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Article

A Review of Wormhole Stabilization in $f(R)$ Gravity Theories

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Abstract: It has been proven that in standard Einstein gravity, exotic matter (i.e. matter violating the point-wise and averaged weak and null energy conditions) is required to stabilize traversable wormholes. Quantum field theory permits these violations due to the quantum coherent effects found in any quantum field [1]. Even reasonable classical scalar fields violate the energy conditions. In the case of the Casimir effect and squeezed vacuum states, these violations have been experimentally proven. It is advantageous to investigate methods of minimizing the use of exotic matter. One such area of interest is extended theories of Einstein gravity. It has been claimed that in some extended theories, stable traversable wormholes solutions can be found without the use of exotic matter. There are many extended theories of gravity and in this review paper we first explore $f(R)$ theories and then explore some wormhole solutions in $f(R)$ theories including Lovelock gravity.

Keywords: traversable wormhole; modified gravity; extended gravity; $f(R)$ theory; wormhole stabilization

1. Introduction

The concept of traversable wormholes which allow inter and intra-universe travel by humans was first introduced by Morris and Thorne in their classic 1988 paper [2]. It is well known that the Weak and Null Energy Conditions (WEC and NEC) [3,4] must be violated as a minimum for a stable human traversable wormhole. The violation of the Null energy condition is needed due to the flare-out condition requirement, i.e., the throat should open up outward as a human travels through the wormhole [2,3]. There is a claim [5] in the published literature about stable traversable wormholes that can be constructed within the framework of Einstein's General Relativity (henceforth called 'Einstein Gravity' in this paper) but without the need for exotic matter. This claim has been refuted by other authors [6]. Some of the suggested reasons in [6] are: (i) possible error in calculations; (ii) failing to check for violations at the throat, such as divergence of the inverse of the metric; and (iii) failing to check for discontinuities in the exterior curvature in the vicinity of the throat, based on the thin shell formalism.

There are many attempts to use various modified gravity theories to check the existence of stable traversable wormholes in these theories. In many of these studies [7–9], the Morris-Thorne (MT) traversable wormhole has been used to see the effects of a modified gravity background on the stability of the wormhole. The various modified gravity theories that we surveyed are $f(R)$ theories, $f(T)$ theories, $f(R, \mathcal{T})$ theories, Lovelock gravity (a special case of $f(R)$ theory), Einstein-Gauss-Bonnet (EGB) gravity (another special case of $f(R)$ theory), Brans-Dicke theory and Kaluza-Klein theory.

Lovelock gravity and EGB gravity are $f(R)$ theories that include higher order curvature terms and are also applicable to higher dimensional spacetimes. For example, in EGB gravity [3], the authors replace the 2-sphere in the MT wormhole with an $(n-2)$ -sphere and thus the MT wormhole line element is modified as follows:

$$ds^2 = -e^{2\phi(r)} dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2 d\Omega_{n-2}^2. \quad (1)$$

The standard Einstein-Hilbert action of General Relativity (GR), is

$$S = \int R \sqrt{-g} d^4x. \quad (2)$$

In $f(R)$ theories [10], the Ricci scalar R is replaced by a function of the Ricci Scalar $f(R)$ as follows:

$$S = \int f(R) \sqrt{-g} d^4x, \quad (3)$$

where $f(R)$ is a function of the Ricci scalar and g is the determinant of the metric. Examples of commonly used functions $f(R)$ are: $f(R) = R$, $f(R) = R + \alpha R^n$, and $f(R) = R + \alpha e^{(\beta R)}$.

In teleparallel gravity [11,12], the Ricci scalar R is replaced by the torsion scalar T as follows:

$$S = \int T \sqrt{-g} d^4x. \quad (4)$$

In GR, we assume spinless particles to follow the geodesic of the underlying spacetime and hence we have only R in the action and no T . In teleparallel gravity, T replaces R . This interpretation holds true in the case where we see teleparallel gravity as a gauge theory [13] for the translation group.

The $f(T)$ gravity [14] is an additional modification to teleparallel gravity, in which T is replaced by the torsion function which is a function of T , namely $f(T)$, as follows:

$$S = \int f(T) \sqrt{-g} d^4x, \quad (5)$$

and we can obtain the modified field equations by varying the action S with respect to the metric in the same way it is done in section 2.1 of this paper for $f(R)$ theory.

T is a fundamental geometric quantity in the context of theories of gravity that involve spacetime torsion. Unlike GR, where spacetime curvature plays a central role in describing gravity, theories that incorporate torsion consider the twisting or non-metricity of spacetime, and the torsion scalar is a measure of this twisting. The torsion scalar is given by the equation

$$T = S_{\alpha\beta\mu} T^{\alpha\beta\mu}, \quad (6)$$

where $S_{\alpha\beta\mu}$ is the contorsion tensor, which is the difference between the affine (Levi-Civita) connection's components and the Weitzenböck connection's components. The contorsion tensor quantifies nonmetricity and describes how spacetime is twisted. The indices α, β, μ refer to spacetime coordinates. In teleparallel gravity theory, the teleparallel connection is used instead of the metric affine connection that is used in GR.

In the $f(R, \mathcal{T})$ theory, $\mathcal{T}$ is the trace of the energy-momentum tensor $T_{\mu\nu}$. Similar to $f(R)$ theories [7], the presence of $f(R, \mathcal{T})$ in the action leads to changes in gravitational dynamics as compared to Einstein gravity. These modifications can have implications on the behaviour of gravitational fields in various contexts. It also has consequences in cosmology and gravitational lensing. An example of an $f(R, \mathcal{T})$ function used to stabilize a wormhole is

$$f(R, \mathcal{T}) = R + \alpha R^2 + \lambda \mathcal{T}, \quad (7)$$

where α and λ are constants. In $f(R, \mathcal{T})$ theory, wormhole solutions with normal matter are feasible when appropriate shape functions are used. The coupling parameters α and λ in the action of $f(R, \mathcal{T})$ theory play an important role in accommodating the composition of matter. According to [7], when $\alpha < 0$, wormholes exist in the presence of exotic matter and when $\alpha > 0$, wormholes exist even in the absence of exotic matter.

In Brans-Dicke theory [15], a scalar field ϕ is introduced to modify the standard Einstein-Hilbert action as follows [16]:

$$S = \int [\phi R - \frac{\omega}{\phi} \phi_{;\mu} \phi^{;\mu} + \mathcal{L}_m] \sqrt{-g} d^4x, \quad (8)$$

where ϕ is the Brans-Dicke scalar field, $\phi_{;\mu}$ is the covariant derivative of ϕ and ω is a coupling constant that couples the Brans-Dicke scalar field ϕ with the gravitational field. $\mathcal{L}_m$ is the Lagrangian density for the matter field(s).

The field equations in Brans-Dicke theory can be obtained as always by varying the action. These field equations are modified as compared to Einstein's field equations in GR, and can be expressed in terms of the scalar field ϕ and its derivatives, as well as the metric tensor, the curvature tensors and scalars. The value of the coupling parameter ω affects the behavior of the theory and can influence the stability of wormholes.

Kaluza-Klein (KK) theory [17] extends the usual four-dimensional spacetime of GR to include one or more extra dimensions. These extra dimensions are compactified or 'curled up' so small that they are not perceptible on macroscopic scales. The total spacetime is a product of the usual four-dimensional spacetime and the compactified extra dimensions.

In KK theory, the metric tensor describes the geometry of the higher-dimensional spacetime. It has components corresponding to both the usual four dimensions denoted by μ and ν and the extra dimensions denoted by a and b . The metric tensor can be decomposed into a 4-dimensional part and an extra-dimensional part. The extra-dimensional part manifests as additional vector or scalar fields, and these fields are typically associated with electromagnetic interactions. This process of decomposing the higher dimensional metric tensor field is called dimensional reduction.

The action of KK theory after dimensional reduction is

$$S = \int [\mathcal{R} - F^{ab} F_{ab} - L_m] \sqrt{-g} d^4x, \quad (9)$$

where d^4x is the 4-dimensional spacetime volume element, $\mathcal{R}$ is the 4-dimensional scalar curvature, F^{ab} is the electromagnetic field strength tensor and L_m is the usual Lagrangian for the matter field(s).

In the remainder of this paper, we will focus on wormholes in $f(R)$ theories including Lovelock gravity. Wormholes in EGB gravity will be covered in a planned follow up paper on wormholes in higher dimensional theories. We will begin with describing $f(R)$ gravity theories in Section 2. In Section 3, we provide a synopsis of the study of wormholes in each of these $f(R)$ theories of gravity.

2. Modified Gravity Theories

There have been efforts to construct stable traversable wormholes in $f(R)$ theories including Lovelock gravity. In this section, we give a brief overview of the basics of these theories to provide us with enough background to explore in the next section the properties of traversable wormholes.

2.1. $f(R)$ Gravity Theories

We started the discussion of $f(R)$ theories in the Introduction. To recap, in $f(R)$ theories [10,18–20] R is replaced by $f(R)$ as follows:

$$S = \int \frac{1}{2\kappa} f(R) \sqrt{-g} d^4x, \quad (10)$$

where $\kappa \equiv \frac{8\pi G}{c^4}$.

In $f(R)$ theories, there are two formalisms [21,22] to derive the Einstein field equations from the action. They are: (i) The metric formalism in which a matter term $S_m(g_{\mu\nu}, \psi)$ is added to the action, where ψ represents the matter field(s). The action is then varied with respect to the metric, by not treating the connections $\Gamma_{\alpha\beta}^\mu$, independently, to obtain the field equations. (ii) The Palatini formalism in which an independent variation is done with respect to the metric and with respect to the connection.

The action is the same but the curvature tensors and Ricci scalar are constructed with this independent connection.

Here we will follow [18,19] and show the derivation of the Einstein field equations in much more detail using the metric formalism. Before we vary the action, we first vary each of the quantities in the action. The variation of the determinant is

$$\delta\sqrt{-g} = -(1/2)\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}. \quad (11)$$

The Ricci Scalar is $R = g^{\mu\nu}R_{\mu\nu}$ and the variation of R with respect to $g^{\mu\nu}$ is

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}, \quad (12)$$

$$= R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}(\nabla_\rho\delta\Gamma_{\nu\mu}^\rho - \nabla_\nu\delta\Gamma_{\rho\mu}^\rho), \quad (13)$$

where

$$\delta\Gamma_{\mu\nu}^\rho = (1/2)g^{\rho\alpha}(\nabla_\mu\delta g_{\alpha\nu} + \nabla_\nu\delta g_{\alpha\mu} - \nabla_\alpha\delta g_{\mu\nu}), \quad (14)$$

$\Gamma_{\mu\nu}^\rho$ is the Christoffel symbol representing the Levi-Civita connection, and ∇_μ is a covariant derivative. Now substituting (14) in (13), we get

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g_{\mu\nu}\square\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu}, \quad (15)$$

where $\square \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ is known as the D'Alembert operator. The variation of $f(R)$ is

$$\delta f(R) = \frac{df(R)}{dR}\delta R. \quad (16)$$

Let $f'(R) \equiv \frac{df(R)}{dR}$. Then,

$$\delta f(R) = f'(R)\delta R. \quad (17)$$

By varying the action (10),

$$\delta S = \int \frac{1}{2\kappa}(\delta f(R)\sqrt{-g} + f(R)\delta\sqrt{-g})d^4x. \quad (18)$$

Substituting for $\delta f(R)$ and $\delta\sqrt{-g}$ from (17) and (11) into (18), we get

$$\delta S = \int \frac{1}{2\kappa}\left(f'(R)\delta R\sqrt{-g} - \frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}f(R)\right)d^4x. \quad (19)$$

Now substitute for δR from (15) into (19), to get

$$\delta S = \int \frac{\sqrt{-g}}{2\kappa}\left[f'(R)\left(R_{\mu\nu}\delta g^{\mu\nu} + (g_{\mu\nu}\square - \nabla_\mu\nabla_\nu)\delta g^{\mu\nu}\right) - \frac{1}{2}g_{\mu\nu}f(R)\delta g^{\mu\nu}\right]d^4x. \quad (20)$$

After Integrating by parts and factoring out $\delta g^{\mu\nu}$,

$$\delta S = \int \frac{1}{2\kappa}\sqrt{-g}\delta g^{\mu\nu}\left[f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + (g_{\mu\nu}\square - \nabla_\mu\nabla_\nu)f'(R)\right]d^4x. \quad (21)$$

Finally, by requiring that the action remain invariant with the variation of the metric, we obtain the field equations for $f(R)$ modified gravity theory,

$$f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + (g_{\mu\nu}\square - \nabla_\mu\nabla_\nu)f'(R) = \kappa T_{\mu\nu}, \quad (22)$$

where,

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}}, \quad (23)$$

and L_m is the Lagrangian for matter.

2.2. Lovelock gravity theory

Introduced in 1971, Lovelock's theory of gravity [23] is considered to be the most generalized extension to the theory of gravitation in n dimensions, because it satisfies the requirements of GR that the field equations be covariant and not include more than the second order derivatives of the metric. The rest of this section mostly follows [8,24].

The Einstein field equations for GR including the cosmological term, are given by,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \quad (24)$$

In Lovelock gravity, higher order curvature terms are added to the left hand side of (24), while usually excluding the cosmological term. Excluding $\Lambda g_{\mu\nu}$ and including the higher order curvature terms up to third order, the generalized field equations for Lovelock gravity in n dimensions are given by,

$$G_{\mu\nu}^{(1)} + \sum_{n=2}^3 \alpha_n (H_{\mu\nu}^{(n)} - (1/2)g_{\mu\nu}L^{(n)}) = \kappa_n T_{\mu\nu}. \quad (25)$$

Here, α_n 's are the Lovelock coefficients and n runs from 2 to 3, indicating second and third order. $T_{\mu\nu}$ is the energy-momentum tensor, $G_{\mu\nu}^{(1)}$ is the Einstein tensor and its superscript (1) indicates order 1.

The second order Lagrangian term is

$$L^{(2)} = R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \quad (26)$$

which is also known as the Gauss-Bonnet Lagrangian. The third order Lovelock Lagrangian is given by

$$L^{(3)} = 2R^{\mu\nu\sigma\kappa}R_{\mu\kappa\rho\tau}R_{\nu\sigma}^{\rho\tau} + 8R_{\sigma\rho}^{\mu\nu}R_{\nu\tau}^{\sigma\kappa}R_{\mu\kappa}^{\rho\tau} + 24R^{\mu\nu\sigma\kappa}R_{\sigma\kappa\nu\rho}R_{\mu}^{\rho} + 3RR^{\mu\nu\sigma\kappa}R_{\sigma\kappa\mu\nu} \\ + 24R^{\mu\nu\sigma\kappa}R_{\sigma\mu}R_{\kappa\nu} + 16R^{\mu\nu}R_{\nu\sigma}R_{\mu}^{\sigma} - 12RR^{\mu\nu}R_{\mu\nu} + R^3, \quad (27)$$

and $H_{\mu\nu}^{(2)}$ and $H_{\mu\nu}^{(3)}$ are defined by

$$H_{\mu\nu}^{(2)} \equiv 2(R_{\mu\sigma\kappa\tau}R_{\nu}^{\sigma\kappa\tau} - 2R_{\mu\rho\nu\sigma}R^{\rho\sigma} - 2R_{\mu\sigma}R_{\nu}^{\sigma} + RR_{\mu\nu}), \quad (28)$$

and

$$H_{\mu\nu}^{(3)} \equiv -3(4R^{\rho\sigma\kappa}R_{\sigma\kappa\lambda\rho}R_{\nu\tau\mu}^{\lambda} - 8R_{\lambda\sigma}^{\tau\rho}R_{\tau\mu}^{\sigma\kappa}R_{\nu\rho\kappa}^{\lambda} \\ + 2R_{\nu}^{\tau\sigma\kappa}R_{\sigma\kappa\lambda\rho}R_{\tau\mu}^{\lambda\rho} - R^{\tau\rho\sigma\kappa}R_{\sigma\kappa\tau\rho}R_{\nu\mu} \\ + 8R_{\nu\sigma\rho}^{\tau}R_{\tau\mu}^{\sigma\kappa}R_{\kappa}^{\rho} + 8R_{\nu\tau\kappa}^{\sigma}R_{\sigma\mu}^{\tau\rho}R_{\rho}^{\kappa} + 4R_{\nu}^{\tau\sigma\kappa}R_{\sigma\kappa\mu\rho}R_{\tau}^{\rho} \\ - 4R_{\nu}^{\tau\sigma\kappa}R_{\sigma\kappa\tau\rho}R_{\mu}^{\rho} + 4R^{\tau\rho\sigma\kappa}R_{\sigma\kappa\tau\mu}R_{\nu\rho} - 4RR_{\nu\mu\rho}^{\tau}R_{\tau}^{\rho} + 4R^{\tau\rho}R_{\rho\tau}R_{\nu\mu} \\ - 8R_{\nu}^{\tau}R_{\tau\rho}R_{\mu}^{\rho} + 4RR_{\nu\rho}R_{\mu}^{\rho} - R^2R_{\mu\nu}) \quad (29)$$

3. Fundamentals of Morris-Thorne (MT) Wormhole Stabilization

In section 3.1, we give a brief background of how the stability of a wormhole (usually MT wormhole) is studied in GR. In section 3.2, we summarize a general methodology to study wormholes in $f(R)$ gravity theories based on the various papers we reviewed on this subject. This background

will be useful to follow section 4, where we review these calculations in greater detail for general $f(R)$ theories and Lovelock gravity theory.

3.1. MT Wormhole Stabilization in GR

In [2], Morris and Thorne clearly explain in great detail why blackholes and Schwarzschild wormholes are not traversable. However, the MT wormhole is designed to be made traversable if it has the following properties that ensures wormhole stability for traversability. The MT metric allows for the realization of faster-than-light interstellar space travel that does not violate the special relativistic light speed limit. The metric should be spherically symmetric and static (time independent). The following metric has these properties [2]:

$$ds^2 = -e^{2\Phi} dt^2 + [1 - (b/r)]^{-1} dr^2 + r^2 [d\theta^2 + \sin^2 \theta d\phi^2]. \quad (30)$$

The solution must obey the Einstein field equations as does (30). The solution must have a throat that connects two asymptotically flat regions of spacetime. The spatial geometry must have a wormhole shape consistent with the well known Flamm diagram for a spherically-symmetric throat. This puts the following constraints on the shape function $b(r)$ and redshift function $\Phi(r)$:

- The throat is at minimum of r , specified as r_0 .
- $b(r)$ is finite, continuous, and differentiable.
- In this spacetime, $(1 - b/r) \geq 0$, which implies $b/r \leq 1$, and so $b(r) \leq r$.
- Proper radial distance is defined by

$$l \equiv \int_{r_0}^r \frac{1}{\sqrt{1 - (b/r)}} dr,$$

and should be real and finite for $r > r_0$.

- As $l \rightarrow \pm\infty$ (asymptotically flat regions of the Universe), $b/r \rightarrow 0$ and so $r \sim |l|$.
- There should be no horizons, since it will prevent 2-way travel through the wormhole. There are no singularities. This implies that Φ is finite, continuous, and differentiable everywhere, and the fact that τ measures proper time in asymptotically flat regions implies $\Phi \rightarrow 0$ as $l \rightarrow \pm\infty$.
- The flare-out condition

$$\frac{(b - b'r)}{2b^2} > 0,$$

and so, $rb' < b$. That is, the throat of the wormhole must expand outward from the central point. The throat of the wormhole must open up as one travels through it.

The tidal gravitational forces experienced by a traveler must be $\leq g$, where g is the acceleration due to Earth's gravity. This condition is not a rigid requirement.

The procedure to check the stability of this traversable wormhole in GR involves the following steps:

Compute Curvature tensors. Here, we give as briefly as possible the method to calculate the curvature tensors. First, using the MT metric written in the form

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta, \quad (31)$$

with $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$, the connection coefficient $\Gamma_{\beta\gamma}^\alpha$ and the components of the Riemann curvature tensor $R_{\beta\gamma\delta}^\alpha$ are calculated using the standard equations

$$\Gamma_{\beta\gamma}^\alpha = \frac{1}{2} g^{\alpha\lambda} (g_{\lambda\beta,\gamma} + g_{\lambda\gamma,\beta} - g_{\beta\gamma,\lambda}), \quad (32)$$

and

$$R_{\beta\gamma\delta}^\alpha = \Gamma_{\beta\delta,\gamma}^\alpha - \Gamma_{\beta\gamma,\delta}^\alpha + \Gamma_{\lambda\gamma}^\alpha \Gamma_{\beta\delta}^\lambda - \Gamma_{\lambda\delta}^\alpha \Gamma_{\beta\gamma}^\lambda, \quad (33)$$

where the comma denote partial derivatives. By applying these equations to the metric, we can get the 24 non-zero components of the Riemann tensor as shown in (5) of [2]. These were obtained using the basis vectors $(e_t, e_r, e_\theta, e_\phi)$.

To **simplify further calculations**, we can switch to the following orthonormal basis vectors,

- $\hat{e}_t = e^{-\Phi} e_t,$
- $\hat{e}_r = (1 - \frac{b}{r})^{-1/2} e_r,$
- $\hat{e}_\theta = r^{-1} e_\theta,$
- $\hat{e}_\phi = (r \sin \theta)^{-1} e_\phi.$

In this basis, the metric coefficients become the same as that of flat (Minkowski) spacetime,

$$g_{\alpha\beta} = \hat{e}_\alpha \hat{e}_\beta = \eta_{\alpha\beta} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (34)$$

and the 24 non-zero components of the Riemann tensor take a much simpler form as seen in (8) of [2].

Contraction. Next, by contracting the Riemann tensor we get the Ricci tensor,

$$R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}, \quad (35)$$

and again, by contracting the Ricci tensor, we get the Ricci scalar R.

Compute Einstein tensor. We finally obtain the Einstein tensor $G_{\mu\nu}$ from the metric, Ricci tensor and the Ricci scalar. This forms the left hand side of the Einstein field equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}. \quad (36)$$

This yields the non-zero components of $G_{\mu\nu}$, in terms of the shape function and red-shift function, namely,

$$G_{tt} = \frac{b'}{r^2}, \quad (37)$$

$$G_{rr} = \frac{-b}{r^3} + 2(1 - \frac{b}{r}) \frac{\Phi'}{r}, \quad (38)$$

and

$$G_{\theta\theta} = G_{\phi\phi} = (1 - \frac{b}{r}) \left[\Phi'' - \frac{(b'r - b)}{2r(r - b)} \Phi' + (\Phi')^2 + \frac{\Phi'}{r} - \frac{(b'r - b)}{2r^2(r - b)} \right]. \quad (39)$$

Compute stress-energy tensor. The next step involves computing the stress-energy tensor (right hand side of Einstein field equations). Based on Birkhoff's theorem, which states that 'any spherically symmetric solution of the vacuum field equations must be static and asymptotically flat', the exterior solution must be given by the Schwarzschild metric (with certain modifications), which is a spherical wormhole. Therefore, we cannot have a vacuum solution for a traversable wormhole, which implies that our wormhole must be threaded by matter with a non-zero stress-energy tensor. Based on Einstein's field equations,

$$G_{\mu\nu} = \kappa T_{\mu\nu}. \quad (40)$$

$T_{\mu\nu}$ must have the same algebraic structure as $G_{\mu\nu}$ in the orthonormal basis that we have chosen. Similar to $G_{\mu\nu}$, only the 4 components T_{tt} , T_{rr} , $T_{\theta\theta}$ and $T_{\phi\phi}$ are non-zero. Based on a remote static observer's measurement, each of the components have a simple physical interpretation as follows:

$$T_{tt} = \rho(r), \quad (41)$$

is the total rest-energy density that the static observer measures,

$$T_{rr} = -\tau(r) \quad (42)$$

is the radial tension measured per unit area, and

$$T_{\theta\theta} = T_{\phi\phi} = p(r) \quad (43)$$

is the tangential pressure measured per unit area in a direction orthogonal to the radial tension $\tau(r)$. This give us the final stress-energy tensor,

$$T_{\mu\nu} = \begin{bmatrix} \rho(r) & 0 & 0 & 0 \\ 0 & -\tau(r) & 0 & 0 \\ 0 & 0 & p(r) & 0 \\ 0 & 0 & 0 & p(r) \end{bmatrix}. \quad (44)$$

Engineer the traversable wormhole. In the next step, we need to ‘engineer’ the traversable wormhole to obtain the properties enumerated earlier in this section. This can be done by controlling the shape function b and the redshift function Φ based on a suitable $T_{\mu\nu}$ that we require. We substitute the $G_{\mu\nu}$ and $T_{\mu\nu}$ found in the previous steps into (40), to solve for ρ , $\tau(r)$, and p in terms of $b(r)$ and $\Phi(r)$ to get,

$$\rho(r) = \frac{b'}{r^2}, \quad (45)$$

$$\tau(r) = \frac{b}{r^3} - \frac{2(r-b)}{r^2} \Phi', \quad (46)$$

$$p(r) = \frac{r}{2} [(\rho - \tau) \Phi' - \tau'] - \tau. \quad (47)$$

Our strategy for stabilization of the MT wormhole in Einstein gravity will involve tailoring $b(r)$ and $\Phi(r)$ to build a wormhole with the required properties. Our choice of $b(r)$ gives us $\rho(r)$. Our choice of $b(r)$ and $\Phi(r)$ will give us the tangential pressure $\tau(r)$. We next find $p(r)$ using $\rho(r)$, $\tau(r)$ and $\Phi(r)$. We then ensure the (averaged) NEC and WEC are violated which means that the source of matter for traversable wormholes must be exotic.

With this process for analyzing stability of the MT wormhole in GR, many authors in our review have used similar techniques to analyze the stability of the Morris-Thorne wormhole in $f(R)$ gravity theories. We will hone into the nuances of these analysis in the following sections.

3.2. A General Methodology for MT Wormhole Stabilization in $f(R)$ Gravity Theories

Based on Section 4, we summarize briefly, the method that can be used to stabilize the MT wormhole in $f(R)$ gravity theories. First, require that matter threading the wormhole satisfy the NEC and WEC. Include the required flare-out condition for traversable wormholes. Delegate the required energy condition violations, and sustenance of the non-standard wormhole geometry to the higher order curvature terms, including the derivative terms ($T_{\mu\nu}^{(c)}$). Consider a redshift function ($\Phi = c$, $\Phi' = 0$), where c is a constant, which simplifies calculations and still provides physically relevant solutions. This condition defines an ultrastatic traversable wormhole and is a very narrow and specific subclass of solutions called zero tidal force (ZTF) solutions. Specify $b(r)$. Some examples used in [18] are $b(r) = \frac{r_0^2}{r}$, $b(r) = \sqrt{r_0 r}$ and $b(r) = r_0 + \gamma^2 r_0 (1 - \frac{r_0}{r})$ with $0 < r < 1$. For each shape function $b(r)$, assume an equation of state such as $p_r = p_r(\rho)$ and $p_t = p_t(\rho)$. Find $F(r)$ (see Section 4.1) from the modified gravitational field equations with Ricci curvature scalar $R(r)$ obtained from the MT metric. Finally, obtain the exact $f(R)$ that we need from the trace equation (48).

4. Wormholes in $f(R)$ Gravity Theories

4.1. Wormholes in $f(R)$ gravity theory

In this section of the review, we mainly follow the references [10,18–20] to analyze the stability of MT wormhole in $f(R)$ gravity theories. In Section 2.1, we obtained (22) for $f(R)$ gravity theory. Contracting this field equation, we get the trace of $T_{\mu\nu}$,

$$FR - 2f(R) + 3\Box F = T, \quad (48)$$

where $F = f'(R)$ is used for convenience, R is the Ricci scalar, and $T = T^\mu_\mu$ is the trace of the stress-energy tensor $T_{\mu\nu}$. Substituting (48) into (22) and rearranging terms, we get an updated field equation,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}^{\text{eff}}. \quad (49)$$

We will call (49) the effective field equations for $f(R)$ gravity theory, where

$$T_{\mu\nu}^{\text{eff}} = T_{\mu\nu}^{(c)} + T_{\mu\nu}^{(m)}. \quad (50)$$

$T_{\mu\nu}^{(c)}$ is the curvature stress-energy tensor for higher order curvature given by

$$T_{\mu\nu}^{(c)} = \frac{1}{F} \left[\nabla_\mu \nabla_\nu F - \frac{1}{4} g_{\mu\nu} (RF + \Box F + T) \right], \quad (51)$$

and $T_{\mu\nu}^{(m)} = \frac{T_{\mu\nu}}{F}$. In analyzing the stability of the MT wormhole in GR, we imposed the condition that matter threading the wormhole satisfy the energy conditions. Here, we impose the same condition and use anisotropic distribution of matter as follows:

$$T_{\mu\nu} = (\rho + p_t)u_\mu u_\nu + p_t g_{\mu\nu} + (p_r - p_t)x_\mu x_\nu, \quad (52)$$

such that $u^\mu u_\mu = -1$ and $x^\mu x_\mu = -1$, where u^μ is a 4-velocity vector and

$$x^\mu = \sqrt{1 - \frac{b'(r)}{r}} \delta_r^\mu$$

is a unit spacelike vector in the radial direction, $\rho(r)$ is the rest-energy density, $p_r(r)$ is the radial tension, and $p_t(r)$ is the tangential pressure orthogonal to x^μ . This gives us

$$T_{\mu\nu} = \begin{bmatrix} -\rho(r) & 0 & 0 & 0 \\ 0 & p_r(r) & 0 & 0 \\ 0 & 0 & p_r(r) & 0 \\ 0 & 0 & 0 & p_t(r) \end{bmatrix} \quad (53)$$

and

$$T = (T^\mu_\mu) = (-\rho + 2p_r + p_t). \quad (54)$$

Substituting this value of T in the trace form of the field equations for $f(R)$ gravity (48), we get

$$FR - 2f + 3\Box F = (-\rho + 2p_r + p_t). \quad (55)$$

We now use the MT wormhole metric (30) just as we did in GR, where $\Phi(r)$ is the redshift function and $b(r)$ is the shape function as before. The radial coordinate r is non-monotonic and decreases from

∞ to a minimum value r_0 at the wormhole throat. At the throat, $b(r_0) = r_0$, and then increases from r_0 back to ∞ . We recall the flare-out condition, which is important for traversability

$$\frac{(b - b'(r))}{b^2} > 0.$$

At the throat, since $b(r_0) = r = r_0$, $b(r_0) > b'(r_0)r$ or $r > b'(r_0)r$ or $b'(r_0) < 1$ is the flare-out condition that we need for a traversable wormhole solution. For the wormhole to be traversable, no horizon should be present. Horizons are surfaces with $e^{2\Phi} \rightarrow 0$, so we want $\Phi(r)$ to be finite everywhere. This implies that we can use a constant redshift function $\Phi(r)$, $\Phi = c$, and $\Phi' = 0$. This will help simplify calculations and avoid 4th order differential equations. The following steps can be used to calculate $\rho(r)$, p_r , and p_t in terms of the shape and redshift functions. Substitute for $T_{\mu\nu}$ in the effective field equation (52). Use the MT metric (30) to obtain $R = \frac{2b'}{r}$. Use

$$\square F = (1 - b/r) \left[F'' - \frac{b'r - b}{2r^2(1 - b/r)} F' + \frac{2F'}{r} \right]. \quad (56)$$

Define

$$H(r) \equiv \frac{1}{4}(FR + \square F + T). \quad (57)$$

Note that $F' = \frac{d}{dR} \left(\frac{df(R)}{dR} \right)$ and $F'' = \frac{d(F'(R))}{dR}$. Thus, we get the following simplified equations from the effective field equation (49)

$$\rho = \frac{Fb'}{r^2} \quad (58)$$

$$p_r = \frac{-bF}{r^3} + \frac{F'}{2r^2}(b'r - b) - F''(1 - b/r) \quad (59)$$

$$p_t = -\frac{F'}{r}(1 - b/r) + \frac{F}{2r^3(b - b'r)}. \quad (60)$$

These are the required simplified equations for matter threading the wormhole as a function of $b(r)$ and $F(r)$. We will use them later in this paper. We can now determine the matter content by choosing an appropriate shape function and a specific form of $F(r)$. At this point, let us recap the strategy for stabilizing a wormhole in $f(R)$ gravity. We choose a shape function $b(r)$. Then specify an equation of state such as $p_r = p_r(\rho)$ or $p_t = p_t(\rho)$. This will let us compute $F(r)$ from the effective field equations (49). We can also find the Ricci scalar $R(r)$ from the MT wormhole metric (30). Then obtain $T = T_{\mu}^{\mu}$ as a function of r . Finally, we compute the function $f(R)$ from the trace of the field equation (48).

4.2. Violation of Energy Conditions

Just as the motion of a single particle is governed by the geodesic equation, the equation of motion of a family of particles, also known as a congruence, is governed by the Raychaudhuri equation [25]. From this equation the following focusing condition in terms of the Ricci tensor arises:

$$R_{\mu\nu}k^{\mu}k^{\nu} \geq 0,$$

where k^{μ} is a null vector. If this condition is satisfied, then the geodesic congruences focus (diverge) into a finite value of the parameter for labeling points on the geodesics. In GR, this is written as the null energy condition (NEC) in terms of the stress-energy tensor, as follows

$$T_{\mu\nu}k^{\mu}k^{\nu} \geq 0.$$

In modified gravity, in particular in $f(R)$ theories, we could first assume $T_{\mu\nu}^{(m)}$ satisfies this energy condition and violation of the energy condition can be assumed to come from the higher order curvature terms $T_{\mu\nu}^{(c)}$. Note that this condition applied to $f(R)$ gravity theory does not mean geodesics are focused as required by the Raychaudhuri equation. In terms of the radial null vector, violation of NEC requires

$$T_{\mu\nu}^{\text{eff}} k^\mu k^\nu < 0,$$

and takes the form

$$\rho^{\text{eff}} + p_r^{\text{eff}} = \frac{\rho + p_r}{F} = \frac{1}{F} (1 - b/r) [F'' - F' \frac{b'r - b}{2r^2(1 - b/r)}], \quad (61)$$

where

$$\rho^{\text{eff}} + p_r^{\text{eff}} < 0.$$

Using the gravitational field equation (49), we get

$$\rho^{\text{eff}} + p_r^{\text{eff}} = \frac{b'r - b}{r^3}. \quad (62)$$

Applying the flare-out condition,

$$\frac{b'r - b}{b^2} < 0, \\ \rho^{\text{eff}} + p_r^{\text{eff}} < 0.$$

At the throat, with $r = r_0$,

$$\rho^{\text{eff}} + p_r^{\text{eff}}|_{r_0} = \frac{\rho + p_r}{F}|_{r_0} + \frac{1 - b'(r_0)}{2r_0} \frac{F'}{F}|_{r_0} < 0. \quad (63)$$

At the throat, this gives us

$$F'_0 < -2r_0 \frac{(\rho + p_r)}{(1 - b')}|_{r_0}; F > 0 \quad (64)$$

$$F'_0 > -2r_0 \frac{(\rho + p_r)}{(1 - b')}|_{r_0}; F < 0 \quad (65)$$

We have been highlighting that matter threading the wormhole should obey the WEC, i.e., $\rho \geq 0$ and $\rho + p_r \geq 0$. Applying these WECs to the simplified equations (58), (59), (60) for ρ , p_r and p_t , we get

$$\frac{Fb'}{r^2} \geq 0 \quad (66)$$

$$\frac{(2F + rF')(b'r - b)}{2r^2} - F''(1 - b/r) \geq 0. \quad (67)$$

The four inequalities above (64), (65), (66), and (67) must be obeyed by the function $f(R)$ for traversable wormholes in $f(R)$ theories, taking into account that matter threading the wormhole satisfies the NEC and WEC and the flare-out conditions are satisfied at the wormhole throat. The task of maintaining the wormhole geometry (violating the NEC) is delegated to the higher order curvature terms $T_{\mu\nu}^{(c)}$.

4.3. Wormholes in Lovelock Gravity Theory

We will now discuss how the modifications in Lovelock gravity help with stabilization of a wormhole. In this section we mainly follow [8,24]. The second and third order curvature terms in Lovelock gravity are important to be considered, especially in the case of wormholes with smaller

throat diameters, where the curvature is very high. First, the MT wormhole metric is modified for n dimensions as follows:

$$ds^2 = -e^{2\Phi(r)} dt^2 + (1 - b(r)/r)^{-1} dr^2 + r^2 (d\theta_1^2 + \sum_{i=2}^{n-2} \prod_{j=1}^{i-1} \sin^2 \theta_j d\theta_i^2), \quad (68)$$

where $\Phi(r)$ is the redshift function and $b(r)$ is the shape function. WEC requires that matter threading the wormhole have positive energy density $\rho(r)$, positive $(\rho(r) - \tau(r))$, where $\tau(r)$ is the radial tension, and positive $\rho(r) + p(r)$, where $p(r)$ is the tangential pressure orthogonal to the radius. If the WEC

$$\rho = T_{\mu\nu} u^\mu u^\nu \geq 0$$

is satisfied everywhere then the wormhole can be constructed with normal matter without the need for exotic matter. Here u_μ is a timelike velocity of the observer. Note that the stress tensor $T_{\mu\nu}$ used in the WEC is calculated using (25), which includes all the Lovelock gravity modifications to Einstein's field equations.

The NEC is

$$T_{\mu\nu} k^\mu k^\nu \geq 0$$

The authors in [8] use the orthonormal basis set

$$\mathbf{e}_{\hat{t}} = e^{-\Phi} \frac{\partial}{\partial t}, \quad (69)$$

$$\mathbf{e}_{\hat{r}} = \left(1 - \frac{b(r)}{r}\right)^{1/2} \frac{\partial}{\partial r}, \quad (70)$$

$$\mathbf{e}_{\hat{1}} = r^{-1} \frac{\partial}{\partial \theta_1}, \quad (71)$$

and

$$\mathbf{e}_{\hat{i}} = \left(r \prod_{j=1}^{i-1} \sin \theta_j\right)^{-1} \frac{\partial}{\partial \theta_i} \quad (72)$$

to calculate the components of the energy-momentum tensor, which gives us ρ , τ and p . That is,

$$T_{\hat{t}\hat{t}} = \rho, \quad (73)$$

$$T_{\hat{r}\hat{r}} = -\tau, \quad (74)$$

and

$$T_{\hat{i}\hat{i}} = p. \quad (75)$$

Then, by setting the $\Phi(r) = c = 0$, where c is a constant, $(\rho - \tau)$ and $(\rho + p)$ are calculated as follows:

$$(\rho - \tau) = -\frac{(n-2)}{2r^3} (b - rb') \left(1 + \frac{2\alpha_2 b}{r^3} + \frac{3\alpha_3 b^2}{r^6}\right) \quad (76)$$

and

$$(\rho + p) = -\frac{(b - rb')}{2r^3} \left(1 + \frac{6\alpha_2 b}{r^3} + \frac{15\alpha_3 b^2}{r^6}\right) + \frac{b}{r^3} \left\{ (n-3) + (n-5) \frac{2\alpha_2 b}{r^3} + (n-7) \frac{3\alpha_3 b^2}{r^6} \right\}, \quad (77)$$

where α_2 and α_3 are the second order and third order Lovelock coefficients. In order to obtain positive ρ and $(\rho + p)$, the rest of their analysis were done by studying three types of shape functions $b(r)$: the

power law, the logarithmic, and the hyperbolic shape functions. The power law shape function is given by

$$b = \frac{r_0^m}{r^{m-1}} \quad (78)$$

with positive m . The functions ρ and $(\rho + p)$ for the power law above are positive for $r > r_0$, provided $r_0 > r_c$, and r_c is the largest positive real root of certain equations given in [8]. For the logarithmic shape function,

$$b(r) = \frac{r \ln(r_0)}{\ln(r)}, \quad (79)$$

where ρ and $(\rho + p)$ are positive for $r > r_0$ with the condition $r_0 \geq r_c$, and r_c is the largest real root of a second set of equations in [8]. For the hyperbolic shape function

$$b(r) = \frac{r_0 \tanh(r)}{\tanh(r_0)}, \quad (80)$$

where ρ and $(\rho + p)$ are positive if $r_0 > r_c$ with the condition that r_c is the largest root of a third set of equations in [8].

To ensure that $(\rho - \tau)$ is positive, we need

$$1 + \frac{2\alpha_2 b}{r^3} + \frac{3\alpha_3 b^2}{r^6} < 0,$$

where α_2 and α_3 are the Lovelock coefficients. For certain combinations of the Lovelock coefficients contributing to the throat radius, with negative value for either α_2 or α_3 , the above condition is satisfied only in the vicinity of the throat for all three types of shape functions discussed above. The throat radius must be in the range $r_- < r_0 < r_+$ with

$$r_- = \sqrt{\left(-\alpha_2 - \sqrt{\alpha_2^2 - 3\alpha_3}\right)}, \quad (81)$$

and

$$r_+ = \sqrt{\left(-\alpha_2 + \sqrt{\alpha_2^2 - 3\alpha_3}\right)}. \quad (82)$$

The following points summarize why the higher order curvature terms are needed to stabilize a wormhole and how the Gauss-Bonnet (second order) and Lovelock (third order) curvature terms actually help with using normal matter as opposed to exotic matter in the vicinity of the throat. Higher order curvature corrections are used in wormholes with smaller throat radius since the curvature near the throat is very large for such wormholes. The matter near the throat can be normal for the region $r_0 \leq r \leq r_{\max}$, where $r_{\max}$ depends on the Lovelock coefficients and the shape function. We get larger radius (more region) with normal matter in the case of third order Lovelock gravity with negative coupling constant α_3 as compared to the second order Gauss-Bonnet wormholes. There is a lower limit on the throat radius r_0 imposed by the positivity of ρ and $(\rho + p)$ in the case of Lovelock gravity, but not in the case of Einstein gravity. The lower limit depends on the Lovelock coefficients, the dimensionality of the spacetime and the shape function.

5. Conclusion and Further Research

In this review paper, we have reviewed traversable wormholes in various modified gravity theories and focused more on wormholes in $f(R)$ and Lovelock gravity theories. The key takeaways for wormholes in $f(R)$ gravity theory are as follows: In GR, violation of NEC is required for static traversable wormholes. In $f(R)$ theory of gravity, we obtain modified field equations by varying the modified action with respect to the metric. Using these modified field equations, we can require that

matter (stress-energy tensor) threading the wormhole satisfy the NEC and the required violation of NEC can be enabled by the total stress-energy tensor, which includes higher order curvature terms. The higher order curvature terms interpreted as a gravitational fluid supports the non-standard wormhole geometries. A constant redshift function is assumed in these analysis to reduce the complexity of the calculations. In Section 3.2, we have summarized the strategy to solve the modified field equations for wormholes in $f(R)$ theory of gravity.

Based on the review of papers related to wormholes in Lovelock gravity we have the following key takeaways: Higher order curvature terms become useful for the analysis of wormholes with smaller throat radius. There exists a lower limit for the throat radius in Lovelock gravity imposed by the requirement that ρ and $(\rho + p)$ be positive. There is no such limit in Einstein's gravity. The radius of the region with normal matter is higher for wormholes in third order Lovelock gravity, with negative coupling constant (α_3), compared to wormholes in second order Gauss-Bonnet gravity.

We plan to next provide a follow-up review paper that focuses more on review of wormholes in higher dimensional modified gravity theories such as the KK theory and EGB gravity.

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Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	Linear dichroism

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