2 = f(r_1) \in R_2$, $b_2 \in M_2$ such that $r_2^m b_2 \in (N_2 :_{M_2} J)$. Then $r_2^m J b_2 \subseteq N_2$ and so we have $(r_1, r_2)^m (0, Jb_2) \subseteq \overline{N_2}^{\varphi}$. By assumption, $(r_1, r_2)^n \in (\overline{N_2}^{\varphi} :_{R_1 \bowtie^f J} M_1 \bowtie^{\varphi} JM_2)$ or $(0, Jb_2) \subseteq \overline{N_2}^{\varphi}$. If $(r_1, r_2)^n \in (\overline{N_2}^{\varphi} :_{R_1 \bowtie^f J} M_1 \bowtie^{\varphi} JM_2)$, then simple computations give $r_2^n \in (N_2 :_{R_2} JM_2) = ((N_2 :_{M_2} J) :_{R_2} M_2)$. If $(0, Jb_2) \subseteq \overline{N_2}^{\varphi}$, then $Jb_2 \subseteq N_2$ and so $b_2 \in (N_2 :_{M_2} J)$, as required. □

In particular, we have the following corollaries of the previous two theorems.

Corollary 36. *Let N be a submodule of an R-module M, J be an ideal of R and n, m be positive integers. Then $N \bowtie J$ is an (m,n)-closed submodule of $M \bowtie J$ if and only if N is an (m,n)-closed submodule of M.*

Corollary 37. *Let N be a submodule of an R-module M, J be an ideal of R and n, m be positive integers. Then*

1. N is an (m,n)-closed submodule of M if and only if $\overline{N}$ is an (m,n)-closed submodule of $M \bowtie J$.
2. If $\overline{N}$ is an (m,n)-closed submodule of $M \bowtie J$ and $J \nsubseteq (N :_R M)$, then $(N :_M J)$ is an (m,n)-closed submodule of M.

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