

Article

Not peer-reviewed version

Asymptotic Microcausality to Macrocausality: Complete Quantum Electrodynamics and the Regulated S-Matrix

[E. J. Thompson](#) *

Posted Date: 3 February 2026

doi: 10.20944/preprints202602.0207.v1

Keywords: S-matrix; scattering; QED; locality; microcausality



Preprints.org is a free multidisciplinary platform providing preprint service that is dedicated to making early versions of research outputs permanently available and citable. Preprints posted at Preprints.org appear in Web of Science, Crossref, Google Scholar, Scilit, Europe PMC.

Copyright: This open access article is published under a [Creative Commons CC BY 4.0 license](#), which permit the free download, distribution, and reuse, provided that the author and preprint are cited in any reuse.

Disclaimer/Publisher's Note: The statements, opinions, and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions, or products referred to in the content.

Article

Asymptotic Microcausality to Macrocausality: Complete Quantum Electrodynamics and the Regulated S-Matrix

E. J. Thompson

Department of Physics and Astronomy, Trent University, Peterborough, Ontario K9L 0G2, Canada; ethanthompson@trentu.ca

Abstract

We formulate an ultraviolet-complete, entire-function regulated QED in which operational observables form a quasi-local net $\mathcal{A}_F(O)$. The regulator term produces asymptotically-local operators obeying exponential spacelike commutator bounds. We show that this quantitative locality is sufficient for Haag–Ruelle scattering, yielding asymptotic in/out states, an S -matrix, and an LSZ reduction formula with on-shell normalization. Perturbative unitarity and the optical theorem are maintained, while the leading infrared structure of QED is unchanged. Cross sections are expressed as detector click probabilities via quasi-local POVMs, establishing macrocausality in operational terms.

Keywords: S-matrix; scattering; QED; locality; microcausality

1. Introduction

Local relativistic quantum field theory is traditionally organized around exact microcausality, the vanishing of commutators of observables at spacelike separation, and around the existence of asymptotic particle states and an S -matrix describing scattering [1]. In ultraviolet-complete nonlocal or quasi-local formulations, strict microcausality is generically replaced by a quantitative asymptotic notion where commutators do not vanish exactly but are uniformly suppressed outside a fundamental resolution scale. The purpose of this work is to show that this controlled, exponentially small violation of strict microcausality is sufficient to recover the full predictive content of QED at the level of scattering, unitarity, and infrared-safe observables, and to do so in a manner that is asymptotically consistent with the fact that physically realizable detectors couple to quasi-local observables rather than to idealized sharp projectors [2–4].

We work on Minkowski spacetime $\mathbb{R}^{1,3}$ with metric $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, and we denote the d'Alembertian by $\square \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu$. We assume the existence of a Hilbert space $\mathcal{H}$, a vacuum vector $\Omega \in \mathcal{H}$, and a unitary representation of translations $U(a)$, $a \in \mathbb{R}^{1,3}$, with spectrum in the closed forward light cone, as in the standard Wightman framework [5]. The regulated theory is defined by replacing local fields $\Phi(x)$ by quasi-local fields $\Phi^{(F)}(x)$ obtained by acting with an entire function F of $\square/M_*^2$. We will show how asymptotic microcausality implies the existence of scattering states and an S -matrix via an asymptotically-local Haag–Ruelle construction, how LSZ reduction and the optical theorem persist with regulator-dependent modifications that vanish in the IR, and how IR physics in QED is completed by dressed charged states and inclusive observables, all while remaining compatible with a detector-based operational interpretation.

2. Entire-Function Regulation and Quasi-Local Observable Nets

The regulated framework begins by specifying an entire function $F : \mathbb{C} \rightarrow \mathbb{C}$ obeying physical normalization and growth conditions. We interpret F as a covariant UV completion, and we implement it through the functional calculus of $\square$ or, in gauge theories, through a gauge-covariant Laplacian acting on the relevant bundle representation. We first define the regulated scalar case and then indicate the gauge-covariant specialization needed for QED.

Definition 1 (Entire regulator and normalization). Let $M_* > 0$ be a fixed UV scale and let F be an entire function such that

$$F(0) = 1, \quad \overline{F(z)} = F(\bar{z}), \quad (1)$$

and such that F has sufficiently rapid decay along the Euclidean axis to render loop integrals finite, for example $F(-p_E^2/M_*^2)$ decays faster than any power as $p_E^2 \rightarrow +\infty$. We call such F an admissible regulator.

To define regulated fields, we use convolution with a kernel determined by $F(\square/M_*^2)$. Let Φ be an operator-valued distribution on $\mathbb{R}^{1,3}$. Formally:

$$\Phi^{(F)}(x) \equiv (F(\square/M_*^2)\Phi)(x). \quad (2)$$

If F has a Fourier transform representation, then $F(\square/M_*^2)$ becomes multiplication by $F(-p^2/M_*^2)$ in momentum space. Writing $\tilde{\Phi}(p)$ for the Fourier transform distribution, we have:

$$\tilde{\Phi}^{(F)}(p) = F(-p^2/M_*^2) \tilde{\Phi}(p). \quad (3)$$

In position space we can write:

$$\Phi^{(F)}(x) = \int_{\mathbb{R}^{1,3}} d^4y K_F(x-y) \Phi(y), \quad (4)$$

where the kernel K_F is the distribution defined by:

$$K_F(x) \equiv \int_{\mathbb{R}^{1,3}} \frac{d^4p}{(2\pi)^4} e^{-ip \cdot x} F(-p^2/M_*^2), \quad (5)$$

with $p \cdot x \equiv \eta_{\mu\nu} p^\mu x^\nu$.

The operational content of the theory is captured by a net of algebras generated by regulated smearings.

Definition 2 (Regulated smearing and quasi-local net). Let $\mathcal{S}(\mathbb{R}^{1,3})$ denote the Schwartz space and let $f \in \mathcal{S}(\mathbb{R}^{1,3})$. Define the regulated smeared operator

$$\Phi^{(F)}(f) \equiv \int_{\mathbb{R}^{1,3}} d^4x f(x) \Phi^{(F)}(x). \quad (6)$$

For an open bounded region $O \subset \mathbb{R}^{1,3}$, define the regulated algebra

$$\mathcal{A}_F(O) \equiv \text{vN}\{ \Phi^{(F)}(f) : f \in \mathcal{S}(\mathbb{R}^{1,3}), \text{supp}(f) \subset O \}, \quad (7)$$

where $\text{vN}\{\dots\}$ denotes the von Neumann algebra generated by the indicated set.

In QED, gauge covariance requires that F be applied in a way compatible with gauge symmetry. A convenient formulation is background-field or BRST-invariant regularization where F acts on gauge-covariant kinetic operators. For the present scattering-focused work, we only require that $F(0) = 1$ so the soft sector is unchanged, and that F introduces no additional poles, so analytic structure is preserved. We will return to gauge-specific IR subtleties.

3. Asymptotic Microcausality and Asymptotic-Locality

Exact microcausality would assert $[\Phi(f), \Psi(g)] = 0$ whenever $\text{supp}(f)$ and $\text{supp}(g)$ are spacelike separated. In the regulated theory, the smearing kernel K_F has tails extending beyond any bounded region, so commutators are not exactly zero. The fundamental structural replacement is an exponential or faster bound on commutators at spacelike separation.

We write the spacelike separation distance between sets $X, Y \subset \mathbb{R}^{1,3}$ as:

$$\rho(X, Y) \equiv \inf \{ \sqrt{-(x-y)^2} : x \in X, y \in Y, (x-y)^2 < 0 \}, \quad (8)$$

where $(x-y)^2 \equiv \eta_{\mu\nu}(x-y)^\mu(x-y)^\nu$.

Definition 3 (Asymptotic-local operators). *An operator A is called asymptotically local if there exists a family of strictly local operators $\{A_R\}_{R>0}$, with $A_R \in \mathcal{A}(O_R)$ for some bounded region O_R of diameter $\mathcal{O}(R)$, such that for every $N \in \mathbb{N}$ there is a constant C_N with*

$$\|A - A_R\| \leq C_N R^{-N} \quad \text{as } R \rightarrow +\infty. \quad (9)$$

Regulated observables of the form $\Phi^{(F)}(f)$ are asymptotically local under mild conditions on F and f , because K_F decays rapidly outside a scale ℓ_* . The quantitative causal statement is the commutator bound below.

Theorem 1 (Asymptotic microcausality bound). *Let $\Phi^{(F)}(f)$ and $\Psi^{(F)}(g)$ be regulated smeared observables with test functions $f, g \in \mathcal{S}(\mathbb{R}^{1,3})$ and spacelike separated supports $X = \text{supp}(f)$ and $Y = \text{supp}(g)$. Assume F is admissible and f, g are fixed. Then there exist constants $\alpha > 0$ and $C_N > 0$ (depending on F, f, g and N) such that for all $N \in \mathbb{N}$,*

$$\left\| [\Phi^{(F)}(f), \Psi^{(F)}(g)] \right\| \leq C_N (1 + M_* \rho(X, Y))^{-N} e^{-\alpha M_* \rho(X, Y)}. \quad (10)$$

Using the kernel representation (4)–(6):

$$\Phi^{(F)}(f) = \int d^4x d^4y f(x) K_F(x-y) \Phi(y), \quad (11)$$

$$\Psi^{(F)}(g) = \int d^4x' d^4y' g(x') K_F(x'-y') \Psi(y'). \quad (12)$$

Therefore:

$$[\Phi^{(F)}(f), \Psi^{(F)}(g)] = \int d^4y d^4y' f_F(y) g_F(y') [\Phi(y), \Psi(y')], \quad (13)$$

where:

$$f_F(y) \equiv \int d^4x f(x) K_F(x-y), \quad (14)$$

$$g_F(y') \equiv \int d^4x' g(x') K_F(x'-y'). \quad (15)$$

If Φ, Ψ are local fields, then $[\Phi(y), \Psi(y')] = 0$ for spacelike separated $y - y'$. The only contributions come from the subset of (y, y') that are not spacelike separated, which requires y to lie within the nonlocal tails of f_F outside X and y' within the tails of g_F outside Y . For admissible entire F , the kernel K_F decays at least exponentially outside the fundamental scale ℓ_* , implying the same for f_F and g_F away from X and Y . Estimating (13) in operator norm by bounding the kernels and using rapid decrease of Schwartz functions yields (10) with constants C_N and some $\alpha > 0$ determined by the decay of K_F .

The estimate (10) replaces the binary axiom of microcausality by a quantitative, scale-controlled statement [2–4]. For $\rho(X, Y) \gg \ell_*$, spacelike commutators are exponentially small. This is precisely the level of locality required by asymptotic detector models, because no physical detector can realize perfectly sharp bounded-region projectors.

4. Detector Couplings and Asymptotic Macrocausality

Macrocausality is the statement that observable influences do not propagate outside the light cone at the level of experimentally realizable operations. In an operational formulation, a measurement in a region O is described by a completely positive (CP) instrument built from operators in $\mathcal{A}_F(O)$. This viewpoint makes the link between microcausality bounds and measurable causal constraints direct.

Let $\mathcal{H}_S$ denote the system Hilbert space, the QED field degrees of freedom and $\mathcal{H}_D$ the detector Hilbert space. A detector interaction localized by a real switching function $\chi \in C_c^\infty(\mathbb{R}^{1,3})$ with $\text{supp}(\chi) \subset O$ is modeled by an interaction Hamiltonian density:

$$\mathcal{H}_{\text{int}}(x) = g \chi(x) O^{(F)}(x) \otimes P, \quad (16)$$

where $g \in \mathbb{R}$ is a coupling constant, $O^{(F)}(x)$ is a regulated system observable, and P is a bounded self-adjoint operator acting on $\mathcal{H}_D$, for example a pointer momentum. The corresponding evolution operator in the interaction picture is:

$$U \equiv T \exp\left(-i \int_{\mathbb{R}^{1,3}} d^4x \mathcal{H}_{\text{int}}(x)\right), \quad (17)$$

where T denotes time ordering.

A measurement outcome is represented by a detector POVM element $\Pi \geq 0$ on $\mathcal{H}_D$. The induced system effect is then:

$$E \equiv \text{Tr}_D\left[(\mathbb{1} \otimes \rho_D) U^\dagger (\mathbb{1} \otimes \Pi) U\right], \quad (18)$$

where ρ_D is the initial detector state and Tr_D is the partial trace over detector degrees of freedom. By construction, E is positive and belongs to the ultraweak closure generated by $O^{(F)}(\chi)$, hence $E \in \mathcal{A}_F(O)$.

Proposition 1 (Exponential suppression of spacelike detector influence). *Let O_1, O_2 be bounded spacelike separated regions and let $E_1 \in \mathcal{A}_F(O_1)$ and $E_2 \in \mathcal{A}_F(O_2)$ be effects induced by detector couplings of the form (17). Then there exist constants $\alpha > 0$ and $C_N > 0$ such that*

$$\|[E_1, E_2]\| \leq C_N (1 + M_* \rho(O_1, O_2))^{-N} e^{-\alpha M_* \rho(O_1, O_2)}. \quad (19)$$

Each effect E_i lies in $\mathcal{A}_F(O_i)$ and can be approximated in norm by finite polynomials in regulated smeared observables with support contained in O_i . The commutator of such polynomials expands into sums of commutators of regulated observables, each bounded by Theorem 1. The bound (19) follows by multilinearity and norm estimates.

Proposition 1 is the asymptotic macrocausal statement that the influence between spacelike separated detector regions is uniformly bounded by the same exponential suppression that controls asymptotic microcausality. The next task is to prove that this structure is sufficient to build the asymptotic scattering theory.

5. Haag–Ruelle Scattering for Asymptotically-Local Fields

Haag–Ruelle theory constructs asymptotic multiparticle states from local operators by exploiting locality and the spectral condition [6,7]. In our setting, locality is replaced by asymptotically-locality with the commutator bound (10) [8]. The construction survives because the bound is integrable in time once wavepackets are separated by relativistic propagation, and the error terms vanish as $t \rightarrow \pm\infty$.

Let A be an asymptotically-local operator that creates a one-particle state from the vacuum with mass $m > 0$. More precisely, assume the spectral projection E_m onto the isolated one-particle mass

shell exists and that $E_m A\Omega \neq 0$. Let $\tilde{f}(\mathbf{p}) \in \mathcal{S}(\mathbb{R}^3)$ be a momentum wavepacket supported away from $\mathbf{p} = 0$ for simplicity. Define a positive-energy solution of the Klein–Gordon equation:

$$f_t(\mathbf{x}) \equiv \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\mathbf{p})}} \tilde{f}(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x} - i\omega(\mathbf{p})t}, \quad (20)$$

where $\omega(\mathbf{p}) \equiv \sqrt{\mathbf{p}^2 + m^2}$. Define the smeared time-dependent creation operator:

$$A_t(f) \equiv \int_{\mathbb{R}^3} d^3\mathbf{x} f_t(\mathbf{x}) A(t, \mathbf{x}), \quad (21)$$

where $A(t, \mathbf{x}) \equiv U(t, \mathbf{x}) A U(t, \mathbf{x})^\dagger$ denotes the translated operator by the unitary representation of spacetime translations.

The key step in Haag–Ruelle is that products of such operators acting on Ω converge as $t \rightarrow \pm\infty$, provided their associated wavepackets have disjoint velocity supports. Velocity support is defined by:

$$\mathbf{v}(\mathbf{p}) \equiv \nabla_{\mathbf{p}} \omega(\mathbf{p}) = \frac{\mathbf{p}}{\omega(\mathbf{p})}, \quad (22)$$

and disjointness means the sets $\{\mathbf{v}(\mathbf{p}) : \mathbf{p} \in \text{supp}(\tilde{f}_i)\}$ are pairwise disjoint.

Theorem 2 (Existence of scattering states for asymptotically-local creators). *Let $A^{(1)}, \dots, A^{(n)}$ be asymptotically-local operators creating isolated massive one-particle states from Ω , and let $f_1, \dots, f_n$ be wavepackets with pairwise disjoint velocity supports. Define*

$$\Psi_t \equiv A_t^{(1)}(f_1) \cdots A_t^{(n)}(f_n) \Omega. \quad (23)$$

Assume asymptotic microcausality (10). Then the limits

$$\Psi^{\text{out}} \equiv \lim_{t \rightarrow +\infty} \Psi_t, \quad \Psi^{\text{in}} \equiv \lim_{t \rightarrow -\infty} \Psi_t \quad (24)$$

exist in $\mathcal{H}$, and define asymptotic scattering states. Moreover, the dependence on F enters only through exponentially small error terms controlled by M_ .*

We prove Cauchy convergence as $t \rightarrow +\infty$; the $t \rightarrow -\infty$ case is analogous. It suffices to show that $\|\Psi_{t_2} - \Psi_{t_1}\| \rightarrow 0$ as $t_1, t_2 \rightarrow +\infty$. Using the fundamental theorem of calculus:

$$\Psi_{t_2} - \Psi_{t_1} = \int_{t_1}^{t_2} dt \frac{d}{dt} \Psi_t. \quad (25)$$

Differentiating (23) produces a sum of terms where $\dot{A}_t^{(k)}(f_k)$ appears at position k . Each derivative can be expressed as a commutator with the Hamiltonian, generator of time translations plus explicit time-dependence from f_t . The standard Haag–Ruelle argument shows that $\dot{A}_t^{(k)}(f_k)$ is a spatial integral of an asymptotically-local operator weighted by a wavepacket whose support moves with velocities in the support of f_k . Because the velocity supports are disjoint, at large t the effective supports of different factors become spacelike separated with separation growing linearly in t . The commutator bound (10) then yields an integrable estimate:

$$\left\| \frac{d}{dt} \Psi_t \right\| \leq C_N (1 + M_* t)^{-N} e^{-\alpha M_* t}, \quad (26)$$

for suitable constants and sufficiently large t . Inserting (26) into (25) gives:

$$\|\Psi_{t_2} - \Psi_{t_1}\| \leq \int_{t_1}^{t_2} dt C_N (1 + M_* t)^{-N} e^{-\alpha M_* t} \rightarrow 0, \quad (27)$$

as $t_1, t_2 \rightarrow +\infty$, which proves convergence and thus existence of Ψ^{out} .

Theorem 2 implies the existence of Møller operators and an S -matrix on the scattering subspace. In particular, defining $\Omega^{\text{in/out}}$ as the vacuum and extending by linearity and completion yields an isometry from the incoming Fock space of asymptotic particles to the interacting Hilbert space.

6. LSZ Reduction with an Entire-Function Regulator

The LSZ reduction formula relates scattering amplitudes to time-ordered correlation functions by isolating the one-particle poles of external legs [5]. In a regulated theory, fields are modified by $\Phi \mapsto \Phi^{(F)} = F(\square/M_*^2)\Phi$. The essential analytic point is that an entire function F introduces no new poles and can be normalized to unity on shell, so the pole structure required by LSZ persists.

We consider a scalar interpolating field $\phi(x)$ for conceptual clarity and later interpret ϕ as a component field in QED, for example, the electron field and the gauge field in covariant gauges. Let G_n be the time-ordered n -point function:

$$G_n(x_1, \dots, x_n) \equiv \langle \Omega | T \phi(x_1) \cdots \phi(x_n) | \Omega \rangle. \quad (28)$$

Define its Fourier transform by:

$$\tilde{G}_n(p_1, \dots, p_n) \equiv \int \prod_{j=1}^n d^4 x_j e^{i \sum_{j=1}^n p_j \cdot x_j} G_n(x_1, \dots, x_n), \quad (29)$$

with momentum conservation $\sum_j p_j = 0$ implicit.

In the regulated theory we use $\phi^{(F)}(x) = F(\square/M_*^2)\phi(x)$. In momentum space this gives:

$$\tilde{\phi}^{(F)}(p) = F(-p^2/M_*^2) \tilde{\phi}(p). \quad (30)$$

Therefore the regulated n -point function satisfies:

$$\tilde{G}_n^{(F)}(p_1, \dots, p_n) = \left(\prod_{j=1}^n F(-p_j^2/M_*^2) \right) \tilde{G}_n(p_1, \dots, p_n). \quad (31)$$

The LSZ amplitude for a $2 \rightarrow 2$ process is obtained by amputating the external propagators and taking on-shell limits. Let $m > 0$ and define the renormalized one-particle pole residue $Z > 0$ by the exact two-point function behavior:

$$\tilde{G}_2(p, -p) = \frac{iZ}{p^2 - m^2 + i0} + \text{regular terms at } p^2 = m^2. \quad (32)$$

The corresponding regulated two-point function is:

$$\tilde{G}_2^{(F)}(p, -p) = \frac{iZ F(-p^2/M_*^2)^2}{p^2 - m^2 + i0} + \text{regular}. \quad (33)$$

To prevent an artificial rescaling of external one-particle states, we impose the on-shell normalization condition:

$$F(-m^2/M_*^2) = 1. \quad (34)$$

With (34), the residue at the physical pole remains Z .

Theorem 3 (LSZ reduction with entire regulation). *Assume an isolated one-particle pole of mass $m > 0$ with residue $Z > 0$ and impose $F(-m^2/M_*^2) = 1$. Then the regulated scattering amplitude for $2 \rightarrow 2$ scalar scattering obeys*

$$\begin{aligned} \mathcal{M}_F(p_1, p_2 \rightarrow p_3, p_4) &= \left(\prod_{j=1}^4 Z^{-1/2} \right) \\ &\times \lim_{\substack{p_j^2 \rightarrow m^2 \\ j=1, \dots, 4}} \left(\prod_{j=1}^4 (p_j^2 - m^2) \right) \times \tilde{G}_4^{(F)}(p_1, p_2, p_3, p_4), \end{aligned} \quad (35)$$

where $\tilde{G}_4^{(F)}$ is the Fourier transform of the regulated time-ordered 4-point function. The dependence on F enters through internal momentum dependence in $\tilde{G}_4^{(F)}$ and through off-shell external leg factors that vanish on shell by (34).

The proof follows the standard LSZ derivation, with the only modification being the replacement $\phi \mapsto \phi^{(F)}$. The key step is that the amputating operator $(\square + m^2)$ acting on an external leg corresponds in momentum space to multiplication by $(p^2 - m^2)$, while the entire factor $F(-p^2/M_*^2)$ is analytic and introduces no new poles. The on-shell normalization (34) ensures that residues match the unregulated theory. Taking limits yields (35).

For massless external legs, such as photons in QED, the appropriate normalization is $F(0) = 1$, already included in (1). In that case external soft factors are unchanged at leading order, which is essential for the IR analysis in Sec. 8.

7. Perturbative Unitarity and the Optical Theorem

A complete scattering framework requires unitarity of the S -matrix. In perturbation theory, unitarity is encoded by Cutkosky cutting rules and by the optical theorem relating the imaginary part of a forward amplitude to a sum over physical intermediate states [13]. The entire-function regulator preserves these structures because it modifies propagators and vertices by analytic form factors but introduces no additional singularities.

We write the S -matrix as $S = \mathbb{1} + iT$, where T is the transition operator. Unitarity $S^\dagger S = \mathbb{1}$ implies:

$$i(T^\dagger - T) = T^\dagger T. \quad (36)$$

Taking matrix elements between asymptotic states $|i\rangle, |f\rangle$ gives:

$$2 \operatorname{Im} \langle f | T | i \rangle = \sum_n \langle f | T^\dagger | n \rangle \langle n | T | i \rangle, \quad (37)$$

where $\{|n\rangle\}$ is a complete orthonormal basis of physical intermediate states in the scattering subspace. Equation (37) is the optical theorem.

In the regulated theory, Feynman rules are modified schematically by inserting factors $F(-p^2/M_*^2)$ in propagators and/or vertices. A representative scalar propagator becomes:

$$\Delta_F(p) \equiv \frac{i F(-p^2/M_*^2)^2}{p^2 - m^2 + i0}, \quad (38)$$

while gauge and spinor propagators in QED acquire analogous form factors consistent with gauge symmetry. The crucial analytic property is:

Lemma 1 (No additional poles from an entire regulator). *Let F be entire. Then the regulated propagator (38) has the same pole locations as the unregulated propagator, namely at $p^2 = m^2$, and introduces no new poles in the finite complex p^0 plane at fixed $\mathbf{p}$.*

The denominator of (38) is unchanged. The factor $F(-p^2/M_*^2)^2$ is entire in p^2 and thus analytic everywhere in the finite complex plane, hence it cannot generate additional poles.

Theorem 4 (Cutkosky rules and perturbative unitarity). *Assume the regulated perturbation theory is defined with the standard $i0$ prescription and an admissible entire regulator. Then the discontinuity of a regulated amplitude across a physical branch cut is obtained by the usual Cutkosky cutting procedure, with the integrand multiplied by the corresponding regulator factors evaluated on the cut momenta. Consequently, the optical theorem (37) holds order-by-order in the regulated perturbation theory.*

The proof follows the standard contour deformation argument: discontinuities arise when internal propagator poles pinch the integration contour. By Lemma 1, pinching singularities occur at the same locations as in the unregulated theory. The regulator factors are analytic and can be pulled through the contour argument, contributing only multiplicative weights on the cut. The resulting cut integral matches the phase-space sum in (37).

8. Infrared Completion of QED: Dressing and Inclusive Observables

Ultraviolet finiteness does not eliminate the infrared subtleties of QED: charged asymptotic states are not sharp mass-shell particles but infraparticles accompanied by clouds of soft photons. The regulated framework preserves the soft sector because $F(0) = 1$ and because $F(-k^2/M_*^2) = 1 + \mathcal{O}(k^2/M_*^2)$ for soft photon momentum k . Therefore the correct completion of QED remains a dressed-state or inclusive-observable formulation, and we show that this completion is compatible with the quasi-local observable net.

We write the QED Lagrangian in covariant gauge as:

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\cancel{\partial} - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2, \quad (39)$$

where A_μ is the gauge field, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength, ψ is the Dirac field of mass $m > 0$, $e \in \mathbb{R}$ is the electric charge, and $\xi > 0$ is the gauge-fixing parameter. We denote $\cancel{\partial} \equiv \gamma^\mu \partial_\mu$ with $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}$.

The universal soft theorem states that, for emission of a soft photon of momentum k^μ and polarization $\varepsilon^\mu(k)$ with $k^2 = 0$ and $k^0 > 0$, an amplitude factorizes as:

$$\mathcal{M}_{n+1}(k; \{p_i\}) = \left[e \sum_{i \in \text{charged}} \eta_i \frac{p_i \cdot \varepsilon(k)}{p_i \cdot k} \right] \mathcal{M}_n(\{p_i\}) + \mathcal{O}(k^0), \quad (40)$$

where $\eta_i = +1$ for outgoing charges and $\eta_i = -1$ for incoming charges, and $\{p_i\}$ denote the hard external momenta. The $1/(p_i \cdot k)$ singularity yields infrared divergences upon phase space integration of k .

In the regulated theory, the photon field is modified by $A_\mu \mapsto A_\mu^{(F)} = F(\square/M_*^2)A_\mu$. In momentum space, soft photons satisfy $k^2 = 0$ and thus $F(-k^2/M_*^2) = F(0) = 1$. Therefore the leading soft factor (40) is unchanged. This observation implies:

Proposition 2 (Soft sector stability under entire regulation). *Let F be admissible with $F(0) = 1$. Then the leading soft photon factorization formula (40) holds in the regulated theory with the same universal soft factor, and regulator corrections are at most $\mathcal{O}(k^2/M_*^2)$.*

The IR-complete scattering description of QED requires either inclusive cross sections, Bloch–Nordsieck and KLN, or dressed asymptotic states (Kulish–Faddeev) [9,10,10–12]. We present the dressed-state formulation, which is particularly natural asymptotically because detectors have finite resolution and cannot distinguish configurations differing by arbitrarily soft radiation.

Let $a_\lambda(\mathbf{k})$ and $a_\lambda^\dagger(\mathbf{k})$ be photon annihilation and creation operators for polarization λ , satisfying:

$$[a_\lambda(\mathbf{k}), a_{\lambda'}^\dagger(\mathbf{k}')] = (2\pi)^3 2|\mathbf{k}| \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{k} - \mathbf{k}'). \quad (41)$$

Define a dressing operator for a charged particle of momentum p^μ by:

$$W(p) \equiv \exp \left\{ -e \sum_\lambda \int_{\mathbb{R}^3} \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|} \left[\frac{p \cdot \varepsilon_\lambda(\mathbf{k})}{p \cdot k} a_\lambda^\dagger(\mathbf{k}) - \frac{p \cdot \varepsilon_\lambda(\mathbf{k})}{p \cdot k} a_\lambda(\mathbf{k}) \right] \right\}, \quad (42)$$

where $k^\mu = (|\mathbf{k}|, \mathbf{k})$ and $\varepsilon_\lambda^\mu(\mathbf{k})$ are polarization vectors satisfying $k \cdot \varepsilon_\lambda = 0$. The dressed one-electron state is:

$$|p\rangle_{\text{dress}} \equiv W(p) |p\rangle \otimes |0_\gamma\rangle, \quad (43)$$

where $|p\rangle$ is the bare one-electron momentum eigenstate and $|0_\gamma\rangle$ is the photon vacuum.

Theorem 5 (IR finiteness of dressed scattering). *In the dressed-state formulation (43), the infrared divergences in exclusive QED amplitudes are cancelled between virtual soft loops and the coherent dressing factors, yielding finite transition probabilities between dressed asymptotic charged states. In the regulated theory with $F(0) = 1$, this cancellation persists, and UV finiteness is simultaneously guaranteed by the entire-function damping at large Euclidean momenta.*

The standard Kulish–Faddeev argument shows that the dressing operator exponentiates precisely the universal soft factors that appear in the virtual corrections, producing cancellation of IR singularities in probabilities. Since the regulator does not alter the soft limit by Proposition 2, the IR exponentiation and cancellation mechanism is unchanged. UV finiteness follows from admissibility of F by construction.

9. Asymptotic Cross Sections from Detector Click Probabilities

A complete scattering theory must connect asymptotic transition amplitudes to experimentally measurable rates. In this framework, the measurable objects are click probabilities for quasi-local detectors and their inclusive statistics under finite resolution. The detector model of Sec. IV yields such probabilities as expectation values of effects $E \in \mathcal{A}_F(O)$.

Let ρ_{in} be an incoming asymptotic density operator prepared at early times. Let $E_{\text{click}} \in \mathcal{A}_F(O)$ be the system effect corresponding to a click outcome of a detector localized in region O . The click probability is:

$$P_{\text{click}} = \text{Tr}_{\mathcal{H}}(\rho_{\text{in}} E_{\text{click}}). \quad (44)$$

If $\rho_{\text{in}} = |\Psi^{\text{in}}\rangle \langle \Psi^{\text{in}}|$ is a pure incoming scattering state and the detector is placed far in the asymptotic region, P_{click} is determined by the outgoing particle content of $|\Psi^{\text{out}}\rangle = S|\Psi^{\text{in}}\rangle$. Operationally, detectors with finite resolution are insensitive to arbitrarily soft photons, which is precisely the condition under which inclusive observables are IR finite.

We formalize this by introducing a detector resolution scale $\Delta E > 0$ and defining the inclusive effect:

$$E_{\text{incl}}(\Delta E) \equiv \sum_{\alpha: E_\alpha < \Delta E} E_\alpha, \quad (45)$$

where $\{E_\alpha\}$ is a POVM decomposition of outcomes carrying energy below the resolution threshold. The inclusive click probability is:

$$P_{\text{incl}}(\Delta E) = \text{Tr}_{\mathcal{H}}(\rho_{\text{in}} E_{\text{incl}}(\Delta E)), \quad (46)$$

which is finite and regulator-independent at leading order in the IR because the soft sector is unchanged and the regulator suppresses only UV momentum regions.

Proposition 3 (IR safety as asymptotic finite resolution). *Let $\Delta E > 0$ be fixed. Then inclusive detector probabilities $P_{\text{incl}}(\Delta E)$ computed in the regulated theory are infrared finite and coincide with standard QED predictions up to corrections of order $\mathcal{O}(\Delta E^2/M_*^2)$ and exponentially small quasi-locality corrections controlled by $M_*\rho$.*

Infrared finiteness follows from the KLN/Bloch–Nordsieck mechanism when summing over soft radiation below detector resolution. The regulator modifies the integrand only through factors $F(-k^2/M_*^2)$ which equal $1 + \mathcal{O}(k^2/M_*^2)$ in the soft region. The quasi-locality corrections arise from replacing idealized sharp localization by effects in $\mathcal{A}_F(O)$ and are bounded by Proposition 1.

10. Macrocausality as a Consequence of Asymptotic Microcausality

We now state explicitly the emergent macrocausality principle implied by the regulated net and the commutator bound. In standard axiomatic QFT, macrocausality in scattering is encoded in analyticity and cluster decomposition properties derived from strict microcausality. In the regulated framework, these properties hold up to exponentially small corrections.

Let O_1, O_2 be spacelike separated regions and $E_1 \in \mathcal{A}_F(O_1), E_2 \in \mathcal{A}_F(O_2)$ be effects corresponding to detector outcomes. Define the correlation function:

$$C(E_1, E_2) \equiv \langle \Omega | E_1 E_2 | \Omega \rangle - \langle \Omega | E_1 | \Omega \rangle \langle \Omega | E_2 | \Omega \rangle. \quad (47)$$

Then:

$$|C(E_1, E_2)| \leq \|[E_1, E_2]\|, \quad (48)$$

because for bounded operators we can bound connected correlations by commutator norms after inserting and subtracting products of expectations. Combining with (19) yields exponentially suppressed spacelike correlations. This is the asymptotic content of macrocausality: correlations cannot be used to transmit signals outside the light cone except at exponentially small levels controlled by the UV scale.

Theorem 6 (asymptotic macrocausality bound). *Let $E_1 \in \mathcal{A}_F(O_1)$ and $E_2 \in \mathcal{A}_F(O_2)$ be bounded effects in spacelike separated regions. Then for all $N \in \mathbb{N}$,*

$$|C(E_1, E_2)| \leq C_N (1 + M_*\rho(O_1, O_2))^{-N} e^{-\alpha M_*\rho(O_1, O_2)}. \quad (49)$$

The proof for this is trivial, combine (48) with Proposition 1.

11. Ultraviolet Finiteness at Arbitrary Loop Order

This section records the precise sense in which the entire-function UV completion renders every fixed-order virtual correction finite [4]. For any connected Feynman graph with L independent loop momenta, the corresponding L -loop amplitude admits a Wick-rotated Euclidean representation whose integral is absolutely convergent. In particular, all ultraviolet divergences and UV subdivergences are absent order-by-order in perturbation theory.

Let G be a connected Feynman graph contributing to a scattering amplitude in the regulated theory. We write:

$$V(G) = \text{vertex set}, \quad (50)$$

$$I(G) = \text{internal line set}, \quad (51)$$

$$E(G) = \text{external line set}. \quad (52)$$

Here $|V(G)|$ is the number of vertices, $|I(G)|$ is the number of internal propagators, and $|E(G)|$ is the number of external legs. Let L denote the loop number of G :

$$L \equiv |I(G)| - |V(G)| + 1.$$

Here L is the dimension of the cycle space of a connected graph. We choose a basis of independent loop momenta $k = (k_1, \dots, k_L)$ with each:

$$k_\ell \in \mathbb{R}^4, \quad \ell = 1, \dots, L,$$

and denote the collection of external momenta by $p = (p_1, \dots, p_{|E(G)|})$. In the Euclidean representation we use the Euclidean inner product $(\cdot, \cdot)_E$ and Euclidean norm:

$$|q|_E^2 \equiv (q, q)_E = \sum_{\mu=1}^4 q_\mu^2,$$

for any $q \in \mathbb{R}^4$. For the full loop vector $k \in \mathbb{R}^{4L}$ we set:

$$|k|_E^2 \equiv \sum_{\ell=1}^L |k_\ell|_E^2.$$

For each internal line $r \in I(G)$, let $\ell_r(k, p)$ denote the Euclidean four-momentum flowing through that line under a fixed momentum routing. Then ℓ_r is an affine-linear function of loop and external momenta:

$$\ell_r(k, p) = \sum_{\ell=1}^L A_{r\ell} k_\ell + q_r(p), \quad (53)$$

where $A_{r\ell} \in \{0, \pm 1\}$ encodes how loop ℓ flows through line r , and $q_r(p)$ is a fixed linear combination of external momenta determined by the routing. In (53), $r = 1, \dots, |I(G)|$ indexes internal lines and $\ell = 1, \dots, L$ indexes the independent loops.

The regulated Euclidean propagators carry a multiplicative entire-function form factor. For definiteness, for a scalar internal line of mass $m_r \geq 0$ we write the Euclidean propagator factor in the integrand as:

$$\Delta_{F,r}^{(E)}(\ell_r) \equiv \frac{F\left(\frac{|\ell_r|_E^2}{M_*^2}\right)^{\sigma_r}}{|\ell_r|_E^2 + m_r^2}, \quad (54)$$

where $M_* > 0$ is the UV scale, $\sigma_r \in \{1, 2\}$ is a model-dependent integer power (e.g. arising from where the regulator is inserted), and $F : \mathbb{C} \rightarrow \mathbb{C}$ is an entire function with $F(0) = 1$.

In gauge and spinor theories, numerator algebra and vertex factors appear. To cover all cases in a single statement, we assume:

After Wick rotation, the Euclidean integrand for a fixed graph G can be written as:

$$\mathcal{I}_G^{(E)}(k, p) = \mathcal{N}_G(k, p) \prod_{r \in I(G)} \Delta_{F,r}^{(E)}(\ell_r(k, p)), \quad (55)$$

where $\mathcal{N}_G(k, p)$ is a rational function of the loop momenta whose absolute value is bounded by a polynomial in $|k|_E$ at fixed external momenta p .

In (55), $\mathcal{N}_G$ captures all numerator factors from spin, gauge fixing, and derivative couplings, as well as any polynomial momentum dependence from vertices, while the product runs over internal lines.

There exist constants $c > 0$, $R \geq 0$, and $C_F \geq 1$ such that for all Euclidean four-momenta $q \in \mathbb{R}^4$ with $|q|_E^2 \geq R$:

$$\left| F\left(\frac{|q|_E^2}{M_*^2}\right) \right| \leq C_F \exp\left(-c \frac{|q|_E^2}{M_*^2}\right). \quad (56)$$

In (56), c sets the strength of the UV suppression and R indicates that the bound is required only for sufficiently large Euclidean momentum.

Assumption B is stronger than and implies the qualitative requirement that $F(|q|_E^2/M_*^2)$ decays faster than any power as $|q|_E \rightarrow \infty$, and it is satisfied by standard entire UV suppressors, Gaussian-type or stronger.

We now isolate the two estimates that yield absolute convergence.

Lemma 2 (Polynomial bound for the rational prefactor). *Fix external momenta p in a compact set or equivalently, fix p . Under Assumption A, there exist constants $C_{G,p} > 0$ and an integer $N_G \geq 0$ such that for all $k \in \mathbb{R}^{4L}$,*

$$|\mathcal{N}_G(k, p)| \leq C_{G,p} (1 + |k|_E)^{N_G}. \quad (57)$$

In (57), $|k|_E$ is the Euclidean norm on $\mathbb{R}^{4L}$ defined above.

By Assumption A, $\mathcal{N}_G(k, p)$ is a rational function of k whose numerator and denominator are polynomials in the components of k with coefficients depending continuously on p . For fixed p , a rational function can grow at most polynomially as $|k|_E \rightarrow \infty$ unless it has an essential singularity at infinity, which rational functions do not. Thus there exist $C_{G,p}$ and N_G such that (57) holds.

Lemma 3 (Coercivity of internal momenta in terms of loop momenta). *Let G be connected and let the internal line momenta be parameterized by (53). Then there exist constants $a_G > 0$ and $b_{G,p} \geq 0$ such that for all $k \in \mathbb{R}^{4L}$,*

$$\sum_{r \in I(G)} |\ell_r(k, p)|_E^2 \geq a_G |k|_E^2 - b_{G,p}. \quad (58)$$

In (58), a_G depends only on the choice of loop basis (equivalently the matrix A), while $b_{G,p}$ depends on the external momenta through the shifts $q_r(p)$.

Define the linear map $T : \mathbb{R}^{4L} \rightarrow \mathbb{R}^{4|I(G)|}$ by:

$$T(k) \equiv \left(\sum_{\ell} A_{r\ell} k_{\ell} \right)_{r \in I(G)}.$$

Here T is the block-extension of the integer matrix $A = (A_{r\ell})$ to four-vectors; equivalently $T = (A \otimes \mathbf{1}_4)$ acting on $k \in \mathbb{R}^{4L}$. The affine momentum routing (53) may be written as:

$$(\ell_r(k, p))_{r \in I(G)} = T(k) + q(p),$$

where $q(p) \in \mathbb{R}^{4|I(G)|}$ collects the shifts $(q_r(p))_{r \in I(G)}$.

Because G is connected and $k_1, \dots, k_L$ form an independent loop basis, the matrix A has full column rank L . Hence $A^T A$ is positive definite on $\mathbb{R}^L$, so there exists $\lambda_{\min}(A^T A) > 0$ such that

$$|T(k)|_E^2 \geq \lambda_{\min}(A^T A) |k|_E^2.$$

Here $|T(k)|_E^2$ denotes the Euclidean norm on $\mathbb{R}^{4|I(G)|}$ induced by summing the Euclidean norms of each four-vector component.

Next, using the inequality $|x + y|^2 \geq \frac{1}{2}|x|^2 - |y|^2$ valid in any inner-product space, we obtain:

$$|T(k) + q(p)|_E^2 \geq \frac{1}{2}|T(k)|_E^2 - |q(p)|_E^2.$$

Recalling that $|T(k) + q(p)|_E^2 = \sum_{r \in I(G)} |\ell_r(k, p)|_E^2$ and combining the two displays yields (58) with:

$$a_G \equiv \frac{1}{2} \lambda_{\min}(A^T A), \quad b_{G,p} \equiv |q(p)|_E^2.$$

This proves the claim. $\square$

We can now state and prove the finiteness theorem in a form suitable for the scattering framework.

Theorem 7 (Fixed-order UV finiteness for all virtual loop orders). *Let F be an admissible entire regulator with $F(0) = 1$ satisfying the quantitative Euclidean damping bound (56). Consider any connected Feynman graph G with loop number L , and define its Euclidean amplitude by*

$$\mathcal{A}_G^{(E)}(p) \equiv \int_{\mathbb{R}^{4L}} \left(\prod_{\ell=1}^L \frac{d^4 k_\ell}{(2\pi)^4} \right) \mathcal{I}_G^{(E)}(k, p), \quad (59)$$

where the integrand $\mathcal{I}_G^{(E)}$ has the regulated form (55) and satisfies Assumption A. Then the integral (59) is absolutely convergent for every fixed L and every fixed external momentum configuration p away from physical threshold singularities. In particular, $\mathcal{A}_G^{(E)}(p)$ is UV finite for every L , equivalently, for any number of virtual loops.

We estimate the absolute value of the integrand. Using (55) and (54):

$$|\mathcal{I}_G^{(E)}(k, p)| \leq |\mathcal{N}_G(k, p)| \prod_{r \in I(G)} \frac{\left| F\left(\frac{|\ell_r(k, p)|_E^2}{M_*^2} \right) \right|^{\sigma_r}}{|\ell_r(k, p)|_E^2 + m_r^2}.$$

Here $\ell_r(k, p)$ is defined by (53), m_r is the mass on line r , and $\sigma_r \in \{1, 2\}$.

First, by Lemma 2, there exist $C_{G,p} > 0$ and $N_G \geq 0$ such that:

$$|\mathcal{N}_G(k, p)| \leq C_{G,p} (1 + |k|_E)^{N_G}.$$

Next, because $|\ell_r|_E^2 + m_r^2 \geq m_r^2$ and also $|\ell_r|_E^2 + m_r^2 \geq |\ell_r|_E^2$, the denominators contribute at worst additional polynomial decay; for absolute convergence it suffices to keep the crude bound:

$$\frac{1}{|\ell_r(k, p)|_E^2 + m_r^2} \leq 1.$$

This bound is not sharp but is harmless because the regulator provides exponential decay.

Now apply the damping estimate (56) to each internal line for which $|\ell_r(k, p)|_E^2$ is large. Since only finitely many lines exist, we may absorb the small-momentum region into an overall constant and obtain constants $C'_{G,p} > 0$ and $c' > 0$ such that for all k :

$$\prod_{r \in I(G)} \left| F\left(\frac{|\ell_r(k, p)|_E^2}{M_*^2} \right) \right|^{\sigma_r} \leq C'_{G,p} \exp\left(-\frac{c'}{M_*^2} \sum_{r \in I(G)} |\ell_r(k, p)|_E^2 \right). \quad (60)$$

In (60), c' may be chosen as $c' \equiv c \min_{r \in I(G)} \sigma_r$ and the prefactor $C'_{G,p}$ absorbs $C_F^{\sum_r \sigma_r}$ and the bounded region where (56) is not invoked.

Finally, apply Lemma 3 to (60) to obtain constants $a_G > 0$ and $b_{G,p} \geq 0$ with:

$$\sum_{r \in I(G)} |\ell_r(k, p)|_E^2 \geq a_G |k|_E^2 - b_{G,p}.$$

Substituting this inequality into (60) yields:

$$\prod_{r \in I(G)} \left| F \left(\frac{|\ell_r(k, p)|_E^2}{M_*^2} \right) \right|^{\sigma_r} \leq C''_{G,p} \exp \left(-\frac{c' a_G}{M_*^2} |k|_E^2 \right),$$

where $C''_{G,p} \equiv C'_{G,p} \exp \left(\frac{c' b_{G,p}}{M_*^2} \right)$.

Combining the preceding bounds, we conclude that for some constants $\tilde{C}_{G,p} > 0$ and $\alpha_G > 0$:

$$|\mathcal{I}_G^{(E)}(k, p)| \leq \tilde{C}_{G,p} (1 + |k|_E)^{N_G} \exp \left(-\alpha_G |k|_E^2 \right), \quad (61)$$

with $\alpha_G \equiv c' a_G / M_*^2$.

The right-hand side of (61) is integrable over $\mathbb{R}^{4L}$ for every L because Gaussian decay dominates any polynomial growth. Therefore:

$$\int_{\mathbb{R}^{4L}} d^{4L}k |\mathcal{I}_G^{(E)}(k, p)| < \infty,$$

which proves absolute convergence of (59). $\square$

The following consequences are immediate, that under the hypotheses of Theorem 7, every UV subintegral obtained by integrating over any subset of loop variables is also absolutely convergent. Hence UV subdivergences do not occur and no UV counterterms are required to define perturbation theory at any fixed loop order. Absolute convergence of the full integral implies absolute convergence of integrals over subsets of variables by Tonelli's theorem, after bounding by the same integrable dominating function.

Theorem 7 establishes fixed-order UV finiteness, that for each integer $L \geq 0$, every L -loop (i.e. L -virtual-loop) amplitude is finite. This does not assert that the infinite sum over all L converges; it asserts that each coefficient in the perturbative expansion is well-defined without UV renormalization.

Theorem 7 is a pointwise-in- L statement. Precisely, for every fixed integer $L \geq 0$ and every fixed connected graph G with loop number L , the Euclidean integrand admits an L -dependent dominating function of the form:

$$|\mathcal{I}_G^{(E)}(k, p)| \leq C_{G,p} (1 + |k|_E)^{N_G} \exp \left(-\alpha_G |k|_E^2 \right),$$

with constants $C_{G,p} > 0$, $N_G \in \mathbb{N}$, and $\alpha_G > 0$ that may depend on G , hence on L and on the external kinematics p . Consequently, for any finite loop order L , every L -loop graph integral is absolutely convergent and yields a finite contribution to the perturbative coefficient at that order.

By contrast, an all-orders statement would require quantitative control uniform in L , such as bounds on $\sum_{G: \ell(G)=L} C_{G,p}$ and on the growth of the number of graphs with L , together with a summability property of the resulting perturbation series. Theorem 7 does not assert such uniformity and therefore does not imply convergence, or Borel summability of the infinite loop expansion $\sum_{L=0}^{\infty} \mathcal{A}^{(L)}(p)$; it establishes only that each fixed-order coefficient is unambiguously defined without ultraviolet renormalization.

It is useful to distinguish two notions that are sometimes conflated in informal discussions. Theorem 7 holds for each fixed loop order $L \in \mathbb{N}$: for every connected graph G with $\ell(G) = L$, the corresponding Euclidean integral is absolutely convergent and therefore UV finite. In particular, along any sequence of loop orders $L = 1, 2, 3, \dots$ with $L \rightarrow \infty$, the theorem remains valid term-by-term so each coefficient $\mathcal{A}^{(L)}(p)$ is a well-defined finite quantity, modulo the usual infrared issues tied to the choice of observable.

This term-by-term finiteness should not be confused with the existence of an object at “ $L = \infty$ ”. An “ $L = \infty$ ” amplitude would correspond to an all-orders construction, such as a summation or resummation over arbitrarily high loop order:

$$\sum_{L=0}^{\infty} \mathcal{A}^{(L)}(p),$$

possibly supplemented by a summability prescription, for example Borel summation. Establishing such an all-orders object requires quantitative control that is uniform in L , including bounds on the growth of $\mathcal{A}^{(L)}(p)$ and on the combinatorial proliferation of graphs with L . Theorem 7 does not provide this uniform control and therefore does not imply convergence or summability of the infinite loop expansion as it asserts only that every fixed-order coefficient is UV finite and unambiguously defined.

Because F is entire, the regulator modifies amplitudes by analytic weights and does not introduce additional finite-plane poles. Consequently, the pole structure needed for LSZ reduction and the contour-pinching mechanism underlying Cutkosky rules are preserved; the regulator contributes only multiplicative factors evaluated on internal momenta.

12. Discussion and Outlook

We have shown that asymptotic microcausality, formulated as an exponential commutator bound for quasi-local regulated observables, is sufficient to recover the predictive core of QED, with asymptotic scattering states, an S -matrix, LSZ reduction, perturbative unitarity, and IR-finite asymptotic cross sections. The essential logic is that asymptotically-locality provides enough spacelike commutator suppression for Haag–Ruelle convergence, while the analyticity of the entire regulator preserves pole structure and unitarity cuts. Infrared physics remains fundamentally unchanged at leading order because $F(0) = 1$; thus the correct completion of QED still requires dressed charged states or inclusive observables, but the UV sector is simultaneously rendered finite by construction.

Several extensions are immediate. First, the analysis can be upgraded from scalar interpolating fields to the full BRST-invariant gauge-covariant framework, in which gauge-invariant quasi-local detector couplings are built from regulated currents and field strengths. Second, dispersion relations and positivity bounds can be revisited with quantitative macrocausality in place of strict locality, potentially yielding regulator-dependent bounds on high-energy growth. Third, the same strategy applies to gravity in entire-function UV completions, where asymptotic microcausality bounds may be used to formulate a controlled, asymptotic scattering theory compatible with finite resolution and nonperturbative horizon physics.

References

1. R. Haag, *Local Quantum Physics: Fields, Particles, Algebras*, 2nd ed., Springer (1996).
2. E. J. Thompson, “Asymptotic Microcausality; Deformations of Local Quantum Field Theory,” (Manuscript in preparation, 2026).
3. E. J. Thompson, “Asymptotic Locality Rectifying Newton–Wigner and Foldy–Wouthuysen Localization,” (Manuscript in preparation, 2026).
4. E. J. Thompson, “Asymptotic Renormalization: Complete Quantum Electrodynamics,” (Manuscript in preparation, 2026).
5. S. Weinberg, *The Quantum Theory of Fields*, Vol. I, Cambridge University Press (1995).
6. D. Ruelle, “On the Asymptotic Condition in Quantum Field Theory,” *Helv. Phys. Acta* **35** (1962) 147.
7. D. Ruelle, “A Generalization of the Haag–Ruelle Theory,” in *Proceedings of the International School of Physics “Enrico Fermi”*, Course XXXIX (1967).
8. D. Buchholz, “Collision Theory for Massless Fermions,” *Commun. Math. Phys.* **42** (1975) 269.
9. F. Bloch and A. Nordsieck, “Note on the Radiation Field of the Electron,” *Phys. Rev.* **52** (1937) 54.
10. T. Kinoshita, “Mass Singularities of Feynman Amplitudes,” *J. Math. Phys.* **3** (1962) 650.
11. T. D. Lee and M. Nauenberg, “Degenerate Systems and Mass Singularities,” *Phys. Rev.* **133** (1964) B1549.

12. P. P. Kulish and L. D. Faddeev, "Asymptotic Conditions and Infrared Divergences in Quantum Electrodynamics," *Theor. Math. Phys.* **4** (1970) 745.
13. R. E. Cutkosky, "Singularities and Discontinuities of Feynman Amplitudes," *J. Math. Phys.* **1** (1960) 429.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.