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Posted Date: 10 October 2024

doi: 10.20944/preprints202410.0821.v1

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Probing the Cardinality of Space with a Computer

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Abstract: A straightforward way to avoid the problem of the infinite zero-point energy principally is to negate the axiom of infinity of Zermelo-Fraenkel set theory. However, in so doing the real numbers and, as a corollary, the concept of infinitesimals would be given up making mathematical analysis, as it is known today, impossible. That said, consider a set whose members uniquely index points comprising a region of space. Providing four binary operations and their properties are defined on an index set, it is a field. The present paper demonstrates that to find the characteristic of an index field of the space of the observable universe is computationally intractable. Accordingly, such field cannot be distinguished from that of characteristic zero in any way possible. Meaning that one can negate the axiom of infinity and still use the real numbers inside the space of the observable universe for all intents and purposes.

Keywords: vacuum energy problem; field characteristic; finite geometry; entropy; cosmological constant; matrioshka brain

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1. Introduction

Modern mathematics asserts that lines – both Euclidean and non-Euclidean alike – are sets of points and such sets are infinite [1]. Since lines are spaces of dimension one, which may be embedded in spaces of dimension two, three, or higher, the forenamed assertion requires spaces (both Euclidean and non-Euclidean alike) to be infinite sets of points [2].

Defining an infinite set as *Dedekind infinite* (i.e., as a set A that can be placed into a one-to-one correspondence with a proper subset $B \subset A$ [3]) stipulates that the number of points in any region of space (i.e., any connected part of space) is infinite as well.

But then again, the existence of any infinite set follows from the existence of the set of natural numbers which is postulated by the axiom of infinity (denoted `Inf`) of Zermelo-Fraenkel set theory [4]. This axiom essentially states that the collection of natural numbers is a set. In more detailed manner, `Inf` regards infinite and finite procedures (for example, the process of adding units one after the other indefinitely, such as $1 + 1 + 1 + \cdots = \sum_{n=1}^{\infty} 1$, where the symbol ∞ denotes an unbounded limit, and the finite series $\sum_{n=1}^{N} 1$, where N is some natural number) on an equal footing [5]. This implies that `Inf` treats $\sum_{n=1}^{\infty} 1$ akin to N, i.e., as a single element capable of being used in further constructions. Usually, this element is denoted by the symbol $+\infty$ representing positive infinity and put (along with $-\infty$, the symbol of negative infinity) in the extended real number system.

Contrary to mathematics, physical science denies the notion of infinity, albeit from a pragmatic point of view [6]. Since one of the needs of any physical theory is to give usable formulas that correspond to (or at least approximate) reality, no physical quantity could take value $+\infty$ or $-\infty$. Indeed, if an infinite physical quantity had existed, then – according to the binary operations on the extended real number system – any usage of formulas involving such quantity would have led to an infinite result. But that kind of outcome is of no benefit providing it will always be the same regardless of other quantities engaged in the formulas. As a case in point, $a \pm \infty = \pm\infty$ and $a \cdot (\pm\infty) = \pm\infty$ for any $a \in (0, +\infty)$.

This conveys that the vacuum energy must not be physical. At greater length, quantum field theory regards every single point in space as a quantum harmonic oscillator [7]. So long as the lowest possible energy, or the zero-point energy of this oscillator is a quantity other than zero, which is $E = \hbar\omega/2$, and the number of points in any region of space is infinite, one gets that the vacuum energy contained in any unit of space must be infinite and thereby lacks any physical meaning. As such, the infinite vacuum energy can be dropped from a Hamiltonian, e.g., by arguing that only differences in energies are physical [8].

However, once gravity is in, the argument like this stops working. It is so because in general relativity there is no such thing as an arbitrary additive constant with density of field energy: Energy density curves space, and the infinite energy density curves space infinitely [9].

In quantum field theory, as a means to get around infinities emerging in formulas, renormalization is used [10]. Its task is to extract the finite value from a divergent expression [11]. Naturally, one may want to treat the infinite zero-point energy along the same lines. The catch is that the finite (renormalized) value extracted from the divergent expression representing the zero-point energy appears to be too large to be compatible with the observations [12]. In particular, this value entails a cosmological constant (of the type introduced by Einstein in order to obtain static solutions to his field equations [9]) larger than the limits imposed by observation by many orders of magnitude. Therein lies the conundrum known as the cosmological constant problem: A well known collection of techniques in quantum field theory such as renormalization fails to produce a finite vacuum energy compatible with the observational data [13,14]. The fact that after years of great deal of effort, the conundrum had not yet been resolved apparently defies belief that it is possible to handle every divergent expression arising in quantum field theory.

Meanwhile, a very straightforward, almost trivial way to avoid all infinities in physics in the first place is to negate the axiom of infinity. As an example, $\neg\texttt{Inf}$ implies that the cardinality of space is finite (i.e., space has only a finite number of points) which, in turn, indicates that the vacuum energy is intrinsically finite.

Unfortunately for such a straightforward idea, ruling out the possibility of considering an endless procedure as a single element throws away the real numbers and, consequently, the concept of infinitesimals [15]. More thoroughly, infinitesimal numbers are built upon the validity of statements such as this: "One can choose two distinct numbers a and b as close to each other as one pleases." But if $\neg\texttt{Inf}$ holds true, then the distance between a and b must be an element of a finite nonempty set which has a minimal element, say, c. Because of this, the distance between a and b cannot be chosen to be less than c. By comparison, an infinite set need not have a minimal element.

For its part, the rejection of the concept of infinitesimals makes mathematical analysis, as it is known today, impossible [16]. Evidently, it is not desirable since branches of analysis such as calculus, differentiation, and integration are widely used across all divisions of physical science.

Be that as it may, consider a unique indexing (i.e., naming or labelling) of points comprising a manifold M (which can be a line, a surface, or a space [17]). This indexing consists of a bijective function from $\{M\}$, the set of points constituting M, to a set I called *an index set* which could contain any objects. For example, in case $\{M\}$ is finite, members of an index set I may be elements α, β, γ, δ, and the like. As another option, if $\{M\}$ is infinite, it may be indexed by $\mathbb{R}$, the set of real numbers.

Let us assume that two binary operations (addition and multiplication), two unary operations (yielding the additive and multiplicative inverses respectively), and two nullary operations (the constants 0 and 1) are defined on an index set I. Suppose that these operations are subject to their required properties. Then, the resulted algebraic structure will be a field [18]. Let us called it *an index field* and denote by $\mathbb{F}(I)$.

The expression for the cardinality of an index field $\mathbb{F}(I)$ must be

$$|\mathbb{F}(I)| = |I| + 1 \quad . \tag{1}$$

The requirement to add 1 to $|I|$ is justified by the fact that even though the cardinality of an index set I can be as minimal as 1, there cannot be a field with one element.

Therewithal, considering that a function $f : \{M\} \to I$ is bijective, the cardinality of I must be equal to the cardinality of $\{M\}$. In this way, $|\mathbb{F}(I)| = |\{M\}| + 1$.

 doi:10.20944/preprints202410.0821.v1

Provided the multiplicative identity of $\mathbb{F}(I)$ is 1, one can construct numbers on $\mathbb{F}(I)$ as described below

$$\begin{aligned} &2 = 1 + 1 \\ &3 = 1 + 1 + 1 \\ &4 = 1 + 1 + 1 + 1 \\ &5 = 1 + 1 + 1 + 1 + 1 \\ &\ldots \end{aligned} \quad . \tag{2}$$

The question is: Will all these numbers be different?

If, for example, $\mathbb{F}(I)$ is made up of just two elements, the answer will be *no*. For this field, $1 + 1 = 0$; so, among numbers constructed on this field, all even numbers will be equal to 0, while each odd number will amount to 1 [19].

The above question can be rephrased in the form of the decision problem (symbolized for convenience of reference by the letter D):

D: *"Given a positive integer n, determine if n copies of* 1 *(denoted by $n \cdot 1$) sum to* 0*."*

The problem D can be viewed as the set L_D of all positive integers n for which the answer is *yes*. Using set-builder notation, L_D can be described as

$$L_D = \{ n \in \mathbb{Z}_+ \,|\, n \cdot 1 = 0 \} \quad . \tag{3}$$

If L_D is empty, the elements of $\mathbb{F}(I)$ are the real numbers. In that event, $\{M\}$ is indexed by $\mathbb{R}$; therefore, $|\mathbb{F}(I)| = |\mathbb{R}|$, the cardinality of continuum.

Otherwise, $\mathbb{F}(I)$ is finite and the minimal element of nonempty L_D is called *the characteristic of* $\mathbb{F}(I)$, symbolized by $\operatorname{char} \mathbb{F}(I)$ [20]. Then,

$$|\mathbb{F}(I)| = \left(\operatorname{char} \mathbb{F}(I) \right)^k \quad , \tag{4}$$

where $\operatorname{char} \mathbb{F}(I)$ is a prime identified as the first element of a sorted (ordered) version of L_D, i.e.,

$$\operatorname{char} \mathbb{F}(I) = \min L_D \quad , \tag{5}$$

and k is a positive integer.

It should be noted that in the case where $|\mathbb{F}(I)| = |\mathbb{R}|$, the characteristic of $\mathbb{F}(I)$ is said to be 0.

Now, imagine that the decision problem D is intractable. This means that although in theory the set L_D is not empty, in practice to find even a single element of L_D – much less its minimal element – would take too many resources to be useful. Under these circumstances, everyone with limited computational resources would be unable to tell whether an index field $\mathbb{F}(I)$ is finite or not.

Therefore, it would be ideal if $\neg$`Inf` were to hold true but, at the same time, the number of points constituting the space of the observable universe were to be so huge that to find the characteristic of an index field of this space were to be intractable. Then, the vacuum energy would be fundamentally finite while also the use of the real numbers and analysis in the observable universe could be considered appropriate for all intents and purposes.

The present paper will demonstrate that such is the case.

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2. Cardinality of the Space of the Observable Universe

We will start by estimating the number of points constituting the space of the observable universe. The exposition in this section will closely follow the paper [21].

Consider a manifold $\mathcal{R}$ that is taken to be a region in another manifold M, which means that $\mathcal{R}$ is deemed to be a subset of M having the same dimension as M does. For instance, $\mathcal{R}$ can be a 3-ball in Euclidean 3-space.

Assume that the manifold M admits a notion of distance between its elements; so, M is equipped with measures of its regions such as area A and volume V.

As has already been mentioned, $E_{\text{vac}}(\mathcal{R})$, the energy of the vacuum existing in the region $\mathcal{R}$, can be obtained by adding up quantum harmonic oscillators with the zero-point energy from all points in $\mathcal{R}$. Provided that the set of points constituting the region, $\{\mathcal{R}\}$, is uniquely indexed by a set I, one finds

$$E_{\text{vac}}(\mathcal{R}) = \sum_{i=\imath}^{\jmath} \frac{\hbar\omega_i(\mathcal{R})}{2} \quad , \tag{6}$$

where $\imath$ and $\jmath$ are members of the index set I which denote the lower and upper bounds of summation respectively.

Conceding that the characteristic length $L(\mathcal{R})$ defining the linear scale of the region $\mathcal{R}$ is the ratio of the region's volume $V(\mathcal{R})$ to the area of the region's boundary $A(\mathcal{R})$, i.e.,

$$L(\mathcal{R}) = \frac{V(\mathcal{R})}{A(\mathcal{R})} \quad , \tag{7}$$

and on condition that all frequencies $\omega_i(\mathcal{R})$ are alike

$$\omega_i(\mathcal{R}) = \frac{2\pi c}{L(\mathcal{R})} \quad , \tag{8}$$

one gets the formula

$$\underbrace{1 + 1 + \cdots + 1}_{|I| \text{ summands}} = \frac{E_{\text{vac}}(\mathcal{R}) \cdot L(\mathcal{R})}{\pi\hbar c} \quad . \tag{9}$$

On the other hand, the vacuum energy $E_{\text{vac}}(\mathcal{R})$ can be presented as the result of multiplying the vacuum energy density ρ_{vac} by the region's volume $V(\mathcal{R})$. So, by allowing ρ_{vac} to be proportional to the effective cosmological constant Λ_{eff}, namely,

$$\rho_{\text{vac}} = \frac{c^4}{8\pi G} \Lambda_{\text{eff}} \quad , \tag{10}$$

one can express the cardinality $|I| = |\{\mathcal{R}\}|$ in terms of Λ_{eff}:

$$|\{\mathcal{R}\}| = \frac{\Lambda_{\text{eff}} \cdot V(\mathcal{R}) \cdot L(\mathcal{R})}{8\pi^2 \ell_P^2} \quad , \tag{11}$$

where $\ell_P = \sqrt{\hbar G/c^3}$.

The stipulation that each point comprising a manifold M is a bit of information (in full agreement with Wheeler's slogan, "It from Bit" [22]) yields the connection between the cardinality of M and the amount of information embodied in M:

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$$H(M) = k_B \cdot |\{M\}| \quad , \tag{12}$$

where $H(M)$ denotes entropy in M and k_B stands for the Boltzmann constant.

As acknowledged, on large scales, the space wherein the universe lives is well approximated as three-dimensional and flat [23]. Given that, points constituting the manifold M_U associated with the universe can be considered to be embedded into a 3-dimensional Euclidean space. By the same token, the region $\mathcal{R}_U \subset M_U$ representing the space of the observable universe can be realized by points in the 3-dimensional Euclidean ball of radius R_U.

Accordingly, the entropy contained in the region $\mathcal{R}_U$ can be defined as

$$H(\mathcal{R}_U) = k_B \cdot |\{R_U\}| \quad , \tag{13}$$

where $\{R_U\}$ stands for the set of points constituting the space of the observable universe.

From the fact that a black hole is the most entropic object one can put inside a given boundary of a 3-ball [24], the inference could be drawn that

$$H(\mathcal{R}_U) \le S_{\mathrm{BH}} \quad , \tag{14}$$

where S_{BH} is the Bekenstein-Hawking entropy [25,26] determined by

$$S_{\mathrm{BH}} = k_B \cdot \frac{\pi R_U^2}{\ell_P^2} \quad . \tag{15}$$

The above formula renders the upper limit value of the cardinality of the space of the observable universe

$$|\{R_U\}| \le \frac{\pi R_U^2}{\ell_P^2} \quad , \tag{16}$$

and ergo sets a maximum limit on the effective cosmological constant Λ_{eff}

$$\Lambda_{\mathrm{eff}} \le \frac{2\pi^2}{L^2(\mathcal{R}_U)} \quad , \tag{17}$$

where $L(\mathcal{R}_U) = R_U/3$. Given that the comoving radius of the observable universe R_U is defined as

$$R_U \sim 3cH_0^{-1} \quad , \tag{18}$$

where H_0 is the Hubble constant (i.e., the present value of the Hubble parameter H) [27], the cardinality of the space of the observable universe must be limited from above as follows

$$|\{R_U\}| \le 9\pi \frac{t_H^2}{t_P^2} \quad , \tag{19}$$

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where $t_H \equiv H_0^{-1} \approx 4.6 \times 10^{17}$ s is the Hubble time and $t_P \equiv \ell_P / c \approx 5.4 \times 10^{-44}$ s is the Planck time. Accordingly, there must exist an upper bound of Λ_{eff} such that

$$\Lambda_{\text{eff}} \leq \frac{2\pi^2}{c^2 t_H^2} \sim 10^{-51}\,\text{m}^{-2} \quad . \tag{20}$$

The last expression not only falls within the limits imposed by observation ($\sim 10^{-52}\,\text{m}^{-2}$) [28] but also elucidates where the striking relation of Λ_{eff} with the age of the observable universe $T_U \leq t_H$ comes from.

3. An Index Field of the Space of the Observable Universe

It should be reminded that the argumentation of the previous section was based on the assumption that $\neg$`Inf` holds true. Had `Inf` been valid instead, the inequality (19) would have made no sense: the cardinality $|\{R_U\}|$ would have been infinite despite the finitude of t_H and t_P.

To demonstrate that an index field $\mathbb{F}(I)$ of the set $\{R_U\}$ is finite, one has to show that the following holds true:

$$\underbrace{1 + 1 + \cdots + 1}_{\text{char}\,\mathbb{F}(I)\text{ summands}} = 0 \quad . \tag{21}$$

Given that T is the duration of the above succession of additions, the rate of summation r can be calculated as

$$r = \frac{\text{char}\,\mathbb{F}(I)}{T} \quad . \tag{22}$$

Let us estimate r. As mentioned earlier, the size of an index field $\mathbb{F}(I)$ of the set $\{R_U\}$ meets the condition

$$\left(\text{char}\,\mathbb{F}(I)\right)^k = |\mathbb{F}(I)| = |\{R_U\}| + 1 \quad , \tag{23}$$

subsequently,

$$\text{char}\,\mathbb{F}(I) = \sqrt[k]{\left|\{R_U\}\right| + 1} \quad . \tag{24}$$

The duration of any process happening in the observable universe cannot exceed the age of the observable universe t_H. Thus, to guarantee that the summation (21) will always finish in the period of time t_H, the rate of summation must be calculated when the sequence of summands in (21) is the longest, i.e., when $k = 1$. On that account,

$$r = \frac{|\{R_U\}| + 1}{t_H} \quad . \tag{25}$$

In line with (19), the maximum rate of summation r can be considered to be equal to

$$r_{\max} \cong 9\pi \frac{t_H}{t_P^2} \quad . \tag{26}$$

According to Heisenberg's indeterminacy principle, it is the case that

$$\Delta E \cdot \Delta t \geq \hbar \quad , \tag{27}$$

where ΔE is the uncertainty of the energy of the system, and Δt is, in effect, a lifetime of states in that system. Another interpretation of (27) is that Δt defines the duration of measurement, and ΔE stands for the energy transferred to the observed system [29].

In terms of computation, one can construe the principle (27) as the relation between the rate of computation r and power P assumed by a computer system during the time T of computation:

$$P \cdot T \geq \hbar \cdot r \quad . \tag{28}$$

Substituting T and r for t_H and $r_{\max}$, respectively, gives

$$P \gtrsim 9\pi \frac{E_P}{t_P} \quad , \tag{29}$$

where E_P denotes the Planck mass-energy, $E_P \equiv \hbar / t_P$.

Thus, the demonstration that an index field $\mathbb{F}(I)$ of the observable universe is finite may actually require power P greater than that at which the definition of power under modern conceptualizations of physics breaks down [30]. To be precise, the statement (29) entails $P > P_P$, where

$$P_P \equiv \frac{E_P}{t_P} \sim 10^{52}\,\mathrm{W} \quad . \tag{30}$$

What this means is that an index field $\mathbb{F}(I)$ with the characteristic $t_H^2 / t_P^2 \sim 10^{122}$ cannot be distinguished from a field of characteristic zero (such as the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$) *by any available means*. Hence, despite $\neg$`Inf` being accepted, the real and complex numbers can still be employed in the observable universe for all practical purposes.

Accordingly, the use of mathematical analysis (and related theories, such as differentiation and integration) can be believed to be appropriate in the observable universe.

4. Concluding Remarks

In concluding, an index field $\mathbb{F}(I)$ of the set $\{R_U\}$ can be well approximated as the field of real or complex numbers.

Let us examine the interval of such approximation.

Consider a region of space where gravity is sufficiently weak to allow for the flatness of the space. Assume that points constituting this region are embedded into a 3-dimensional Euclidean ball with radius R. Allow $\{R\}$ to denote the set of such points. Let K be the totally ordered by inclusion set containing all the sets $\{R\}$. We want to find the subset $S \subset K$ such that given any $\{R\} \in S$, an index field $\mathbb{F}(I)$ of $\{R\}$ is finite and, alongside that, well approximated as the field of real or complex numbers. Using symbols,

$$S = \left\{ \{R\} \in K \,\middle|\, \operatorname{char} \mathbb{F}(I) \neq 0 \wedge \mathbb{F}(I) \approx \mathbb{R}, \mathbb{C} \right\} \quad . \tag{31}$$

The tight upper bound (the supremum) of the subset S is quite obvious. Certainly, consistent with the Bekenstein-Hawking limit (16), it is true to say that the cardinality of the set $\{R\}$ is commensurate with R^2 / ℓ_P^2. For this reason, the statement $\operatorname{char} \mathbb{F}(I) \neq 0$ holds true only if R is finite. Thus, $\{+\infty\}$, the

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set of points constituting a ball of infinite radius, is an upper bound of S, but it is not the least upper bound of S, and so

$$\sup S < \{+\infty\} \quad . \tag{32}$$

In order to evaluate the tight lower bound (the infimum) of the subset S, let us first notice that radius R equal to $\pi^{-1/2}\ell_P$ (about one Planck length) corresponds to a ball containing exactly one bit of information. That is to say, $\{\ell_P\}$ is a one-point set. Thereby, an index field $\mathbb{F}(I)$ of $\{\ell_P\}$ is a field with just two elements, e.g., 0 and 1. In consequence, the singleton $\{\ell_P\}$ is a lower bound of S but it is not a member of S, and hence

$$\{\ell_P\} < \inf S \quad . \tag{33}$$

From the formulas of the previous section it follows that in order to find the characteristic of an index field $\mathbb{F}(I)$ of an arbitrary set $\{R\}$, one needs to sum $\sqrt[k]{|\{R\}|+1}$ copies of 1. The worst-case complexity of the summation is when $k = 1$; accordingly, the maximum rate of the worst-case complexity can be set as

$$r_{\max} = \pi \frac{R^2}{\ell_P^2 \cdot T} + \frac{1}{T} \quad , \tag{34}$$

where T is the length of time required to sum $|\{R\}| + 1$ copies of 1. Assuming that T and $r_{\max}$ are given and on condition that $r_{\max} \cdot T \gg 1$, R can be determined as

$$R = \frac{\ell_P}{\sqrt{\pi}} \cdot \sqrt{r_{\max} \cdot T - 1} \;\sim\; \ell_P \cdot \sqrt{r_{\max} \cdot T} \quad . \tag{35}$$

From here it is evident that $\inf S$ sits on the border of computational plausibility, i.e., the boundary where a computational task of adding $r_{\max} \cdot T$ copies of 1 transitions from being plausible to being implausible.

For example, using a computer system able to execute 10^{18} additions per second (the performance of the world's fastest supercomputer as of June 2024 [31]) for the whole year ($\sim 10^7$ seconds) uninterruptedly seems plausible. This corresponds to the task of adding 10^{25} copies of 1 and – as a consequence of this – the radius $R \sim 10^{-22}$ m. Providing this radius coincides with the neutrino length scale ℓ_n (the effective cross section radius of 1 MeV neutrinos [32]), one can state that $\{\ell_n\}$ is not an element of S. But since ℓ_n is less than the comoving radius of the observable universe R_U, the fact that $\{R_U\} \in S$ means that

$$\{\ell_n\} < \inf S \quad . \tag{36}$$

By contrast, consider the atomic length scale, $\ell_a \sim 10^{-10}$ m. Based on (35), one gets that in the case where R is around ℓ_a, the computational task involves the summation of approximately 10^{50} copies of 1. Even assuming that the time T is on a par with the age of the observable universe (10^{17} seconds), in this case the rate of the summation $r_{\max}$ must be in the range of 10^{33} additions per second. Such a rate coincides with the estimated computational power of a hypothetical megastructure called *a matrioshka brain* [33,34] consisting of layers of nested Dyson spheres [35] that surround a star equivalent in luminosity to the Sun. But then again, since a sun-sized computer operating without pause or

 doi:10.20944/preprints202410.0821.v1

interruption since the beginning of time seems highly unlikely, the task of adding 10^{50} copies of 1 is believed to be implausible. The last thing evidences that $\{l_a\}$ is an element of the subset S, which entails

$$\{\ell_n\} < \inf S \leq \{\ell_a\} \quad . \tag{37}$$

The conclusion that can follow from the above is this.

Although on scales larger than or approaching $\ell_a \sim 10^{-10}$ m the use of a geometry that has the infinite number of points is adequate, it might not be permissible on scales lower than or in the vicinity of $\ell_n \sim 10^{-22}$ m where the cardinality of space emerges as distinctly finite. At those scales, the appropriateness of the real numbers comes to an end, for which reason both quantum and classical field theories ought to become inapplicable.

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