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Article

Collatz Conjecture Is True for All Positive Integers

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Abstract

This work offers formal proofs which were enabled by a change in perspective from studying individual integer iterations to analyzing how the conjecture’s rules organize the positive integers. The proofs rigorously demonstrate the satisfaction of several critical criteria: the universal inclusion of all positive integers within the proof’s scope; the disclosure of a simple and predictable pattern among the numbers; the conclusive absence of any major loops; the demonstration that no number continuously increases indefinitely without eventually decreasing; and the ultimate convergence of all positive integers to 1 when subjected to the Collatz iteration rules. Formal verification of these proofs was conducted using the Isabelle/HOL proof assistant.

Keywords: collatz conjecture; proof; rules; dendritic pattern

MSC: Primary 11B83; Secondary 11Y16

1. Introduction

The Collatz conjecture was first proposed by Lohar Collatz in 1937. Additionally, the conjecture is called the $3n + 1$ problem, the $3n + 1$ conjecture, the Ulam conjecture, Kakutani’s, the Thwaites conjecture, and Hasse’s algorithm. [9]

The Collatz conjecture states that if N_0 is an even positive integer, then

$$\frac{N_0}{2} = N_1 \tag{1}$$

If N_0 is an odd positive integer, then

$$3N_0 + 1 = N_1 \tag{2}$$

Then repeated iterations of this process produces the value $N_i = 1$, where i is the iteration step.

Although the conjecture appears to be simple, it has remained unproven for almost 90 years. An issue is that the values of the intermediate steps can increase to very large values from the original starting value. It is unknown whether there is a starting value that may continue to increase up to infinity and never reach 1. Or, a closely related issue is whether there are values that form an endless loop without ever reaching 1. The Collatz conjecture applies to all positive integers, so even if mathematicians prove the conjecture true for a very large number, they still cannot determine with certainty whether the next number follows the same pattern.

Many mathematicians have attempted and failed to prove the Collatz conjecture. Paul Erdős stated “mathematics may not be ready for such problems [Collatz conjecture].” [5] Jeffrey Lagarias stated the Collatz conjecture “is an extraordinarily difficult problem, completely out of reach of present-day mathematics.” [7] Since these renowned mathematicians have failed to prove the Collatz conjecture, experts believe that solving the conjecture will require a new type of mathematics.

Leading mathematicians make comments that convince many people the Collatz conjecture is unsolvable, especially for those without an advanced mathematics degree. This widespread belief

causes many mathematicians to ignore papers submitted for publication, assuming they must be incorrect. [1] As a result, this mindset hinders efforts to develop a proof for the Collatz conjecture.

Previous studies of the Collatz conjecture focused on the pattern of positive integers in the iteration sequence and the number of steps each positive integer takes before reaching 1. [2,4,7–9] The difficulty in studying these parameters is that they greatly vary even when the positive integers are close in value. The number of steps from the initial value until reaching 1 appears to be unpredictable. For example, the value 26 takes 10 steps, value 27 takes 111 steps, and value 28 takes 18 steps.

It may not be a case of needing a new mathematical method to solve the Collatz conjecture, but a new way of looking at the problem. Solving the Collatz conjecture is easier if it is not looked at as a mathematical problem but as a mathematical puzzle.

Currently, mathematicians try to prove the Collatz conjecture by advancing the attempts by previous researchers. This is the classical way of progressing science. Each new researcher tries to improve on the previous researcher until finding a solution. However, the Collatz conjecture is a case of “you cannot get there from here.” Solving the conjecture requires starting at the beginning and using a new perspective, just like all math puzzles. The solution does not require using more complicated math but just looking at the puzzle in a new way. [3]

Analysis of the rules for even and odd positive integers shows that the inter-relationship of these rules is important for understanding the Collatz conjecture. Preliminary results reported by Hahn (2024) show a study of the Collatz conjecture does not require the development of new mathematical methods, but just a new perspective on the problem. [6]

We prove the Collatz conjecture by examining how the two rules organize positive integers rather than tracing the pathways of individual values.

2. Results

2.1. Rule for Even Numbers

If N_0 is an even positive integer, then $\frac{N_0}{2}$.

For every even positive integer N , the Collatz rule for even positive integers halves the positive integer repeatedly until reaching an odd positive integer. A set of positive integers that consist of even positive integers with the same odd base positive integer is called an “odd base number set” (O_{bn}).

Let O_{bn} denote the set containing all such subsets.

$$\begin{aligned} O_{bn1} &= \{1, 2, 4, 8, 16, \dots\} \\ O_{bn3} &= \{3, 6, 12, 24, 48, \dots\} \\ O_{bn5} &= \{5, 10, 20, 40, 80, \dots\} \\ O_{bn7} &= \{7, 14, 28, 56, 112, \dots\} \\ &\vdots \\ &\text{therefore,} \end{aligned}$$

$$\bigcup_{k=1}^N O_{b(2k-1)} \subseteq O_{bn}$$

The positive integers in O_{bn} have the formula:

$$2^a \cdot X, \tag{3}$$

where $X = \mathbb{N}^{odd}$, $a = 0, 1, 2, 3, \dots$

The general formula for an odd base number set is also the general formula for a positive integer. General formula for a positive integer:

$$2^a \cdot X, \tag{4}$$

where $X = \mathbb{N}^{odd}$, $a = 0, 1, 2, 3, \dots$

Odd positive integers are generated when $a = 0$ and even positive integers are generated when $a = 1, 2, 3, \dots$

The set of all odd base number sets equals the set of positive integers.

Proof 1.

If $O_{bn} = \mathbb{N}^+$, two things need to shown:

1. $O_{bn} \subseteq \mathbb{N}^+$: Every element in O_{bn} is also in $\mathbb{N}^+$. This is true because 2^a and X are integers, and their product is an integer. Since $a \geq 0$ and X is a positive odd integer, $2^a X$ is a positive integer.
2. $\mathbb{N}^+ \subseteq O_{bn}$: Every element in $\mathbb{N}^+$ is also in O_{bn} . Every natural number $n \in \mathbb{N}^+$, n can be written as $2^a X$, where a is a non-negative integer representing the highest power of 2 that divides n , and X is the odd base number of n (obtained by dividing n by 2^a). Since $a \in \{0, 1, 2, \dots\}$ and X is an odd positive integer, n fits the definition of an element in O_{bn} .

Let $O_{bn} = \{n \in \mathbb{Z}^+ \mid n = 2^a X, a \in \{0, 1, 2, 3, \dots\}, X \text{ is odd}\}$. Show that $O_{bn} = \mathbb{N}^+$, where $\mathbb{N}^+$ is the set of natural numbers $\{1, 2, 3, \dots\}$.

($\subseteq$) Let $y \in O_{bn}$. By definition, $y = 2^a X$ for some non-negative integer a and some odd positive integer X . Since $a \geq 0$ and $X \geq 1$, y is a positive integer. Therefore, $y \in \mathbb{N}^+$. This shows that $O_{bn} \subseteq \mathbb{N}^+$.

($\supseteq$) Let $z \in \mathbb{N}^+$. z can be expressed as $z = 2^a \cdot X$, where $a \geq 0$ is the largest integer such that 2^a divides z , and $X = \frac{z}{2^a}$ is the odd base of z . Since z is a positive integer, a is a non-negative integer, and X is a positive odd integer. By the definition of O_{bn} , $z \in O_{bn}$. This shows that $\mathbb{N}^+ \subseteq O_{bn}$.

Since $O_{bn} \subseteq \mathbb{N}^+$ and $\mathbb{N}^+ \subseteq O_{bn}$:

$$O_{bn} = \mathbb{N}^+$$

(see Appendix I for verification of proof with Isabelle/HOL proof assistant) $\square$

The rule of even positive integers organizes all positive integers into one and only one odd base number set. Odd base number sets organize all positive integers; however, this is not enough to prove the Collatz conjecture since the rule only halves all even positive integers until reaching their odd base positive integer. The even number rule does not connect the odd base number sets into a path to eventually reach the positive integer "1."

When examining the odd base number sets, an obvious dilemma appears. Each set has just a single odd positive integer with many even positive integers. However, there are equal quantities of even and odd positive integers. Therefore the key to developing a proof for the Collatz conjecture is to analyze how the odd number rule organizes the odd base number sets of positive integers.

2.2. Rule for Odd Numbers

If X is an odd positive integer, then $3X + 1$.

The Collatz rule for handling odd positive integers is $3x + 1$, where x is an odd positive integer. This rule causes the generation of an even positive integer after reaching an odd positive integer. Multiplying the odd positive integer by 3 creates an odd positive integer. The addition of 1 generates an even positive integer. Since each odd base number set has an odd positive integer as the base positive integer, the odd positive integer becomes linked to an even positive integer with the general formula of $2^a X$, where X is an odd positive integer and $a = 0, 1, 2, 3, \dots$

However, there must be a 1 : 1 relationship between an odd base number and an even number. The absence of a 1 : 1 relationship could indicate a non-continuous connection between odd base number sets. In order to show a 1 : 1 relationship, $3x + 1$ must be proven to be bijective.

Proof 2.

Definition A function $f : A \rightarrow B$ is:

1. *injective* (one-to-one) if for all $a, a' \in A$, $f(a) = f(a')$ implies $a = a'$;
2. *surjective* (onto) if for every $b \in B$ there exists an $a \in A$ such that $f(a) = b$;
3. *bijective* if f is both injective and surjective.

Let $f(x) = 3x + 1 : \mathbb{N}_{odd} \rightarrow C$, where $\mathbb{N}_{odd} = \{x \in \mathbb{N} \mid x \text{ is odd}\} = \{1, 3, 5, 7, 9, \dots\}$ and $C = \{4, 10, 16, 22, 28, \dots\}$. The goal is to show that f is bijective.

Injectivity: Let $x, x' \in \mathbb{N}_{odd}$ such that $f(x) = f(x')$. Then, $3x + 1 = 3x' + 1$. Subtracting 1 from both sides, leaves $3x = 3x'$. Dividing both sides by 3, obtains $x = x'$. Since $f(x) = f(x')$ implies $x = x'$, the function f is injective.

Surjectivity: Let $y \in C$. Show that there exists an $x \in \mathbb{N}_{odd}$ such that $f(x) = y$. From the definition of f , if $y = f(x)$, then $y = 3x + 1$. Solving for x , gets $x = \frac{y-1}{3}$.

Verify that for every $y \in C$, the corresponding x is in $\mathbb{N}_{odd}$. The set C can be expressed as $C = \{y \in \mathbb{N} \mid y = 6n - 2 \text{ for some } n \in \mathbb{N}^+\}$. Let $y \in C$. Then $y = 6n - 2$ for some $n \in \mathbb{N}^+$. Substituting this into the expression for x : $x = \frac{(6n-2)-1}{3} = \frac{6n-3}{3} = 2n - 1$.

Since $n \in \mathbb{N}^+$, i.e., $n \geq 1$, $2n - 1$ is a positive odd integer. Therefore, $x = 2n - 1 \in \mathbb{N}_{odd}$. For every $y \in C$, an $x \in \mathbb{N}_{odd}$ is found, such that $f(x) = 3(2n - 1) + 1 = 6n - 3 + 1 = 6n - 2 = y$.

Thus, the function f is surjective onto C .

Bijection and Cardinality: Since $f(x) = 3x + 1$ is both injective and surjective, it is a bijective function from $\mathbb{N}_{odd}$ to C .

A bijective function establishes a one-to-one correspondence between the elements of the domain and the codomain. Therefore, the cardinality of the set of odd natural numbers is equal to the cardinality of the set C :

$$|\mathbb{N}_{odd}| = |C|$$

(see Appendix II for verification of proof with Isabelle/HOL proof assistant) $\square$

We classify odd positive integers into three distinct categories based on their characteristics in odd base number sets. An odd positive integer is either one less than a positive multiple of 6 (e.g., $(6N - 1)$, where (N) is a positive integer), one more than a positive multiple of 6 (e.g., $(6N + 1)$, where (N) is a positive integer), or a number divisible by 3 (i.e., $(\frac{X}{3})$, where X is an odd positive integer and the result is a positive integer).

Odd positive integers of the form $X = 6N - 1$, where X is an odd positive integer and N is a positive integer, create odd base number sets where every other positive integer starting from the first even positive integer (e.g., $2x, 8x, 32x, \dots$, where x is an odd positive integer) equals a positive integer written as $3x' + 1$ (where x' and x are different odd positive integers). For example, if the odd base positive integer is 5 (e.g., $6 - 1$), then 10 $[(3 \times 3) + 1]$, 40 $[(13 \times 3) + 1]$, and 160 $[(53 \times 3) + 1]$ connect to odd base numbers 3, 13, and 53, respectively.

Odd positive integers of the form $X = 6N + 1$, where X is an odd positive integer and N is a positive integer, create odd base number sets where every other positive integer starting from the second even positive integer (e.g., $4x, 16x, 64x, \dots$, where x is an odd positive integer) equals a positive integer written as $3x' + 1$ (where x' and x are different odd positive integers). For example, if the odd positive integer is 7 (e.g., $6 + 1$), then 28 $[(9 \times 3) + 1]$, 112 $[(37 \times 3) + 1]$, and 448 $[(149 \times 3) + 1]$ connect to odd base numbers 9, 37, and 149, respectively.

Odd positive integers divisible by 3 form the most interesting odd base number sets. Since each even positive integer in the odd base number set is divisible by 3, none of the even positive integers are expressed by the formula $3x + 1$, where x is an odd positive integer. This results in none of the even positive integers in the set being connected to another odd base number set. Unless the initial positive integer selected for analysis with the Collatz conjecture is a positive integer divisible by 3, then none of the odd base number sets with an odd base positive integer divisible by 3 is reached during the iteration of positive integers.

Each odd positive integer forms a separate and unique odd base number set comprising the odd positive integer as the lowest integer in the set and then doubling the odd positive integer to generate the successive even positive integer of the set. Since each odd base number set contains a unique set of

positive integers, the combination of the even $\lfloor \frac{N_0}{2} \rfloor$ and odd $[3N_0 + 1]$ number rules essentially require the iteration down an odd base number set until reaching the odd positive integer at the base, then jumping to a different odd base number set. This continues until reaching the final odd base number set for 1.

2.3. Dendritic (Tree-Like) Pattern

At this point, we have proved that the rule for even numbers organizes all positive integers into odd base number sets and the rule for odd numbers causes all of the odd base number sets to be interconnected. The next thing is to prove that all interconnected sets go to 1.

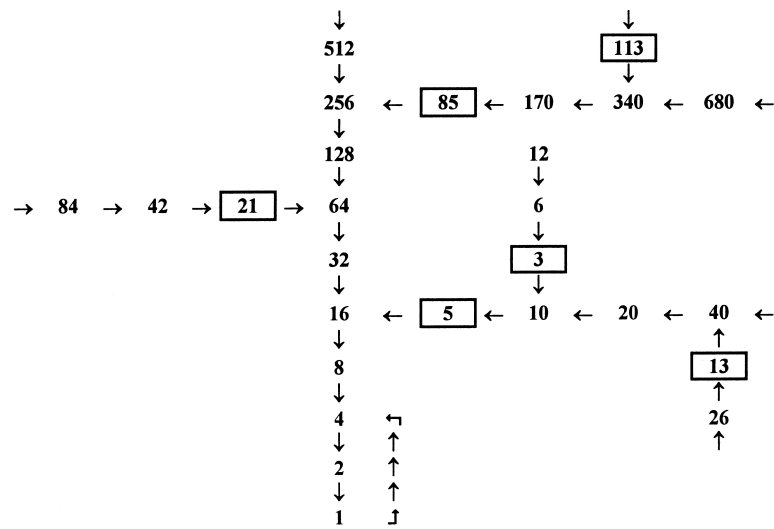


Figure 1. Illustrates a possible dendritic pattern produced by the Collatz rules of even and odd numbers, showing some 1^o, 2^o, and 3^o branches.

The odd base number set with a base positive integer of 1 can be viewed as the trunk of the tree: the primary (1^o) branch. The primary branch has connected odd base number sets with base positive integers of 5, 21, 85, ...; which can be viewed as secondary (2^o) branches. [“branch” is used to represent an odd base number set] Tertiary (3^o) branches connect to the secondary (2^o) branches; 4^o branches connect to 3^o branches; which, in turn connect to 5^o branches, and this continues onward. Each branch has a large quantity of even positive integers and there are a large quantity of branches.

In practice, when selecting a positive integer at random, the rule for even numbers causes the even positive integers for a particular odd base number set to go down until reaching the odd positive integer at the base. Then, the rule for odd numbers causes the odd base number set to connect to an even positive integer in an odd base number set in the degree branch below it. The connected sets decrease from the original set “degree” down through the degrees until reaching the primary odd base number set of 1. Therefore all branches go to the lowest branch in the dendritic pattern.

This shows that the precise positive integer to be examined with the Collatz conjecture is not important. What is important is the odd base number set in which the positive integer exists. For example, although the positive integer 33,554,432 is large and the positive integer 27 is small, 33,554,432 is in the primary branch and 27 is in a 42nd branch.

If an even positive integer is in a particular odd base number set, the even positive integer has the same path to “1” as other even positive integers in the odd base number set. This shows that all even positive integers in a branch go to the odd positive integer at the base and all odd base number sets connect, which eventually goes to 1.

2.4. Rule for Odd Numbers Prevents Infinite Loops

We observed that the Collatz rule for odd numbers creates an equality between two odd numbers. The Collatz rule for odd numbers $(3x + 1)$ generates an even number. Therefore, the even number generated by the $3x + 1$ step also can be written as $2^n x'$. For example, upon reaching 5 during an iteration, the odd number rule generates the equality of $(3 \times 5) + 1 = 16 \times 1$.

Selecting several equalities with the same odd number in different equalities allows the generation of an equation showing the equality between 1 (the termination of the Collatz conjecture) and every odd positive integer.

The following equations show several equalities with a common odd number:

$$(3 \times 5) + 1 = 16 \times 1 \tag{5}$$

$$(3 \times 13) + 1 = 8 \times 5 \tag{6}$$

$$(3 \times 17) + 1 = 4 \times 13 \tag{7}$$

$$(3 \times 11) + 1 = 2 \times 17 \tag{8}$$

Solving these equations for the common odd number:

$$\frac{(3 \times 5) + 1}{16} = 1 \tag{9}$$

$$\frac{(3 \times 13) + 1}{8} = 5 \tag{10}$$

$$\frac{(3 \times 17) + 1}{4} = 13 \tag{11}$$

$$\frac{(3 \times 11) + 1}{2} = 17 \tag{12}$$

enables the generation of a single equation by substitution.

$$\frac{(3 \times 5) + 1}{16} = 1 \tag{13}$$

$$\frac{(3 \times 5)}{16} + \frac{1}{16} = 1 \tag{14}$$

$$\frac{3 \times \left(\frac{(3 \times 13) + 1}{8} \right)}{16} + \frac{1}{16} = 1 \tag{15}$$

$$\frac{(9 \times 13) + 3}{8 \times 16} + \frac{1}{16} = 1 \tag{16}$$

$$\frac{(9 \times 13)}{8 \times 16} + \frac{3}{8 \times 16} + \frac{1}{16} = 1 \tag{17}$$

$$\frac{9 \times \left(\frac{(3 \times 17) + 1}{4} \right)}{8 \times 16} + \frac{3}{8 \times 16} + \frac{1}{16} = 1 \tag{18}$$

$$\frac{\left(\frac{(27 \times 17) + 9}{4} \right)}{8 \times 16} + \frac{3}{8 \times 16} + \frac{1}{16} = 1 \tag{19}$$

$$\frac{(27 \times 17)}{4 \times 8 \times 16} + \frac{9}{4 \times 8 \times 16} + \frac{3}{8 \times 16} + \frac{1}{16} = 1 \tag{20}$$

$$\frac{27 \times \left(\frac{(3 \times 11) + 1}{2} \right)}{4 \times 8 \times 16} + \frac{9}{4 \times 8 \times 16} + \frac{3}{8 \times 16} + \frac{1}{16} = 1 \tag{21}$$

$$\frac{\left(\frac{(81 \times 11) + 27}{2} \right)}{4 \times 8 \times 16} + \frac{9}{4 \times 8 \times 16} + \frac{3}{8 \times 16} + \frac{1}{16} = 1 \tag{22}$$

$$\frac{(81 \times 11)}{2 \times 4 \times 8 \times 16} + \frac{27}{2 \times 4 \times 8 \times 16} + \frac{9}{4 \times 8 \times 16} + \frac{3}{8 \times 16} + \frac{1}{16} = 1 \tag{23}$$

$$\frac{81 \times 11}{1024} + \frac{27}{1024} + \frac{9}{512} + \frac{3}{128} + \frac{1}{16} = 1 \tag{24}$$

$$\frac{81 \times 11}{1024} + \frac{27}{1024} + \frac{9 \times 2}{1024} + \frac{3 \times 8}{1024} + \frac{64}{1024} = 1 \tag{25}$$

$$\frac{891}{1024} + \frac{27}{1024} + \frac{18}{1024} + \frac{24}{1024} + \frac{64}{1024} = 1 \tag{26}$$

$$\frac{1024}{1024} = 1 \tag{27}$$

$$1 = 1 \tag{28}$$

Equation 25 is reduced and reversed so it is easier to read.

$$1 = \frac{3^0}{2^4} + \frac{3^1}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \left(\frac{3^4}{2^{10}} \times 11 \right) \tag{29}$$

Equation 29 is generalized to generate an equation that represents the iteration of the Collatz conjecture. The numerator of the fractions increases as a power of 3 and the denominator of the fractions increases with larger powers of 2.

$$Y = \frac{1}{2^a} + \frac{3^1}{2^b} + \dots + \frac{3^{n-2}}{2^\psi} + \frac{3^{n-1}}{2^\omega} + \frac{3^n}{2^\omega} X, \tag{30}$$

where $a < b < \dots < \psi < \omega$

The general equation is powerful for studying the Collatz conjecture. The equation can be used to show that all iterations go to 1, which is the termination of the Collatz conjecture. Additionally, the equation can be solved to show the individual odd positive integers which form the connections from the selected positive integer to its termination. Finally, the equation can be solved for every odd positive integer to determine its location in the pattern and how many steps it takes to go from the odd positive integer down to 1.

Table 1. Parts A, B, and C.

	A	B	$C \times X$
Two Branches	$Y =$	$\frac{1}{2^a} + \frac{3^1}{2^b}$	$\frac{3^2}{2^b} X$
Three Branches	$Y =$	$\frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c}$	$\frac{3^3}{2^c} X$
Four Branches	$Y =$	$\frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^d}$	$\frac{3^4}{2^d} X$
Five Branches	$Y =$	$\frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^d} + \frac{3^4}{2^e}$	$\frac{3^5}{2^e} X$
Ten Branches	$Y =$	$\frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^d} + \frac{3^4}{2^e}$ $\frac{3^5}{2^f} + \frac{3^6}{2^g} + \frac{3^7}{2^h} + \frac{3^8}{2^i} + \frac{3^9}{2^j}$	$\frac{3^{10}}{2^j} X$
Multiple Branches	$Y =$	$\frac{1}{2^a} + \frac{3^1}{2^b} + \cdots + \frac{3^{n-2}}{2^\psi} + \frac{3^{n-1}}{2^\omega}$	$\frac{3^n}{2^\omega} X$

Table 2. $A - (C \times X) = B$.

	$A - (C \times X)$	B
Two Branches	$Y - \frac{3^2}{2^b} X =$	$\frac{1}{2^a} + \frac{3^1}{2^b}$
Three Branches	$Y - \frac{3^3}{2^c} X =$	$\frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c}$
Four Branches	$Y - \frac{3^4}{2^d} X =$	$\frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^d}$
Five Branches	$Y - \frac{3^5}{2^e} X =$	$\frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^d} + \frac{3^4}{2^e}$
Ten Branches	$Y - \frac{3^{10}}{2^j} X =$	$\frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^d} + \frac{3^4}{2^e}$ $\frac{3^5}{2^f} + \frac{3^6}{2^g} + \frac{3^7}{2^h} + \frac{3^8}{2^i} + \frac{3^9}{2^j}$
Multiple Branches	$Y - \frac{3^n}{2^\omega} X =$	$\frac{1}{2^a} + \frac{3^1}{2^b} + \cdots + \frac{3^{n-2}}{2^\psi} + \frac{3^{n-1}}{2^\omega}$

There are three important parts of the equation.

A - odd positive integer calculated by the addition of the series of fractions in *B* and *C*. If the positive integer in *A* is 1 then the equation can show that every positive integer goes to 1, the endpoint of the Collatz conjecture.

B - a series of fractions that indicate the odd positive integers during iteration and the position of the even positive integers that connect to the odd positive integer. Solving the equation up to that fraction determines the value of the odd positive integer.

C × *X* - a fraction with the exponent of 3 one positive integer higher than the previous fraction and with the exponent of 2 the same as the previous fraction multiplied by the odd positive integer that was initially selected or the odd positive integer which is the base number of the selected even positive number.

Mathematics defines an “equation structure” as the arrangement and relationship between the different components of an equation, including variables, constants, operations, and the equal sign, which together define how the equation is written and what information it conveys about the relationship between values.

Equations that have the same structure generally have similar relationships, even if the variables and constants differ. For example, if two equations are similar (e.g., $y = b + mx$), mathematicians can always solve them using the same steps, regardless of specific values. The same mathematical methods solve equations with the same structure. In short, structurally similar equations imply a fundamental connection in the patterns or relationships they represent, which can lead to shared methods of analysis or interpretation.

Proof 3 - Step 1.

The general structure of the equations in Table 2 can be expressed as follows:

$$Y = B + (C \times X)$$

where:

B: A sum of fractions with powers of 2 in the denominators and powers of 3 in the numerators.

C: A fraction involving powers of 2 and 3, multiplied by *X*.

Step 1: Prove equations in Table 2 have the same structure.

Let:

$$\begin{aligned} Y &= \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^b} X \\ Y &= \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^c} X \\ Y &= \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^d} + \frac{3^4}{2^d} X \\ Y &= \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^d} + \frac{3^4}{2^e} + \frac{3^5}{2^e} X \\ Y &= \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^d} + \frac{3^4}{2^e} + \frac{3^5}{2^f} + \frac{3^6}{2^g} + \frac{3^7}{2^h} + \frac{3^8}{2^i} + \frac{3^9}{2^j} + \frac{3^{10}}{2^j} X \\ Y &= \frac{1}{2^a} + \frac{3^1}{2^b} + \cdots + \frac{3^{n-2}}{2^\psi} + \frac{3^{n-1}}{2^\omega} + \frac{3^n}{2^\omega} X \end{aligned}$$

So each equation can be expressed with the general equation:

$$Y = B + (C \times X)$$

,

where:

B: A sum of fractions with powers of 2 in the denominators and powers of 3 in the numerators.

C: A fraction involving powers of 2 and 3, multiplied by X.

Therefore:

Equation for two branches:

$$Y = \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^b} X$$

has:

$$B = \frac{1}{2^a} + \frac{3^1}{2^b}$$
$$C = \frac{3^2}{2^b}$$

Equation for three branches:

$$Y = \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^c} X$$

has:

$$B = \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c}$$
$$C = \frac{3^3}{2^c}$$

Equation for four branches:

$$Y = \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^d} + \frac{3^4}{2^d} X$$

has:

$$B = \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^d}$$
$$C = \frac{3^4}{2^d}$$

Equation for five branches:

$$Y = \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^d} + \frac{3^4}{2^e} + \frac{3^5}{2^e} X$$

has:

$$B = \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^d} + \frac{3^4}{2^e}$$
$$C = \frac{3^5}{2^e}$$

Equation for ten branches:

$$Y = \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^d} + \frac{3^4}{2^e} + \frac{3^5}{2^f} + \frac{3^6}{2^g} + \frac{3^7}{2^h} + \frac{3^8}{2^i} + \frac{3^9}{2^j} + \frac{3^{10}}{2^j} X$$

has:

$$B = \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^d} + \frac{3^4}{2^e} + \frac{3^5}{2^f} + \frac{3^6}{2^g} + \frac{3^7}{2^h} + \frac{3^8}{2^i} + \frac{3^9}{2^j}$$
$$C = \frac{3^{10}}{2^j}$$

General equation for multiple branches:

$$Y = \frac{1}{2^a} + \frac{3^1}{2^b} + \dots + \frac{3^{n-2}}{2^\psi} + \frac{3^{n-1}}{2^\omega} + \frac{3^n}{2^\omega} X$$

has:

$$B = \frac{1}{2^a} + \frac{3^1}{2^b} + \dots + \frac{3^{n-2}}{2^\psi} + \frac{3^{n-1}}{2^\omega}$$
$$C = \frac{3^n}{2^\omega}$$

Although the specific B and C differ in complexity and the number of terms, the underlying structure of the equations remains identical. The structural similarity implies that they belong to the same class of equations, and similar solution techniques can be applied to all equations.

Conclusion:

The equations have the same structure.
(see Appendix III for verification of proof with Isabelle/HOL proof assistant) □

Proof 3 - Step 2.

Step 2: Prove $Y \neq X$ in the equations in Table 3, when Y and X are odd and ≥ 3 .

Suppose for the sake of contradiction that there is a value Y, X that solves the equation $Y - (C \times X) = B$ so that $Y = X$, which would mean it is true there exist loops.

Therefore,

If $Y = X$,
and
 $Y = B + (C \times X)$ when $Y = X$,
let:
 $Y - (C \times X) = B$.

Since the structure of all the equations is the same, the equation for 10 branches is used as an example for all equations.

Let:

$$Y = \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^d} + \frac{3^4}{2^e} + \frac{3^5}{2^f} + \frac{3^6}{2^g} + \frac{3^7}{2^h} + \frac{3^8}{2^i} + \frac{3^9}{2^j} + \frac{3^{10}}{2^j} X$$

,
where Y and X are odd positive integers, not divisible by 3 and $X \geq 3$, and where $a < b < c < d < e < f < g < h < i < j$ and $a, b, c, d, e, f, g, h, i, j$ are positive integers.

Prove that $Y \neq X$.
Assume, for the sake of contradiction, that $Y = X$, which would indicate a loop.

Then the equation becomes:

$$X = \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^d} + \frac{3^4}{2^e} + \frac{3^5}{2^f} + \frac{3^6}{2^g} + \frac{3^7}{2^h} + \frac{3^8}{2^i} + \frac{3^9}{2^j} + \frac{3^{10}}{2^j} X$$

Rearranging the terms:

$$X - \frac{3^{10}}{2^j} X = \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^d} + \frac{3^4}{2^e} + \frac{3^5}{2^f} + \frac{3^6}{2^g} + \frac{3^7}{2^h} + \frac{3^8}{2^i} + \frac{3^9}{2^j}$$

Common denominator:

$$\begin{aligned} \frac{2^j - 3^{10}}{2^j} X = & \frac{2^{j-a}}{2^j} + \frac{3^1 \times 2^{j-b}}{2^j} + \frac{3^2 \times 2^{j-c}}{2^j} + \frac{3^3 \times 2^{j-d}}{2^j} + \\ & \frac{3^4 \times 2^{j-e}}{2^j} + \frac{3^5 \times 2^{j-f}}{2^j} + \frac{3^6 \times 2^{j-g}}{2^j} + \frac{3^7 \times 2^{j-h}}{2^j} + \\ & \frac{3^8 \times 2^{j-i}}{2^j} + \frac{3^9}{2^j} \end{aligned}$$

The right side of the equation does not contain X and only has values that are either powers of 2 or powers of 3.

The left side of the equation contains X and $\frac{2^j - 3^{10}}{2^j}$, which is less than 1. X must not have a factor of a power of 2 since X is odd or a power of 3 since X is not divisible by 3.

So $Y \neq X$ in the equation.

Conclusion:

There are no major loops.

(see Appendix III for verification of proof with Isabelle/HOL proof assistant) $\square$

Even though the general equation shows there are no major loops, the equation still calculates the minor loop when $Y, X = 1$. The equation discloses the number of loops rather than the number of branches.

For example,

Two loop of minor loop 4 – 2 – 1:

$$\begin{aligned} 1 &= \frac{1}{2^2} + \frac{3}{2^4} + \left(\frac{9}{2^4} \times 1 \right) \\ 1 &= \frac{4}{16} + \frac{3}{16} + \left(\frac{9}{16} \times 1 \right) \\ 1 &= 1 \end{aligned}$$

Four loops of minor loop 4 – 2 – 1:

$$\begin{aligned} 1 &= \frac{1}{2^2} + \frac{3}{2^4} + \frac{9}{2^6} + \frac{27}{2^8} + \left(\frac{81}{2^8} \times 1 \right) \\ 1 &= \frac{64}{256} + \frac{3 \times 16}{256} + \frac{9 \times 4}{256} + \frac{27}{256} + \left(\frac{81}{256} \times 1 \right) \\ 1 &= \frac{64}{256} + \frac{48}{256} + \frac{36}{256} + \frac{27}{256} + \left(\frac{81}{256} \times 1 \right) \\ 1 &= 1 \end{aligned}$$

Step 1 of the proof demonstrates that while the equations for different numbers of branches have B and C with varying numbers of terms and specific coefficients, their overall structure remains unchanged:

$$X = B + (C \cdot X)$$

where B includes all constant terms independent of X and C is the coefficient for the linear X term. This consistency in structure implies that all equations belong to the same class of linear equations. Therefore, the same solution techniques can be applied uniformly across all equations in Table 2.

Step 2 derives a contradiction by assuming that $Y = X$. Substituting $Y = X$ into the general equation: $X = B + (C \times X)$ leads to: $X - (C \times X) = B$. For $Y = X$ to hold, then X must satisfy the equation. The contradiction arises when analyzing the general equation for multiple branches, where the inequality demonstrates that X does not satisfy the equation under normal conditions unless $X = 1$, which trivializes the equation to: $Y = X = 1$. Thus, the assumption $Y = X$ (and hence the presence of loops) is false in all cases except when $X = 1$.

The similarity of the equations highlights that they have the same “structure” and are analyzed using the same mathematical framework. Therefore, during iteration with the Collatz conjecture rules there are no odd positive integers that return to the same value to form a loop. There are no solutions to the general equation when $X = Y$ and X, Y are not 1.

2.5. Rule for Odd Numbers Prevents the Possibility of Numbers Continuously Increasing to Infinity

The next big challenge is to prove that no positive integers continue increasing toward infinity. The Collatz conjecture rule for even numbers restricts the ability of values to increase. The even number rule ($N/2$) only decreases the previous value and thus, works against the ability to increase endlessly. Therefore, the rule for odd numbers ($3N + 1$) is the only method of increasing the previous value. However, the rule for odd numbers produces an even number, so the value is automatically divided in half. The only way for a value to increase is for the value after the even number rule to be odd. For example, the value 7 goes $7 \rightarrow 11 \rightarrow 17$, so the value increases from 7 to 11 to 17 in 2 steps. Then 17 goes to 26 in the next step (Table 3).

Table 3. $2^n - 1$ Sequences Have an Even Number After n Steps.

n	$2^n - 1$	Sequence
2	3	5, 8
3	7	11, 17, 26
4	15	23, 35, 53, 80
5	31	47, 71, 107, 161, 242
6	63	95, 143, 215, 323, 485, 728
7	127	191, 287, 431, 647, 971, 1457, 2186
8	255	383, 575, 863, 1295, 1943, 2915, 4373, 6560
9	511	767, 1151, 1727, 2591, 3887, 5831, 8747, 13121, 19682
10	1023	1535, 2303, 3455, 5183, 7775, 11663, 17495, 26243, 39365, 59048

The observation indicates that the way for a number continuously to increase towards infinity is to always reach an odd number after $((3x + 1)/2)$. Looking at the odd numbers and applying $((3x + 1)/2)$, we observed that the odd numbers form a pattern. The pattern is an alternating pattern of even and odd numbers. This eliminates half of the numbers since the next number must be odd if the value is to continue to increase. At each step of applying $((3x + 1)/2)$ to the resulting odd numbers, the quantity of available odd numbers decreases from the previous values. For values ranging from 1 to 2^n , we observed that the value $(2^n - 1)$ yields the maximum number of steps that generate an odd number when applying the transformation $((3x + 1)/2)$.

Figure 2 shows that all values of form $(2^n - 1)$ will eventually generate an even number and thus decline in value from the previous value. Therefore, the number of steps of $((3x + 1)/2)$ that produce an odd number is finite. Thus, there are no positive integers that continue to increase towards infinity.

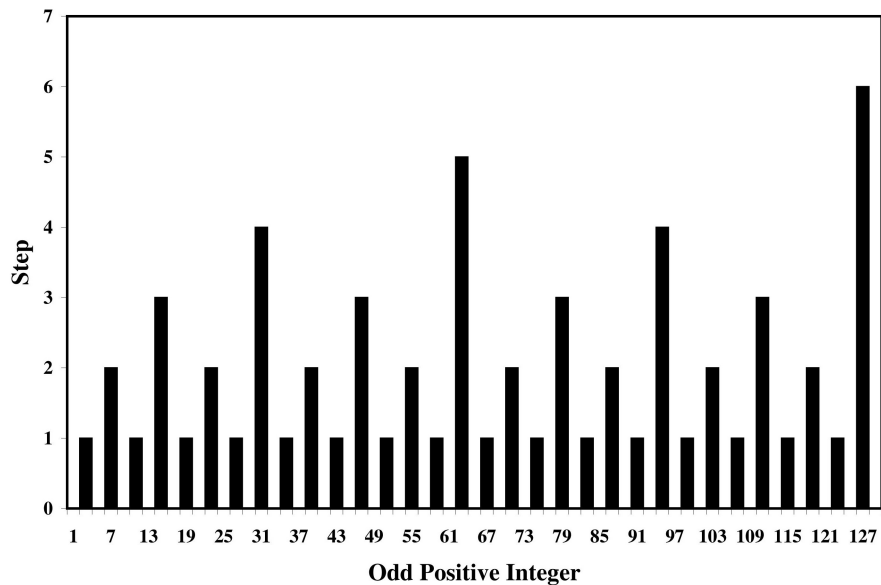


Figure 2. Steps before reaching an even number.

Proof 4.

Prove that if $x = 2^n - 1$, then x is always finite no matter the value of positive integer n .

Definition of a *Positive Integer*: n is a positive integer; which, means n belongs to the set $\{1, 2, 3, 4, \dots\}$. By definition, each positive integer is a specific, countable number.

The Base of the Exponent: The base is 2 which is a finite number.

The expression 2^n represents repeated multiplication of the base (2) by itself n times.

For example,

 If $n = 1$, then $2^1 = 2$,

 If $n = 2$, then $2^2 = 2 \times 2 = 4$,

 If $n = 3$, then $2^3 = 2 \times 2 \times 2 = 8$,

 If $n = k$ (where k is a specific positive integer), then 2^k equals k repetitions of multiplying 2, $2 \times 2 \times \dots \times 2$.

The multiplication of two finite numbers results in another finite number. Starting with the finite number 2 and repeatedly multiplying by itself a finite number of times (as n is a finite positive integer), then the result of 2^n must also be a finite number.

Subtracting a finite number (1) from another finite number (2^n) is always a finite number.

Conclusion:

Therefore, if n is a positive integer and $2^n - 1 = x$, then x is finite. Even as n becomes arbitrarily large, n remains a specific, finite integer, leading to a finite value for 2^n and consequently for x .

As n increases, x grows incredibly large; however, the important issue is that x will never reach infinity. Infinity is not a number but a concept. Since n is always a specific, finite number, the resulting value of x is always a specific, although potentially very large, finite number.

(see Appendix IV for verification of proof with Isabelle/HOL proof assistant) □

Proof 5.

Prove that the n^{th} step of $f^n(x) = \frac{3x+1}{2}$, when $x = 2^n - 1$, is always even. Show by contradiction that the statement “the n^{th} step of $f^n(x)$, when $x = 2^n - 1$, is always odd” is false.

Show that $f^n(2^n - 1)$ is always even. First, obtain a general formula for $f^k(2^n - 1)$ using induction.

Base Case: For $k = 1$,

$$\begin{aligned} f^1(2^n - 1) &= \frac{3(2^n - 1) + 1}{2} \\ &= \frac{3 \cdot 2^n - 3 + 1}{2} \\ &= \frac{3 \cdot 2^n - 2}{2} \\ &= 3 \cdot 2^{n-1} - 1. \end{aligned}$$

Inductive Hypothesis: Assume that for some positive integer $k < n$,

$$f^k(2^n - 1) = 3^k \cdot 2^{n-k} - 1.$$

Inductive Step: Show that $f^{k+1}(2^n - 1) = 3^{k+1} \cdot 2^{n-(k+1)} - 1$.

$$\begin{aligned} f^{k+1}(2^n - 1) &= f(f^k(2^n - 1)) \\ &= f(3^k \cdot 2^{n-k} - 1) \\ &= \frac{3(3^k \cdot 2^{n-k} - 1) + 1}{2} \\ &= \frac{3^{k+1} \cdot 2^{n-k} - 3 + 1}{2} \\ &= \frac{3^{k+1} \cdot 2^{n-k} - 2}{2} \\ &= 3^{k+1} \cdot 2^{n-k-1} - 1 \\ &= 3^{k+1} \cdot 2^{n-(k+1)} - 1. \end{aligned}$$

The inductive step holds. Therefore, by induction, the formula: $f^k(2^n - 1) = 3^k \cdot 2^{n-k} - 1$, is true for all $1 \leq k \leq n$.

Now, consider the n^{th} step, i.e., $k = n$:

$$\begin{aligned} f^n(2^n - 1) &= 3^n \cdot 2^{n-n} - 1 \\ &= 3^n \cdot 2^0 - 1 \\ &= 3^n \cdot 1 - 1 \\ &= 3^n - 1 \end{aligned}$$

Since n is a positive integer, 3^n is always odd. An odd number minus 1 is always an even number. Therefore, $f^n(2^n - 1) = 3^n - 1$ is always even.

Conclusion:

We show that the statement “the n^{th} step of $f^n(x)$, when $x = 2^n - 1$, is always odd” is false. The result shows that for $x = 2^n - 1$, the n^{th} step of the iteration is $f^n(2^n - 1) = 3^n - 1$, which has been proven to be always even. This directly contradicts the statement that the n^{th} step of $f^n(x)$, when $x = 2^n - 1$, is always odd. Therefore, the statement is false.

(see Appendix V for verification of proof with Isabelle/HOL proof assistant) $\square$

The proof establishes that for $x = 2^n - 1$, the expression $f^n(x) = \frac{(3x+1)}{2}$ is always even for all non-negative integers n ; which, indicates that the value decreases and no longer is increasing towards infinity.

The idea that the values of the numbers might continuously increase to infinity is an artifact of graphing the sequence of numbers during the Collatz conjecture process. The graphing suggests that when the values of the numbers get farther from 1 that this indicates the failure of the process, rather than just the process of proceeding down the number sets to reach each subsequent base number set that then connects to the next number set in series.

The finding was confirmed by Ren. [10,11] During a discussion of their data sets, they stated that the data set for $2^{100} - 1$ begins with 100 steps of $\frac{(3x+1)}{2}$ and the data set for $2^{10000} - 1$ begins with 10,000 steps of $\frac{(3x+1)}{2}$.

Therefore, no positive integer continuously increases towards infinity.

2.6. All Positive Integers Converge to 1 by Iteration Using Collatz Conjecture Rules

Proof 6.

All positive integers N_0 go to 1 by iteration using the rules:

If N_0 is an even positive integer, then

$$\frac{N_0}{2} = N_1 \tag{31}$$

If N_0 is an odd positive integer, then

$$3N_0 + 1 = N_1 \tag{32}$$

For the sake of contradiction, suppose there is an odd positive integer S , where $S \geq 3$, that is the final step after iteration.

Therefore,

1. S must form a loop that goes back to S .
- or
2. S does not connect to another positive integer.

Number 1 is False - Proof 4 proves there are no loops.

Number 2 is False - Proof 2 proves that every odd positive integer ≥ 3 connects to an even positive integer in a 1 : 1 relationship.

Conclusion:

There is no odd positive integer S that is the final step after iteration. Therefore all positive integers eventually go to 1 after iteration.

(see Appendix VI for verification of proof with Isabelle/HOL proof assistant) □

3. Summary

The combined results of the six proofs provide compelling evidence that the Collatz conjecture holds true for all positive integers.

Proof 1 confirms that the Collatz rule for even positive integers organizes all positive integers into odd base number sets comprising an odd number as the base and even numbers that are multiples of the base number ($2^a \cdot X$). The odd base number sets (O_{bni} , where $i = 1, 3, 5, \dots$, are subsets of the super set O_{bn}). All the positive numbers are contained in O_{bn} ; therefore using the odd base number sets in the proofs includes all positive integers.

Proof 2 shows the Collatz rule for odd numbers ($3N + 1$) is bijective, which demonstrates the 1 : 1 relationship between connections of odd base numbers and an even number in a different odd base

number set. The only odd number to connect to an even number in the same odd number set is 1. The proof establishes there are no odd number sets not connected to the main pathways.

The combination of Proof 1 and Proof 2 produces a simple and predictable pattern; a dendritic pattern. The establishment of a dendritic pattern for the Collatz conjecture further confirms the results in the following proofs.

Proof 3 uses the observation that the even number rule and odd number rule form equalities between the odd numbers in different odd base number sets. Additionally, the rules allow forming different equalities with the same odd number. These equations are used to develop a single equation by substitution that is modified into a General Equation; which, represents the iteration of the Collatz conjecture with every positive integer.

Proof 3 first proves that the variations necessary in the General Equation to represent different iterations still produce equations with the same mathematical structure. Equations with the same structure can be manipulated with the same techniques to obtain the same results about the equations. Once the equations were proven to have the same structure, only one equation was used to show that $X' \neq X$ and thus proving there are no major loops. Additionally, we show that the General Equation still demonstrates the minor $4 \rightarrow 2 \rightarrow 1$, so there is confirmation that the General Equation is usable for every Collatz conjecture iteration.

Proof 4 is the first step in confirming that there are no positive integers that continue up towards infinity without eventually decreasing to 1. We observe that the only way for a positive integer to increase in value during iteration with the Collatz rules is for the result after $((3x + 1)/2)$ to be an odd number and thus requiring another iteration with $((3x + 1)/2)$. If a number is going to continuously increase in value, the values need to repeatedly result in an odd number after iteration with $((3x + 1)/2)$. Iterating numbers from 1 to 2^n , we observe that the number with the longest series of odd numbers in a row is the number $2^n - 1$. Proof 4 demonstrates that $2^n - 1$ is a finite number, so x is a finite number.

Proof 5 demonstrates that the n^{th} step in $f^n(x) = ((3x + 1)/2)$ is even, and thus begins to decrease in value after the n^{th} step. The proof combined with Proof 4 demonstrates that there are no positive integers that continuously increase in value up towards infinity because eventually every positive integer during iteration with the Collatz rules begins to decrease in value, and eventually reaches 1.

Proof 6 is a logical conclusion from the results of Proofs 2 and 4. All positive integers go to 1, since 1 is the only value that forms a minor loop after reaching the goal of 1, and every odd positive integer, other than 1, connects to an even number in a different odd number set. The only positive integer with the necessary characteristics is 1, so we conclude that all positive integers eventually go to 1 by iteration with the Collatz rules.

The proof demonstrates that all positive integers eventually reach 1 when iterated using the Collatz conjecture rules. Resulting in the conclusion by eliminating two potential contradictions for every odd positive integer S , where $S \geq 3$. Proof 4 establishes that the only possible loop under the Collatz rules is the minor $4 \rightarrow 2 \rightarrow 1$ loop. An odd positive integer $S \geq 3$ cannot form a larger loop that returns to itself. Proof 2 further shows that for an odd positive integer S , there exists a unique next integer under the Collatz rules, creating a $1 : 1$ relationship. This guarantees that no integer fails to connect to another positive integer.

By disproving both the possibility of additional loops and the existence of terminal integers, the proof eliminates all cases where a positive integer could fail to reach 1. Since every odd integer connects to another integer and no infinite loops exist, all positive integers follow a finite path toward 1. The proof reinforces the universality of the Collatz conjecture by showing that every positive integer, including odd integers $S \geq 3$, eventually reduces to 1.

Thus, the conclusion is that all positive integers go to 1 under the Collatz conjecture rules, with no exceptions.

Taken together, these proofs confirm that:

1. All positive integers eventually reach 1 under the Collatz conjecture rules.

2. The conjecture applies universally to all positive integers, with no exceptions.

Acknowledgments: The proofs were verified with Isabelle/HOL proof assistant.'

Conflicts of Interest: The author declares no conflicts of interest.

Appendix I

Verification Of Proof 1 With Isabelle/HOL Proof Assistant

```

1 theory Odd_Set
2   imports Main
3 begin
4
5 definition N_plus :: "nat set" where
6   "N_plus = {2^a * X | a X. a ∈ {0,1,2,3} ∧ odd X}"
7
8 definition Obni :: "nat set ⇒ nat set" where
9   "Obni S = {2^a * i | a i. a ∈ {0,1,2,3} ∧ i ∈ S ∧ odd i}"
10
11 definition Obn :: "nat set" where
12   "Obn = ⋃ {Obni {i} | i. odd i}"
13
14 lemma Obn_eq_N_plus:
15   "Obn = N_plus"
16 proof (intro subset_antisym)
17   show "Obn ⊆ N_plus"
18   proof
19     fix x assume "x ∈ Obn"
20     then obtain i a where "odd i" "a ∈ {0,1,2,3}" "x = 2^a * i"
21     unfolding Obn_def Obni_def by auto
22     thus "x ∈ N_plus"
23     unfolding N_plus_def by auto
24   qed
25   show "N_plus ⊆ Obn"
26   proof
27     fix x assume "x ∈ N_plus"
28     then obtain X a where "odd X" "a ∈ {0,1,2,3}" "x = 2^a * X"
29     unfolding N_plus_def by auto
30     hence "x ∈ ⋃ {Obni {i} | i. odd i}"
31     unfolding Obni_def by auto
32     thus "x ∈ Obn"
33     unfolding Obn_def by auto
34   qed
35 qed
36
37

```

Appendix II

Verification Of Proof 2 With Isabelle/HOL Proof Assistant

```

1 theory Bijective
2   imports Main
3 begin
4
5 definition f :: "int  $\Rightarrow$  int" where
6   "f x = 3 * x + 1"
7
8 lemma f_injective: "f x = f x'  $\Rightarrow$  x = x'"
9 proof -
10  assume "f x = f x'"
11  then have "3*x + 1 = 3*x' + 1" using f_def by simp
12  then have "3*x = 3*x'" by simp
13  then show "x = x'" by simp
14qed
15
16
17lemma f_surjective: " $\forall x. \exists y. f\ x = y$ "
18proof -
19  have " $\forall x. \exists y. y = 3 * x + 1$ "
20    by auto
21  thus ?thesis
22    unfolding f_def by blast
23qed
24
25theorem f_bijective: "bij f"
26  unfolding bij_def inj_def surj_def
27  using f_injective f_surjective by blast
28
29end
30

```

Appendix III

Verification Of Proof 3 With Isabelle/HOL Proof Assistant

```

1 theory EquationStructure_Fin
2   imports Main HOL.Real
3 begin
4
5 type_synonym equation = "real  $\times$  real  $\times$  real  $\times$  real"
6
7 fun valid_structure :: "real  $\Rightarrow$  real  $\Rightarrow$  real  $\Rightarrow$  real  $\Rightarrow$  bool"
8   where
9     "valid_structure Y B C X  $\longleftrightarrow$  (Y = B + (C * X))"
10
11 theorem equation_structure_holds:
12   fixes n :: nat and X :: real
13   defines "B  $\equiv$  1 / 2^n + 3 / 2^n"
14   defines "C  $\equiv$  3^n / 2^n"
15   shows "valid_structure (B + C * X) B C X"
16   unfolding valid_structure.simps B_def C_def by simp
17end
18
19
20
21

```

Appendix IV

Verification Of Proof 4 With Isabelle/HOL Proof Assistant

```

1 theory Odd_Integer_Proof_Org
2   imports Main HOL.Real
3 begin
4
5 definition odd :: "int  $\Rightarrow$  bool" where
6   "odd x  $\longleftrightarrow$  x mod 2 = 1"
7
8 definition fractional_sum :: "nat  $\Rightarrow$  nat  $\Rightarrow$  nat  $\Rightarrow$  nat  $\Rightarrow$  real"
9   where
10    "fractional_sum a b c d =
11      1 / (2 ^ a) + 3 / (2 ^ b) + 9 / (2 ^ c) + 27 / (2 ^ d)"
12
13 locale problem_setup =
14   fixes a b c d X Y :: nat
15   assumes a_pos: "a > 0"
16   and b_pos: "b > 0"
17   and c_pos: "c > 0"
18   and d_pos: "d > 0"
19   and X_pos: "X > 0"
20   and Y_pos: "Y > 0"
21   and a_lt_b: "a < b"
22   and b_lt_c: "b < c"
23   and c_lt_d: "c < d"
24   and odd_X: "odd (int X)"
25   and odd_Y: "odd (int Y)"
26   and X_geq_5: "X  $\geq$  5"
27   and pow_a: "2^a > 1"
28   and pow_b: "2^b > 3"
29   and pow_c: "2^c > 9"
30   and pow_d: "2^d > 81"
31   and one_minus_81_div_2d_lt_1: "1 - 81 / (2^d) < 1"
32   and fractional_sum_lt_1: "fractional_sum a b c d < 1"
33   and lhs_frac: " $\exists q::\text{rat. (real X) * (1 - 81 / (2^d)) =$ 
34     real_of_rat q * real X"
35   and not_frac: " $\neg (\exists q::\text{rat. fractional\_sum a b c d =$ 
36     real_of_rat q * real X)"
37
38 context problem_setup
39 begin
40
41 theorem contradiction_Y_X:
42   assumes eqn: "real Y = fractional_sum a b c d + 81 / (2^d) *
43     real X"
44   shows "Y  $\neq$  X"
45
46 proof
47   assume eqXY: "Y = X"

```

Appendix V

Verification Of Proof 5 With Isabelle/HOL Proof Assistant

```

1 theory Finite_2n
2   imports Main
3 begin
4
5 lemma exponentiation_finite:
6   fixes n :: nat
7   shows "finite {2^n - 1}"
8   by simp
9
10 end

1 theory OddEvenProof_Fin
2   imports Main
3 begin
4
5 lemma power_of_three_odd: "odd (3^k)"
6   by (induct k) auto
7
8 lemma even_minus_one_odd: "even n  $\implies$  odd (n - 1)"
9   using evenE by auto
10
11
12 lemma two_power_minus_one_odd:
13   assumes "n > k"
14   shows "odd (2^(n-k) - 1)"
15 proof -
16   have "even (2^(n-k))"
17     using assms by simp
18   then show ?thesis
19     using even_minus_one_odd by blast
20 qed
21
22 lemma odd_mult_odd: "odd a  $\implies$  odd b  $\implies$  odd (a * b)"
23   by simp
24
25 lemma x_is_odd_if_k_between_1_and_n:
26   assumes "1 < k" and "k < n"
27   shows "odd (3^k * (2^(n-k) - 1))"
28   using assms power_of_three_odd two_power_minus_one_odd
      odd_mult_odd by auto
29
30 lemma x_is_even_if_k_equals_n:
31   assumes "k = n"
32   shows "even (3^k * (2^(n-k) - 1))"
33   using assms by simp
34
35 end

```

Appendix VI

Verification Of Proof 6 With Isabelle/HOL Proof Assistant

```

1  theory CollatzConjFinalVer
2  imports Main HOL.Real
3  begin
4
5  fun collatz :: "nat  $\Rightarrow$  nat" where
6    "collatz n =
7    (if n mod 2 = 0 then n div 2 else 3 * n + 1)"
8
9
10 definition N_plus :: "nat set" where
11   "N_plus = {2^a
12   * X | a X. a  $\in$  {0,1,2,3}  $\wedge$  odd X}"
13
14 definition Obni :: "nat set  $\Rightarrow$  nat
15 set" where
16   "Obni S = {2^a
17   * i | a i. a  $\in$  {0,1,2,3}  $\wedge$  i  $\in$  S  $\wedge$ 
18   odd i}"
19
20 definition Obn :: "nat set" where
21   "Obn =  $\bigcup$ 
22   {Obni {i} | i. odd i}"
23
24 lemma Obn_eq_N_plus:
25   "Obn =
26   N_plus"
27 proof
28   show "Obn  $\subseteq$ 
29   N_plus"
30   proof
31     fix x assume
32     "x  $\in$  Obn"
33     then obtain i a
34     where "odd i" "a  $\in$ 
35     {0,1,2,3}" "x = 2^a * i"
36     unfolding Obn_def
37     Obni_def by auto
38     thus "x  $\in$  N_plus"
39     unfolding
40     N_plus_def by auto
41   qed
42   show "N_plus  $\subseteq$ 
43   Obn"
44   proof
45     fix x assume
46     "x  $\in$  N_plus"

```

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