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Optimal Design of Multi-Asset Options

Alejandro Balbás [1], **Beatriz Balbás** [2] and **Raquel Balbás** [3]

[1] University Carlos III of Madrid; alejandro.balbas@uc3m.es.
[2] University of Alcalá; beatriz.balbas@uah.es
[3] University Complutense of Madrid; raquel.balbas@ccee.ucm.es

Abstract: The combination of stochastic derivative pricing models and downside risk measures often leads to the paradox (risk,return)=(-infinity,+infinity) in a portfolio choice problem. The construction of a portfolio of derivatives with high expected return and very negative downside risk (henceforth "golden strategy") has been only studied if all the involved derivatives have the same underlying asset. This paper also considers multi-asset derivatives, gives practical methods to build multi-asset golden strategies for both the expected shortfall and the expectile risk measure, and shows that the use of multi-asset options makes it more efficient the performance of the obtained golden strategy. Practical rules are given under the Black-Scholes-Merton multi-dimensional pricing model.

Keywords: multi-asset derivative; downside risk measure; unbounded market price of risk; golden strategy

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JEL Classification: G13; G11; C61

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1. Introduction

Since the monetary tail risk measures became more popular at the end of the last century, they have been used to revisit many classical actuarial and financial topics. Particular attention has deserved the portfolio selection problem (Alexander *et al.*, 2006, Stoyanov *et al.*, 2007, or Mansini *et al.*, 2007, to name a few) where the classical role of the standard deviation has been replaced by a tail risk measure. Nevertheless, theoretical approaches have shown that the portfolio selection problem may become unbounded if one accepts the assumptions of many arbitrage free stochastic pricing models (Black-Scholes-Merton, Heston, etc.) and minimizes the downside risk under a minimum expected return. The paradoxical consequence is that one is able to create a sequence of investment strategies composed of derivative securities whose expected return tends to $+\infty$ whereas the downside risk tens to $-\infty$. Although there are former studies, an easy-to-understand theoretical exposition may found in Balbás *et al.* (2019), where the authors present closed formulas to create the sequences above even in a buy and hold framework and only involving European options and the riskless asset. Later, Balbás and Serna (2024) have empirically tested the strategies of Balbás *et al.* (2019) in the American and the German markets, and have shown that in both cases the corresponding benchmark (*i.e.*, the $S\&P500$ index and the DAX index, respectively) is clearly beaten. In Balbás *et al.* (2023*a*) one can also find methods to build these sequences in a more efficient way but still in a buy and hold framework. The expression "more efficient" means that the divergence to $(+\infty, -\infty)$ is faster.

All the portfolios analyzed in the papers above contain derivatives with a unique underlying asset. An obvious *Question Q* arises: can one construct more efficient sequences if one deals with multi-asset options? The answer to this question is the main objective of this paper. Since organized markets rarely quotes multi-asset derivatives, one has to deal with derivatives that can be replicated by dynamically trading their underlying assets, so we decided to present the approach in a complete market in order to simplify the mathematical exposition. Actually, though the empirical test of Balbás and Serna (2024)

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only involved quoted European options, a former paper by Balbás *et al.* (2016) already reported an empirical study indicating that very important international stock indices may be outperformed by dynamically trading the corresponding index future contract.

The outline of this paper is as follows. Notations and some theoretical background are presented in Section 2. The focus is on two tail risk measures, namely, the expected shortfall (Rockafellar and Uryasev, 2000) and the expectile (Newey and Powell, 1987). The expected shortfall (or conditional value at risk) has been selected because it is very well-known and reflects the downside risk in a very intuitive manner. The expectile is not so intuitive, but it is coherent (Artzner *et al.*, 1999) and expectation bounded (Rockafellar *et al.*, 2006) and it is also elicitable, making it easier a back testing implementation (recall that the expected shortfall is not elicitable). Moreover, both the expected shortfall and the expectile are very closely related (Bellini *et al.*, 2014, and Tadese and Drapeau, 2020, for further details) and the divergence of the expectile to $-\infty$ frequently implies the expected shortfall divergence. The value at risk could be also studied but it has been discarded because the expected shortfall divergence obviously implies the value at risk divergence. Indeed, recall than the value at risk is never higher than the expected shortfall with the same level of confidence.

The main notion to create the sequences above, that is, the notion of "golden strategy" of Balbás *et al.* (2019), is presented in Section 3. Roughly speaking, a golden strategy can be sold for a price which is strictly higher than the downside risk generated by this sale. It is also proved that the presence of golden strategies implies the absence of efficient portfolios in a return/risk approach. Indeed, every portfolio is beaten by the involved portfolio plus the sale of the golden strategy. Theorems 1 and 2 characterize the existence and practical estimation of golden strategies in a complete pricing model and for an arbitrary coherent and expectation bounded risk measures. These theorems are particularized for the expected shortfall (Theorem 3) and the expectile (Theorem 6) in Section 4. For the expected shortfall one can give a closed formula providing us with the optimal golden strategy (if it exists), but an alternative closed formula is not achievable for the expectile. At any rate, the lack of any closed formula does not prevent the calculation of the optimal expectile linked golden strategy by means of tractable linear programming methods (Theorem 6). It is worth pointing out that the existence of golden strategies for the expectile implies this existence for the expected shortfall, but the converse may fail.

The multi-dimensional Black-Scholes-Merton model (BSM) has been selected in Section 5. There are other models to price multi-asset derivatives, but BSM is good enough an simplifies the mathematical exposition. Theorem 8 particularizes for this model the closed formula of Theorem 3, and therefore it yields the optimal golden strategy, which is really a multi-asset derivative. This is an indisputable answer to *Question Q* above: the answer is "yes". If one deals with multi-asset derivatives then the optimal golden strategy is more efficient than those obtained by combining derivatives with a unique underlying asset. Section 6 presents the main conclusions of this paper.

2. Preliminaries and Notation

Fix a future planning period T, a set $\mathcal{T}$ of trading dates such that $\{0, T\} \subset \mathcal{T} \subset [0, T]$, a filtered probability space $\left(\Omega, (\mathcal{F}_t)_{t \in \mathcal{T}}, \mathcal{F}, \mathbb{P}\right)$ composed of the set of states of nature Ω, the filtration $(\mathcal{F}_t)_{t \in \mathcal{T}}$ yielding the arrival of information and such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F} = \mathcal{F}_T$, and the probability measure $\mathbb{P}$ on $\mathcal{F}$. If $1 \leq p < \infty$, then L^p ($L^p(\mathcal{F}, \mathbb{P})$, if necessary) denotes the space of random variables y such that $\mathbb{E}\left(|y|^p\right) < \infty$ endowed with the norm $\|y\| = \left(\mathbb{E}\left(|y|^p\right)\right)^{1/p}$, where $\mathbb{E}(.)$ represents "mathematical expectation". Similarly, L^∞ ($L^\infty(\mathcal{F}, \mathbb{P})$, if necessary) denotes the space of essentially bounded random variables y and it is endowed with the norm $\|y\| = Ess\_Sup(|y|)$, where $Ess\_Sup(.)$ represents "essential supremum". We deal with a complete financial market, that is, we consider a framework such that every (pay-off at T, or marketed claim at T) $y \in L^2$ can be replicated by means of a self-financing strategy composed of the finitely many available assets. The pricing rule $\Pi : L^2 \longrightarrow \mathbb{R}$ yields the initial (at $t = 0$) price $\Pi(y)$ of the marketed claim $y \in L^2$, and it is known that the absence

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of arbitrage implies the existence of a unique stochastic discount factor (*SDF*, Duffie, 1988), that is, a unique $z_\Pi \in L^2$ such that

$$\mathbb{P}(z_\Pi > 0) = 1 \tag{1}$$

and

$$\Pi(y) = e^{-rT}\mathbb{E}(z_\Pi y)$$

holds for every $y \in L^2$, where r denotes the continuously compounded riskless rate. In particular, $e^{-rT} = \Pi(1) = e^{-rT}\mathbb{E}(z_\Pi)$ trivially implies that

$$\mathbb{E}(z_\Pi) = 1. \tag{2}$$

$\varphi : L^2 \longrightarrow \mathbb{R}$ is a coherent (Artzner *et al.*, 1999) and expectation bounded (Rockafellar *et al.*, 2006) risk measure, that is, φ is sub-additive ($\varphi(y_1 + y_2) \leq \varphi(y_1) + \varphi(y_2)$ if $y_j \in L^2$, $j = 1,2$), positively homogeneous ($\varphi(\varkappa y) = \varkappa\varphi(y)$ if $\varkappa \geq 0$ in $\mathbb{R}$ and $y \in L^2$), decreasing ($\varphi(y_1) \leq \varphi(y_2)$ if $y_j \in L^2$, $\mathbb{P}(y_1 \geq y_2) = 1$), translation invariant ($\varphi(y + \varkappa) = \varphi(y) - \varkappa$ if $\varkappa \in \mathbb{R}$ and $y \in L^2$) and mean dominating ($\varphi(y_1) \geq -\mathbb{E}(y)$ if $y \in L^2$). With general convex analysis linked methods (Zalinescu, 2002), it has been proved that the properties above imply the norm-continuity of φ and are also equivalent to the fulfillment of

$$\varphi(y) = Max\left\{-\mathbb{E}(zy); z \in \partial_\varphi\right\}, \tag{3}$$

for every $y \in L^2$, where

$$\partial_\varphi = \left\{z \in L^2; -\mathbb{E}(zy) \leq \varphi(y)\ \forall y \in L^2\right\} \tag{4}$$

is the sub-gradient (or sub-gradient at $y = 0$) of φ and satisfies

$$\begin{cases} 1 \in \partial_\varphi \\ \mathbb{E}(z) = 1,\ \forall z \in \partial_\varphi. \end{cases} \tag{5}$$

Moreover, the norm-continuity of φ implies the weak-compactness of ∂_φ in L^2 and therefore the lower-semi-continuity of φ if L^2 is endowed with the weak topology (see Kopp, 1984, or Zalinescu, 2002, for further details about all of these concepts).

There are many examples of risk measures satisfying the properties above (Artzner *et al.*, 1999, Rockafellar and Uryasev, 2000, Hamada and Sherris, 2003, Rockafellar *et al.*, 2006, Mansini *et al.*, 2007, Ahmadi-Javid, 2012, Chen and Hu, 2018, etc.). Two particular cases will play an important role in this paper, namely, the expected shortfall with the confidence level $1-\beta^*$ for $0 < \beta^* < 1$ ($\varphi = ES_{1-\beta^*}$), and the expectile with parameter β for $0 < \beta < 1/2$ ($\varphi = \mathcal{E}_\beta$). Although the special focus will be on $ES_{1-\beta^*}$, some specific aspects of $\mathcal{E}_\beta$ will also be analyzed. The properties below of both measures may be found in Balbás *et al.* (2023*b*) among many others. In particular, the set ∂_φ of (4) becomes

$$\partial_{ES_{1-\beta^*}} = \left\{z \in L^2; \mathbb{E}(z) = 1,\ 0 \leq z \leq 1/\beta^*\right\} \tag{6}$$

and

$$\partial_{\mathcal{E}_\beta} = \left\{z \in L^2; \mathbb{E}(z) = 1,\ \xi \leq z \leq \xi\frac{1-\beta}{\beta},\ \xi \in \mathbb{R}\right\} \tag{7}$$

respectively. Moreover, taking expectations in (7) it is easy to see that $\xi \leq \mathbb{E}(z) = 1 \leq \xi(1-\beta)/\beta$, and therefore (7) is equivalent to

$$\partial_{\mathcal{E}_\beta} = \left\{z \in L^2; \mathbb{E}(z) = 1,\ \xi \leq z \leq \xi\frac{1-\beta}{\beta},\ \xi \in [\beta/(1-\beta), 1]\right\}. \tag{8}$$

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Evidently, (3) becomes

$$\begin{cases} Max \ -\mathbb{E}(zy) \\ \mathbb{E}(z) = 1 \\ z \in L^2, \ 0 \leq z \leq 1/\beta^* \end{cases} \tag{9}$$

for $ES_{1-\beta^*}$ and

$$\begin{cases} Max \ -\mathbb{E}(zy) \\ \mathbb{E}(z) = 1 \\ z \in L^2, \ \xi \in \mathbb{R}, \ \xi \leq z \leq (\xi(1-\beta))/\beta \end{cases} \tag{10}$$

for $\mathcal{E}_\beta$. Both (9) and (10) are linear optimization problems. According to Balbás *et al.* (2023*b*), their duals do not reflect any duality gap and provide us with another representation of both $ES_{1-\beta^*}(y)$ and $\mathcal{E}_\beta(y)$. The duals are

$$\begin{cases} Min \ \lambda + \mathbb{E}(\lambda_M)/\beta^* \\ y = \lambda_m - \lambda_M - \lambda \\ \lambda_m \in L^2, \ \lambda_M \in L^2, \ 0 \leq \lambda_m, \ 0 \leq \lambda_M, \ \lambda \in \mathbb{R} \end{cases} \tag{11}$$

and

$$\begin{cases} Min \ \lambda \\ y = \lambda_m - \lambda_M - \lambda \\ \beta\mathbb{E}(\lambda_m) - (1-\beta)\mathbb{E}(\lambda_M) = 0 \\ \lambda_m \in L^2, \ \lambda_M \in L^2, \ 0 \leq \lambda_m, \ 0 \leq \lambda_M, \ \lambda \in \mathbb{R} \end{cases} \tag{12}$$

respectively. The complementary slackness conditions below, along with the feasibility, characterize the optimal solutions of both (9)-(11)

$$z\lambda_m = (1/\beta^* - z)\lambda_M = 0.$$

Similarly, the solutions of (10)-(12) are characterized by feasibility and the fulfillment of

$$(z-\xi)\lambda_m = \left(\xi\frac{1-\beta}{\beta} - z\right)\lambda_M = 0.$$

Furthermore, in both cases one can show that a couple of primal and the dual solutions satisfy

$$\begin{cases} \lambda_m = (y - \mathbb{E}(zy))^+ \\ \lambda_M = (\mathbb{E}(zy) - y)^+. \end{cases} \tag{13}$$

Lastly, the inequality

$$ES_{1-\beta^*}(y) \leq \mathcal{E}_\beta(y) + \frac{\beta}{\beta^*(1-2\beta)}\left(\mathcal{E}_\beta(y) + \mathbb{E}(y)\right) \tag{14}$$

holds for every $y \in L^2$, every $0 < \beta < 1/2$ and every $0 < \beta^* < 1$.

3. Golden Strategies

As it has been shown in Balbás *et al.* (2019) among others, the absence of arbitrage is compatible with the existence of marketed claims $y \in L^2$ such that

$$\varphi(-y) < \mathbb{E}(z_\Pi y). \tag{15}$$

Following these authors, let us use the term "golden strategy" or "$\varphi-$golden strategy" to refer to the strategy above. Since φ is translation invariant,

$$\varphi(-y + \mathbb{E}(z_\Pi y)) = \varphi(-y) - \mathbb{E}(z_\Pi y) < 0,$$

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that is, the sale of y along with the investment of the received price $e^{-rT}\mathbb{E}(z_{\Pi}y)$ in a riskless asset, is a self-financing strategy leading to the pay-off $-y+\mathbb{E}(z_{\Pi}y)$ whose global risk is strictly negative. Consequently, for every marketed claim $u \in L^2$ one has

$$\mathbb{E}(z_{\Pi}(u-y+\mathbb{E}(z_{\Pi}y))) = \mathbb{E}(z_{\Pi}u),$$

$$\varphi(u-y+\mathbb{E}(z_{\Pi}y)) \leq \varphi(u) + \varphi(-y+\mathbb{E}(z_{\Pi}y)) < \varphi(u)$$

because φ is sub-additive, and

$$\mathbb{E}(u-y+\mathbb{E}(z_{\Pi}y)) = \mathbb{E}(u) + \mathbb{E}(-y+\mathbb{E}(z_{\Pi}y)) > \mathbb{E}(u)$$

because φ is mean dominating and therefore $\mathbb{E}(-y+\mathbb{E}(z_{\Pi}y)) = \mathbb{E}(-y)+\mathbb{E}(z_{\Pi}y) \geq -\varphi(-y)+\mathbb{E}(z_{\Pi}y) > 0$. In other words, every u is outperformed by $u-y+\mathbb{E}(z_{\Pi}y)$ because this second strategy has identical price, strictly higher expected pay-off (and thus strictly higher expected return) and strictly lower risk. $-y+\mathbb{E}(z_{\Pi}y)$ allows us to beat every position. Lastly, since φ is positively homogeneous and mean dominating,

$$\begin{cases} \lim_{\varkappa\to+\infty} \varphi(\varkappa(-y+\mathbb{E}(z_{\Pi}y))) = \lim_{\varkappa\to+\infty} \varkappa\varphi((-y+\mathbb{E}(z_{\Pi}y))) - \infty \\ \lim_{\varkappa\to+\infty} \mathbb{E}(\varkappa(-y+\mathbb{E}(z_{\Pi}y))) \geq -\lim_{\varkappa\to+\infty} \varphi(\varkappa(-y+\mathbb{E}(z_{\Pi}y))) = +\infty, \end{cases}$$

that is, if $-y+\mathbb{E}(z_{\Pi}y)$ is repeated over and over, with no limit, then one can construct a sequence of self-financing strategies whose risk tends to $-\infty$ whereas its expected pay-off tends to $+\infty$. Henceforth, let us focus on the existence of y satisfying (15).

Theorem 1. *Suppose that $z_{\Pi} \notin \partial_{\varphi}$. Then:*

a) There are golden strategies y such that $y \geq a$ for every $a \in \mathbb{R}$. In particular, for $a = 0$ one has that the prohibition of short-sales does not impede the existence of golden strategies.

b) There are golden strategies y such that $y \leq a$ for every $a \in \mathbb{R}$.

Proof. a) Suppose for a few moments that $a = 0$ and consider the optimization problem

$$\begin{cases} Min\ \varphi(-y) - \mathbb{E}(z_{\Pi}y) \\ y \geq 0, \end{cases} \tag{16}$$

$y \in L^2$ being the decision variable. As Balbás *et al.* (2016) did for quite similar optimization problems, if (16) is bounded then one can prove that the dual of (16) does not generate any duality gap and becomes

$$\begin{cases} Max\ 0 \\ \Lambda = z - z_{\Pi},\ z \in \partial_{\varphi},\ \Lambda \geq 0, \end{cases} \tag{17}$$

$(z,\Lambda) \in L^2 \times L^2$ being the decision variable. Taking expectations in the constraint of (17) one has $\mathbb{E}(\Lambda) = 0$ (see (2) and (5)), and therefore $\Lambda \geq 0$ leads to $\Lambda = 0$ and $z = z_{\Pi}$. This equality is unfeasible because $z_{\Pi} \notin \partial_{\varphi}$ and therefore the feasible set of (17) becomes void. Accordingly, (16) becomes unbounded.

Consider now a general $a \in \mathbb{R}$. Take $y \in L^2$ such that $\varphi(-y) - \mathbb{E}(z_{\Pi}y) < 0$ and $y \geq 0$. Hence, (recall that φ is translation invariant and see (2))

$$\varphi(-(y+a)) - \mathbb{E}(z_{\Pi}(y+a)) = \varphi(-y) - \mathbb{E}(z_{\Pi}y) < 0.$$

b) Similar to a) if the constraint $y \geq 0$ of (16) is replaced with $y \leq 0$. □

Suppose that $z_\Pi \notin \partial_\varphi$. Then, the proof of Theorem 1 implies that (16) is unbounded, at it remains unbounded if the constraint $y \geq 0$ is removed or replaced by $y \geq a$ or $y \leq a$. It could be also replaced by $b \leq y \leq c$, but the change of variable

$$y' = \frac{y-a}{b-a}$$

replaces $b \leq y \leq c$ with $0 \leq y' \leq 1$. In other words, $0 \leq y \leq 1$ is as general as the most general constraint if one looks for self-financing essentially bounded marketed claims with non-positive risk.

Theorem 2. *a) Problem*

$$\begin{cases} Min\ \varphi(-y) - \mathbb{E}(z_\Pi y) \\ 0 \leq y \leq 1 \end{cases} \tag{18}$$

is bounded and solvable, its optimal value is negative or zero, and it is strictly negative if and only if $z_\Pi \notin \partial_\varphi$, in which case the solution of (18) is a golden strategy. If $z_\Pi \in \partial_\varphi$, then there are no golden strategies and the optimal value of (18) vanishes.

b) Problem

$$\begin{cases} Min\ \mathbb{E}(\Lambda_M) \\ \Lambda_m - \Lambda_M = z - z_\Pi,\ z \in \partial_\varphi,\ \Lambda_m \geq 0,\ \Lambda_M \geq 0, \end{cases} \tag{19}$$

is the dual of (18), $(z, \Lambda_m, \Lambda_M) \in L^2 \times L^2 \times L^2$ being the decision variable. It is feasible and solvable, and the optimal values of both (18) and (19) have identical absolute value and opposite sign.

c) If $z_\Pi \notin \partial_\varphi$ and $\left(\tilde{z}, \tilde{\Lambda}_m, \tilde{\Lambda}_M\right)$ solves (19), then $\mathbb{P}(z_\Pi > \tilde{z}) > 0$ and $\mathbb{P}(\tilde{z} > z_\Pi) > 0$.

d) If $\tilde{y}$ is (18)-feasible and $\left(\tilde{z}, \tilde{\Lambda}_m, \tilde{\Lambda}_M\right)$ is (19)-feasible, then they solve the corresponding problem if and only if

$$\begin{cases} \tilde{y}\tilde{\Lambda}_m = 0 \\ (\tilde{y} - 1)\tilde{\Lambda}_M = 0 \\ \mathbb{E}(\tilde{z}\tilde{y}) \geq \mathbb{E}(z\tilde{y})\ \forall z \in \partial_\varphi. \end{cases} \tag{20}$$

e) The solutions $\tilde{y}$ and $\left(\tilde{z}, \tilde{\Lambda}_m, \tilde{\Lambda}_M\right)$ of (18) and (19) satisfy

$$\begin{cases} \tilde{\Lambda}_m = (\tilde{z} - z_\Pi)^+ \\ \tilde{\Lambda}_M = (z_\Pi - \tilde{z})^+ \\ \tilde{z} > z_\Pi \implies \tilde{y} = 0 \\ \tilde{z} < z_\Pi \implies \tilde{y} = 1 \\ 0 < \tilde{y} < 1 \implies \tilde{z} = z_\Pi, \end{cases} \tag{21}$$

where the three implications hold for $\omega \in \Omega$ out of a $\mathbb{P}-$null set.

Proof. The Alaoglus' theorem (Luenberger, 1969) implies that the set $0 \leq y \leq 1$ is weakly compact in L^2. Thus, the existence of solution of (18) follows from the lower-semi-continuity of φ for the weak topology of L^2. Furthermore, since $y = 0$ is (18)-feasible, the optimal value can never be strictly positive. One can proceed as in Balbás *et al.* (2016) in order to prove that (19) is the dual problem of (18), and similar arguments show that (20) characterizes the solutions of (18) and (19). Take a solution $\left(\tilde{z}, \tilde{\Lambda}_m, \tilde{\Lambda}_M\right)$, of (19) whose existence is guaranteed by the usual primal-dual relationships (Luenberger, 1969), suppose that $z_\Pi \notin \partial_\varphi$ and let us prove both $\mathbb{P}(z_\Pi > \tilde{z}) > 0$ and $\mathbb{P}(\tilde{z} > z_\Pi) > 0$. If $\tilde{z} \leq z_\Pi$, then $\mathbb{E}(\tilde{z}) = \mathbb{E}(z_\Pi) = 1$ (see (2) and (5)) leads to $\tilde{z} = z_\Pi$, which is a contradiction because $\tilde{z} \in \partial_\varphi$ and $z_\Pi \notin \partial_\varphi$.The proof of $\mathbb{P}(\tilde{z} > z_\Pi) > 0$ is analogous. Take again a solution $\left(\tilde{z}, \tilde{\Lambda}_m, \tilde{\Lambda}_M\right)$ of (19). If there are no golden strategies then $\tilde{y} = 0$ is an obvious solution of (18), so $\tilde{\Lambda}_m = \tilde{z} - z_\Pi$ and $\tilde{\Lambda}_M = 0$ must be satisfied by the solution of (19) due to (20). Taking expectations, $\mathbb{E}\left(\tilde{\Lambda}_m\right) = \mathbb{E}(\tilde{z}) - \mathbb{E}(z_\Pi) = 1 - 1 = 0$ (see (2) and (5)), so $\tilde{\Lambda}_m \geq 0$ leads to $\tilde{\Lambda}_m = 0$. The constraints of (19) imply that $0 = \tilde{z} - z_\Pi$, that is, $z_\Pi \in \partial_\varphi$. Conversely, if $z_\Pi \in \partial_\varphi$, then one can take $\left(\tilde{z}, \tilde{\Lambda}_m, \tilde{\Lambda}_M\right) = (z_\Pi, 0, 0)$ so $\tilde{y} = 0$ solves (18) and there are no golden strategies. The constraints $\Lambda_m - \Lambda_M = z - z_\Pi$, $\Lambda_m \geq 0$ and $\Lambda_M \geq 0$ of (19) easily

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imply that the minimum of $\mathbb{E}(\Lambda_M)$ must satisfy $\tilde{\Lambda}_m = (\tilde{z} - z_\Pi)^+$ and $\tilde{\Lambda}_M = (z_\Pi - \tilde{z})^+$. Finally, the rest of conditions in (21) trivially follow from $\tilde{\Lambda}_m = (\tilde{z} - z_\Pi)^+$, $\tilde{\Lambda}_M = (z_\Pi - \tilde{z})^+$ and (20). □

4. Focusing on the Expected Shortfall and the Expectile

As already said, $ES_{1-\beta^*}$ and $\mathcal{E}_\beta$ are two important examples of risk measures satisfying the imposed conditions. Let us focus on them. Henceforth $\mathbb{I}_A : \Omega \longrightarrow \mathbb{R}$ will denote the usual indicator for every measurable set $A \in \mathcal{F}$, that is, $\mathbb{I}_A(\omega) = 1$ if $\omega \in A$ and $\mathbb{I}_A(\omega) = 0$ otherwise. Moreover, similar notation will apply if $(\Omega, \mathcal{F}, \mathbb{P})$ is replaced by another probability space.

Theorem 3. *Consider $0 < \beta^* < 1$ and $\rho = ES_{1-\beta^*}$.*

a) There exist $ES_{1-\beta^*}-$golden strategies if and only if the inequality $\|z_\Pi\|_\infty = Ess\_Sup(z_\Pi) > 1/\beta^*$ holds.

b) There is a linear dual problem of (19) (bidual of (18)) given by

$$\begin{cases} Max\ \mathbb{E}(z_\Pi y_1) - \mathbb{E}(y_2)/\beta^* + y_3 \\ y_2 \geq y_1 + y_3 \\ y_1 \leq 1 \\ y_1 \geq 0,\ y_2 \geq 0, \end{cases} \tag{22}$$

$(y_1, y_2, y_3) \in L^2 \times L^2 \times \mathbb{R}$ being the decision variable. (22) is solvable and there is no duality gap between (19) and (22).

c) If $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ solves (22), then $\tilde{y} = \tilde{y}_1$ solves (18).

d) If $\|z_\Pi\|_\infty > 1/\beta^*$, then $\tilde{y} = \mathbb{I}_{z_\Pi > 1/\beta^*}$ solves (18) and is a $ES_{1-\beta^*}-$golden strategy.

Proof. *a*) This is an obvious consequence of (6) and Theorem 2*a*.

b) Bearing in mind (3) and (6), Problem (19) becomes the linear problem

$$\begin{cases} Min\ \mathbb{E}(\Lambda_M) \\ \Lambda_m - \Lambda_M = z - z_\Pi \\ z \leq 1/\beta^* \\ \mathbb{E}(z) = 1 \\ z \geq 0,\ \Lambda_m \geq 0,\ \Lambda_M \geq 0. \end{cases} \tag{23}$$

Thus, according to duality-methods of Anderson and Nash (1987), (23) is the dual of (23). Although (22) is bounded, neither its solvability nor the absence of duality gap with (23) are guaranteed. Nevertheless, both properties will be proved if one finds a (23)-feasible element and a (22)-feasible one that make the objectives of both problems identical. Consider a solution $\tilde{y}$ of (18) and a solution $\left(\tilde{z}, (\tilde{z} - z_\Pi)^+, (z_\Pi - \tilde{z})^+\right)$ of (23) (recall Theorem 2 and (21)). Consider also Problems $(9| -\tilde{y})$ and $(11| -\tilde{y})$, that is, Problems (9) and (11) once y has been replaced by $-\tilde{y}$. The third condition in (20) shows that $\tilde{z}$ solves $(9| -\tilde{y})$. Consider finally a solution $\left(\tilde{\lambda}, \tilde{\lambda}_m, \tilde{\lambda}_M\right)$ of $(11| -\tilde{y})$, and take

$$(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3) = \left(\tilde{y}, \tilde{\lambda}_M, -\tilde{\lambda}\right) \tag{24}$$

It is sufficient to verify the second constraint of (22) in order to show that (24) is (22)-feasible, since the rest of restrictions trivially follow from the restrictions of (18) and $(11| -\tilde{y})$. One has that

$$\tilde{y}_1 + \tilde{y}_3 = -\tilde{\lambda} + \tilde{y} = \tilde{\lambda}_M - \tilde{\lambda}_m \leq \tilde{\lambda}_M = \tilde{y}_2.$$

Hence, the solvability of (22) and the absence of duality gap trivially follow from

$$\begin{cases} \mathbb{E}(z_\Pi \tilde{y}_1) - \mathbb{E}(\tilde{y}_2)/\beta^* + \tilde{y}_3 = \mathbb{E}(z_\Pi \tilde{y}) - \mathbb{E}\left(\tilde{\lambda}_M\right)/\beta^* - \tilde{\lambda} \\ = \mathbb{E}(z_\Pi \tilde{y}) - \varphi(-\tilde{y}) = \mathbb{E}\left((z_\Pi - \tilde{z})^+\right), \end{cases}$$

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where the second and third equalities are implied by the absence of duality gap between the pairs (9| $-\tilde{y}$)-(11| $-\tilde{y}$) and (18)-(23).

$c)$ Take the solutions $\left(\tilde{z}, (\tilde{z}-z_\Pi)^+, (z_\Pi-\tilde{z})^+\right)$ of (23) and $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ of (22), and let us prove that $\tilde{y}=\tilde{y}_1$ satisfies (20). The complementary slackness conditions of Linear Programming (Anderson and Nash, 1987) become

$$\begin{cases} \tilde{z}(\tilde{y}_2-(\tilde{y}_1+\tilde{y}_3))=0 \\ (\tilde{z}-z_\Pi)^+\tilde{y}_1=0 \\ (z_\Pi-\tilde{z})^+(1-\tilde{y}_1)=0 \\ \tilde{y}_2(1/\beta^*-\tilde{z})=0. \end{cases} \tag{25}$$

The second equality of (25) implies the first one of (20), whereas the third equality of (25) implies the second one of (20). The first and fourth equalities of (25) lead to

$$\begin{cases} \mathbb{E}(\tilde{z}\tilde{y}_2)=\mathbb{E}(\tilde{z}(\tilde{y}+\tilde{y}_3)) \\ \mathbb{E}(\tilde{z}\tilde{y}_2)=\mathbb{E}(\tilde{y}_2)/\beta^* \end{cases}$$

i.e. (recall (5)), $\mathbb{E}(\tilde{z}\tilde{y})=\mathbb{E}(\tilde{z}\tilde{y}_2)-\tilde{y}_3=\mathbb{E}(\tilde{y}_2)/\beta^*-\tilde{y}_3$. It only remains to see that

$$\mathbb{E}(z\tilde{y})\leq \mathbb{E}(\tilde{y}_2)/\beta^*-\tilde{y}_3 \tag{26}$$

for every $z\in\partial_{ES_{1-\beta^*}}$. Indeed, if $\tilde{y}_4=\tilde{y}_2-(\tilde{y}+\tilde{y}_3)$, then $(\tilde{y}_4,\tilde{y}_2,-\tilde{y}_3)$ is (11| $-\tilde{y}$)-feasible, and therefore (26) holds because z is (9| $-\tilde{y}$)-feasible.

$d)$ According to Theorem 2, it is sufficient to show that $\tilde{y}=\mathbb{I}_{z_\Pi>1/\beta^*}$ solves (18). Thus, according to $c)$, it is sufficient to show that $\left(\mathbb{I}_{z_\Pi>1/\beta^*}, \mathbb{I}_{z_\Pi>1/\beta^*}, 0\right)$ solves (22). Thus, it is enough to find $\tilde{z}\in\partial_{ES_{1-\beta^*}}$ such that $\left(\mathbb{I}_{z_\Pi>1/\beta^*},0,0\right)$ and $\left(\tilde{z}, (\tilde{z}-z_\Pi)^+, (z_\Pi-\tilde{z})^+\right)$ satisfy (25). First of all, let us show that $\gamma=\mathbb{P}(z_\Pi>1/\beta^*)<\beta^*$. Indeed, $\gamma\geq\beta^*$ would lead to the contradiction with (2)

$$\mathbb{E}(z_\Pi)=\int_\Omega z_\Pi(\omega)\mathbb{P}(d\omega)\geq\int_{z_\Pi>1/\beta^*} z_\Pi(\omega)\mathbb{P}(d\omega)>\frac{\gamma}{\beta^*}\geq 1.$$

For $\alpha\in\mathbb{R}$, $\alpha\geq 0$, consider

$$z_\alpha(\omega)=\begin{cases} 1/\beta^*, & if\ \ z_\Pi(\omega)>1/\beta^* \\ Min\{z_\Pi(\omega)+\alpha, 1/\beta^*\}, & otherwise. \end{cases}$$

Obviously, $0\leq z_\alpha\leq 1/\beta^*$, so $z_\alpha\in\partial_{ES_{1-\beta^*}}$ if $\mathbb{E}(z_\alpha)=1$ (recall (6)). The Dominated Convergence Theorem obviously implies that $[0,\infty)\ni\alpha\longrightarrow z_\alpha\in L^1$ is continuous, and therefore so is $[0,\infty)\ni\alpha\longrightarrow\mathbb{E}(z_\alpha)\in\mathbb{R}$. Furthermore,

$$\mathbb{E}(z_0)=\int_{z_\Pi\leq 1/\beta^*} z_\Pi(\omega)\mathbb{P}(d\omega)+\frac{\gamma}{\beta^*}<\mathbb{E}(z_\Pi)=1$$

and $\mathbb{E}\left(z_{1/\beta^*}\right)=1/\beta^*>1$, so the Bolzano's Theorem implies the existence of $\alpha\in(0,1/\beta^*)$ such that $\mathbb{E}(z_\alpha)=1$ and $z_\alpha\in\partial_{ES_{1-\beta^*}}$. Take $\tilde{z}=z_\alpha$ and notice that $\left(\tilde{z}, (\tilde{z}-z_\Pi)^+, (z_\Pi-\tilde{z})^+\right)$ and $\left(\mathbb{I}_{z_\Pi>1/\beta^*}, \mathbb{I}_{z_\Pi>1/\beta^*}, 0\right)$ obviously satisfy (25). □

Corollary 1. *If $\rho=ES_{1-\beta^*}$ and z_Π is not essentially bounded, then $\tilde{y}=\mathbb{I}_{z_\Pi>1/\beta^*}$ solves (18) and is a $ES_{1-\beta^*}-$golden strategy.*

Proof. $\|z_\Pi\|=+\infty$ obviously implies $\|z_\Pi\|_\infty>1/\beta^*$. □

Remark 1. *Notice that, under the conditions of Theorem 3 or Corollary 4, a digital (or binary) option is an optimal $ES_{1-\beta^*}-$golden strategy. Things are a little bit different for expectiles. Indeed, let us show that $\mathcal{E}_\beta-$golden strategies do also exist under the absence of strictly positive lower bounds of z_Π. Accordingly, a general simple expression such as $\tilde{y} = \mathbb{1}_{z_\Pi > 1/\beta^*}$ cannot be given for the expectile risk measure.* □

Theorem 4. *Consider $0 < \beta < 1/2$ and $\rho = \mathcal{E}_\beta$.*

a) If $\|z_\Pi\|_\infty > (1-\beta)/\beta$ or $Ess\_Inf(z_\Pi) < \beta/(1-\beta)$, then there are $\mathcal{E}_\beta-$golden strategies.

b) Problem (19) becomes the linear problem

$$\begin{cases} Min\ \mathbb{E}(\Lambda_M) \\ \Lambda_m - \Lambda_M = z - z_\Pi \\ \xi \le z \le \xi \dfrac{1-\beta}{\beta} \\ \mathbb{E}(z) = 1 \\ z \in L^2,\ \Lambda_m \ge 0,\ \Lambda_M \ge 0,\ \xi \in \mathbb{R}, \end{cases} \tag{27}$$

$(\xi, z, \Lambda_m, \Lambda_M) \in \mathbb{R} \times L^2 \times L^2 \times L^2$ being the decision variable. Consider the optimization problem

$$\begin{cases} Max\ \mathbb{E}(z_\Pi y_1) + y_4 \\ -y_1 = y_2 - y_3 + y_4 \\ \beta \mathbb{E}(y_2) = (1-\beta)\mathbb{E}(y_3) \\ y_1 \le 1 \\ y_1 \ge 0,\ y_2 \ge 0, y_3 \ge 0,\ y_4 \in \mathbb{R} \end{cases} \tag{28}$$

$(y_1, y_2, y_3, y_4) \in L^2 \times L^2 \times L^2 \times \mathbb{R}$ being the decision variable. (28) is linear dual of (27) (or bidual of (18)), bounded, solvable, and its optimal value equals the optimal value of (19) and (27).

c) If $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)$ solves (28), then $\tilde{y} = \tilde{y}_1$ solves (18). If $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)$ solves (28), then $\tilde{y} = \tilde{y}_1$ is a $\mathcal{E}_\beta-$golden strategy if and only if $\mathbb{E}(z_\Pi \tilde{y}_1) + \tilde{y}_4 > 0$.

Proof. *a*) If $\|z_\Pi\|_\infty > (1-\beta)/\beta$ or $Ess\_Inf(z_\Pi) < \beta/(1-\beta)$, then $z_\Pi \notin \partial_{\mathcal{E}_\beta}$ due to (8), and Theorem 2*a* applies.

b) Bearing in mind (3) and (7), Problem (19) becomes the linear problem (27), whose linear dual is (28) (Anderson and Nash, 1987). As in Theorem 3*b*, consider a solution $\tilde{y}$ of (18) and a solution $\left(\tilde{\xi}, \tilde{z}, (\tilde{z} - z_\Pi)^+, (z_\Pi - \tilde{z})^+\right)$ of (27), and one must find a (28)-feasible element $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)$ such that $\mathbb{E}\left((z_\Pi - \tilde{z})^+\right) = \mathbb{E}(z_\Pi \tilde{y}_1) + \tilde{y}_4$. With similar notations as in the proof of Theorem 3, the third condition in (20) shows that $(\tilde{\xi}, \tilde{z})$ solves $(10-\tilde{y})$. Consider finally a solution$(\tilde{\lambda}, \tilde{\lambda}_m, \tilde{\lambda}_M)$ of $(12| -\tilde{y})$, and take $\tilde{y}_1 = \tilde{y}, \tilde{y}_2 = \tilde{\lambda}_m, \tilde{y}_3 = \tilde{\lambda}_M$, and $\tilde{y}_4 = -\tilde{\lambda}$. The constraints of (18) and $(12-\tilde{y})$ show that $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)$ is (28)-feasible. Moreover, since $\tilde{\lambda}$ is the optimal value of (28),

$$\mathbb{E}(z_\Pi \tilde{y}_1) + \tilde{y}_4 = \mathbb{E}(z_\Pi \tilde{y}) - \tilde{\lambda} = \mathbb{E}(z_\Pi \tilde{y}) - \mathcal{E}_\beta(-\tilde{y}) = \mathbb{E}\left((z_\Pi - \tilde{z})^+\right),$$

where the last equality is implied by Theorem 2.

c) Take the solutions $\left(\tilde{z}, (\tilde{z} - z_\Pi)^+, (z_\Pi - \tilde{z})^+\right)$ of (27) and $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)$ of (28), and let us prove that $\tilde{y} = \tilde{y}_1$ satisfies (20). The complementary slackness conditions of Linear Programming become

$$\begin{cases} \tilde{y}_1 (\tilde{z} - z_\Pi)^+ = 0 \\ (1 - \tilde{y}_1)(z_\Pi - \tilde{z})^+ = 0 \\ \tilde{y}_2 (\tilde{z} - \tilde{\xi}) = 0 \\ \tilde{y}_3 \left(\tilde{\xi} \dfrac{1-\beta}{\beta} - \tilde{z}\right) = 0. \end{cases} \tag{29}$$

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The first and second equalities of (29) and (20) coincide. Besides, for $z \in \partial_{\mathcal{E}_\beta}$ the first and second constraints of (28) lead to (recall (5))

$$\mathbb{E}(z\tilde{y}_1) = -\mathbb{E}(z\tilde{y}_2) + \mathbb{E}(z\tilde{y}_3) - \tilde{y}_4 = \mathbb{E}(z(\tilde{y}_3 - \tilde{y}_2)) - \tilde{y}_4.$$

Take $\xi > 0$ such that (ξ, z) satisfy the conditions of (8). Since $y_2 \geq 0$ and $y_3 \geq 0$,

$$\begin{cases} \mathbb{E}(z\tilde{y}_3) \leq \xi \dfrac{1-\beta}{\beta} \mathbb{E}(\tilde{y}_3) \\ \mathbb{E}(z\tilde{y}_2) \geq \xi \mathbb{E}(\tilde{y}_2). \end{cases}$$

Hence,

$$\mathbb{E}(z(\tilde{y}_3 - \tilde{y}_2)) \leq \xi \left(\frac{1-\beta}{\beta} \mathbb{E}(\tilde{y}_3) - \mathbb{E}(\tilde{y}_2) \right) = 0$$

due to the second constraint of (28). Besides, if $z = \tilde{z}$, the third and fourth conditions in (29), along with the second constraint of (28), lead to

$$\mathbb{E}(\tilde{z}(\tilde{y}_3 - \tilde{y}_2)) = \tilde{\xi} \frac{1-\beta}{\beta} \mathbb{E}(\tilde{y}_3) - \tilde{\xi}\mathbb{E}(\tilde{y}_2) = 0.$$

Finally, the solution $\tilde{y} = \tilde{y}_1$ of (18) is a golden strategy if and only if the optimal objective value of (18) (of (28)) is strictly negative (positive) owing to Theorem 2. □

Remark 2. *Notice that there are important differences between Theorems* 3 *and* 6. *Although the optimal* $ES_{1-\beta^*}-$*golden strategy may be computed by solving the linear problem (22), it is actually enough to know* z_Π, *and one does not need to optimize any linear problem. Indeed, though Theorems* 3*a*), 3*b*) *and* 3*c*) *were needed because they have to be used in order to prove Theorem* 3*d*), *once* z_Π *is known* $\mathbb{I}_{z_\Pi > 1/\beta^*}$ *will solve (18), and it will be non null if and only if* $\|z_\Pi\|_\infty > 1/\beta^*$, *the unique case leading to the existence of* $ES_{1-\beta^*}-$*golden strategies. In contrast, Theorem* 6 *does not provide us with any closed formula for the optimal* $\mathcal{E}_\beta-$*golden strategy, but it will be known if one optimizes the linear problem (28).* □

5. Focusing on the Black-Scholes-Merton Multi-Dimensional Model

Let us focus on the *BSM* multi-dimensional model as a particular relevant case. Accordingly, first of all let us summarize the most important properties of this model, which may be found in Contreras *et al.* (2016), amongst many others. There are alternative models to price multi-asset derivatives (Wu *et al.*, 2023, Zhou *et al.*, 2024, etc.) but let us focus on the most usual one and simplify the mathematical exposition.

5.1. Model Summary

Consider a continuously compounded riskless rate r and n risky assets, $S_1, ..., S_n$, whose stochastic behavior is given by the Geometric Brownian Motions (*GBM*)

$$dS_j = S_j \Big((\mu_j - \gamma_j) dt + \sigma_j dW_j^* \Big), \tag{30}$$

μ_j being "drift", γ_j being "dividend yield", σ_j being "volatility" and W_j^* being a Standard Brownian Motion (*SBM*), $j = 1, 2, ..., n$. The (symmetric) correlation matrix of $\{W_1^*, ..., W_n^*\}$ will be denoted by

$$\rho = \begin{pmatrix} 1, & \rho_{1,2}, & ... & \rho_{1,n} \\ \rho_{2,1}, & 1, & ... & \rho_{2,n} \\ ... & ... & ... & ... \\ \rho_{n,1}, & \rho_{n,2}, & ... & 1 \end{pmatrix}$$

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and we will assume that ρ is regular (and therefore positive definite) in order to prevent the existence of redundant (replicable) securities in the set $\{S_1, ..., S_n\}$. If one fixes the time horizon T, it is known that the explicit solution of (30) becomes

$$S_j(T) = S_j(0)Exp\left(\left(\mu_j - \gamma_j - \frac{\sigma_j^2}{2}\right)T + \sigma_j\sqrt{T}W_j\right) \tag{31}$$

$(W_1, ..., W_n)$ being a $n-$dimensional standard normal random variable whose correlation and covariance matrix equals ρ. In order to price and hedge European derivatives with maturity at T, notice that (31) allows us to simplify the probability space and suppose that

$$\Omega = \mathbb{R}_+^n = \{(\omega_1, ..., \omega_n) \in \mathbb{R}^n;\ \omega_j > 0,\ j = 1, ..., n\},$$

$\mathcal{F}$ is the Borel $\sigma-$algebra of $\mathbb{R}_+^n$ and $\mathbb{P}$ is the probability measure induced on $\mathcal{F}$ by the log-normal random variables $\{S_1(T), ..., S_n(T)\}$. Obviously,

$$\mathbb{P}(S_j(T) \le \omega_j) = \mathbb{P}\left(W_j \le \frac{\log(\omega_j/S_j(0)) - \left(\mu_j - \gamma_j - \frac{\sigma_j^2}{2}\right)T}{\sigma_j\sqrt{T}}\right)$$

for $j = 1, ..., n$ and $\omega_j > 0$, and the joint cumulative distribution function of $\{S_1(T), ..., S_n(T)\}$ becomes

$$F\left((\omega_j)_{j=1}^n\right) = \mathbb{P}\left((W_j)_{j=1}^n \le \left(\frac{\log(\omega_j/S_j(0)) - \left(\mu_j - \gamma_j - \frac{\sigma_j^2}{2}\right)T}{\sigma_j\sqrt{T}}\right)_{j=1}^n\right).$$

Hence, the joint density function becomes

$$\begin{cases} \dfrac{\partial^n F}{\partial\omega_1...\partial\omega_n} = \\ f_\rho\left(\left(\dfrac{\log(\omega_j/S_j(0)) - \left(\mu_j - \gamma_j - \frac{\sigma_j^2}{2}\right)T}{\sigma_j\sqrt{T}}\right)_{j=1}^n\right)\sqrt{T}^{-n}\prod_{j=1}^n\left(\dfrac{S_j(0)}{\sigma_j\omega_j}\right) \end{cases}$$

where

$$f_\rho(u) = \frac{1}{\sqrt{2\pi}^n\sqrt{|\rho|}}Exp\left(-\frac{1}{2}u\rho^{-1}u'\right) \tag{32}$$

is the joint density function of $\{W_1, ..., W_n\}$, $|\rho|$ is the determinant of ρ, $u = (u_1, ..., u_n)$ and A' denotes the transposed of an arbitrary matrix A. Moreover, if $\mathbb{L}_n$ denotes the Lebesgue measure on $\mathcal{F}$, it is known that the Radon–Nikodym derivative of $\mathbb{P}$ with respect to $\mathbb{L}_m$ is the density function above, that is,

$$\begin{cases} \dfrac{d\mathbb{P}}{d\mathbb{L}_n} = \\ f_\rho\left(\left(\dfrac{\log(\omega_j/S_j(0)) - \left(\mu_j - \gamma_j - \frac{\sigma_j^2}{2}\right)T}{\sigma_j\sqrt{T}}\right)_{j=1}^n\right)\sqrt{T}^{-n}\prod_{j=1}^n\left(\dfrac{S_j(0)}{\sigma_j\omega_j}\right) \end{cases} \tag{33}$$

on $\mathbb{R}_+^n$.

5.2. The Stochastic Discount Factor

Consider the family of Arithmetic Brownian Motions (ABM)

$$\tilde{W}_j^* = W_j^* + \frac{\mu_j - r}{\sigma_j} t,$$

$j = 1, ..., n$, $t \geq 0$. Straightforward manipulations imply that (30) and (31) become

$$\begin{cases} dS_j = S_j\Big((r - \gamma_j)dt + \sigma_j d\tilde{W}_j^*\Big) \\ S_j(T) = S_j(0)Exp\left(\left(r - \gamma_j - \frac{\sigma_j^2}{2}\right)T + \sigma_j\sqrt{T}\tilde{W}_j\right) \end{cases} \tag{34}$$

Thus, both (30) and (31) remain the same if every μ_j is replaced by r and every W_j^* is replaced by $\tilde{W}_j^*$. It is known that the Girsanov Theorem guarantees the existence of an equivalent to $\mathbb{P}$ probability measure $\mathbb{Q}$ making $\tilde{W}_j^*$ a SBM for $j = 1, ..., n$. Therefore, $(\tilde{W}_1, ..., \tilde{W}_n)$ becomes a $n-$dimensional standard normal random variable under $\mathbb{Q}$ whose correlation matrix is still ρ. Moreover, the current price of every marketed claim P_T with maturity at T will be $e^{-rT}\mathbb{E}(z_\Pi P_T) = e^{-rT}\mathbb{E}_{\mathbb{Q}}(P_T)$, $\mathbb{E}_{\mathbb{Q}}$ denoting "expectation under $\mathbb{Q}$". Proceeding as above,

$$\begin{cases} \frac{d\mathbb{Q}}{d\mathbb{L}_n} = \\ f_\rho\left(\left(\frac{\log(\omega_j/S_j(0)) - \left(r - \gamma_j - \frac{\sigma_j^2}{2}\right)T}{\sigma_j\sqrt{T}}\right)_{j=1}^n\right)\sqrt{T}^{-n}\prod_{j=1}^n\left(\frac{S_j(0)}{\sigma_j\omega_j}\right). \end{cases} \tag{35}$$

Hence, since the stochastic discount factor satisfies (Duffie, 1988)

$$z_\Pi = \frac{d\mathbb{Q}}{d\mathbb{P}} = \left(\frac{d\mathbb{Q}}{d\mathbb{L}_n}\right)\left(\frac{d\mathbb{L}_n}{d\mathbb{P}}\right),$$

(33) and (35) lead to

$$z_\Pi = \frac{f_\rho\left(\left(\frac{\log(\omega_j/S_j(0)) - \left(r - \gamma_j - \frac{\sigma_j^2}{2}\right)T}{\sigma_j\sqrt{T}}\right)_{j=1}^n\right)}{f_\rho\left(\left(\frac{\log(\omega_j/S_j(0)) - \left(\mu_j - \gamma_j - \frac{\sigma_j^2}{2}\right)T}{\sigma_j\sqrt{T}}\right)_{j=1}^n\right)}$$

for every $\omega = (\omega_1, ..., \omega_n) \in \mathbb{R}_+^n$. Consequently, (32) leads to

$$z_\Pi = Exp\left(\frac{1}{2}\left(u\rho^{-1}u' - v\rho^{-1}v'\right)\right) \tag{36}$$

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with

$$u_j = \frac{\log(\omega_j/S_j(0)) - \left(\mu_j - \gamma_j - \frac{\sigma_j^2}{2}\right)T}{\sigma_j\sqrt{T}}$$

$$v_j = \frac{\log(\omega_j/S_j(0)) - \left(r - \gamma_j - \frac{\sigma_j^2}{2}\right)T}{\sigma_j\sqrt{T}},$$

$j = 1, ..., n$. Bearing in mind that expressions inside the second parenthesis of (36) remains the same if u_j is replaced by $-u_j$ and v_j is replaced by $-v_j$, one can take

$$v_j = \frac{\log(S_j(0)/\omega_j) + \left(r - \gamma_j - \frac{\sigma_j^2}{2}\right)T}{\sigma_j\sqrt{T}} \tag{37}$$

$$u_j = \frac{\log(S_j(0)/\omega_j) + \left(\mu_j - \gamma_j - \frac{\sigma_j^2}{2}\right)T}{\sigma_j\sqrt{T}} = v_j + \frac{\mu_j - r}{\sigma_j}\sqrt{T} \tag{38}$$

$j = 1, ..., n$, $\omega_j > 0$. If one considers the Sharpe ratios

$$\mathcal{R} = (\mathcal{R}_1, ..., \mathcal{R}_n) = \left(\frac{\mu_1 - r}{\sigma_1}, ..., \frac{\mu_n - r}{\sigma_n}\right),$$

then, bearing in mind that ρ^{-1} is symmetric ($\rho^{-1} = (\rho^{-1})'$), (38) implies that

$$\begin{cases} u\rho^{-1}u' - v\rho^{-1}v' = \left(v + \sqrt{T}\mathcal{R}\right)\rho^{-1}\left(v + \sqrt{T}\mathcal{R}\right)' - v\rho^{-1}v' \\ = 2\sqrt{T}v\rho^{-1}\mathcal{R}' + T\mathcal{R}\rho^{-1}\mathcal{R}', \end{cases}$$

and therefore (36), (37) and (38) imply that

$$\begin{cases} z_\Pi = Exp\left(\sqrt{T}v\rho^{-1}\mathcal{R}' + \frac{T}{2}\mathcal{R}\rho^{-1}\mathcal{R}'\right) \\ v = (v_1, ...v_n) \\ v_j = \frac{\log(S_j(0)/\omega_j) + \left(r - \gamma_j - \sigma_j^2/2\right)T}{\sigma_j\sqrt{T}}, \quad j = 1, ..., n, \quad \omega_j > 0 \\ \mathcal{R} = (\mathcal{R}_1, ..., \mathcal{R}_n) = \left(\frac{\mu_1 - r}{\sigma_1}, ..., \frac{\mu_n - r}{\sigma_n}\right), \quad j = 1, ..., n \end{cases} \tag{39}$$

provides us with the *SDF* of the model.

5.3. The Optimal Expected Shortfall-Linked Golden Strategy

(39) obviously implies that all the conditions imposed in Theorem 3 are satisfied unless $\mu_j = r$, $j = 1, ..., n$, in which case $z_\Pi = 1$ (or $\mathbb{Q} = \mathbb{P}$) and the model is risk-neutral. Let us suppose that $\mu_j \neq r$ for at least one risky asset. The equality

$$\|z_\Pi\| = Ess\_Sup(z_\Pi) = +\infty \tag{40}$$

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easily follows from (39), and therefore Corollary 4 implies the existence of $ES_{1-\beta^*}-$golden strategies for every $0<\beta^*<1$. More accurately, one has:

Theorem 5. *Suppose that $\mu_j \neq r$ for at least one risky asset. Then:*

a) There are $ES_{1-\beta^*}-$golden strategies for every $0<\beta^*<1$, and $\tilde{y}_{\beta^*}=\mathbb{1}_{z_\Pi>1/\beta^*}$ is the optimal one, where z_Π is given by (39).

b) Consider the row matrix $\mathcal{R}^{(\rho}=\left(\mathcal{R}_1^{(\rho},...,\mathcal{R}_n^{(\rho}\right)=\mathcal{R}\rho^{-1}$. There exists $j\in\{1,...,n\}$ such that $\mathcal{R}_j^{(\rho}\neq 0$. If $\mathcal{R}_j^{(\rho}>0$, then $\tilde{y}_{\beta^*}(\omega)=1$ if and only if

$$\begin{cases} \log(\omega_j) < \\ \log(S_j(0)) + \left(r-\gamma_j-\sigma_j^2/2\right)T \\ +\dfrac{\sigma_j}{\mathcal{R}_j^{(\rho}}\sum_{i\neq j}\dfrac{\mathcal{R}_i^{(\rho}}{\sigma_i}\left(\log(S_i(0)/\omega_i)+(r-\gamma_i-\sigma_i^2/2)T\right) \\ +\dfrac{\sigma_j}{\mathcal{R}_j^{(\rho}}\left(\dfrac{T}{2}\mathcal{R}\rho^{-1}\mathcal{R}'+\log(\beta^*)\right) \end{cases} \tag{41}$$

for $\omega=(\omega_1,...,\omega_n)\in\mathbb{R}^n_+$. If $\mathcal{R}_j^{(\rho}<0$, then $\tilde{y}_{\beta^*}(\omega)=1$ if and only if

$$\begin{cases} \log(\omega_j) > \\ \log(S_j(0)) + \left(r-\gamma_j-\sigma_j^2/2\right)T \\ +\dfrac{\sigma_j}{\mathcal{R}_j^{(\rho}}\sum_{i\neq j}\dfrac{\mathcal{R}_i^{(\rho}}{\sigma_i}\left(\log(S_i(0)/\omega_i)+(r-\gamma_i-\sigma_i^2/2)T\right) \\ +\dfrac{\sigma_j}{\mathcal{R}_j^{(\rho}}\left(\dfrac{T}{2}\mathcal{R}\rho^{-1}\mathcal{R}'+\log(\beta^*)\right) \end{cases} \tag{42}$$

for $\omega=(\omega_1,...,\omega_n)\in\mathbb{R}^n_+$.

Proof. *a*) Obvious consequence of Corollary 4 and (40).

b) Obviously, $\mathcal{R}\rho^{-1}=0 \implies \mathcal{R}=0 \implies \mu_j=r, j=1,...,n$, against the assumptions. Suppose that $\mathcal{R}_j^{(\rho}>0$.

$$\tilde{y}_{\beta^*}(\omega)=1 \Longleftrightarrow z_\Pi > 1/\beta^* \Longleftrightarrow \log(z_\Pi) > -\log(\beta^*),$$

and therefore (39) leads to

$$\tilde{y}_{\beta^*}(\omega)=1 \Longleftrightarrow v\left(\mathcal{R}^{(\rho}\right)' > -\left(\frac{\sqrt{T}}{2}\mathcal{R}^{(\rho}\mathcal{R}'+\frac{1}{\sqrt{T}}\log(\beta^*)\right).$$

The third equality in (39) and straightforward manipulations imply that (41) is equivalent to $\tilde{y}_{\beta^*}(\omega)=1$. Besides, the proof of (42) is similar. □

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Remark 3. *If one were dealing with future derivatives rather than spot ones, then it is known that (30), (31) and (34) become*

$$\begin{cases} dF_j = F_j\Big((\mu_j - r)dt + \sigma_j dW_j^*\Big) \\ F_j(T) = F_j(0)Exp\left(\left(\mu_j - r - \dfrac{\sigma_j^2}{2}\right)T + \sigma_j\sqrt{T}W_j\right) \\ dF_j = F_j\sigma_j d\tilde{W}_j^* \\ F_j(T) = F_j(0)Exp\left(\left(-\dfrac{\sigma_j^2}{2}\right)T + \sigma_j\sqrt{T}W_j\right) \end{cases}$$

Thus, straightforward modifications of the arguments above imply that the right hand side of (41) and (42) will become

$$\begin{cases} \log\big(F_j(0)\big) - \left(\sigma_j^2/2\right)T \\ +\dfrac{\sigma_j}{\mathcal{R}_j^{(\rho}} \sum_{i\neq j} \dfrac{\mathcal{R}_i^{(\rho}}{\sigma_i}\left(\log(F_i(0)/\omega_i) - (\sigma_i^2/2)T\right) \\ +\dfrac{\sigma_j}{\mathcal{R}_j^{(\rho}}\left(\dfrac{T}{2}\mathcal{R}\rho^{-1}\mathcal{R}' + \log(\beta^*)\right) \end{cases} \tag{43}$$

and one has the optimal $ES_{1-\beta^*}-$golden strategy for future derivatives. □

Remark 4. *If $n = 1$, then straightforward manipulations of (41) or (42) easily imply that, under the obvious notation, $\tilde{y}_{\beta^*}$ is the binary put (respectively, call) with strike*

$$k = S(0)e^{\left(\frac{\mu + r - \sigma^2}{2} - \gamma\right)T}(\beta^*)^{\sigma^2/(\mu - r)} \tag{44}$$

if $\mu > r$ (respectively, $\mu < r$). Notice also that (44) leads to

$$\beta^* = \left[(k/S(0))e^{\left(\frac{\sigma^2 - \mu - r}{2} + \gamma\right)T}\right]^{(\mu - r)/\sigma^2}, \tag{45}$$

that is, if (45) generates a value $\beta^* \in (0,1)$, then, given the strike of a digital option (put if $\mu > r$, call if $\mu < r$), one can compute the level of confidence making this option an optimal $ES_{1-\beta^*}-$golden strategy. Besides, (44) and (45) become

$$\begin{cases} k = F(0)e^{\left(\frac{\mu - r - \sigma^2}{2}\right)T}(\beta^*)^{\sigma^2/(\mu - r)} \\ \beta^* = \left[(k/F(0))e^{\left(\frac{\sigma^2 - \mu + r}{2}\right)T}\right]^{(\mu - r)/\sigma^2} \end{cases}$$

for future options. □

Remark 5. *If $n = 2$, then*

$$\begin{cases} \rho^{-1} = \dfrac{1}{1-\rho_{1,2}^2}\begin{pmatrix} 1, & -\rho_{1,2} \\ -\rho_{1,2}, & 1 \end{pmatrix} \\ \\ \mathcal{R}^{(\rho} = \dfrac{1}{1-\rho_{1,2}^2}(\mathcal{R}_1 - \rho_{1,2}\mathcal{R}_2, \mathcal{R}_2 - \rho_{1,2}\mathcal{R}_1)' \\ \\ \mathcal{R}\rho^{-1}\mathcal{R}' = \dfrac{\mathcal{R}_1^2 + \mathcal{R}_2^2 - 2\rho_{1,2}\mathcal{R}_1\mathcal{R}_2}{1-\rho_{1,2}^2}. \end{cases} \tag{46}$$

Since there are several potential scenarios, let us shorten the exposition and consider the most common case $\mathcal{R}_2 > \mathcal{R}_1 > 0$. In order to simplify the notation, suppose also that one is dealing with future derivatives. (41), (43) and (46) easily lead to

$$\frac{\omega_2}{F_2(0)} < C\left(\frac{\omega_1}{F_1(0)}\right)^{-\frac{\sigma_2(\mathcal{R}_1 - \rho_{1,2}\mathcal{R}_2)}{\sigma_1(\mathcal{R}_2 - \rho_{1,2}\mathcal{R}_1)}}(\beta^*)^{\frac{\sigma_2\left(1-\rho_{1,2}^2\right)}{\mathcal{R}_2 - \rho_{1,2}\mathcal{R}_1}} \tag{47}$$

where the parameter $C > 0$ depends on $(r, \mu_1, \mu_2, \sigma_1, \sigma_2, \rho_{1,2}, T)$, that is, C is not affected by $\omega_j / F_j(0)$, $j = 1, 2$. The subset of $\mathbb{R}_+^2$ generated by (47) clearly depends on the sign of $\mathcal{R}_1 - \rho_{1,2}\mathcal{R}_2$. If $\mathcal{R}_1 - \rho_{1,2}\mathcal{R}_2 > 0$ (for instance, if $\rho_{1,2}$ vanishes), then $\omega_2 / F_2(0)$ must be lying under a curve tending to infinity as $\omega_1 / F_1(0)$ tends to zero. If If $\mathcal{R}_1 - \rho_{1,2}\mathcal{R}_2 = 0$, then $\omega_2 / F_2(0)$ must belong to the interval

$$\left(0, C(\beta^*)^{\frac{\sigma_2\left(1-\rho_{1,2}^2\right)}{\mathcal{R}_2 - \rho_{1,2}\mathcal{R}_1}}\right),$$

that is, $\tilde{y}$ is a digital put whose unique underlying asset is that with the highest Sharpe ratio. Lastly, if $\mathcal{R}_1 - \rho_{1,2}\mathcal{R}_2 < 0$, then $\omega_2 / F_2(0)$ must be lying under a curve tending to zero as $\omega_1 / F_1(0)$ tends to zero. □

Remark 6. *Remarks 10 and 11 show that it is worth involving several underlying securities. If $n = 2$, $\mathcal{R}_2 > \mathcal{R}_1 > 0$ and one separately deals with S_1 and S_2, then Remark 10 implies that the best choice is a couple of digital puts, one per asset. By contrast, if S_1 and S_2 are integrated, then (47) shows that one can beat the use of a digital put per security.* □

6. Conclusions

The existence of expected shortfall-linked and expectile-linked golden strategies has been deeply studied, and it has been pointed out that this existence often holds. These strategies are very important for practitioners because they allow us to create self-financing positions with negative risk. If a golden strategy is implemented jointly with another one, both risk and return are improved. Tractable (probably infinite-dimensional) linear programming problems have been presented to detect the expectile-linked golden strategies, and a closed formula has been given for the expected shortfall. This closed formula has been particularized for the Black-Scholes-Merton multi-dimensional model, and an important consequence has been obtained: the optimal golden strategy is a multi-asset option, that is, multi-asset-options allow us to beat portfolios composed of options with a single underlying asset.

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