

Review

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Review

On Lyapunov Stability and Attractivity of Fractional Order Systems

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Abstract

Lyapunov stability is addressed here, which expands new knowledge of identifying continuous trajectories of fractional order systems that develops the ultimate goal to reach near or converge to its equilibria whenever one convincingly chooses the right Lyapunov functions. The notions of asymptotic stability, stability, and multi-order Mittag-Leffler stability were discussed for complicated nonlinear fractional order systems whenever associated different orders that may lie in $(0, 1]$ and begin with the initial position posed at a random initial time take values on the real number line. The overview of this work is to give readers an enlightening insight into the so-called fractional Lyapunov direct method, which asserts how amazingly one can think of scalar Lyapunov functions to reasonably predict stability dynamics in large time, especially when time t tends to ∞ . We also establish some new sufficient conditions for stability and introduce a new notion of attractiveness of any bounded fixed solution or solution pairs that can be visualized in many such systems. The consequences of some results were adequate in exemplary models.

Keywords: fractional order systems; fractional Lyapunov stability theorems; Lyapunov functions; inequalities; multi-order Mittag-Leffler attractive

JEL Classification: 26A33; 33E12; 34A08; 34D05; 34D20; 34D23

1. Introduction

Fractional calculus concerns new mathematical insights into elegant expressions of constructing potential global derivative and integral operators whose orders may take any appropriate values on the real number line that might be visualizable. The aim of this paper is to give the reader the latest survey of mathematical theory concerning the progress of fractional Lyapunov theorems of Caputo-type fractional order systems.

The study of Lyapunov theory is very essential in fractional order systems as it upholds new theoretical knowledge in mathematics. In short, it concerns stability of equilibrium analysis in different systems by adopting huge field of fractional calculus (see, e.g., Kiryakova [1], Diethelm and Ford [2]). The matter of Lyapunov's second method (it is also known as the Lyapunov direct method) is famous in deciding stability of continuous-time dynamical systems that associate standard integer order derivatives (see, e.g., Khalil [3], Slotine and Li [4]). The idea of the Lyapunov direct method cannot be considered as mature for stability analysis of fractional order systems as of date. However, it has made significant progress in the last decade, and many interesting results are still expected.

The emphasis of this work is mostly on new mathematics with a small number of mathematical results from literature. Throughout this paper, we consider the standard expression of fractional order system:

$$\begin{aligned} {}^C D_{b+,t}^{\delta} x(t) &= f(t, x(t)) \\ x(b) &= x_b \end{aligned} \quad (1)$$

where state variable $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in \mathbb{R}^n$, initial time $b \in \mathbb{R}$, operator ${}^C\mathcal{D}_{b+,t}^{\hat{\delta}} x(t) = \left({}^C\mathcal{D}_{b+,t}^{\delta_1} x_1(t), {}^C\mathcal{D}_{b+,t}^{\delta_2} x_2(t), \dots, {}^C\mathcal{D}_{b+,t}^{\delta_n} x_n(t)\right)^T \in \mathbb{R}^n$ (see Section 2), order-index $\hat{\delta} = (\delta_1, \delta_2, \dots, \delta_n) \in (0, 1] \times (0, 1] \times \dots \times (0, 1]$, and function $f = (f_1, f_2, \dots, f_n)^T : [b, \infty) \times \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous, uniformly Lipschitz and satisfy $f(t, 0) = 0$ for all $t \geq b$. The system (1) is called of commensurate order when all involved fractional orders $\delta_1, \delta_2, \dots, \delta_n$ become equal; otherwise, we call it of incommensurate order. Moreover, the system (1) is called autonomous if $f(t, x(t)) = f(x(t))$; otherwise it is called nonautonomous.

The Lyapunov direct method is quite unknown to many mathematicians, scientists, and engineers still to date. The original concept introduced by A. M. Lyapunov [5] illustrates that it is possible to predict the stability of equilibrium and its asymptotic stability without needing to know the exact solutions of integer-order differential systems. This is based on a simple analogue of an energy-like function and computes its derivative along the mentioned system's nontrivial solutions. Moreover, this approach allows one to deduce exponential asymptotic stability (exponential stability is often widely used in the literature) under appropriate knowledge of the Lyapunov function when the derivative of such a function is sought. The applications of Lyapunov theory enhance our understanding of control systems design, and numerous nonlinear systems emerge across various fields of science and engineering.

There has been a significant growing interest in the scientific literature concerning the idea of extending the integer order differential systems to non-integer order differential systems (we call them fractional-order systems), and possible explanations of such things are widely puzzling in the notion of rational orders, irrational orders, real orders, etc. (see Podlubny [6], Kilbas et al. [7]). It is the concern that roughly mathematicians understood there could be different notations of non-integer order derivatives (we call them fractional derivatives) and non-integer order integrals (we call them fractional integrals) that can possibly violate many well-known properties of classical derivatives and classical integrals (see [8,9]). This is an apparent paradox: the construction of differential and integral operators with attachment to real numbers remains underdeveloped and quite puzzling. But some progress has been made in the last few decades as of date, and more is still expected considering the views of the famous Newton-Leibniz derivative and Cauchy integral using reasonable functions. To give an illustration, for example, we thought of the Riemann-Liouville fractional integral and the Caputo fractional derivative, which develop singular kernels that induce memory in the evolution of fractional order systems that often relay all the past history-dependent states beginning from the initial state position (see Definitions 2 and 3). Note that this is one of the peculiar, interesting phenomena fractional calculus provides that concerns global fractional derivative and integral operators. However, these are not the only kinds of fractional operators in fractional calculus, and many possible generalized ones have been of interest in recent decades remains not seen in the literature. Some peculiar operators can be found in mathematical works suggested by different authors [10–12].

In the description of fractional order systems, one often concerns the stability of systems in the sense of roughly capturing or guessing an intuitive solution of such systems that can be constant (whenever it occurs, we call it an equilibrium solution) or nonconstant (we may call it a nonequilibrium solution or perhaps a better predicted distinguished solution). Stability is an intuitive concept that concerns properties of solutions of such fractional order systems that can be used to predict whether these are staying nearby to an observer solution (e.g., equilibrium solution) forever or eventually converge to it later and never leave again in a large time, especially at symbolic ∞ . Matignon [13] first addressed asymptotic stability and developed a sector condition $|\arg(\text{spec}(A))| > \frac{\pi}{2}\delta$ for an autonomous linear commensurate fractional order system of order $\delta \in (0, 1)$ associated with a coefficient constant matrix $A \in \mathbb{R}^{n \times n}$. His approach was based on a famous Laplace integral transform, and later, Deng et al. [14] generalized new sector conditions for autonomous linear incommensurate fractional order systems. But both the mentioned approaches do not provide adequate reasoning for the case when the mentioned class of systems starts at any initial position that is posed at a random initial time, which is not restricted only to 0. Some fascinating progress has been made in this direction

to tackle such fractional order systems discussed by authors in works [15–17]. This phenomenon often bothers many students and scientific scholars when it comes to various exemplary nonlinear systems as well as realistic models that adapt to the knowledge of global fractional derivatives. This is indeed true in the sense that many different experiments that seek mathematical systems concern fractional derivative at different places might require different initial time supplemented by initial position in order to predict reasonable system dynamics. This observation shows in a realistic situation a real-world scenario, fractional order systems that are demanding not necessarily always to be started at initial time 0. Thus, random initial time fractional order systems can be fascinating to both theory and applications in future owing to restricted zero initial time widely known for physical interest. On the other hand, because of difficulty and limited knowledge of new mathematical tools, the ideas of the fractional comparison method [18–20] and Lyapunov direct method [21,22] altogether have shown some new knowledge for an effective stability analysis of equilibria of complicated nonlinear fractional order systems. The ideas of the Lyapunov direct method indicate new insights that undertake a negative definite characterization of a scalar Lyapunov function acted upon by possible fractional derivatives, making stability dynamics understanding simpler. We thought of it and wish to give a survey that highlights important results of fractional Lyapunov theory that concerns the fractional Lyapunov direct method that appeared in the recent literature under many different scientific works.

The goal of fractional Lyapunov theory is to provide a basic foundation for the stability of going beyond analyzing nonconstant solutions of nonlinear fractional orders, including linear ones. Although it can be imagined in the absence of unknown solutions of many such systems, it might be very crucial for generating a scalar energy function to hold the stability dynamics under appropriate circumstances. But how to take fractional derivatives of such an energy function remains crucial and unknown. Lakshmikantham et al. [23] introduced the Dini-like fractional derivative of the continuous Lyapunov function and formulated abstract Lyapunov theorems for commensurate fractional order systems. In [24], Agarwal et al. introduced the Caputo fractional Dini derivative of the Lyapunov function and studied the stability of commensurate fractional order systems. However, the notion of Dini-type fractional derivatives of the Lyapunov function is not the same as the Caputo fractional derivative of the Lyapunov function for Caputo-type fractional order systems learned by different scholars. Apparently, the progress of the global fractional derivative of the Lyapunov function for similar kinds of systems remains underdeveloped and seems promising for future directions.

The fractional Lyapunov direct method is important because it allows one to think of discovering or constructing a potential Lyapunov function in the future and calculate the fractional Lyapunov function along the unknown trajectories of fractional order systems owing to the observed solution. It can have huge applications in physics, biology, and engineering, and still more can be expected in the future. We may give a few examples of the possibility of such mentioned scenarios in the realization of recent decade. For instance, many physical systems demonstrating fractional nonlinear dynamics concern finding the largest invariant sets to cover all possible bounded dynamics of peculiar systems that use fractional operators (see, e.g., [25,26]). In biology, convergence and boundedness of non-negative responses of fractional-order epidemic systems are essential, and the Lyapunov function can be useful for the construction of a reasonable non-negative orthant in Euclidean space $\mathbb{R}^n$ [27–29]. In engineering, the design of adaptive control schemes and observers of fractional order reference models seems quite challenging, and the Lyapunov function-based approach seems quite convincing [30,31].

We may remark that the Lyapunov function can be categorized into two types: (i) continuous Lyapunov function (see, e.g., [23],[24]) and (ii) continuously differentiable Lyapunov function (see, e.g., [32–34]). A nice reasoning of what would be the cases of the Lyapunov function and the fractional derivative of the Lyapunov function can give precise conclusions to the stability and asymptotic stability of at least the equilibrium solution of such systems remains unknown. But in the literature, the progress of fractional Lyapunov theory has made interesting, and there have been some concerns about what the right Lyapunov theorems can be (see, e.g., Trigeassou et al. [35], Chen et al. [36], Tuan and

Trinh [37], Ren and Wu [38], Lenka and Upadhyay [39], Lenka and Bora [40], Wu [41], and Gallegos and Duarte-Mermoud [42]). These works develop some challenging issues in stability theory and quite puzzling new prospects to date. On the other hand, it was believed that fractional Lyapunov theory remains less focused on the qualitative stability theory dealing with incommensurate fractional order systems. The existing mathematical tools in the literature do not adequately provide new insights into how to simultaneously compute various orders of fractional derivatives of scalar Lyapunov functions. This mentioned issue often bothers many students and scholars, who think of such difficult incommensurate fractional-order systems and the fact that no adequate new mathematical knowledge has been developed so far in the literature. Recently, Lenka [43] introduced some discussions on a novel mathematical framework concerning the formulations of Lyapunov stability theorems for incommensurate fractional order systems. This framework adds some new perspectives in developing fractional Lyapunov theory in the pathways to progress in the understanding of qualitative stability theory of fractional order systems.

It has been visualized that in many situations different solutions of any given fractional order system might attract each other as time $t \rightarrow \infty$, irrespective of constant equilibrium points. The ideas of Lyapunov functions can be meaningful in this direction. We are interested and wish to give an enlightening survey in this context to Lyapunov theory.

In short, this paper contributes a rigorous framework to address some fascinating advanced Lyapunov-based stability results that have been developed by different scientific authors in the literature and introduces new results that concern a notion of attractive solutions of fractional order systems as follows.

- i) In [21,22], the authors have introduced a notion of Mittag-Leffler stability and fractional Lyapunov theorems by using a **class- $\mathcal{K}$** function that concerns equilibrium stability of commensurate fractional order systems. Their work does not address how to predict stability and asymptotic stability when such systems begin at non-zero initial time. In Section 3.1, we address some new progress of the fractional Lyapunov direct method that has been extensively studied by many different notable authors.
- ii) A famous problem that concerns how to go beyond such insightful concepts to incommensurate fractional order systems remains open to date. In Section 3.2, we address some recent progress of the fractional Lyapunov direct method that concerns a new notion of multi-order Mittag-Leffler stability and multi-order Mittag-Leffler asymptotic stability. Some fundamental new Lyapunov theorems, the latest in the literature, were addressed.
- iii) The applications of the fractional Lyapunov direct method are actually quite difficult in many different senses. It might require correct guesses of the Lyapunov function and computation of fractional derivatives of such a function whenever one thinks of a system like that given in (1). We recall some new inequalities in Section 4 that build new insights into many possible guesses for Lyapunov functions and reasonable use of applicable potential Lyapunov theorems.
- iv) In many applications of interest, one might think of some small class of linear and nonlinear systems to characterize conditions for both asymptotic stability and stability of equilibrium points. In Section 5, we establish new sufficient conditions for the mentioned class of systems in Subsection 5.1 and Subsection 5.2, respectively. These new results build reasonable matrix inequality conditions under some basic reasonable permissible bounds to nonlinear functions associated with such systems.
- v) Having the knowledge of stability and asymptotic stability of equilibrium may not necessarily imply what rate the non-trivial solutions of system (1) should march to equilibria or stay near tolerable equilibria. It is the concern that the total energy (think of the Lyapunov function) may be useful to establish sharp bounds to Mittag-Leffler decay or multi-order Mittag-Leffler decay under certain reasonable assumptions. In Section 6, we give a new bound to multi-order Mittag-Leffler decay that has shown interest for the system (1).

- vi) From the basics of fractional Lyapunov theory, it is intuitively clear that many different solutions of system (1) can be attracted to each other or converge to any fixed solution that might be non-constant in nature. We introduce a novel concept of attractiveness of any fixed bounded solution or solution pair that is crucial in understanding beyond existing knowledge of fractional Lyapunov theory. In Section 7, we give definitions of Mittag-Leffler attractive and Mittag-Leffler asymptotically attractive of the mentioned kind of solutions that emerge in the commensurate system (5) that has appeared in a recent work of Lenka and Upadhyay [44]. We introduce the new definitions of multi-order Mittag-Leffler attractive and multi-order Mittag-Leffler asymptotically attractive of a fixed solution or solution pair arising in the incommensurate system (1). Some fundamental new Lyapunov theorems were formulated to end this discussion.
- vii) Predicting stability dynamics in both commensurate and incommensurate fractional order systems can be challenging and complex. In Section 8, we present various examples that demonstrate the diversity achieved through the use of a novel fractional derivative. Additionally, we show how some Lyapunov theorems can be applied to real-world scientific models, including the Van der Pol oscillator system and the Lorenz system. This demonstration offers readers new insights into the fundamental principles of fractional Lyapunov theory, which could be advantageous for developing future complex models.

Paper structure: The symbolic notions and basics of fractional calculus are given in Section 2. Fractional Lyapunov theorems for both commensurate and incommensurate order systems were addressed in Section 3. Some Caputo fractional derivatives of Lyapunov inequalities were recalled in Section 4. Some new stability conditions for small classes of systems were formulated in Section 5. New bounds to energy decay associated with fractional order systems are established in Section 6. A new notion of an attractive concept and some new Lyapunov attractive theorems were formulated in Section 7. Applications of some peculiar fractional Lyapunov theorems are demonstrated in Section 8. Conclusions, roughly close to this progress, are given in Section 9.

2. Notations and Basics of Fractional Calculus

Throughout this paper, we fix the following standard notations. We denote by $\mathbb{N}$ the set of natural numbers, $\mathbb{R}$ the set of real numbers, $\mathbb{R}_+$ the set of positive real numbers, $\mathbb{R}^n$ Euclidean space, $\|v\|$ the Euclidean norm of a vector $v \in \mathbb{R}^n$, v_1^T the transpose of vector or matrix $v_1 \in \mathbb{R}^{m \times n}$, $\text{spec}(A)$ the set of all eigenvalues of constant matrix $A \in \mathbb{R}^{n \times n}$, $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denotes the minimal and maximal eigenvalues of real symmetric matrix $P \in \mathbb{R}^{n \times n}$, respectively, and C^n be the n -th continuously differentiable function.

The Riemann-Liouville fractional integral and Caputo fractional derivative are basic in fractional calculus. We begin with the following:

Definition 1. [6,7] Let $b \in \mathbb{R}$. The Riemann-Liouville fractional integral of an integrable function $\chi : [b, \infty) \rightarrow \mathbb{R}$ is given by

$${}^{\text{RL}}\mathcal{I}_{b^+,t}^{-\beta}\chi(t) = \frac{1}{\Gamma(\beta)} \int_b^t (t-u)^{\beta-1} \chi(u) du, \quad t > b, \quad (2)$$

where $\beta \in \mathbb{R}_+$.

Definition 2. [6,7] Let $b \in \mathbb{R}$. The Caputo fractional derivative of a continuous function $\chi : [b, \infty) \rightarrow \mathbb{R}$ which is C^n on (b, ∞) is given by

$${}^{\text{CD}}\mathcal{D}_{b^+,t}^{\beta}\chi(t) = \begin{cases} {}^{\text{RL}}\mathcal{I}_{b^+,t}^{-(n-\beta)} \frac{d^n}{dt^n} \chi(t), & \text{if } n-1 < \beta < n, \\ \frac{d^n}{dt^n} \chi(t), & \text{if } \beta = n, \end{cases} \quad (3)$$

where $n \in \mathbb{N}$, and $\beta \in \mathbb{R}_+$.

The expression of the Mittag-Leffler function below is standard in the theory of fractional differential equations. This function is known as the queen function of fractional calculus, as it extends the widely known exponential function.

Definition 3. [45] The two parameter Mittag-Leffler function $E_{\gamma_1, \gamma_2} : \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$E_{\gamma_1, \gamma_2}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\gamma_1 k + \gamma_2)}, \quad \gamma_1, \gamma_2 > 0. \quad (4)$$

A class function that gives wings to Lyapunov theory is recalled in the definition below. We shall use it in frontier fractional Lyapunov theory.

Definition 4. [3] A continuous function $\tilde{\alpha} : [0, r) \rightarrow [0, \infty)$ is called **class- $\mathcal{K}$** if it is strictly increasing and satisfy $\tilde{\alpha}(0) = 0$. Whenever $r = \infty$, and $\tilde{\alpha}(r) \rightarrow \infty$ as $r \rightarrow \infty$, it is called as **class- $\mathcal{K}_\infty$ function**.

3. Some Results in Fractional Lyapunov Theory

This section progress some recent new results of fractional Lyapunov stability theorems for both commensurate order and incommensurate order systems. In Subsection 3.1, we discuss some well-known Lyapunov theorems dealt with commensurate system (5) described below. In Subsection 3.2, we present some new extensions of Lyapunov theorems that concerns to the incommensurate system (1).

3.1. Lyapunov Stability Results for Commensurate Order Systems

For $\delta_1 = \delta_2 = \dots = \delta_n = \tilde{\delta} \in (0, 1]$, the system (1) reduces to commensurate nonautonomous fractional order system:

$$\begin{aligned} {}^C\mathcal{D}_{b+,t}^{\tilde{\delta}} x(t) &= f(t, x(t)) \\ x(b) &= x_b. \end{aligned} \quad (5)$$

The following small fundamental theorem appeared in a work of Li et al. [22] that gives a new tool for Mittag-Leffler stability of the equilibrium point of the commensurate order system (5) that may begin only at $b = 0$. The Theorem 1 below develops a deep insight into the unknown-order fractional derivative of the Lyapunov function and builds a linear inequality via rigorous assumptions.

Definition 5. [21] The equilibrium point $x = 0$ of system (5) is said to be Mittag-Leffler stable if

$$\|\xi(t)\| \leq \{m[x(b)]E_{\alpha,1}(-\lambda(t-b)^\alpha)\}^q, \quad (6)$$

where $\alpha \in (0, 1)$, $t \geq b$, $\lambda \geq 0$, $q > 0$, $m(0) = 0$, $m(x) \geq 0$, and $m(x)$ is locally Lipschitz on $x \in \mathbb{B} \in \mathbb{R}^n$ with Lipschitz constant m_0 .

Remark 1. When $\lambda > 0$ in inequality (6), one may assures that Mittag-Leffler stability implies asymptotic stability.

Theorem 1. [21,22] Let $x = 0$ be an equilibrium point of system (5) with $b = 0$, and let $\mathbb{D}$ be the domain containing the origin. Let $V : [0, \infty) \times \mathbb{D} \rightarrow \mathbb{R}$ be a continuously differentiable function and locally Lipschitz with respect to x such that

- $A_1)$ $q_1 \|x\|^p \leq V(t, x) \leq q_2 \|x\|^{pq}$,
- $A_2)$ along system (5) solution $\xi(t)$:

$${}^C\mathcal{D}_{0+,t}^\alpha V(t, \xi(t)) \leq -q_3 \|\xi(t)\|^{pq} \quad (7)$$

where $t \geq 0$, $x \in \mathbb{D}$, $\alpha \in (0, 1)$, q_1, q_2, q_3, p, q are arbitrary positive constants.

Then $x = 0$ is Mittag-Leffler stable. If the assumptions hold globally on $\mathbb{R}^n$, then $x = 0$ is globally Mittag-Leffler stable.

How to extend Theorem 1 to a generalized one in the sense of Mittag-Leffler stability remains unknown. But Li et al. [21,22] put forward asymptotic stability by using **class- $\mathcal{K}$** function bounds to the Lyapunov function and a negative sign multiplied by **class- $\mathcal{K}$** function majorization to the fractional derivative of the Lyapunov function, which is given below.

We first give the following definition below that has importance in the conclusion of Theorem 2.

Definition 6. [21,22] The zero equilibrium of (5) is said to be:

- (i) *stable*, if for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for every $\|x_0\| < \delta$ we have $\|\xi(t)\| < \epsilon$, for all $t \geq 0$.
- (ii) *asymptotically stable* if it is stable and there exists $\delta_+ > 0$ such that $\lim_{t \rightarrow \infty} \xi(t) = 0$ whenever $\|x_0\| < \delta_+$.

Theorem 2. [21,22] Let $x = 0$ be an equilibrium point of system (5) with $b = 0$, and let $\mathbb{D}$ be the domain containing the origin. Let $V : [0, \infty) \times \mathbb{D} \rightarrow \mathbb{R}$ be a Lyapunov function and **class- $\mathcal{K}$** functions $\beta_i (i = 1, 2, 3)$ satisfying

$$A_1) \quad \beta_1(\|x\|) \leq V(t, x) \leq \beta_2(\|x\|),$$

$$A_2) \quad \text{along system (5) solution } \xi(t):$$

$${}^{\mathcal{C}}\mathcal{D}_{0^+, t}^{\alpha} V(t, \xi(t)) \leq -\beta_3(\|\xi(t)\|) \quad (8)$$

where $t \geq 0$, and $\alpha \in (0, 1)$.

Then $x = 0$ is asymptotically stable.

Remark 2. It may be noted that Theorem 1 and Theorem 2 cannot be suitable when any commensurate order version of system (5) begins at initial time $b \neq 0$. The relationship between order α in the mentioned theorems and the actual given order $\tilde{\delta}$ in the system (1) remains unknown, but when they become equal, it indicates an exact relationship with how to take the fractional derivative of the Lyapunov function.

The following stability result proved by Duarte-Mermoud et al. [31] concerns the finding of a continuous Lyapunov function bounded below by a **class- $\mathcal{K}$** function and a fractional derivative of the Lyapunov function majorized by constant 0.

First, we recall the following definition that is valid for the commensurate order version of system (5).

Definition 7. [31] To say an equilibrium point $x = 0$ to (5) is Lyapunov stable (stable) if any $\epsilon > 0$, there exists a $\delta = \delta(b, \epsilon) > 0$ such that the non-trivial solution $\xi(t)$ of (5) obeys an inequality

$$\|\xi(b)\| \leq \delta \implies \|\xi(t)\| \leq \epsilon, \quad \forall t \geq b. \quad (9)$$

Moreover, if there is a $\delta = \delta(\epsilon) > 0$ such that

$$\|\xi(b)\| \leq \delta \implies \|\xi(t)\| \leq \epsilon, \quad \forall t \geq b, \quad (10)$$

then $x = 0$ to (5) is Lyapunov uniformly stable.

Theorem 3. [31] Let $x = 0$ be an equilibrium point of system (5) with $b \in \mathbb{R}$, and let $\mathbb{D}$ be the domain containing the origin. Let $V : [b, \infty) \times \mathbb{D} \rightarrow \mathbb{R}$ be a continuous Lyapunov function and **class- $\mathcal{K}$** function $\beta_1(\cdot)$ such that, $\forall x \neq 0$

- $A_1)$ $\beta_1(\|x\|) \leq V(t, x)$,
 $A_2)$ along system (5) solution $\xi(t)$:

$${}^C\mathcal{D}_{b^+,t}^\alpha V(t, \xi(t)) \leq 0 \quad (11)$$

where $t \geq b$, and $\alpha \in (0, 1]$.

Then $x = 0$ is Lyapunov stable. Furthermore, if there is a **class- $\mathcal{K}$** function $\beta_2(\cdot)$ satisfying

$$V(t, x) \leq \beta_2(\|x\|) \quad (12)$$

then $x = 0$ is Lyapunov uniformly stable.

Remark 3. Note that Theorem 3 does not guarantee the trajectory of the commensurate order system (5) must tend to equilibrium $x = 0 \in \mathbb{R}^n$. But this result assures that the trajectory cannot escape to ∞ at any finite time.

Note that Theorems 1, 2, and 3 seek knowledge of computing fractional derivatives of the Lyapunov function, which is itself a challenging issue. In [34], Chen et al. developed a convex Lyapunov function and introduced the simplified Lyapunov theorem, which is stated below.

Theorem 4. [34] Let $x = 0$ be an equilibrium point of system (5) with $b = 0$. Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a convex Lyapunov function and locally Lipschitz with respect to x such that

- $A_1)$ $q_1\|x\|^p \leq V(x) \leq q_2\|x\|^{pq}$,
 $A_2)$ along system (5) solution $\xi(t)$:

$$\left(\frac{\partial V(\xi(t))}{\partial \xi(t)} \right)^T f(t, \xi(t)) \leq -q_3\|\xi(t)\|^{pq} \quad (13)$$

where $t \geq 0$, q_1, q_2, q_3, p, q are arbitrary positive constants.

Then $x = 0$ is globally Mittag-Leffler stable.

A typical generalization of Theorem 4 by using Theorem 2 and a convex Lyapunov function [34] is stated in Theorem 5.

Theorem 5. Let $x = 0$ be an equilibrium point of system (5) with $b = 0$. Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a convex Lyapunov function and **class- $\mathcal{K}$** functions $\beta_i (i = 1, 2, 3)$ satisfying

- $A_1)$ $\beta_1(\|x\|) \leq V(x) \leq \beta_2(\|x\|)$,
 $A_2)$ along system (5) solution $\xi(t)$:

$$\left(\frac{\partial V(\xi(t))}{\partial \xi(t)} \right)^T f(t, \xi(t)) \leq -\beta_3(\|\xi(t)\|) \quad (14)$$

where $t \geq 0$, and $\alpha \in (0, 1)$.

Then $x = 0$ is asymptotically stable.

Consider the autonomous version of fractional order system (5):

$$\begin{aligned} {}^C\mathcal{D}_{b^+,t}^\delta x(t) &= f(x(t)) \\ x(b) &= x_b. \end{aligned} \quad (15)$$

Recently, Tuan and Trinh [37] considered the autonomous system (15) when $b = 0$ and introduced a fundamental Lyapunov stability theorem stated below. First, we recall the definition that is linked to Theorem 6.

Definition 8. [37] The zero equilibrium of (15) is called:

- (i) *stable*, if for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for every $\|x_0\| < \delta$ we have $t_{\max}(x_0) = \infty$ and $\|\xi(t, x_0)\| < \epsilon$, for all $t \geq 0$.
- (ii) *asymptotically stable*, if it is stable and there exists $\delta_+ > 0$ such that $\lim_{t \rightarrow \infty} \xi(t, x_0) = 0$ whenever $\|x_0\| < \delta_+$.
- (iii) *weakly asymptotically stable*, if it is stable and there exists $\delta_+ > 0$ and a positive sequence $\{t_k\}_{k=1}^{\infty}$, where $t_k \rightarrow \infty$ as $k \rightarrow \infty$ such that $\lim_{k \rightarrow \infty} \xi(t_k, x_0) = 0$ whenever $\|x_0\| < \delta_+$.

Theorem 6. [37] Consider the system (15) with $b = 0$. Assume there is a Lyapunov function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying

- A₁) V is convex, differentiable on $\mathbb{R}^n$ and $V(0) = 0$,
- A₂) there exist constants $q_1, q_2, C_1, C_2, r > 0$ such that

$$C_1 \|x\|^{q_1} \leq V(x) \leq C_2 \|x\|^{q_2}, \quad (16)$$

for all $x \in B_r(0)$,

- A₃) there are constants $C_3 \geq 0$ and $q_3 \geq q_2$ such that

$$\langle \nabla V(x), f(x) \rangle \leq -C_3 \|x\|^{q_3}, \quad (17)$$

for all $x \in B_r(0)$.

Then

- i) the zero equilibrium of (15) is stable if $C_3 = 0$,
- ii) the zero equilibrium of (15) is asymptotically stable if $C_3 > 0$ and $q_2 = q_3$,
- iii) the zero equilibrium of (15) is weakly asymptotically stable if $C_3 > 0$ and $q_2 < q_3$.

Remark 4. The advantage of Theorem 6 is that it does not allow us to compute the fractional derivative of the Lyapunov function, but it does allow us to compute the gradient of the Lyapunov function as compared to Theorem 1. However, a sharp demerit lies in the fact that the Lyapunov function $V(x)$ should not be time-varying for the system (15).

It may be noted that these aforementioned results do not concern Mittag-Leffler stability and asymptotic stability of commensurate fractional order systems whenever they begin at initial time $b \neq 0$. The difficulty is obvious in light of the limited number of new mathematical tools so far available in the literature. Quite surprisingly, Wu [46] worked on positive initial time commensurate fractional order systems and developed some new progress on Lyapunov theorems that concern asymptotic stability and uniform asymptotic stability of equilibrium points.

Definition 9. [46] Let $b \geq 0$. Let $x(t) = x(t, b, x_b)$ denote the solution of (5). Then the zero equilibrium $x = 0$ of (5) is said to be:

- (i) *stable*, if for any $\epsilon > 0$, $b \geq 0$, there exists $\delta(b, \epsilon) > 0$ such that $\|x(b)\| < \delta$ implies $\|x(t)\| < \epsilon$, for all $t \geq b$,
- (ii) *asymptotically stable*, if it is stable, and for any $b \geq 0$, there exists a $\eta(b) > 0$ such that $\|x(b)\| < \eta$ implies $\lim_{t \rightarrow \infty} \|x(t)\| = 0$,
- (iii) *globally uniformly asymptotically stable*, if for any $\eta > 0$ and $\epsilon > 0$, there exists $T(\eta, \epsilon) > 0$ such that $\|x(b)\| < \eta$ implies $\|x(t)\| < \epsilon$, for all $t \geq b + T$.

Theorem 7. [46] Consider the system (5) with $b \geq 0$. Assume $f : \mathbb{R}_+ \times B(r) \rightarrow \mathbb{R}^n$ is continuous and $f(t, 0) = 0$. Let $V : \mathbb{R}_+ \times B(r) \rightarrow \mathbb{R}$. If V is continuously differentiable and positive definite, and the Caputo fractional derivative of V along the solutions of (5), ${}^{\mathcal{C}}\mathcal{D}_{b^+, t}^{\delta} V(t, x)$ is negative definite, then

- (i) the zero equilibrium of (5) is stable, i.e. for any $\epsilon > 0$ and $b \geq 0$, there exists a $\delta = \delta(\epsilon, b) > 0$ such that $\|x(b)\| < \delta$ implies $\|x(t)\| < \epsilon$ for all $t \geq b$,
- (ii) let $\eta = \delta$ then $\|x(b)\| < \eta$ implies $\lim_{t \rightarrow \infty} \|x(t)\| = 0$.

Note that Theorem 7 does assert asymptotic stability of zero equilibrium in general. Some extra assumptions may be needed to arrive the convergence of solutions to the limiting value zero equilibrium of (5). The following Theorem 8 develops an integral upper bound of negative definitive function for Caputo fractional derivative of Lyapunov function.

Theorem 8. [46] Consider the system (5) with $b \geq 0$. Assume $f : \mathbb{R}_+ \times B(r) \rightarrow \mathbb{R}^n$ is continuous, $f(t, 0) = 0$ and f is bounded. Let $V : \mathbb{R}_+ \times B(r) \rightarrow \mathbb{R}$. If V is continuously differentiable and positive definite, and there exists a positive definite function $W : \mathbb{R}_+ \times B(r) \rightarrow \mathbb{R}$ such that the Caputo fractional derivative of V and Riemann-Liouville fractional integral of W both along the solutions of (5) satisfies

$${}^C\mathcal{D}_{b+,t}^\delta V(t, x) \leq {}^{\text{RL}}\mathcal{I}_{b+,t}^{-(1-\delta)}(-W(t, x)) \quad (18)$$

for all $(t, x) \in \mathbb{R}_+ \times B(r)$, then the zero equilibrium of (5) is asymptotically stable.

The notion of uniform asymptotic stability is quite challenging for the system (5), but it can be obtained from the below mentioned result.

Theorem 9. [46] Consider the system (5) with $b \geq 0$. Assume $f : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous, and $f(t, 0) = 0$. Let $V : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$. If V is continuously differentiable, positive definite, and decrescent, and there exists a constant $c > 0$ such that the Caputo fractional derivative of V along the solutions of (5) satisfies

$${}^C\mathcal{D}_{b+,t}^\delta V(t, x) \leq -cV(t, x) \quad (19)$$

for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, then the zero equilibrium of (5) is globally uniformly asymptotically stable.

It can be observed that Theorems 7, 8, and 9 are limited to the commensurate order system (5) whenever $b \geq 0$. In the case when $b < 0$, it was not known whether the solutions of system (5) stay near equilibrium or eventually converge to it in large time, especially whenever $t \rightarrow \infty$. In short, there are no existing Lyapunov theorems available for the case when $b \in \mathbb{R}$ in (5). Despite the challenges, recently, Lenka and Bora [40] as well as Lenka and Upadhyay [39] have addressed this issue and proposed new results that appear promising for effectively analyzing the stability of equilibrium points.

Definition 10. [39] Throughout this section, we say $x = 0$ to system (5) is Mittag-Leffler asymptotically stable if it is asymptotically stable and the inequality

$$\|x(t)\| \leq [KE_{\alpha,1}(-\lambda(t-b)^\alpha)\|x(b)\|^{r_1}]^{r_2} \quad (20)$$

holds, where $b \in \mathbb{R}$, $\alpha \in (0, 1]$, $\lambda > 0$, $K \geq 1$ and $r_1, r_2 > 0$.

Definition 11. [39] Throughout this section, we say $x = 0$ to system (5) is Mittag-Leffler stable if

$$\|x(t)\| \leq [KE_{\alpha,1}(-\lambda(t-b)^\alpha)\|x(b)\|^{r_1}]^{r_2} \quad (21)$$

where $b \in \mathbb{R}$, $\alpha \in (0, 1]$, $\lambda \geq 0$, $K \geq 1$ and $r_1, r_2 > 0$.

We give here Lyapunov theorems by settling some preliminary assumptions on the notion of Lyapunov function $V(t, x)$.

Theorem 10. [39] Assume that $x = 0$ is an equilibrium point of (5). Suppose that $\exists$ a continuously differentiable function $V : [b, \infty) \times \Omega \subseteq \mathbb{R}^n \rightarrow [0, \infty)$ that satisfies:

(i) inequality

$$k_1 \|x\|^p \leq V(t, x) \leq \mu(t) \|x\|^q, \quad (22)$$

where constants k_1, p, q are positive and continuous function $\mu(t) : [b, \infty) \rightarrow [1, \infty)$ satisfies $\mu(t) - k_1 \geq 0$, $\mu(t) - 1 \geq 0$ and $p - q \leq 0$ with $q - 1 \geq 0$,

(ii) if $x(t)$ is any solution of (5),

$${}^C\mathcal{D}_{b^+,t}^\alpha V(t, x(t)) \leq -\lambda V(t, x(t)), \quad \forall t > b, \quad (23)$$

where $\lambda > 0$, $\alpha \in (0, 1]$, $\forall x \in \Omega \subseteq \mathbb{R}^n - \{0\}$.

Then, the equilibrium point $x = 0$ to system (5) is asymptotically stable (Mittag-Leffler asymptotically stable). If $\Omega = \mathbb{R}^n$, and $V(t, x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$, then $x = 0$ is globally asymptotically stable (Mittag-Leffler asymptotically stable).

Theorem 11. [39] Assume that $x = 0$ is an equilibrium point of system (5). Suppose that $\exists$ a continuously differentiable function $V : [b, \infty) \times \Omega \subseteq \mathbb{R}^n \rightarrow [0, \infty)$ that satisfies:

(i) inequality

$$k_1 \|x\|^p \leq V(t, x) \leq \mu(t) \|x\|^q, \quad (24)$$

where constants k_1, p, q are positive and continuous function $\mu(t) : [b, \infty) \rightarrow [1, \infty)$ satisfies $\mu(t) - k_1 \geq 0$, $\mu(t) - 1 \geq 0$ and $p - q \leq 0$ with $q - 1 \geq 0$,

(ii) if $x(t)$ is any solution of (5),

$${}^C\mathcal{D}_{b^+,t}^\alpha V(t, x(t)) \leq -\lambda V(t, x(t)), \quad \forall t > b, \quad (25)$$

where $\lambda \geq 0$, $\alpha \in (0, 1]$, $\forall x \in \Omega \subseteq \mathbb{R}^n$.

Then, the equilibrium point $x = 0$ to system (5) is stable (Mittag-Leffler stable). If $\Omega = \mathbb{R}^n$, and $V(t, x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$, then $x = 0$ is globally stable (Mittag-Leffler stable).

Remark 5. Note that Theorems 10 and 11 improves the existing Theorems 1 and 9, and can be efficiently useful for the system (5).

The following definitions are useful in the notion of existence of reasonable Lyapunov function.

Definition 12. [39] (*V-asymptotic stability*) Throughout this paper, we say $x = 0$ to system (5) is *V-asymptotic stable* if, for any $x(b) \in \Omega_1$, $\exists$ a Lyapunov function $V(t, x)$ such that the norm of solution $\|x(t)\| \rightarrow 0$ as $t \rightarrow \infty$.

Definition 13. [39] (*V-stability*) Throughout this paper, we say $x = 0$ to system (5) is *V-stable* if, for any $x(b) \in \Omega$, $\exists$ a Lyapunov function $V(t, x)$ such that $\|x(t)\| \leq \epsilon$ where $\epsilon > 0$.

Theorem 12. [39] Assume that $x = 0$ is an equilibrium point of (5). Suppose that $\exists$ a continuously differentiable function $V : [b, \infty) \times \Omega \subseteq \mathbb{R}^n \rightarrow [0, \infty)$ with $V(t, 0) = 0$, $\forall t \geq a$ that satisfies:

(i) $k_1 \|x\|^p \leq V(t, x)$, where the constants k_1 and p are positive,

(ii) if $x(t)$ is any solution of system (5), ${}^C\mathcal{D}_{b^+,t}^\alpha V(t, x(t))$ is uniformly negative semi-definite, for all $t > b$, where $\alpha \in (0, 1]$.

Then, $x = 0$ to system (5) is V -stable. If $\Omega = \mathbb{R}^n$, and $V(t, x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$, then $x = 0$ to system (5) is globally V -stable.

Theorem 13. [39] Assume that $x = 0$ is an equilibrium point of system (5). Suppose that $\exists$ a continuously differentiable function $V : [b, \infty) \times \Omega \subseteq \mathbb{R}^n \rightarrow [0, \infty)$ with $V(t, 0) = 0, \forall t \geq a$ that satisfies:

- (i) $k_1 \|x\|^p \leq V(t, x)$, where constants k_1 and p are positive,
- (ii) $V(t, x)$ is uniformly decrescent,
- (iii) if $x(t)$ is any solution of system (5), ${}^C\mathcal{D}_{b^+, t}^\alpha V(t, x(t))$ is uniformly negative definite, for all $t > b$, where $\alpha \in (0, 1]$.

Then, the $x = 0$ to system (5) is V -asymptotically stable. If $\Omega = \mathbb{R}^n$, and $V(t, x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$, then $x = 0$ to system (5) is globally V -asymptotically stable.

Note that Theorem 12 and Theorem 13 develop sharp improvements to existing Lyapunov theorems in literature in the sense that the fractional derivative of the Lyapunov function is negative-definite or negative semi-definite.

In [40], Lenka and Bora suggested the following definitions for their introductory Lyapunov theorems under small assumptions on the Lyapunov function.

Definition 14. [40] Throughout this paper, the solution or stationary point (SP) $x = 0$ to (5) is said to be point asymptotically stable (PAS) if $\exists$ a Ω such that $\forall x(b) \in \Omega$, the Euclidean nontrivial measure $\|x(t)\| \rightarrow 0$ as $t \rightarrow \infty$.

Definition 15. [40] Throughout this paper, the solution or stationary point (SP) $x = 0$ to (5) is said to be point stable (PS) if there exists a Ω such that, for any $x(b) \in \Omega_1$, there is a $\delta > 0$ so that $\|x(b)\| \leq \delta \implies \|x(t)\| \leq \epsilon$ where $\epsilon > 0$.

Theorem 14. [40] Let $x = 0$ be the SP of (5), and $x = 0$ in the domain D . Assume that $\exists$ a C^1 function $V(t, x) : [b, \infty) \times D \subseteq \mathbb{R}^n \rightarrow [0, \infty)$ satisfying

- $A_1)$ $m_1 \|x\|^{r_1} \leq V(t, x) \leq \mu(t) \|x\|^{r_2}$, where $m_1, r_1, r_2 > 0$ with $\mu(t)$ continuous on $[a, \infty)$, and $m_1 \leq \mu(t)$, $1 \leq \mu(t)$, $r_1 \leq r_2$ and $r_2 \geq 1$,
- $A_2)$ ${}^C\mathcal{D}_{b^+, t}^\alpha V(t, x(t))$ is uniformly negative definite on the nontrivial solution $x(t)$ of (5), i.e.,

$${}^C\mathcal{D}_{b^+, t}^\alpha V(t, x(t)) \leq -\theta < 0, \forall x \in D - \{0\}, \forall t > b, \quad (26)$$

where $0 < \alpha \leq 1$ and some $\theta > 0$.

Then, the SP $x = 0$ to (5) is PAS. When $D = \mathbb{R}^n$ and the result holds, then the SP $x = 0$ is globally PAS.

Theorem 15. [40] Let $x = 0$ be the SP of (5), and $x = 0$ in the domain D . Assume that $\exists$ a C^1 function $V(t, x) : [b, \infty) \times D \subseteq \mathbb{R}^n \rightarrow [0, \infty)$ satisfying

- $A_1)$ $m_1 \|x\|^{r_1} \leq V(t, x) \leq \mu(t) \|x\|^{r_2}$, where $m_1, r_1, r_2 > 0$ with $\mu(t)$ continuous on $[a, \infty)$, and $m_1 \leq \mu(t)$, $1 \leq \mu(t)$, $r_1 \leq r_2$ and $r_2 \geq 1$,
- $A_2)$ ${}^C\mathcal{D}_{b^+, t}^\beta V(t, x(t))$ is uniformly negative semidefinite on the nontrivial solution $x(t)$ of (5), i.e.,

$${}^C\mathcal{D}_{b^+, t}^\alpha V(t, x(t)) \leq -\theta \leq 0, \forall x \in D, \forall t > b \quad (27)$$

where $0 < \alpha \leq 1$ and some $\theta \geq 0$.

Then, the SP $x = 0$ to (5) is PS. When $D = \mathbb{R}^n$ and the result holds, then the SP $x = 0$ is globally PS.

Although the topic of fractional Lyapunov theorems has undergone fascinating progress in the last decade for commensurate fractional order system (5), more is expected for in-depth understanding, and many different new proofs are missing [47,48]. It is hoped that the aforementioned collected

results will give new insights to the readers into the further progress of the fractional Lyapunov direct method in future directions.

To end this part, we give the following open problems.

- Q₁ Is it possible to strengthen the assumption A_1) in Theorems 12, 13, 14, and 15?
 Q₂ Assume A_1) and A_2) in Theorems 13 and 14. Can the zero equilibrium $x = 0$ of (5) be Mittag-Leffler asymptotically stable?

3.2. Lyapunov Stability Results for Incommensurate Order Systems

In this section we introduce some new definitions of conceptual stability notions and Lyapunov stability results for incommensurate system (1) arising in qualitative stability theory. It has undergone promising developments in the recent literature, and more is expected.

Yu et al. [49] examined the fractional Lyapunov direct method for the system (1) and proposed several definitive results. These results share similarities with the findings presented by Li et al. in [21,22].

Definition 16. [49] The equilibrium point $x = 0$ of system (1) is said to be Mittag-Leffler stable if

$$\|\xi(t)\| \leq \{m[x(b)]E_{\varsigma,1}(-\lambda(t-b)^{\varsigma})\}^q, \quad (28)$$

where $\varsigma \in (0,1)$, $t \geq b$, $\lambda \geq 0$, $q > 0$, $m(0) = 0$, $m(x) \geq 0$, and $m(x)$ is locally Lipschitz on $x \in \mathbb{B} \in \mathbb{R}^n$ with Lipschitz constant ℓ .

Theorem 16. [49] Let $x = 0$ be an equilibrium point of system (1) with $b = 0$, and let $\mathbb{D}$ be the domain containing the origin. Let $V : [0, \infty) \times \mathbb{D} \rightarrow \mathbb{R}$ be a continuously differentiable function and locally Lipschitz with respect to x such that

$$A_1) \quad q_1 \|x\|^p \leq V(t, x) \leq q_2 \|x\|^{pq},$$

$$A_2) \quad \text{along system (1) solution } \xi(t):$$

$${}^{\mathcal{C}}\mathcal{D}_{0+,t}^{\varsigma} V(t, \xi(t)) \leq -q_3 \|\xi(t)\|^{pq} \quad (29)$$

where $t \geq 0$, $x \in \mathbb{D}$, $\varsigma \in (0,1)$, q_1, q_2, q_3, p, q are arbitrary positive constants.

Then $x = 0$ is Mittag-Leffler stable. If the assumptions hold globally on $\mathbb{R}^n$, then $x = 0$ is globally Mittag-Leffler stable.

Remark 6. Due to the presence of many different orders $\delta_1, \delta_2, \dots, \delta_n$ in system (1), Theorem 16 does not provide effective strategies to compute the fractional derivative of the Lyapunov function ${}^{\mathcal{C}}\mathcal{D}_{0+,t}^{\varsigma} V(t, \xi(t))$. However, it was believed that the symbolic Caputo fractional derivative ${}^{\mathcal{C}}\mathcal{D}_{0+,t}^{\varsigma}$ might pose interesting platforms for pathways to the relationships between system orders $\delta_1, \delta_2, \dots, \delta_n$ and computational order ς in future directions.

It is quite evident that how to compute simultaneously many different fractional derivatives of any suitable scalar Lyapunov function remains unknown. Recently, Lenka [43] found curiosity and attempted generalization of Mittag-Leffler stability and proposed multi-order Mittag-Leffler stability concepts. Moreover, he has proposed some distinctive versions of the fractional Lyapunov direct method for the system (1).

Definition 17. [43] The equilibrium solution $x = 0$ to system (1) is said to be multi-order Mittag-Leffler asymptotically stable if the non-trivial solution satisfy $x(t)$ tends to 0 as $t \rightarrow \infty$ and the inequality

$$\rho_1(\|x(t)\|) \leq C^* \mu(b) \sum_{i=1}^n E_{\gamma_i,1}(-\lambda_i(t-b)^{\gamma_i}) \rho_2(\|x(b)\|), \quad (30)$$

for all $t \geq b$, where $\rho_1(\cdot)$ and $\rho_2(\cdot)$ are of **class- $\mathcal{K}_\infty$** , $\gamma_1, \gamma_2, \dots, \gamma_n \in (0, 1]$, constants $C^* \geq 1$, $\lambda_i > 0$ for $i = 1, 2, \dots, n$, and $\mu : [b, \infty) \rightarrow [1, \infty)$ is continuous.

Definition 18. [43] The equilibrium solution $x = 0$ to system (1) is said to be multi-order Mittag-Leffler stable whenever non-trivial solution satisfy the inequality

$$\rho_1(\|x(t)\|) \leq C^* \mu(b) \sum_{i=1}^n E_{\gamma_i, 1}(-\lambda_i(t-b)^{\gamma_i}) \rho_2(\|x(b)\|), \quad (31)$$

for all $t \geq b$, where $\rho_1(\cdot)$ and $\rho_2(\cdot)$ are of **class- $\mathcal{K}_\infty$** , $\gamma_1, \gamma_2, \dots, \gamma_n \in (0, 1]$, constants $C^* \geq 1$ and λ_i 's are non-negative constants for $i = 1, 2, \dots, n$, and $\mu : [b, \infty) \rightarrow [1, \infty)$ is continuous..

Theorem 17. [43] Consider the system (1). Let $x = 0$ be an equilibrium solution in a domain $\Omega \subseteq \mathbb{R}^n$. Suppose there exist a $\mathcal{C}^1$ function $V : [b, \infty) \times \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ such that

- (i) $\rho_1(\|x\|) \leq V(t, x) = \sum_{i=1}^n V_i(t, x_i) \leq \mu(t) \rho_2(\|x\|)$, where $\rho_1(\cdot)$ and $\rho_2(\cdot)$ are of **class- $\mathcal{K}_\infty$** , and $\mu : [b, \infty) \rightarrow [1, \infty)$ is continuous,
- (ii) along the non-trivial solution $x(t)$ of (1):

$$\sum_{i=1}^n {}^{\mathcal{C}}\mathcal{D}_{b^+, t}^{\gamma_i} V_i(t, x_i(t)) \leq - \sum_{i=1}^n \lambda_i V_i(t, x_i(t)), \quad \forall x \in \Omega - \{0\}, \quad \forall t > b, \quad (32)$$

where constants λ_i 's are positive for $i = 1, 2, \dots, n$, and orders $\gamma_1, \gamma_2, \dots, \gamma_n \in (0, 1]$.

Then, the equilibrium $x = 0$ to (1) is multi-order Mittag-Leffler asymptotically stable. When $\Omega = \mathbb{R}^n$, the equilibrium $x = 0$ to (1) is globally multi-order Mittag-Leffler asymptotically stable.

Here we state a typical corollary of Theorem 17 by making use of special **class- $\mathcal{K}_\infty$** functions.

Corollary 1. Consider the system (1). Let $x = 0$ be an equilibrium solution in a domain $\Omega \subseteq \mathbb{R}^n$. Suppose there exist a $\mathcal{C}^1$ function $V : [b, \infty) \times \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ such that

- (i) $q_1 \|x\|^p \leq V(t, x) = \sum_{i=1}^n V_i(t, x_i) \leq q_2 \mu(t) \|x\|^q$, where p, q, q_1, q_2 are arbitrary positive constants, and $\mu : [b, \infty) \rightarrow [1, \infty)$ is continuous,
- (ii) along the non-trivial solution $x(t)$ of (1):

$$\sum_{i=1}^n {}^{\mathcal{C}}\mathcal{D}_{b^+, t}^{\gamma_i} V_i(t, x_i(t)) \leq - \sum_{i=1}^n \lambda_i V_i(t, x_i(t)), \quad \forall x \in \Omega - \{0\}, \quad \forall t > b, \quad (33)$$

where constants λ_i 's are positive for $i = 1, 2, \dots, n$, and orders $\gamma_1, \gamma_2, \dots, \gamma_n \in (0, 1]$.

Then, the equilibrium $x = 0$ to (1) is multi-order Mittag-Leffler asymptotically stable. When $\Omega = \mathbb{R}^n$, the equilibrium $x = 0$ to (1) is globally multi-order Mittag-Leffler asymptotically stable.

Theorem 18. [43] Consider the system (1). Let $x = 0$ be an equilibrium solution in a domain $\Omega \subseteq \mathbb{R}^n$. Suppose there exist a $\mathcal{C}^1$ function $V : [b, \infty) \times \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ such that

- (i) $\rho_1(\|x\|) \leq V(t, x) = \sum_{i=1}^n V_i(t, x_i) \leq \mu(t) \rho_2(\|x\|)$, where $\rho_1(\cdot)$ and $\rho_2(\cdot)$ are of **class- $\mathcal{K}_\infty$** , and $\mu : [b, \infty) \rightarrow [1, \infty)$ is continuous,
- (ii) along the non-trivial solution $x(t)$ of (1):

$$\sum_{i=1}^n {}^{\mathcal{C}}\mathcal{D}_{b^+, t}^{\gamma_i} V_i(t, x_i(t)) \leq - \sum_{i=1}^n \lambda_i V_i(t, x_i(t)), \quad \forall x \in \Omega, \quad \forall t > b, \quad (34)$$

where constants λ_i 's are non-negative for $i = 1, 2, \dots, n$, and orders $\gamma_1, \gamma_2, \dots, \gamma_n \in (0, 1]$.

Then, the equilibrium $x = 0$ to (1) is multi-order Mittag-Leffler stable. When $\Omega = \mathbb{R}^n$, the equilibrium $x = 0$ to (1) is globally multi-order Mittag-Leffler stable.

Corollary 2. Consider the system (1). Let $x = 0$ be an equilibrium solution in a domain $\Omega \subseteq \mathbb{R}^n$. Suppose there exist a C^1 function $V : [b, \infty) \times \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ such that

- (i) $q_1 \|x\|^p \leq V(t, x) = \sum_{i=1}^n V_i(t, x_i) \leq q_2 \mu(t) \|x\|^q$, where p, q, q_1, q_2 are arbitrary positive constants, and $\mu : [b, \infty) \rightarrow [1, \infty)$ is continuous,
- (ii) along the non-trivial solution $x(t)$ of (1):

$$\sum_{i=1}^n {}^C\mathcal{D}_{b^+, t}^{\gamma_i} V_i(t, x_i(t)) \leq - \sum_{i=1}^n \lambda_i V_i(t, x_i(t)), \quad \forall x \in \Omega - \{0\}, \quad \forall t > b, \quad (35)$$

where constants λ_i 's are non-negative for $i = 1, 2, \dots, n$, and orders $\gamma_1, \gamma_2, \dots, \gamma_n \in (0, 1]$.

Then, the equilibrium $x = 0$ to (1) is multi-order Mittag-Leffler stable. When $\Omega = \mathbb{R}^n$, the equilibrium $x = 0$ to (1) is globally multi-order Mittag-Leffler stable.

Remark 7. Theorems 17 and 18 develop new knowledge for how to take simultaneously many different fractional derivatives of scalar Lyapunov function as compared to Theorem 16 for the system (1).

The following results show new interest in the progress of the extended fractional Lyapunov direct method as compared to the elementary Lyapunov theorems Theorems 17 and 18.

Theorem 19. [43] Consider the system (1). Let $x = 0$ be an equilibrium solution in a domain $\Omega \subseteq \mathbb{R}^n$. Suppose there exist a C^1 function $V : [b, \infty) \times \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ such that

- A₁) $\rho_1(\|x\|) \leq V(t, x) = \sum_{i=1}^n V_i(t, x_i) \leq \mu(t) \rho_2(\|x\|)$, where $\rho_1(\cdot)$ and $\rho_2(\cdot)$ are of **class- $\mathcal{K}_\infty$** , and $\mu : [b, \infty) \rightarrow [1, \infty)$ is continuous,
- A₂) along the non-trivial solution $x(t)$ of (1):

$$\sum_{i=1}^n {}^C\mathcal{D}_{b^+, t}^{\gamma_i} V_i(t, x_i(t)) \leq - \sum_{i=1}^n \beta_i(\|x_i(t)\|), \quad \forall x \in \Omega - \{0\}, \quad \forall t > b, \quad (36)$$

where $\beta_1(\cdot), \dots, \beta_n(\cdot)$ are **class- $\mathcal{K}_\infty$** and orders $\gamma_1, \dots, \gamma_n \in (0, 1]$.

Then, the equilibrium $x = 0$ to (1) is multi-order Mittag-Leffler asymptotically stable. When $\Omega = \mathbb{R}^n$, the equilibrium $x = 0$ to (1) is globally multi-order Mittag-Leffler asymptotically stable.

A typical result that seems immediate from Theorem 19 is stated below. It considers the case when the Lyapunov function is bounded between curves **class- $\mathcal{K}_\infty$** functions.

Corollary 3. Consider the system (1). Let $x = 0$ be an equilibrium solution in a domain $\Omega \subseteq \mathbb{R}^n$. Suppose there exist a C^1 function $V : [b, \infty) \times \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ such that

- A₁) $\rho_1(\|x\|) \leq V(t, x) = \sum_{i=1}^n V_i(t, x_i) \leq \rho_2(\|x\|)$, where $\rho_1(\cdot)$ and $\rho_2(\cdot)$ are of **class- $\mathcal{K}_\infty$** ,
- A₂) along the non-trivial solution $x(t)$ of (1):

$$\sum_{i=1}^n {}^C\mathcal{D}_{b^+, t}^{\gamma_i} V_i(t, x_i(t)) \leq - \sum_{i=1}^n \beta_i(\|x_i(t)\|), \quad \forall x \in \Omega - \{0\}, \quad \forall t > b, \quad (37)$$

where $\beta_1(\cdot), \dots, \beta_n(\cdot)$ are **class- $\mathcal{K}_\infty$** and orders $\gamma_1, \dots, \gamma_n \in (0, 1]$.

Then, the equilibrium $x = 0$ to (1) is multi-order Mittag-Leffler asymptotically stable. When $\Omega = \mathbb{R}^n$, the equilibrium $x = 0$ to (1) is globally multi-order Mittag-Leffler asymptotically stable.

Theorem 20. [43] Consider the system (1). Let $x = 0$ be an equilibrium solution in a domain $\Omega \subseteq \mathbb{R}^n$. Suppose there exist a C^1 function $V : [b, \infty) \times \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ such that

- $A_1)$ $\rho_1(\|x\|) \leq V(t, x) = \sum_{i=1}^n V_i(t, x_i) \leq \mu(t)\rho_2(\|x\|)$, where $\rho_1(\cdot)$ and $\rho_2(\cdot)$ are of **class- $\mathcal{K}_\infty$** , and $\mu : [b, \infty) \rightarrow [1, \infty)$ is continuous,
 $A_2)$ along the non-trivial solution $x(t)$ of (1):

$$\sum_{i=1}^n {}^C\mathcal{D}_{b^+,t}^{\gamma_i} V_i(t, x_i(t)) \leq - \sum_{i=1}^n \lambda_i V_i(t, x_i(t)) - \sum_{i=1}^n \beta_i(\|x_i(t)\|), \quad \forall x \in \Omega, \quad \forall t > b, \quad (38)$$

where λ_i 's are non-negative constants, $\beta_1(\cdot), \dots, \beta_n(\cdot)$ are **class- $\mathcal{K}_\infty$** and orders $\gamma_1, \dots, \gamma_n \in (0, 1]$.

Then, the equilibrium $x = 0$ to (1) is multi-order Mittag-Leffler stable. When $\Omega = \mathbb{R}^n$, the equilibrium $x = 0$ to (1) is globally multi-order Mittag-Leffler stable.

Corollary 4. Consider the system (1). Let $x = 0$ be an equilibrium solution in a domain $\Omega \subseteq \mathbb{R}^n$. Suppose there exist a C^1 function $V : [b, \infty) \times \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ such that

- $A_1)$ $\rho_1(\|x\|) \leq V(t, x) = \sum_{i=1}^n V_i(t, x_i) \leq \rho_2(\|x\|)$, where $\rho_1(\cdot)$ and $\rho_2(\cdot)$ are of **class- $\mathcal{K}_\infty$** ,
 $A_2)$ along the non-trivial solution $x(t)$ of (1):

$$\sum_{i=1}^n {}^C\mathcal{D}_{b^+,t}^{\gamma_i} V_i(t, x_i(t)) \leq - \sum_{i=1}^n \lambda_i V_i(t, x_i(t)) - \sum_{i=1}^n \beta_i(\|x_i(t)\|), \quad \forall x \in \Omega, \quad \forall t > b, \quad (39)$$

where λ_i 's are non-negative constants, $\beta_1(\cdot), \dots, \beta_n(\cdot)$ are **class- $\mathcal{K}_\infty$** and orders $\gamma_1, \dots, \gamma_n \in (0, 1]$.

Then, the equilibrium $x = 0$ to (1) is multi-order Mittag-Leffler stable. When $\Omega = \mathbb{R}^n$, the equilibrium $x = 0$ to (1) is globally multi-order Mittag-Leffler stable.

We close this part while learning that the previously mentioned results demand extensive knowledge of the Lyapunov function and the computation of the fractional derivative of the Lyapunov function while looking for non-trivial solutions of the incommensurate system (1) or commensurate systems (5) and (15). On the other hand, these sophisticated results do not assert how to discover or construct some adequate Lyapunov functions. This is one of the biggest limitations: there are no general methods known for how to find such an interesting function. The next section facilitates some new knowledge of computing fractional derivatives of Lyapunov functions whenever they exist.

4. Some Lyapunov Function Inequalities

Inequalities are crucial in the successful use of appropriate Lyapunov stability results that appear in the literature. There exist a lot of published works dealing with inequalities concerning reasonable Lyapunov functions (see, e.g., [29,34,37,46,51,52]).

We give below a few inequalities that sharpened new thought for the promising fractional Lyapunov direct method.

Lemma 1. [32] Let $x(t) \in \mathbb{R}$ be a continuous and derivable function. Then, for any time instant $t \geq b$

$$\frac{1}{2} {}^C\mathcal{D}_{b^+,t}^\omega x^2(t) \leq x(t) {}^C\mathcal{D}_{b^+,t}^\omega x(t), \quad \forall \omega \in (0, 1). \quad (40)$$

Lemma 2. [31] Let $x(t) \in \mathbb{R}^n$ be a vector of differentiable functions. Then, for any time instant $t \geq b$, the following relationship holds

$${}^C\mathcal{D}_{b^+,t}^\omega \left(x^T(t) P x(t) \right) \leq 2x^T(t) P {}^C\mathcal{D}_{b^+,t}^\omega x(t), \quad \forall \omega \in (0, 1], \quad (41)$$

where $P \in \mathbb{R}^{n \times n}$ is a constant, square, symmetric and positive definite matrix.

Lemma 3. [40] Suppose that V be a real-valued $\mathcal{C}^1$ function on $[b, \infty) \times \Omega \subseteq \mathbb{R}^n$ which is convex w.r.t. its arguments. Let $x : [b, \infty) \rightarrow \Omega \subseteq \mathbb{R}^n$ be a continuous function which is differentiable on (b, ∞) . Then, we have

$$\mathcal{C}_{b^+,t}^\omega V(t, x(t)) \leq \frac{\partial V(t, x(t))}{\partial t} \mathcal{C}_{b^+,t}^\omega + \left(\frac{\partial V(t, x(t))}{\partial x(t)} \right)^T \mathcal{C}_{b^+,t}^\omega x(t), \quad (42)$$

$\forall t > b$ and $\forall \omega \in (0, 1]$.

Remark 8. Lemma 3 can be found in the work by Wu [46], where the author has not considered initial-time b to the interval $(-\infty, 0)$. As a result, the version of Lemma 3 of [46] cannot be applied to system (5) if initial time b is placed on $(-\infty, 0)$. In comparison to the inequality of Wu [46], Lemma 3 provides promising implications to random initial-time system (5).

Lemma 4. [40] Suppose that V is a real-valued $\mathcal{C}^1$ function on $[b, \infty) \times \Omega \subseteq \mathbb{R}^n$ which is convex w.r.t. Ω . Let $x : [b, \infty) \rightarrow \Omega \subseteq \mathbb{R}^n$ be a continuous function which is differentiable on (b, ∞) . Then, we have

$$\mathcal{C}_{b^+,t}^\omega V(t, x(t)) \leq \left(\frac{\partial V(t, x(t))}{\partial x(t)} \right)^T \mathcal{C}_{b^+,t}^\omega x(t), \quad (43)$$

$\forall t > b$ and $\forall \omega \in (0, 1]$.

Lemma 5. [39] Assume $x \in \Omega$ is continuous on $[b, \infty)$ and differentiable on (b, ∞) . Suppose that $\exists$ a continuously differentiable $V : [b, \infty) \times \Omega \subseteq \mathbb{R}^n \rightarrow [0, \infty)$ which is convex in Ω . Assume that $\frac{\partial V(t, x)}{\partial t} \geq 0$, for all $t \geq b$. Then, for any $t > b$, the inequality

$$\mathcal{C}_{b^+,t}^\omega V(t, x(t)) \leq \frac{\partial V(t, x(t))}{\partial t} + \left(\frac{\partial V(t, x(t))}{\partial x(t)} \right)^T \mathcal{C}_{b^+,t}^\omega x(t), \quad (44)$$

holds, where $\omega \in (0, 1]$.

Lemma 6. [43] Suppose $V_i : [b, \infty) \times \Omega_i \subseteq \mathbb{R}^{n_i} \rightarrow \mathbb{R}$ are continuously differentiable for $i = 1, 2, \dots, n$. Assume that $x_i : [b, \infty) \rightarrow \Omega_i \subseteq \mathbb{R}^{n_i}$ are continuous on $[b, \infty)$ and differentiable on (b, ∞) for $i = 1, 2, \dots, n$. Let $V_1, \dots, V_n$ be convex with respect to its corresponding arguments in $[b, \infty) \times \Omega_1, \dots, [b, \infty) \times \Omega_n$, respectively. Then, for any $t > b$, the inequality

$$\begin{aligned} \sum_{i=1}^n \mathcal{C}_{b^+,t}^{\alpha_i} V_i(t, x_i(t)) &\leq \sum_{i=1}^n \frac{\partial V_i(t, x_i(t))}{\partial t} \mathcal{C}_{b^+,t}^{\alpha_i} (t - b) \\ &\quad + \sum_{i=1}^n \left(\frac{\partial V_i(t, x_i(t))}{\partial x_i(t)} \right)^T \mathcal{C}_{b^+,t}^{\alpha_i} x_i(t), \end{aligned} \quad (45)$$

holds, $\forall \alpha_1, \dots, \alpha_n \in (0, 1]$.

Lemma 7. [43] Suppose $V_i : [b, \infty) \times \Omega_i \subseteq \mathbb{R}^{n_i} \rightarrow \mathbb{R}$ are continuously differentiable for $i = 1, 2, \dots, n$. Assume that $x_i : [t^*, \infty) \rightarrow \Omega_i \subseteq \mathbb{R}^{n_i}$ are continuous on $[b, \infty)$ and differentiable on (b, ∞) for $i = 1, 2, \dots, n$. Let $V_1, \dots, V_n$ be convex with respect to the arguments in $\Omega_1, \dots, \Omega_n$, respectively. Assume that $\frac{\partial V_i(t, x_i(t))}{\partial t} \geq 0$, for all $t \geq b$, for $i = 1, 2, \dots, n$. Then, for any $t > b$, the inequality

$$\sum_{i=1}^n \mathcal{C}_{b^+,t}^{\alpha_i} V_i(t, x_i(t)) \leq \sum_{i=1}^n \frac{\partial V_i(t, x_i(t))}{\partial t} + \sum_{i=1}^n \left(\frac{\partial V_i(t, x_i(t))}{\partial x_i(t)} \right)^T \mathcal{C}_{b^+,t}^{\alpha_i} x_i(t), \quad (46)$$

holds, $\forall \alpha_1, \dots, \alpha_n \in (0, 1]$.

Lemma 8. [43] Suppose $V_i : [b, \infty) \times \Omega_i \subseteq \mathbb{R}^{n_i} \rightarrow \mathbb{R}$ are continuously differentiable for $i = 1, 2, \dots, n$. Assume that $x_i : [t^*, \infty) \rightarrow \Omega_i \subseteq \mathbb{R}^{n_i}$ are continuous on $[b, \infty)$ and differentiable on (b, ∞) for $i = 1, 2, \dots, n$. Let $V_1, \dots, V_n$ be convex with respect to the arguments in $\Omega_1, \dots, \Omega_n$, respectively. Then, for any $t > b$, the inequality

$$\sum_{i=1}^n {}^C\mathcal{D}_{b^+,t}^{\alpha_i} V_i(t, x_i(t)) \leq \sum_{i=1}^n \left(\frac{\partial V_i(t, x_i(t))}{\partial x_i(t)} \right)^T {}^C\mathcal{D}_{b^+,t}^{\alpha_i} x_i(t), \quad (47)$$

holds, $\forall \alpha_1, \dots, \alpha_n \in (0, 1]$.

Lemma 9. [43] Let $x_1, x_2, \dots, x_n \in \mathbb{R}$ be continuous on $[b, \infty)$ and differentiable on (b, ∞) . Then, the inequality

$$\sum_{j=1}^n {}^C\mathcal{D}_{b^+,t}^{\alpha_j} x_j^2(t) \leq 2 \sum_{j=1}^n x_j(t) {}^C\mathcal{D}_{b^+,t}^{\alpha_j} x_j(t), \quad \forall t > b, \quad (48)$$

holds, $\forall \alpha_1, \dots, \alpha_n \in (0, 1]$.

Lemma 10. [43] Let $x_i \in \mathbb{R}^{n_i}$ be continuous on $[b, \infty)$ and differentiable on (b, ∞) for $i = 1, 2, \dots, n$. Then, the inequality

$$\sum_{j=1}^n {}^C\mathcal{D}_{b^+,t}^{\alpha_j} \left(x_j^T(t) P_j x_j(t) \right) \leq 2 \sum_{j=1}^n x_j^T(t) P_j {}^C\mathcal{D}_{b^+,t}^{\alpha_j} x_j(t), \quad \forall t > b, \quad (49)$$

holds, $\forall \alpha_1, \alpha_2, \dots, \alpha_n \in (0, 1]$, where $P_i \in \mathbb{R}^{n_i \times n_i}$ is constant, symmetric and positive definite matrix for $i = 1, 2, \dots, n$.

5. New Stability Conditions in Fractional Order Systems

This section has two goals. The first goal is concerned with identifying conditions for stability of the linear version of system (1). The second goal is to identify conditions for stability of the nonlinear version of system (1). Some new conditions were addressed in Subsection 5.1 and Subsection 5.2, separately.

5.1. Class of Linear Fractional Order Systems

This part discusses new conditions for Mittag-Leffler asymptotic stability and multi-order Mittag-Leffler asymptotic stability of zero equilibrium of linear version of system (1).

When $f(t, x(t)) = A(t)x(t)$, the system (1) becomes linear fractional order system:

$$\begin{aligned} {}^C\mathcal{D}_{b^+,t}^{\hat{\delta}} x(t) &= A(t)x(t) \\ x(b) &= x_b \end{aligned} \quad (50)$$

where state variable $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in \mathbb{R}^n$, initial time $b \in \mathbb{R}$, operator ${}^C\mathcal{D}_{b^+,t}^{\hat{\delta}} x(t) = \left({}^C\mathcal{D}_{b^+,t}^{\delta_1} x_1(t), {}^C\mathcal{D}_{b^+,t}^{\delta_2} x_2(t), \dots, {}^C\mathcal{D}_{b^+,t}^{\delta_n} x_n(t) \right)^T \in \mathbb{R}^n$, order-index $\hat{\delta} = (\delta_1, \delta_2, \dots, \delta_n) \in (0, 1] \times (0, 1] \times \dots \times (0, 1]$, and $A(t) = [a_{ij}(t)] : [b, \infty) \rightarrow \mathbb{R}^{n \times n}$ is continuous.

Theorem 21. Consider the commensurate order system (50) with $\delta_1 = \delta_2 = \dots = \delta_n = \tilde{\delta}$. If there exists a constant, symmetric, and positive definite matrix $P = [p_{ij}] \in \mathbb{R}^{n \times n}$ such that the condition

$$A^T(t)P + PA(t) \leq -\lambda^+ I_n, \quad \forall t \geq b, \quad (51)$$

holds where constant $\lambda^+ > 0$, then the zero equilibrium of (50) is globally Mittag-Leffler asymptotically stable.

Proof. Take $V(t, x) = x^T P x$, where $P = [p_{ij}] \in \mathbb{R}^{n \times n}$ is a constant, symmetric, and positive definite matrix. Note that

$$\lambda_{\min}(P) \|x\|^2 \leq V(t, x) \leq \lambda_{\max}(P) \|x\|^2. \quad (52)$$

Set $\alpha = \tilde{\delta}$. Then, by using Lemma 10, along system (50) solution $x(t)$:

$$\begin{aligned} {}^C\mathcal{D}_{b^+,t}^\alpha V(t, x(t)) &\leq 2x^T(t) P {}^C\mathcal{D}_{b^+,t}^\alpha x(t) \\ &= 2x^T(t) P A(t) x(t) \\ &\leq x^T(t) \left(P A(t) + A^T(t) P \right) x(t) \\ &\leq -\frac{\lambda^+}{\lambda_{\max}(P)} V(t, x(t)). \end{aligned} \quad (53)$$

Set $\lambda = \frac{\lambda^+}{\lambda_{\max}(P)} > 0$. Then, we see that $A_1)$ and $A_2)$ of Theorem 10 are satisfied. Thus, the zero equilibrium of (50) should be globally Mittag-Leffler asymptotically stable. $\square$

Theorem 22. Consider the commensurate order system (50) with $\delta_1 = \delta_2 = \dots = \delta_n = \tilde{\delta}$. If there exists a constant, symmetric, and positive definite matrix $P = [p_{ij}] \in \mathbb{R}^{n \times n}$ such that the condition

$$A^T(t)P + PA(t) \leq -\lambda^\# I_n, \quad \forall t \geq b, \quad (54)$$

holds where constant $\lambda^\# \geq 0$, then the zero equilibrium of (50) is globally Mittag-Leffler stable.

Proof. The is quite similar to Theorem 21. $\square$

Theorem 23. Consider the incommensurate order system (50). If there exists a constant, diagonal, and positive definite matrix $P = \text{diag}[p_{11}, p_{22}, \dots, p_{nn}] \in \mathbb{R}^{n \times n}$ such that the condition

$$A^T(t)P + PA(t) \leq -\lambda^* I_n, \quad \forall t \geq b, \quad (55)$$

holds where constant $\lambda^* > 0$, then the zero equilibrium of (50) is globally multi-order Mittag-Leffler asymptotically stable.

Proof. Take $V(t, x) = \sum_{j=1}^n V_j(t, x_j)$, where $V_j(t, x_j) = p_{jj} x_j^2$ and $p_{jj} > 0$ are constants for $j = 1, 2, \dots, n$. Let $P = \text{diag}[p_{11}, p_{22}, \dots, p_{nn}] \in \mathbb{R}^{n \times n}$. Then

$$\lambda_{\min}(P) \|x\|^2 \leq V(t, x) \leq \lambda_{\max}(P) \|x\|^2. \quad (56)$$

Set $\gamma_j = \delta_j$ for $j = 1, 2, \dots, n$. By using Lemma 10, along system (50) solution $x(t)$:

$$\begin{aligned} \sum_{j=1}^n {}^C\mathcal{D}_{b^+,t}^{\gamma_j} p_{jj} x_j^2(t) &\leq 2 \sum_{j=1}^n p_{jj} x_j(t) {}^C\mathcal{D}_{b^+,t}^{\gamma_j} x_j(t) \\ &= 2x^T(t) P \left({}^C\mathcal{D}_{b^+,t}^{\hat{\gamma}} x(t) \right) \\ &= 2x^T(t) P A(t) x(t) \\ &\leq x^T(t) \left(P A(t) + A^T(t) P \right) x(t) \\ &\leq -\frac{\lambda^*}{\lambda_{\max}(P)} V(t, x(t)). \end{aligned} \quad (57)$$

Set $\lambda_j = \frac{\lambda^*}{\lambda_{\max}(P)} > 0$ for $j = 1, 2, \dots, n$. Then, we see that $A_1)$ and $A_2)$ of Corollary 1 are satisfied. Thus, the zero equilibrium of (50) should be globally multi-order Mittag-Leffler asymptotically stable. $\square$

Theorem 24. Consider the incommensurate order system (50). If there exists a constant, diagonal, and positive definite matrix $P = \text{diag}[p_{11}, p_{22}, \dots, p_{nn}] \in \mathbb{R}^{n \times n}$ such that the condition

$$A^T(t)P + PA(t) \leq -\lambda^+ I_n, \quad \forall t \geq b, \quad (58)$$

holds where constant $\lambda^+ \geq 0$, then the zero equilibrium of (50) is globally multi-order Mittag-Leffler stable.

Proof. The proof is similar to Theorem 23. $\square$

Remark 9. In Theorems 21 and 22, one can use $P = I_n \in \mathbb{R}^{n \times n}$, where I_n is an identity matrix.

5.2. Class of Nonlinear Fractional Order Systems

In this part, we generalize the previous section results for nonlinear systems.

Note that when $f(t, x(t)) = A(t)x(t) + g(t, x(t))$, the general nonlinear system (1) becomes

$$\begin{aligned} {}^C\mathcal{D}_{b^+,t}^{\hat{\delta}} x(t) &= A(t)x(t) + g(t, x(t)) \\ x(b) &= x_b \end{aligned} \quad (59)$$

where state variable $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in \mathbb{R}^n$, initial time $b \in \mathbb{R}$, operator ${}^C\mathcal{D}_{b^+,t}^{\hat{\delta}} x(t) = \left({}^C\mathcal{D}_{b^+,t}^{\delta_1} x_1(t), {}^C\mathcal{D}_{b^+,t}^{\delta_2} x_2(t), \dots, {}^C\mathcal{D}_{b^+,t}^{\delta_n} x_n(t) \right)^T \in \mathbb{R}^n$, order-index $\hat{\delta} = (\delta_1, \delta_2, \dots, \delta_n) \in (0, 1] \times (0, 1] \times \dots \times (0, 1]$, $A(t) = [a_{ij}(t)] : [b, \infty) \rightarrow \mathbb{R}^{n \times n}$ is continuous, and $g = (g_1, g_2, \dots, g_n)^T : [b, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous, uniformly Lipschitz and $g(t, 0) = 0$, for all $t \geq b$.

We discuss first stability results for the commensurate order version of system (59). The following theorems develop new conditions whenever the nonlinear function in (59) obeys norm bound and global Lipschitz properties.

Theorem 25. Consider the commensurate order system (59) with $\delta_1 = \delta_2 = \dots = \delta_n = \tilde{\delta}$. Assume that C_1) the function g obeys norm bound condition

$$\|g(t, u)\| \leq L^+ \|u\|, \quad \forall t \geq b, \quad \forall u \in \Omega \subset \mathbb{R}^n, \quad (60)$$

where constant $L^+ > 0$,

C_2) there exists a constant, symmetric, and positive definite matrix $P = [p_{ij}] \in \mathbb{R}^{n \times n}$ such that

$$PA(t) + A^T(t)P + (L^+)^2 I_n + P^T P \leq -\lambda^+ I_n, \quad \forall t \geq b, \quad (61)$$

where constant $\lambda^+ > 0$ and identity matrix $I_n \in \mathbb{R}^{n \times n}$.

Then, the zero equilibrium of (59) is locally Mittag-Leffler asymptotically stable.

Proof. Take the Lyaunov function $V(t, x) = x^T P x$, where $P = [p_{ij}] \in \mathbb{R}^{n \times n}$ is a constant, symmetric, and positive definite matrix. Note that

$$\lambda_{\min}(P) \|x\|^2 \leq V(t, x) \leq \lambda_{\max}(P) \|x\|^2. \quad (62)$$

Set $\alpha = \tilde{\delta}$. Then, by using Lemma 10, along system (59) solution $x(t)$:

$$\begin{aligned}
 {}^C\mathcal{D}_{b^+,t}^\alpha V(t, x(t)) &\leq 2x^T(t)P {}^C\mathcal{D}_{b^+,t}^\alpha x(t) \\
 &= 2x^T(t)P(A(t)x(t) + g(t, x(t))) \\
 &= 2x^T(t)PA(t)x(t) + 2x^T(t)Pg(t, x(t)) \\
 &\leq x^T(t)(PA(t) + A^T(t)P)x(t) + x^T(t)P^T Px(t) + \|g(t, x(t))\|^2 \\
 &\leq x^T(t)(PA(t) + A^T(t)P + (L^+)^2 I_n + P^T P)x(t) \\
 &\leq -\frac{\lambda^+}{\lambda_{\max}(P)} V(t, x(t)),
 \end{aligned} \tag{63}$$

where C_1) and C_2) were used. Set $\lambda = \frac{\lambda^+}{\lambda_{\max}(P)} > 0$. Then, we see that A_1) and A_2) of Theorem 10 are satisfied. Thus, the zero equilibrium of (59) should be locally Mittag-Leffler asymptotically stable. $\square$

Theorem 26. Consider the commensurate order system (59) with $\delta_1 = \delta_2 = \dots = \delta_n = \tilde{\delta}$. Assume that C_1) the function g obeys norm bound condition

$$\|g(t, u)\| \leq L^\# \|u\|, \quad \forall t \geq b, \quad \forall u \in \Omega \subset \mathbb{R}^n, \tag{64}$$

where constant $L^\# > 0$,

C_2) there exists a constant, symmetric, and positive definite matrix $P = [p_{ij}] \in \mathbb{R}^{n \times n}$ such that

$$PA(t) + A^T(t)P + (L^\#)^2 I_n + P^T P \leq -\lambda^\# I_n, \quad \forall t \geq b, \tag{65}$$

where constant $\lambda^\# \geq 0$ and identity matrix $I_n \in \mathbb{R}^{n \times n}$.

Then, the zero equilibrium of (59) is locally Mittag-Leffler stable.

Theorem 27. Consider the commensurate order system (59) with $\delta_1 = \delta_2 = \dots = \delta_n = \tilde{\delta}$. Assume that C_1) the function g obeys global uniformly Lipschitz condition

$$\|g(t, u) - g(t, w)\| \leq L^* \|u - w\|, \quad \forall t \geq b, \quad \forall u, w \in \mathbb{R}^n, \tag{66}$$

where constant $L^* > 0$,

C_2) there exists a constant, symmetric, and positive definite matrix $P = [p_{ij}] \in \mathbb{R}^{n \times n}$ such that

$$PA(t) + A^T(t)P + (L^*)^2 I_n + P^T P \leq -\lambda^* I_n, \quad \forall t \geq b, \tag{67}$$

where constant $\lambda^* > 0$ and identity matrix $I_n \in \mathbb{R}^{n \times n}$.

Then, the zero equilibrium of (59) is globally Mittag-Leffler asymptotically stable.

Proof. We let the Lyaunov function $V(t, x) = x^T P x$, where $P = [p_{ij}] \in \mathbb{R}^{n \times n}$ is a constant, symmetric, and positive definite matrix. Observe that

$$\lambda_{\min}(P) \|x\|^2 \leq V(t, x) \leq \lambda_{\max}(P) \|x\|^2. \tag{68}$$

Set $\alpha = \tilde{\delta}$. Then, by invoking Lemma 10, along system (59) solution $x(t)$:

$$\begin{aligned} {}^C\mathcal{D}_{b^+,t}^\alpha V(t, x(t)) &\leq 2x^T(t)P {}^C\mathcal{D}_{b^+,t}^\alpha x(t) \\ &= 2x^T(t)PA(t)x(t) + 2x^T(t)Pg(t, x(t)) \\ &\leq x^T(t)\left(PA(t) + A^T(t)P\right)x(t) + x^T(t)P^TPx(t) + \|g(t, x(t))\|^2 \\ &\leq x^T(t)\left(PA(t) + A^T(t)P + (L^*)^2I_n + P^TP\right)x(t) \\ &\leq -\frac{\lambda^*}{\lambda_{\max}(P)}V(t, x(t)), \end{aligned} \quad (69)$$

where the conditions $C_1)$ and $C_2)$ were used. Set $\lambda = \frac{\lambda^*}{\lambda_{\max}(P)} > 0$. Then, we notice that $A_1)$ and $A_2)$ of Theorem 10 are satisfied. Thus, the zero equilibrium of (59) should be globally Mittag-Leffler asymptotically stable. $\square$

Theorem 28. Consider the commensurate order system (59) with $\delta_1 = \delta_2 = \dots = \delta_n = \tilde{\delta}$. Assume that $C_1)$ the function g obeys global uniformly Lipschitz condition

$$\|g(t, u) - g(t, w)\| \leq L^+ \|u - w\|, \quad \forall t \geq b, \quad \forall u, w \in \mathbb{R}^n, \quad (70)$$

where constant $L^+ > 0$,

$C_2)$ there exists a constant, symmetric, and positive definite matrix $P = [p_{ij}] \in \mathbb{R}^{n \times n}$ such that

$$PA(t) + A^T(t)P + (L^+)^2I_n + P^TP \leq -\lambda^+ I_n, \quad \forall t \geq b, \quad (71)$$

where constant $\lambda^+ \geq 0$ and identity matrix $I_n \in \mathbb{R}^{n \times n}$.

Then, the zero equilibrium of (59) is globally Mittag-Leffler stable.

Remark 10. Theorems 26 and 28 build conditions where the trajectory of the commensurate order version of system (59) does not necessarily converge to the origin but rather stays close to it in large time at ∞ . On the other hand, Theorems 25 and 27 develop conditions for which the mentioned trajectory of the system ultimately converges to the origin whenever $t \rightarrow \infty$ in the sense of Mittag-Leffler decay. Liu et al. [53] discussed some new stability conditions for the class of system (59) by using the S-procedure and Lyapunov function inequality technique. Tunç [54] considered a new class of perturbed nonlinear commensurate fractional systems and formulated some reasonable conditions of boundedness, uniform stability, and Mittag-Leffler stability in the sense of Li et al. [21,22].

Next, we construct new conditions for the incommensurate fractional order system (59). Note that in this case the orders involved in (59) need not necessarily be equal.

Theorem 29. Consider the incommensurate order system (59). Assume that

$C_1)$ the function g obeys norm bound condition

$$\|g(t, u)\| \leq L^+ \|u\|, \quad \forall t \geq b, \quad \forall u \in \Omega \subset \mathbb{R}^n, \quad (72)$$

where constant $L^+ > 0$,

$C_2)$ there exists a constant, diagonal matrix $P = \text{diag}[p_{11}, p_{22}, \dots, p_{nn}] \in \mathbb{R}^{n \times n}$ such that

$$PA(t) + A^T(t)P + (L^+)^2I_n + P^TP \leq -\lambda^+ I_n, \quad \forall t \geq b, \quad (73)$$

where constant $\lambda^+ > 0$ and identity matrix $I_n \in \mathbb{R}^{n \times n}$.

Then, the zero equilibrium of (59) is locally multi-order Mittag-Leffler asymptotically stable.

Proof. Take the Lyapunov function $V(t, x) = \sum_{j=1}^n V_j(t, x_j)$, where $V_j(t, x_j) = p_{jj}x_j^2$ and $p_{jj} > 0$ are constants for $j = 1, 2, \dots, n$. Let $P = \text{diag}[p_{11}, p_{22}, \dots, p_{nn}] \in \mathbb{R}^{n \times n}$. Then, we have

$$\lambda_{\min}(P)\|x\|^2 \leq V(t, x) \leq \lambda_{\max}(P)\|x\|^2. \quad (74)$$

Set $\gamma_j = \delta_j$ for $j = 1, 2, \dots, n$. By using Lemma 10, along system (59) solution $x(t)$:

$$\begin{aligned} \sum_{j=1}^n {}^C\mathcal{D}_{b^+, t}^{\gamma_j} p_{jj}x_j^2(t) &\leq 2 \sum_{j=1}^n p_{jj}x_j(t) {}^C\mathcal{D}_{b^+, t}^{\gamma_j} x_j(t) \\ &= 2x^T(t)P \left({}^C\mathcal{D}_{b^+, t}^{\hat{\gamma}} x(t) \right) \\ &= 2x^T(t)(PA(t)x(t) + g(t, x(t))) \\ &\leq x^T(t) \left(PA(t) + A^T(t)P \right) x(t) + x^T(t)P^T Px(t) + \|g(t, x(t))\|^2 \\ &\leq x^T(t) \left(PA(t) + A^T(t)P + (L^+)^2 I_n + P^T P \right) x(t) \\ &\leq -\frac{\lambda^+}{\lambda_{\max}(P)} V(t, x(t)), \end{aligned} \quad (75)$$

where we have used the condition C_1) and C_2). Set $\lambda_j = \frac{\lambda^+}{\lambda_{\max}(P)} > 0$ for $j = 1, 2, \dots, n$. Then, we see that A_1) and A_2) of Corollary 1 are satisfied. Thus, the zero equilibrium of (50) should be globally multi-order Mittag-Leffler asymptotically stable. $\square$

Theorem 30. Consider the incommensurate order system (59). Assume that C_1) the function g obeys norm bound condition

$$\|g(t, u)\| \leq L^\# \|u\|, \quad \forall t \geq b, \quad \forall u \in \Omega \subset \mathbb{R}^n, \quad (76)$$

where constant $L^\# > 0$,

C_2) there exists a constant, diagonal matrix $P = \text{diag}[p_{11}, p_{22}, \dots, p_{nn}] \in \mathbb{R}^{n \times n}$ such that

$$PA(t) + A^T(t)P + (L^\#)^2 I_n + P^T P \leq -\lambda^\# I_n, \quad \forall t \geq b, \quad (77)$$

where constant $\lambda^\# \geq 0$ and identity matrix $I_n \in \mathbb{R}^{n \times n}$.

Then, the zero equilibrium of (59) is locally multi-order Mittag-Leffler stable.

Theorem 31. Consider the incommensurate order system (59). Assume that C_1) the function g obeys global Lipschitz condition

$$\|g(t, u) - g(t, w)\| \leq L^* \|u - w\|, \quad \forall t \geq b, \quad \forall u, w \in \mathbb{R}^n, \quad (78)$$

where constant $L^* > 0$,

C_2) there exists a constant, diagonal matrix $P = \text{diag}[p_{11}, p_{22}, \dots, p_{nn}] \in \mathbb{R}^{n \times n}$ such that

$$PA(t) + A^T(t)P + (L^*)^2 I_n + P^T P \leq -\lambda^* I_n, \quad \forall t \geq b, \quad (79)$$

where constant $\lambda^* > 0$ and identity matrix $I_n \in \mathbb{R}^{n \times n}$.

Then, the zero equilibrium of (59) is globally multi-order Mittag-Leffler asymptotically stable.

Proof. Take the Lyapunov function $V(t, x) = \sum_{j=1}^n V_j(t, x_j)$, where $V_j(t, x_j) = p_{jj}x_j^2$ and $p_{jj} > 0$ are constants for $j = 1, 2, \dots, n$. Let $P = \text{diag}[p_{11}, p_{22}, \dots, p_{nn}] \in \mathbb{R}^{n \times n}$. Then, one has

$$\lambda_{\min}(P)\|x\|^2 \leq V(t, x) \leq \lambda_{\max}(P)\|x\|^2. \quad (80)$$

Set $\gamma_j = \delta_j$ for $j = 1, 2, \dots, n$. By using Lemma 10, along system (59) solution $x(t)$:

$$\begin{aligned} \sum_{j=1}^n {}^C\mathcal{D}_{b^+, t}^{\gamma_j} p_{jj}x_j^2(t) &\leq 2 \sum_{j=1}^n p_{jj}x_j(t) {}^C\mathcal{D}_{b^+, t}^{\gamma_j} x_j(t) \\ &= 2x^T(t)P \left({}^C\mathcal{D}_{b^+, t}^{\hat{\gamma}} x(t) \right) \\ &= 2x^T(t)(PA(t)x(t) + g(t, x(t))) \\ &\leq x^T(t) \left(PA(t) + A^T(t)P \right) x(t) + x^T(t)P^T Px(t) + \|g(t, x(t))\|^2 \\ &\leq x^T(t) \left(PA(t) + A^T(t)P + (L^*)^2 I_n + P^T P \right) x(t) \\ &\leq -\frac{\lambda^*}{\lambda_{\max}(P)} V(t, x(t)), \end{aligned} \quad (81)$$

where we have used the condition C_1) and C_2). Set $\lambda_j = \frac{\lambda^*}{\lambda_{\max}(P)} > 0$ for $j = 1, 2, \dots, n$. Then, we see that A_1) and A_2) of Corollary 1 are satisfied. Thus, the zero equilibrium of (50) should be globally multi-order Mittag-Leffler asymptotically stable. $\square$

Theorem 32. Consider the incommensurate order system (59). Assume that

C_1) the function g obeys global Lipschitz condition

$$\|g(t, u) - g(t, w)\| \leq L^+ \|u - w\|, \quad \forall t \geq b, \quad \forall u, w \in \mathbb{R}^n, \quad (82)$$

where constant $L^+ > 0$,

C_2) there exists a constant, diagonal matrix $P = \text{diag}[p_{11}, p_{22}, \dots, p_{nn}] \in \mathbb{R}^{n \times n}$ such that

$$PA(t) + A^T(t)P + (L^+)^2 I_n + P^T P \leq -\lambda^+ I_n, \quad \forall t \geq b, \quad (83)$$

where constant $\lambda^+ \geq 0$ and identity matrix $I_n \in \mathbb{R}^{n \times n}$.

Then, the zero equilibrium of (59) is globally multi-order Mittag-Leffler stable.

Remark 11. Theorems 29, 30, 31, and 32 do not provide an arbitrary symmetric, positive definite matrix $P \in \mathbb{R}^{n \times n}$, making some strict conditions for the incommensurate system (59) as compared to its commensurate order counterparts. There exist only a few studies in the literature that concern finding reasonable new conditions that can make the system (59) not only asymptotically stable but also stable.

6. Growth or Decay of Solutions in Fractional Order Systems

In this section, we discuss the rate of growth or decay of solutions arising in system (1) by using the knowledge of the Lyapunov function.

At first, the below-mentioned establishes bounds for the nontrivial solution to the commensurate fractional order system (5). In [39], the authors have observed solutions of the mentioned system can obey in the sense of Mittag-Leffler. In short, it will tell whether the norm measure of solutions of system (5) grows, is bound by a constant, or decays at the Mittag-Leffler rate with a constant scaling factor in its argument.

Lemma 11. [39] Consider the commensurate fractional order system (5). Suppose that $\exists$ a continuously differentiable $V : [b, \infty) \times \Omega \subseteq \mathbb{R}^n \rightarrow [0, \infty)$ that satisfies:

(i) inequality

$$k_1 \|x\|^p \leq V(t, x) \leq \mu(t) \|x\|^q, \quad (84)$$

where constants k_1, p, q are positive and continuous function $\mu(t) : [b, \infty) \rightarrow [1, \infty)$ satisfies $\mu(t) - k_1 \geq 0$, $\mu(t) - 1 \geq 0$ and $p - q \leq 0$ with $q - 1 \geq 0$,

(ii) if $x(t)$ is any solution of (5),

$${}^C\mathcal{D}_{b^+,t}^{\tilde{\delta}} V(t, x(t)) \leq \lambda V(t, x(t)), \quad \forall t > b, \quad (85)$$

where $\lambda \in \mathbb{R}$, $\tilde{\delta} \in (0, 1]$, $\forall x \in \Omega \subseteq \mathbb{R}^n$.

Then, the inequality

$$\|x(t)\| \leq \left[\frac{\mu(b)}{k_1} E_{\tilde{\delta},1}(\lambda(t-b)^{\tilde{\delta}}) \|x(b)\|^q \right]^{1/p}, \quad \forall t \geq b. \quad (86)$$

Lemma 12. [50] Consider the inequality

$$\sum_{i=1}^n {}^C\mathcal{D}_{b^+,t}^{\gamma_i} y_i(t) \leq \sum_{i=1}^n \lambda_i y_i(t), \quad \forall t > b, \quad (87)$$

with $y_i(b) = y_i^*$, where constants $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$, and orders $\gamma_1, \gamma_2, \dots, \gamma_n \in (0, 1]$. Then, we have

$$\sum_{i=1}^n y_i(t) \leq \sum_{i=1}^n E_{\gamma_i,1}(\lambda_i(t-b)^{\gamma_i}) y_i(b), \quad \forall t \geq b. \quad (88)$$

In view of Lemma 11, we thought of following result below that develops a new possible bound to the growth, or decay in the sense of several order Mittag-Leffler functions.

Lemma 13. Consider the incommensurate fractional order system (1). Suppose there exist a C^1 function $V : [b, \infty) \times \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ such that

- (i) $q_1 \|x\|^p \leq V(t, x) = \sum_{i=1}^n V_i(t, x_i) \leq q_2 \mu(t) \|x\|^q$, where p, q, q_1, q_2 are arbitrary positive constants, and $\mu : [b, \infty) \rightarrow [1, \infty)$ is continuous,
- (ii) along the non-trivial solution $x(t)$ of (1):

$$\sum_{i=1}^n {}^C\mathcal{D}_{b^+,t}^{\gamma_i} V_i(t, x_i(t)) \leq \sum_{i=1}^n \lambda_i V_i(t, x_i(t)), \quad \forall x \in \Omega, \quad \forall t > b, \quad (89)$$

where constants $\lambda_i \in \mathbb{R}$ for $i = 1, 2, \dots, n$, and orders $\gamma_1, \gamma_2, \dots, \gamma_n \in (0, 1]$.

Then, the inequality

$$\|x(t)\| \leq \left[\frac{q_2 \mu(b)}{q_1} \sum_{i=1}^n E_{\gamma_i,1}(\lambda_i(t-b)^{\gamma_i}) \|x(b)\|^q \right]^{1/p}, \quad \forall t \geq b. \quad (90)$$

Proof. It follows inequality (89) of (ii), and Lemma 12 that

$$\sum_{i=1}^n V_i(t, x_i(t)) \leq \sum_{i=1}^n E_{\gamma_i,1}(\lambda_i(t-b)^{\gamma_i}) V_i(b, x_i(b)), \quad \forall t \geq b. \quad (91)$$

Consequently, by using (i), one has

$$\|x(t)\| \leq \left[\frac{q_2 \mu(b)}{q_1} \sum_{i=1}^n E_{\gamma_i,1}(\lambda_i(t-b)^{\gamma_i}) \|x(b)\|^q \right]^{1/p}, \quad \forall t \geq b. \quad (92)$$

□

Remark 12. Lemma 11 and Lemma 13 provides possibility to estimate growth or decay of energy associated with systems (5) and (1), respectively.

7. Attractive of Solutions in Fractional Order Systems

This section introduces a novel concept to attractive of solutions arising in fractional order systems. It is a distinguishing insightful property that can be observed in system (1) when any bounded fixed solution or any bounded solution pair attracts to each other in large time, especially whenever $t \rightarrow \infty$. Recently, the notion of attractive of solutions beyond constant equilibria has been introduced by Lenka and Upadhyay in work [44]. It gives a deeper aspect to such a scenario for a commensurate fractional order system (5). In Subsection 7.1, we discuss some important definitions and results for the commensurate order system (5). In Subsection 7.2, we introduce some important new definitions and results for the incommensurate order system (1).

7.1. Lyapunov Attractive for Commensurate Order Systems

The following definitions are appeared in the work of Lenka and Upadhyay [44].

Definition 19. [44] Let (φ, ψ) be any pair of solutions to (5). We say the pair (φ, ψ) is Mittag-Leffler asymptotically attractive if $\lim_{t \rightarrow \infty} \|\varphi(t) - \psi(t)\| = 0$ and the inequality

$$\|\varphi(t) - \psi(t)\| \leq C^* \left[C^\# E_{\omega,1}(-\lambda(t-b)^\omega) \right]^{p_1} \|\varphi(b) - \psi(b)\|^{p_2}, \quad \forall t \geq b, \quad (93)$$

holds, where constants $C^* \geq 1$, $C^\# > 0$, $\lambda > 0$, $p_1 > 0$, $p_2 > 0$ and $\omega \in (0, 1]$.

Definition 20. [44] Let (φ, ψ) be any pair of solutions to (5). We say the pair (φ, ψ) is Mittag-Leffler attractive if the inequality

$$\|\varphi(t) - \psi(t)\| \leq C^* \left[C^\# E_{\omega,1}(-\lambda(t-b)^\omega) \right]^{p_1} \|\varphi(b) - \psi(b)\|^{p_2}, \quad \forall t \geq b, \quad (94)$$

holds, where constants $C^* \geq 1$, $C^\# > 0$, $\lambda \geq 0$, $p_1 > 0$, $p_2 > 0$ and $\omega \in (0, 1]$.

Theorem 33. [44] Let (φ, ψ) be any solution pair to system (5) in a domain $\Omega \times \Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n$. Suppose there exists a C^1 function $V : [b, \infty) \times \Omega \times \Omega \rightarrow \mathbb{R}$ such that

A₁) $k_1 \|y_1 - y_2\|^{p_1} \leq V(t, y_1, y_2) \leq \Theta(t) \|y_1 - y_2\|^{p_1 p_2}$, where k_1, p_1, p_2 are positive constants and $\Theta : [b, \infty) \rightarrow [1, \infty)$ is continuous with $p_2 \geq p_1$, $\Theta(t) \geq k_1$ and $p_2 \geq 1$,

A₂) along system (5) non-trivial solutions $(\varphi(t), \psi(t))$:

$${}^C D_{b^+,t}^\omega V(t, \varphi(t), \psi(t)) \leq -\lambda V(t, \varphi(t), \psi(t)), \quad \forall \varphi - \psi \neq 0 \in \Omega, \quad \forall t > b, \quad (95)$$

where constant λ is positive, and $\omega \in (0, 1]$.

Then, the pair (φ, ψ) to system (5) is Mittag-Leffler asymptotically attractive. When $\Omega = \mathbb{R}^n$, the pair (φ, ψ) to (5) is globally Mittag-Leffler asymptotically attractive.

Theorem 34. [44] Let (φ, ψ) be any solution pair to system (5) in a domain $\Omega \times \Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n$. Suppose there exists a C^1 function $V : [b, \infty) \times \Omega \times \Omega \rightarrow \mathbb{R}$ such that

- $A_1)$ $k_1 \|y_1 - y_2\|^{p_1} \leq V(t, y_1, y_2) \leq \Theta(t) \|y_1 - y_2\|^{p_1 p_2}$, where k_1, p_1, p_2 are positive constants and $\Theta : [t^*, \infty) \rightarrow [1, \infty)$ is continuous with $p_2 \geq p_1$, $\Theta(t) \geq k_1$ and $p_2 \geq 1$,
 $A_2)$ along system (5) non-trivial solutions $(\varphi(t), \psi(t))$:

$${}^C\mathcal{D}_{b+,t}^\omega V(t, \varphi(t), \psi(t)) \leq -\lambda V(t, \varphi(t), \psi(t)), \forall \varphi - \psi \in \Omega, \forall t > b, \quad (96)$$

where constant λ is non-negative, and $\omega \in (0, 1]$.

Then, the pair (φ, ψ) to system (5) is Mittag-Leffler attractive. When $\Omega = \mathbb{R}^n$, the pair (φ, ψ) to system (5) is globally Mittag-Leffler attractive.

Definition 21. [44] Let $\varphi(t)$ be any solution to (5). We say the solution $\varphi(t)$ is Mittag-Leffler asymptotically attractive if the solutions $y(t)$ different to $\varphi(t)$ satisfy $\lim_{t \rightarrow \infty} \|y(t) - \varphi(t)\| = 0$ and the inequality

$$\|y(t) - \varphi(t)\| \leq C^* \left[C^\# E_{\omega,1}(-\lambda(t-b)^\omega) \right]^{p_1} \|y(b) - \varphi(b)\|^{p_2}, \forall t \geq b, \quad (97)$$

holds, where constants $C^* \geq 1$, $C^\# > 0$, $\lambda > 0$, $p_1 > 0$, $p_2 > 0$ and $\omega \in (0, 1]$.

Definition 22. [44] Let $\varphi(t)$ be any solution to (5). We say the solution $\varphi(t)$ is Mittag-Leffler attractive if the solutions $y(t)$ different to $\varphi(t)$ satisfy the inequality

$$\|y(t) - \varphi(t)\| \leq C^* \left[C^\# E_{\omega,1}(-\lambda(t-b)^\omega) \right]^{p_1} \|y(b) - \varphi(b)\|^{p_2}, \forall t \geq b, \quad (98)$$

holds, where $C^* \geq 1$, $C^\# > 0$, $\lambda \geq 0$, $p_1 > 0$, $p_2 > 0$ and $\omega \in (0, 1]$.

Theorem 35. [44] Let $\tilde{\varphi}$ be any solution to system (5) in a domain $\Omega \subseteq \mathbb{R}^n$. Suppose there exists a C^1 function $V : [b, \infty) \times \Omega \times \Omega \rightarrow \mathbb{R}$ such that

- $A_1)$ $k_1 \|y - \tilde{\varphi}\|^{p_1} \leq V(t, y, \tilde{\varphi}) \leq \Theta(t) \|y - \tilde{\varphi}\|^{p_1 p_2}$, where k_1, p_1, p_2 are positive constants and $\Theta : [b, \infty) \rightarrow [1, \infty)$ is continuous with $p_2 \geq p_1$, $\Theta(t) \geq k_1$ and $p_2 \geq 1$,
 $A_2)$ along system (5) non-trivial solution $y(t)$:

$${}^C\mathcal{D}_{b+,t}^\omega V(t, y(t), \tilde{\varphi}(t)) \leq -\lambda V(t, y(t), \tilde{\varphi}(t)), \forall y - \tilde{\varphi} \neq 0 \in \Omega, \forall t > b, \quad (99)$$

where constant $\lambda > 0$ and $\omega \in (0, 1]$.

Then, the solution $\tilde{\varphi}$ to system (5) is Mittag-Leffler asymptotically attractive. When $\Omega = \mathbb{R}^n$, the solution $\tilde{\varphi}$ to system (5) is globally Mittag-Leffler asymptotically attractive.

Theorem 36. [44] Let $\tilde{\varphi}$ be any solution to system (5) in a domain $\Omega \subseteq \mathbb{R}^n$. Suppose there exists a C^1 function $V : [b, \infty) \times \Omega \times \Omega \rightarrow \mathbb{R}$ such that

- $A_1)$ $k_1 \|y - \tilde{\varphi}\|^{p_1} \leq V(t, y, \tilde{\varphi}) \leq \Theta(t) \|y - \tilde{\varphi}\|^{p_1 p_2}$, where k_1, p_1, p_2 are positive constants and $\Theta : [b, \infty) \rightarrow [1, \infty)$ is continuous with $p_2 \geq p_1$, $\Theta(t) \geq k_1$ and $p_2 \geq 1$,
 $A_2)$ along system (5) non-trivial solution $y(t)$:

$${}^C\mathcal{D}_{b+,t}^\omega V(t, y(t), \tilde{\varphi}(t)) \leq -\lambda V(t, y(t), \tilde{\varphi}(t)), \forall y - \tilde{\varphi} \neq 0 \in \Omega, \forall t > b, \quad (100)$$

where constant $\lambda \geq 0$ and $\omega \in (0, 1]$.

Then, the solution $\tilde{\varphi}$ to system (5) is Mittag-Leffler attractive. When $\Omega = \mathbb{R}^n$, the solution $\tilde{\varphi}$ to system (5) is globally Mittag-Leffler attractive.

7.2. Lyapunov Attractive for Incommensurate Order Systems

Here we generalize previous subsection results to the incommensurate system (1). These results may be viewed as Lyapunov attractive theorems. First, we give the following definitions.

Definition 23. Let (φ, ψ) be any pair of solutions to (1). We say the pair (φ, ψ) is multi-order Mittag-Leffler asymptotically attractive if $\lim_{t \rightarrow \infty} \|\varphi(t) - \psi(t)\| = 0$ and the inequality

$$\|\varphi(t) - \psi(t)\| \leq C^* \left[C^\# \sum_{j=1}^n E_{\omega_j,1}(-\lambda_j(t-b)^{\omega_j}) \|\varphi(b) - \psi(b)\|^{p_2} \right]^{p_1}, \quad \forall t \geq b, \quad (101)$$

holds, where constants $C^* \geq 1$, $C^\# > 0$, $\lambda_1, \lambda_2, \dots, \lambda_n > 0$, $p_1 > 0$, $p_2 > 0$ and $\omega_1, \omega_2, \dots, \omega_n \in (0, 1]$.

Definition 24. Let (φ, ψ) be any pair of solutions to (1). We say the pair (φ, ψ) is multi-order Mittag-Leffler attractive if the inequality

$$\|\varphi(t) - \psi(t)\| \leq C^* \left[C^\# \sum_{j=1}^n E_{\omega_j,1}(-\lambda_j(t-b)^{\omega_j}) \|\varphi(b) - \psi(b)\|^{p_2} \right]^{p_1}, \quad \forall t \geq b, \quad (102)$$

holds, where constants $C^* \geq 1$, $C^\# > 0$, $\lambda_1, \lambda_2, \dots, \lambda_n \geq 0$, $p_1 > 0$, $p_2 > 0$ and $\omega_1, \omega_2, \dots, \omega_n \in (0, 1]$.

Theorem 37. Let (φ, ψ) be any solution pair to system (1) in a domain $\Omega \times \Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n$. Suppose there exists a C^1 function $V : [b, \infty) \times \Omega \times \Omega \rightarrow \mathbb{R}$ such that

- $A_1)$ $k_1 \|y_1 - y_2\|^{p_1} \leq V(t, y_1, y_2) = \sum_{i=1}^n V_i(t, y_{1i}(t), y_{2i}(t)) \leq \Theta(t) \|y_1 - y_2\|^{p_1 p_2}$, where k_1, p_1, p_2 are positive constants and $\Theta : [b, \infty) \rightarrow [1, \infty)$ is continuous with $p_2 \geq p_1$, $\Theta(t) \geq k_1$ and $p_2 \geq 1$,
 $A_2)$ along system (1) non-trivial solutions $(\varphi(t), \psi(t))$:

$$\sum_{i=1}^n \mathcal{D}_{b^+, t}^{\gamma_i} V_i(t, \varphi_i(t), \psi_i(t)) \leq - \sum_{i=1}^n \lambda_i V_i(t, \varphi_i(t), \psi_i(t)), \quad \forall \varphi - \psi \neq 0 \in \Omega, \quad \forall t > b, \quad (103)$$

where constants $\lambda_i > 0$ for $i = 1, 2, \dots, n$, and orders $\gamma_1, \gamma_2, \dots, \gamma_n \in (0, 1]$.

Then, the pair (φ, ψ) to system (1) is multi-order Mittag-Leffler asymptotically attractive. When $\Omega = \mathbb{R}^n$, the pair (φ, ψ) to (1) is globally multi-order Mittag-Leffler asymptotically attractive.

Proof. It is immediate from inequality (103) of (ii), and Lemma 12 that

$$\sum_{i=1}^n V_i(t, \varphi_i(t), \psi_i(t)) \leq \sum_{i=1}^n E_{\gamma_i,1}(-\lambda_i(t-b)^{\gamma_i}) V_i(b, \varphi_i(b), \psi_i(b)), \quad \forall t \geq b. \quad (104)$$

Consequently, by using (i), one has

$$\|\varphi(t) - \psi(t)\| \leq \left[\frac{q_2 \mu(b)}{q_1} \sum_{i=1}^n E_{\gamma_i,1}(-\lambda_i(t-b)^{\gamma_i}) \|\varphi(b) - \psi(b)\|^q \right]^{1/p}, \quad \forall t \geq b, \quad (105)$$

and the limit

$$\lim_{t \rightarrow \infty} \|\varphi(t) - \psi(t)\| = 0. \quad (106)$$

This completes the proof. $\square$

Theorem 38. Let (φ, ψ) be any solution pair to system (1) in a domain $\Omega \times \Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n$. Suppose there exists a C^1 function $V : [b, \infty) \times \Omega \times \Omega \rightarrow \mathbb{R}$ such that

- $A_1)$ $k_1 \|y_1 - y_2\|^{p_1} \leq V(t, y_1, y_2) = \sum_{i=1}^n V_i(t, y_{1i}(t), y_{2i}(t)) \leq \Theta(t) \|y_1 - y_2\|^{p_1 p_2}$, where k_1, p_1, p_2 are positive constants and $\Theta : [b, \infty) \rightarrow [1, \infty)$ is continuous with $p_2 \geq p_1$, $\Theta(t) \geq k_1$ and $p_2 \geq 1$,

$A_2)$ along system (1) non-trivial solutions $(\varphi(t), \psi(t))$:

$$\sum_{i=1}^n {}^C\mathcal{D}_{b^+,t}^{\gamma_i} V_i(t, \varphi_i(t), \psi_i(t)) \leq - \sum_{i=1}^n \lambda_i V_i(t, \varphi_i(t), \psi_i(t)), \quad \forall \varphi - \psi \neq 0 \in \Omega, \quad \forall t > b, \quad (107)$$

where constants $\lambda_i \geq 0$ for $i = 1, 2, \dots, n$, and orders $\gamma_1, \gamma_2, \dots, \gamma_n \in (0, 1]$.

Then, the pair (φ, ψ) to system (1) is multi-order Mittag-Leffler attractive. When $\Omega = \mathbb{R}^n$, the pair (φ, ψ) to (1) is globally multi-order Mittag-Leffler attractive.

Proof. The proof is similar to Theorem 37. $\square$

Definition 25. Let $\varphi(t)$ be any solution to (1). We say the solution $\varphi(t)$ is multi-order Mittag-Leffler asymptotically attractive if the solutions $y(t)$ different to $\varphi(t)$ satisfy $\lim_{t \rightarrow \infty} \|y(t) - \varphi(t)\| = 0$ and the inequality

$$\|y(t) - \varphi(t)\| \leq C^* \left[C^\# \sum_{j=1}^n E_{\omega_j,1}(-\lambda_j(t-b)^{\omega_j}) \|\varphi(b) - \psi(b)\|^{p_2} \right]^{p_1}, \quad \forall t \geq b, \quad (108)$$

holds, where constants $C^* \geq 1$, $C^\# > 0$, $\lambda_1, \lambda_2, \dots, \lambda_n > 0$, $p_1 > 0$, $p_2 > 0$ and $\omega_1, \omega_2, \dots, \omega_n \in (0, 1]$.

Definition 26. Let $\varphi(t)$ be any solution to (1). We say the solution $\varphi(t)$ is multi-order Mittag-Leffler attractive if the solutions $y(t)$ different to $\varphi(t)$ satisfy the inequality

$$\|y(t) - \varphi(t)\| \leq C^* \left[C^\# \sum_{j=1}^n E_{\omega_j,1}(-\lambda_j(t-b)^{\omega_j}) \|\varphi(b) - \psi(b)\|^{p_2} \right]^{p_1}, \quad \forall t \geq b, \quad (109)$$

holds, where $C^* \geq 1$, $C^\# > 0$, $\lambda_1, \lambda_2, \dots, \lambda_n \geq 0$, $p_1 > 0$, $p_2 > 0$ and $\omega_1, \omega_2, \dots, \omega_n \in (0, 1]$.

In light of Theorem 37, we give the following two results.

Theorem 39. Let $\tilde{\varphi}$ be any solution to system (1) in a domain $\Omega \subseteq \mathbb{R}^n$. Suppose there exists a $\mathcal{C}^1$ function $V : [b, \infty) \times \Omega \times \Omega \rightarrow \mathbb{R}$ such that

$A_1)$ $k_1 \|y - \tilde{\varphi}\|^{p_1} \leq V(t, y, \tilde{\varphi}) \leq \Theta(t) \|y - \tilde{\varphi}\|^{p_1 p_2}$, where k_1, p_1, p_2 are positive constants and $\Theta : [b, \infty) \rightarrow [1, \infty)$ is continuous with $p_2 \geq p_1$, $\Theta(t) \geq k_1$ and $p_2 \geq 1$,

$A_2)$ along system (1) non-trivial solution $y(t)$:

$$\sum_{i=1}^n {}^C\mathcal{D}_{b^+,t}^{\gamma_i} V_i(t, y_i(t), \tilde{\varphi}_i(t)) \leq - \sum_{i=1}^n \lambda_i V_i(t, y_i(t), \tilde{\varphi}_i(t)), \quad \forall y - \tilde{\varphi} \neq 0 \in \Omega, \quad \forall t > b, \quad (110)$$

where constants $\lambda_i > 0$ for $i = 1, 2, \dots, n$, and orders $\gamma_1, \gamma_2, \dots, \gamma_n \in (0, 1]$.

Then, the solution $\tilde{\varphi}$ to system (1) is multi-order Mittag-Leffler asymptotically attractive. When $\Omega = \mathbb{R}^n$, the solution $\tilde{\varphi}$ to system (1) is globally multi-order Mittag-Leffler asymptotically attractive.

Proof. The proof is same lines of argument in Theorem 37. $\square$

Theorem 40. Let $\tilde{\varphi}$ be any solution to system (1) in a domain $\Omega \subseteq \mathbb{R}^n$. Suppose there exists a $\mathcal{C}^1$ function $V : [b, \infty) \times \Omega \times \Omega \rightarrow \mathbb{R}$ such that

$A_1)$ $k_1 \|y - \tilde{\varphi}\|^{p_1} \leq V(t, y, \tilde{\varphi}) \leq \Theta(t) \|y - \tilde{\varphi}\|^{p_1 p_2}$, where k_1, p_1, p_2 are positive constants and $\Theta : [b, \infty) \rightarrow [1, \infty)$ is continuous with $p_2 \geq p_1$, $\Theta(t) \geq k_1$ and $p_2 \geq 1$,

$A_2)$ along system (1) non-trivial solution $y(t)$:

$$\sum_{i=1}^n {}^C\mathcal{D}_{b^+,t}^{\gamma_i} V_i(t, y_i(t), \tilde{\varphi}_i(t)) \leq - \sum_{i=1}^n \lambda_i V_i(t, y_i(t), \tilde{\varphi}_i(t)), \quad \forall y - \tilde{\varphi} \neq 0 \in \Omega, \quad \forall t > b, \quad (111)$$

where constants $\lambda_i \geq 0$ for $i = 1, 2, \dots, n$, and orders $\gamma_1, \gamma_2, \dots, \gamma_n \in (0, 1]$.

Then, the solution $\tilde{\varphi}$ to system (1) is multi-order Mittag-Leffler attractive. When $\Omega = \mathbb{R}^n$, the solution $\tilde{\varphi}$ to system (1) is globally multi-order Mittag-Leffler attractive.

Proof. The proof is same lines of argument in Theorem 37. $\square$

8. Applications to Some Dynamical Models

This section presents a stability analysis of interest systems modeled using Caputo fractional derivatives, which appears to be quite challenging.

Example 1. Let us consider an autonomous fractional order nonlinear system

$$\begin{aligned} {}^C\mathcal{D}_{-50^+,t}^{\delta} x_1(t) &= -x_1(t) + x_2^3(t) \\ {}^C\mathcal{D}_{-50^+,t}^{\delta} x_2(t) &= -x_1(t) - x_2(t) \end{aligned} \quad (112)$$

subject to $x_1(-50) = c_1$ and $x_2(-50) = c_2$, where $\delta \in (0, 1]$.

Define a Lyapunov function by $V(t, x) = 2x_1^2 + x_2^4$, where $x = (x_1, x_2)^T$. Set $\alpha = \delta$. Using Lemma 5, it can be obtained that

$$\begin{aligned} {}^C\mathcal{D}_{-50^+,t}^{\alpha} V(t, x(t)) &\leq 4x_1(t) {}^C\mathcal{D}_{-50^+,t}^{\alpha} x_1(t) + 4x_2^3(t) {}^C\mathcal{D}_{-50^+,t}^{\alpha} x_2(t) \\ &= -4x_1^2(t) - 4x_2^4(t) \\ &< 0, \quad \forall t > -50, \end{aligned} \quad (113)$$

$\forall x = (x_1, x_2)^T \in \mathbb{R}^2 - \{(0, 0)^T\}$. Thus, by Theorem 13 the zero equilibrium of (112) is globally V -asymptotically stable.

Example 2 demonstrates a situation where the fractional order Van der Pol oscillator system [55] starts at positive initial time $b = 25$. It is shown that the zero equilibrium of such a system can be Mittag-Leffler stable.

Example 2. We consider the following fractional order Van der Pol oscillator system [55] whose state aims to evolve in $\mathbb{R}^2$:

$$\begin{aligned} {}^C\mathcal{D}_{25^+,t}^{\delta} x_1(t) &= x_2(t), \\ {}^C\mathcal{D}_{25^+,t}^{\delta} x_2(t) &= -x_1(t) + \mu(1 - x_1^2(t))x_2(t), \end{aligned} \quad (114)$$

subject to $x_1(25) = x_{25}^1$ and $x_2(25) = x_{25}^2$, where order $\delta \in (0, 1]$, and the system parameter μ is negative.

Since the system (114) begins at initial time $b = 25$, the known Lyapunov theorems [21,22,37] cannot be applicable. Here we give an exhibition by selecting a candidate Lyapunov function $V(t, x) = \frac{1}{2}(x_1^2 + x_2^2)$, where $x = (x_1, x_2)^T \in \mathbb{R}^2$. Notice that

$$\frac{1}{4}(x_1^2 + x_2^2) \leq V(t, x) \leq (x_1^2 + x_2^2), \quad \forall t \geq 25, \quad \forall x \in \mathbb{R}^2. \quad (115)$$

Set $\alpha = \tilde{\delta}$. By employing Lemma 4 along with system (114) solution $x(t)$, we obtain

$$\begin{aligned} {}^C\mathcal{D}_{25+,t}^\alpha V(t, x(t)) &\leq x_1(t) {}^C\mathcal{D}_{25+,t}^\alpha x_1(t) + x_2(t) {}^C\mathcal{D}_{25+,t}^\alpha x_2(t) \\ &= \mu \left(1 - x_1^2(t)\right) x_2^2(t) \\ &\leq 0, \quad \forall x \in \mathbb{D} - \{(0,0)^T\}, \quad \forall t > 25, \end{aligned} \quad (116)$$

where $\mathbb{D} = \{(x_1, x_2)^T : x_1^2 + x_2^2 \leq 1\}$. Consequently, we see that (i) and (ii) of Theorem 11 are satisfied on domain $\mathbb{D} \subset \mathbb{R}^2$. Thus we conclude, by Theorem 11 that the equilibrium point $x = 0$ to system (114) should be locally Mittag-Leffler stable on $\mathbb{D}$.

In Example 3, we give a demonstration of the fractional Lyapunov direct method for predicting dynamics of equilibrium when the fractional order Lorenz system state uses many different orders and random initial times. The fractional order Lorenz system [56–58] is known as a paradigm of memory chaos in understanding fractional nonlinear dynamics. This system has the ability to describe the evolution of a state that can be all past history dependent, leading to memory-dependent phenomena for some suitable choices of chosen initial values. How to predict global dynamics of the equilibrium state of such a system remains an open problem.

Example 3. Let us consider the modified nonlinear fractional order Lorenz system [40]:

$$\begin{aligned} {}^C\mathcal{D}_{50+,t}^{0.77} x_1(t) &= \sigma(x_2(t) - x_1(t)) \\ {}^C\mathcal{D}_{50+,t}^{0.87} x_2(t) &= x_1(t)(\rho - x_3(t)) - x_2(t) \\ {}^C\mathcal{D}_{50+,t}^{0.93} x_3(t) &= x_1(t)x_2(t) - \beta x_3(t) \end{aligned} \quad (117)$$

with initial position at $(x_1(50), x_2(50), x_3(50))^T = (v_1, v_2, v_3)^T \in \mathbb{R}^3$, where constants σ, ρ, β are positive.

We thought of Lyapunov function $V(t, x) = \frac{1}{\sigma}x_1^2 + x_2^2 + x_3^2$, where $x = (x_1, x_2, x_3)^T$. Define $V(t, x) = \sum_{j=1}^3 V_j(t, x_j)$, where $V_1(t, x_1) = \frac{1}{\sigma}x_1^2$ and $V_j(t, x_j) = x_j^2$ for $j = 2, 3$. Note that $\min\{\frac{1}{\sigma}, 1\} \|x\|^2 \leq V(t, x) = \sum_{j=1}^3 V_j(t, x_j) \leq \max\{\frac{1}{\sigma}, 1\} \|x\|^2$, $\forall t \geq b$, $\forall x \in \mathbb{R}^3$. Here the known fractional Lyapunov theorems in literature [21,22,38,47] are not suitable for predicting asymptotic stability of zero solution $x(t) = (x_1(t), x_2(t), x_3(t))^T = (0, 0, 0)^T$ of (117). The difficulty lies in the fact that the evolution of system (117) begins at initial time $b = 50$. To overcome such a difficulty, we wish to apply Corollary 1 to predict asymptotic dynamics of mentioned case scenario. Set $\gamma_1 = 0.77$, $\gamma_2 = 0.87$, $\gamma_3 = 0.93$. We use Lemma 5 to obtain along $\xi(t) = (\xi_1(t), \xi_2(t), \xi_3(t))^T \neq (0, 0, 0)^T$:

$$\begin{aligned} \sum_{j=1}^3 {}^C\mathcal{D}_{50+,t}^{\gamma_j} V_j(t, \xi_j(t)) &\leq \frac{2}{\sigma} \xi_1(t) {}^C\mathcal{D}_{50+,t}^{\gamma_1} \xi_1(t) + 2\xi_2(t) {}^C\mathcal{D}_{50+,t}^{\gamma_2} \xi_2(t) + 2\xi_3(t) {}^C\mathcal{D}_{50+,t}^{\gamma_3} \xi_3(t) \\ &= 2\xi_1(t)(\xi_2(t) - \xi_1(t)) + 2\xi_2(t)(\rho\xi_1(t) - \xi_1(t)\xi_3(t) - \xi_2(t)) \\ &\quad + 2\xi_1(t)\xi_2(t)\xi_3(t) - 2\beta\xi_3^2(t) \\ &\leq (\rho - 1)\xi_1^2(t) + (\rho - 1)\xi_2^2(t) - \beta\xi_3^2(t) \\ &\leq -\frac{\lambda}{\max\{\frac{1}{\sigma}, 1\}} V(t, \xi(t)), \quad \forall t > 50, \quad \forall x \in \mathbb{R}^3 - \{(0,0,0)^T\}, \end{aligned} \quad (118)$$

where the condition $0 < \rho < 1$ and constant $\lambda = \min\{1 - \rho, \beta\} > 0$ were utilized. Thus, the conditions (i) and (ii) of Corollary 1 are satisfied. Therefore, the solution $x(t) = (0, 0, 0)^T$ to (117) should be globally multi-order Mittag-Leffler asymptotically stable provided the condition $0 < \rho < 1$ must hold.

This closes the demonstration.

The standard economic integer-order finance system was recently discussed by authors in [59]. Chen [61] uncovered some computational dynamics of fractional-order finance systems by using the Caputo fractional derivative. Škovránek et al. [60] considered a fractional-order finance system and discussed a new dynamical macroeconomic model of the national economies.

Example 4. Consider the modified incommensurate fractional order financial system [60]:

$$\begin{aligned} {}^C\mathcal{D}_{b^+,t}^{\delta_1} x_1(t) &= x_3(t) + (x_2(t) - \tilde{a})x_1(t), \\ {}^C\mathcal{D}_{b^+,t}^{\delta_2} x_2(t) &= 1 - \tilde{b}x_2(t) - x_1^2(t), \\ {}^C\mathcal{D}_{b^+,t}^{\delta_3} x_3(t) &= -x_1(t) - \tilde{c}x_3(t), \end{aligned} \quad (119)$$

subject to initial position $x_1(b) = x_b^1$, $x_2(b) = x_b^2$ and $x_3(b) = x_b^3$, where initial time $b \in (-\infty, \infty)$, orders $\delta_1, \delta_2, \delta_3 \in (0, 1]$, and parameters $\tilde{a}, \tilde{b}, \tilde{c} > 0$. In the system (119), $\tilde{a}$ is the saving amount, $\tilde{b}$ is the cost per investment, and $\tilde{c}$ is the elasticity of demand of the commercial market. The state variables are $x_1(t)$, which is the interest rate; $x_2(t)$, which is the investment demand; and $x_3(t)$, which is the price index, respectively.

Observe that $(0, \frac{1}{\tilde{b}}, 0)^T \in \mathbb{R}^3$ is an equilibrium point of system (119). We introduce here a new transformation $\eta_1(t) = x_1(t)$, $\eta_2(t) = x_2(t) - \frac{1}{\tilde{b}}$ and $\eta_3(t) = x_3(t)$. Then, the system (119) becomes

$$\begin{aligned} {}^C\mathcal{D}_{b^+,t}^{\delta_1} \eta_1(t) &= \eta_3(t) + \left(\eta_2(t) + \frac{1}{\tilde{b}} - \tilde{a} \right) \eta_1(t), \\ {}^C\mathcal{D}_{b^+,t}^{\delta_2} \eta_2(t) &= -\tilde{b}\eta_2(t) - \eta_1^2(t), \\ {}^C\mathcal{D}_{b^+,t}^{\delta_3} \eta_3(t) &= -\eta_1(t) - \tilde{c}\eta_3(t). \end{aligned} \quad (120)$$

Consequently, we have $(0, 0, 0)^T \in \mathbb{R}^3$ is an equilibrium of obtained system (120).

We take the quadratic Lyapunov function $V(t, \eta) = \eta_1^2 + \eta_2^2 + \eta_3^2$, where $\eta = (\eta_1, \eta_2, \eta_3)^T \in \mathbb{R}^3$. Define $V(t, \eta) = \sum_{j=1}^3 V_j(t, \eta_j)$, where $V_j(t, \eta_j) = \eta_j^2$ for $j = 1, 2, 3$. Since

$$\frac{1}{2} \|\eta\|^2 \leq V(t, \eta) \leq 2 \|\eta\|^2, \quad \forall t \geq b, \quad \forall \eta \in \mathbb{R}^3, \quad (121)$$

Set $\gamma_j = \delta_j$ for $j = 1, 2, 3$. We think of Corollary 1 for asymptotic stability analysis of $(0, 0, 0)^T \in \mathbb{R}^3$ in system (120). By using Lemma 5 along system (120) solution $\eta(t)$, we obtain

$$\begin{aligned} \sum_{j=1}^3 {}^C\mathcal{D}_{b^+,t}^{\gamma_j} V_j(t, \eta_j(t)) &\leq 2\eta_1(t) {}^C\mathcal{D}_{b^+,t}^{\gamma_1} \eta_1(t) + 2\eta_2(t) {}^C\mathcal{D}_{b^+,t}^{\gamma_2} \eta_2(t) + 2\eta_3(t) {}^C\mathcal{D}_{b^+,t}^{\gamma_3} \eta_3(t) \\ &= 2\eta_1(t) \left[\eta_3(t) + \left(\eta_2(t) + \frac{1}{\tilde{b}} - \tilde{a} \right) \eta_1(t) \right] + 2\eta_2(t) \left[-\tilde{b}\eta_2(t) - \eta_1^2(t) \right] \\ &\quad + 2\eta_3(t) [-\eta_1(t) - \tilde{c}\eta_3(t)] \\ &= -2 \left(-\frac{1}{\tilde{b}} + \tilde{a} \right) \eta_1^2(t) - 2\tilde{b}\eta_2^2(t) - 2\tilde{c}\eta_3^2(t), \quad \forall t > b, \end{aligned} \quad (122)$$

$\forall \eta \in \mathbb{R}^3 - \{(0, 0, 0)^T\}$. We assume

$$\tilde{a} > \frac{1}{\tilde{b}}, \quad \tilde{b} > 0, \quad \tilde{c} > 0, \quad (123)$$

and define $\lambda = \min\{2\left(\tilde{a} - \frac{1}{b}\right), 2\tilde{c}, 2\tilde{b}\} > 0$. In view of (123), the inequality (122) simplifies to

$$\sum_{j=1}^3 c\mathcal{D}_{b^+,t}^{\gamma_j} V_j(t,\eta_j(t)) \leq -\lambda V(t,\eta(t)), \quad \forall \eta \in \mathbb{R}^3 - \{(0,0,0)^T\}, \quad \forall t > b. \tag{124}$$

Therefore, we see that (i) and (ii) of Corollary 1 are satisfied provided (123) must hold. Immediately, we conclude that $(0,0,0)^T \in \mathbb{R}^3$ should be globally multi-order Mittag-Leffler asymptotically stable. It implies that $\|\eta(t)\| \rightarrow 0$ as $t \rightarrow \infty$ whenever (123) holds. As a result, one has $(x_1(t), x_2(t), x_3(t))^T \rightarrow \left(0, \frac{1}{b}, 0\right)^T$ as $t \rightarrow \infty$ in system (119) provided the condition (123) should hold. This ends the demonstration.

9. Conclusions

In fractional order systems, how to predict stability and asymptotic stability dynamics of equilibrium remains less understood in the literature. The fractional Lyapunov theory discussed in this work gives some new knowledge to predict the behavior of equilibrium in many complicated systems in large time, especially whenever time approaches symbolic ∞ . A typical mathematical insight that has been noticed is as follows: Do there exist possible nearby trajectories that might attract to each other or stay near while equilibrium never aims to bend toward them? A framework that develops extending beyond Lyapunov stability understanding to the attractivity of any fixed bounded solution, or any pair of fixed bounded solutions, which seems quite challenging and puzzling, has been addressed. Some conceptual notions of multi-order Mittag-Leffler attractive and multi-order Mittag-Leffler asymptotically attractive of any fixed solution or solution pair were proposed. New elementary versions of Lyapunov attractive theorems that concern convergence to a fixed solution or attractiveness between solution pairs in the sense of multi-order Mittag-Leffler resemble this work. It is suggested that fractional Lyapunov theory can bring huge potential in numerous studies in incommensurate fractional-order dynamical systems and applications pertaining to diverse control system designs when adapted to such systems.

To end this note, it has been pointed out that the ideas of Lyapunov functions can have advantages in diverse mechanical processes such as:

- to translate any big systems to a single equation or inequality for rigorous mathematical analysis,
- to estimate the theoretical region of attraction of equilibrium or non-equilibrium solutions of complicated fractional order systems,
- to predict how fast nearby trajectories grow or decay in the mentioned systems.

But the mentioned function has the following drawbacks:

- It can be extremely difficult to locate for many typical fractional order systems.
- It can be extremely challenging to construct for fractional order systems.

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