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Approximation of the Mittag-Leffler Functions by Elementary Functions with Physics Applications

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Abstract: In this paper we give approximations to the Mittag-Leffler functions in terms of elementary functions using different methods. This allowed us to establish a practical method we called integerization principle. This principle states that many fractional nonlinear oscillators may be solved by means of the solution to some integer-order Duffing oscillator equation. The accuracy of the obtained results is illsutrated in concrete examples. Formulas for estimating the errors in the approximations are also provided.

Keywords: fractional oscillator; caputo derivative; nonlinear fractional oscillator; duffing equatio; fractional pendulum; fractional Van der Pol equation; duffing; mathieu fractional oscillator

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1. Introduction.

Fractional calculus generalizes the differential and integral operators to non-integer orders, for this reason it is also called arbitrary order calculation. This idea is as old as calculus itself, but its development has been largely conditioned by the absence of a physical interpretation and convincing geometry and also by the numerous definition proposals. We could say that it did not have a real development until the second half of the 20th century, that is why we find here a classical and at the same time modern branch of mathematics.

Currently, a large number of articles are published on the subject, and we find applications in most of the sciences, this is because fractional operators are non-local operators, that is, what occurs at a point depends on an average over an interval containing the point, and this makes fractional calculus an exceptional tool for non-local phenomena such as ecological processes such as accumulation of metals, problems of population evolution, problems of radiation, economy, etc. They also play a very important role in relaxation processes such as those associated with viscoelastic materials.

In 1969, the Italian mathematician and physicist Michele Caputo gave a new definition derived from fractional order that allowed to interpret physically the initial conditions of the increasingly numerous applied problems. Caputo (1969) defined the fractional derivative as

$$\begin{aligned} \mathcal{D}_t^{\alpha} x(t) &= \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{x''(s)ds}{(t-s)^{\alpha-1}}, 1 < \alpha < 2. \\ \mathcal{D}_t^{\alpha} x(t) &= x''(t) \text{ when } \alpha = 2. \end{aligned} \tag{1}$$

where $n-1 < a < n$ and $f^{(n)}(s)$ is an ordinary derivative.

In 1974, the first international conference on Fractional Calculus was held in Connecticut, which served as a stimulus to numerous publications. The second conference took place in 1984 in Scotland, and the third in 1989 in Tokyo. Nowadays it is difficult to find a field of science or engineering that does not consider concepts of the Fractional Calculus, and every year there are several events that show it.

Most nonlinear differential equations do not admit an analytical procedure that describe its solution. For this reason it is necessary to resort to numerical methods such as Runge Kutta, Newton, Euler etc., generally developed in software tools for applications in mathematics and engineering; these methods are a quite practical solution tool that allows approaching in an approximate way the solution of differential equations non linear; another resource contemplated when describing the

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solution of this type of equations refers to the qualitative information of the general behavior of the solutions that are mactually obtains without solving these equations, basically through methods and geometric analysis. The interest of studying nonlinear differential equations lies mainly in that most physical systems, whether electrical, magnetic, biological, chemical, geological, economic, etc., present a non-linear behavior by nature; the procedure of linearizing the equations segments the knowledge about the behavior of systems around an equilibrium point, while the study of systems by means of of nonlinear theory makes it possible to be aware of the behavior of the system at all points within which it is defined; another reason to study nonlinear systems and differential equations that describe them lies in the existence of natural phenomena and surprising mathematical representations that have no place in linear theory.

Suppose we are given that

$$\mathcal{D}_t^{\alpha} x(t) + \omega_0^2 x(t) = 0,\, x(0) = x_0 \text{ and } x'(0) = \dot{x}_0, \tag{2}$$

The exact solutuion to the i.v.p. (2) reads

$$x(t) = x_0 E_{\alpha}(-\omega_0^2 t^{\alpha}) + \dot{x}_0 t E_{\alpha,2}(-\omega_0^2 t^{\alpha}), \tag{3}$$

where $E_{\alpha}(-t^{\alpha})$ and $E_{\alpha,2}(-t^{\alpha})$ are the Mittag-Leffler functions. These functions are defined as follows :

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{4}$$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}. \tag{5}$$

Our aim is to approximate the Mittag-Leffler functions $E_{\alpha}(-t^{\alpha})$ and $E_{\alpha,2}(-t^{\alpha})$ by means of elementary functions. We also will apply the obtained results in the solution of fractional differential equations. The idea is to replace the fractional ode by means of some suitable integer-order oscillator. We will call this method the Integerization Principle.

2. Approximation of Mittag-Leffler functions by means of elementary functions. .

Assume that $1 < \alpha \leq 2$. Let us consider the Mittag-Leffler function $E_{\alpha}(t)$. To begin with, let $\alpha = 1.9$. The function $E_{1.9}(t)$ is plotted in Figure 1. From that figure we see that the function $E_{1.9}(t)$ behaves like a damped oscillator. So, we expect thtat this function may be approximated by means of the solution to some damped Duffing equation

$$\ddot{x} + 2\varepsilon\dot{x} + (\varepsilon^2 + \kappa^2)x + \sum_{r=1}^{N} \mu_{2r+1} x^{2r+1} = 0,\, x(0) = 1 \text{ and } x'(0) = 0. \tag{6}$$

Let us examine several possibilities for the approximation. Choose some positive number T.

2.1. First Method.

We look for suitable positive numbers ε and κ such that $E_{\alpha}(t) \approx x(t)$ for $0 \leq t \leq T$, where

$$\ddot{x} + 2\varepsilon\dot{x} + (\varepsilon^2 + \kappa^2)x = 0,\, x(0) = 0 \text{ and } x'(0) = 0. \tag{7}$$

The solution to the ode (7) is easy to obtain and it reads

$$x_{\varepsilon,\kappa}(t) = \exp(-\varepsilon t)\left[\cos(\kappa t) + \frac{\varepsilon}{\kappa}\sin(\kappa t)\right]. \tag{8}$$

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We choose the numbers ε and κ so that

$$F(\varepsilon,\kappa) = \min_{\epsilon,k} \left\| E_\alpha - x_{\epsilon,k} \right\|, \tag{9}$$

where

$$\left\| E_\alpha - x_{\epsilon,k} \right\| = \max_{0 \le t \le T} \left| E_\alpha(t) - x_{\epsilon,k}(t) \right|. \tag{10}$$

For example,

$$E_{1.9}(t) \approx x_{\varepsilon,\kappa}(t) \text{ for } \varepsilon = 0.0746892 \text{ and } \kappa = 1.01239 \text{ with error } \mathcal{E} = 0.0577757 \text{ for } 0 \le t \le 100 \tag{11}$$

See Figure 1.

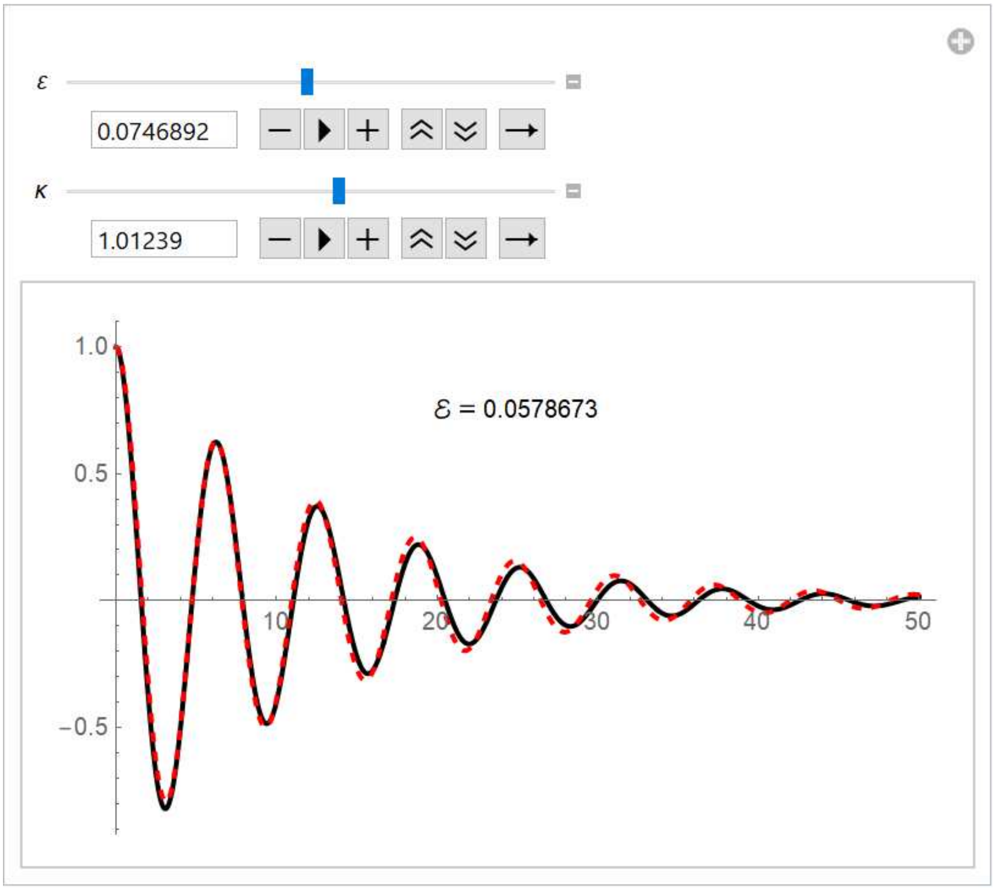

Figure 1. The approximation (11).

The optimal values for ε and κ are presented in Tables 1 and 2.

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Table 1. $E_\alpha(t) \approx \exp(-\varepsilon t)\left[\cos(\kappa t) + \frac{\varepsilon}{\kappa}\sin(\kappa t)\right]$ for $0 \le t \le 100$.

α	ε	κ	$\mathcal{E}$	α	ε	κ	$\mathcal{E}$
1.01	2.39859	0.0036231	0.0940093	1.26	1.15593	1.19478	0.0885624
1.03	2.40063	0.0169119	0.0876904	1.27	1.0848	1.20676	0.0912522
1.04	2.40192	0.00715623	0.0845684	1.28	1.03706	1.20556	0.0937868
1.05	2.40309	0.00916501	0.0814816	1.3	0.979565	1.18393	0.0981991
1.06	2.40234	0.00668212	0.0785424	1.31	0.935677	1.18204	0.100107
1.07	2.40169	0.00691769	0.0756304	1.32	0.886398	1.18551	0.102085
1.08	2.4011	0.0125906	0.0727469	1.33	0.863911	1.17756	0.103876
1.09	2.39893	0.0527265	0.069955	1.34	0.841852	1.1699	0.105489
1.1	2.39543	0.0694989	0.0672848	1.35	0.820175	1.16254	0.106924
1.11	2.39249	0.0742624	0.0646352	1.36	0.784163	1.16107	0.108192
1.12	2.39035	0.0650688	0.0620016	1.37	0.749985	1.16037	0.109503
1.13	2.38793	0.0323902	0.059447	1.38	0.731703	1.15474	0.110688
1.14	2.38281	0.0384946	0.057035	1.39	0.713689	1.14931	0.111716
1.15	2.37801	0.032931	0.0546518	1.4	0.695933	1.14408	0.112586
1.16	2.26175	0.53631	0.0546059	1.41	0.675129	1.13994	0.113302
1.17	2.00951	0.913557	0.057925	1.42	0.639924	1.14016	0.114003
1.18	1.83	1.07467	0.0616554	1.43	0.624375	1.13603	0.114659
1.19	1.66229	1.1809	0.0654689	1.44	0.60902	1.13201	0.115174
1.2	1.54323	1.21544	0.0693225	1.45	0.593852	1.12812	0.115548
1.21	1.45654	1.21547	0.072937	1.46	0.578863	1.12435	0.115782
1.22	1.3619	1.22748	0.0763258	1.47	0.547557	1.12323	0.115898
1.23	1.29742	1.22762	0.0797563	1.48	0.533931	1.12005	0.116077
1.24	1.24888	1.21647	0.0829392	1.49	0.520452	1.11694	0.116126
1.25	1.20518	1.20301	0.0858778	1.5	0.507115	1.11389	0.116047

Table 2. $E_\alpha(t) \approx \exp(-\varepsilon t)\left[\cos(\kappa t) + \frac{\varepsilon}{\kappa}\sin(\kappa t)\right]$ for $0 \le t \le 100$.

α	ε	κ	E	α	ε	κ	E
1.51	0.493917	1.11091	0.115841	1.76	0.196581	1.0534	0.0839411
1.52	0.480244	1.10803	0.115511	1.77	0.187986	1.05035	0.0830348
1.53	0.455723	1.10588	0.115131	1.78	0.17522	1.04589	0.0818397
1.54	0.443631	1.10334	0.114745	1.79	0.166964	1.04317	0.08038
1.55	0.431651	1.10083	0.11424	1.8	0.158227	1.03968	0.0795217
1.56	0.41978	1.09835	0.113617	1.81	0.150397	1.03669	0.078626
1.57	0.408019	1.0959	0.112877	1.82	0.13947	1.0325	0.0773324
1.58	0.394599	1.09341	0.112031	1.83	0.131793	1.0299	0.0756785
1.59	0.375763	1.09074	0.111139	1.84	0.121701	1.02631	0.0737473
1.6	0.364827	1.08856	0.110222	1.85	0.114138	1.02412	0.0715135
1.61	0.353981	1.08639	0.109194	1.86	0.106528	1.02201	0.0689346
1.62	0.343224	1.08422	0.108055	1.87	0.0986547	1.01985	0.0660508
1.63	0.332556	1.08206	0.106807	1.88	0.0901836	1.01709	0.0634756
1.64	0.321616	1.07985	0.105454	1.89	0.0818237	1.01431	0.0608104
1.65	0.304842	1.07683	0.104047	1.9	0.0746892	1.01239	0.0577757
1.66	0.294814	1.07483	0.102617	1.91	0.0665708	1.01006	0.05419
1.67	0.28486	1.07283	0.10108	1.92	0.0590778	1.00816	0.0504512
1.68	0.274983	1.07081	0.0994379	1.93	0.0520227	1.00661	0.0464086
1.69	0.26518	1.06878	0.0976915	1.94	0.0446918	1.00506	0.0418105
1.7	0.255453	1.06674	0.095842	1.95	0.0373087	1.00361	0.0367877
1.71	0.242183	1.06384	0.0939223	1.96	0.0299157	1.00232	0.0311719
1.72	0.231759	1.06162	0.0919596	1.97	0.02262	1.00137	0.0248345
1.73	0.222536	1.05967	0.0899164	1.98	0.0152285	1.00063	0.017689
1.74	0.21338	1.0577	0.0877706	1.99	0.00770126	1.00019	0.00940854
1.75	0.204291	1.05571	0.0855233	2.	0	1	0

See Figure 2.

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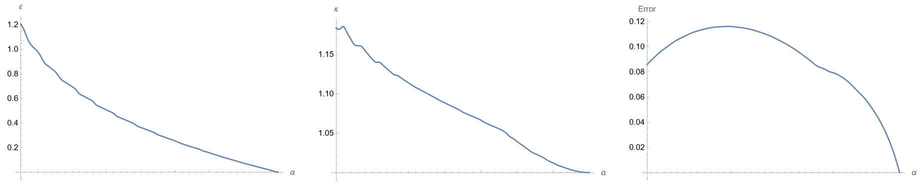

Figure 2. Pots of ε , κ and Error as functions og α .

We obtained the following experimental formulas for $1.5 \leq \alpha \leq 2$:

$E_\alpha(t) \approx \exp(-\varepsilon t)\left[\cos(\kappa t) + \frac{\varepsilon}{\kappa}\sin(\kappa t)\right]$ for $0 \leq t \leq 100$, where

$$\varepsilon = 38.5892(\alpha - 2.2531)(\alpha - 2)\left(\alpha^2 - 3.85082\alpha + 3.84914\right)\left(\alpha^2 - 2.70744\alpha + 1.91924\right).$$

$$\kappa = 153.169(2.22838 - \alpha)(\alpha - 1.25676)\left(\alpha^2 - 3.98123\alpha + 4.06822\right)\left(\alpha^2 - 3.0097\alpha + 2.38305\right). \tag{12}$$

with error

$$\mathcal{E} = 158.157(\alpha - 2.14235)(\alpha - 2)\left(\alpha^2 - 3.53553\alpha + 3.17211\right)\left(\alpha^2 - 2.86208\alpha + 2.06234\right).$$

Thus, the solution (3) to the i.v.p. (2) may be approximated as follows:

$$\begin{aligned} x_{\text{approx}}(t) &= c\exp\left(-\varepsilon\omega_0^{2/\alpha}t\right)\cos\left(\kappa\omega_0^{2/\alpha}t + \cos^{-1}\left(\frac{x_0}{c}\right)\right), \\ \text{where } c &= \pm\frac{1}{\kappa}\sqrt{\dot{x}_0\omega_0^{-4/\alpha}\left(2\varepsilon x_0\omega_0^{2/\alpha} + \dot{x}_0\right) + x_0^2\left(\varepsilon^2 + \kappa^2\right)}. \end{aligned} \tag{13}$$

In particular,

$$E_{\alpha,2}(t) \approx \frac{\sin(\kappa t)}{\kappa t}\exp(-\varepsilon t). \tag{14}$$

Observe that the function (13) is a solution to the linear ode

$$\ddot{x} + 2\varepsilon\omega_0^{2/\alpha}\dot{x} + (\varepsilon^2 + \kappa^2)\omega_0^{4/\alpha}x = 0,\ x(0) = x_0 \text{ and } x'(0) = \dot{x}_0. \tag{15}$$

This says that the solution to the fractional linear ode

$$\mathcal{D}_t^\alpha x(t) + \omega_0^2 x(t) = f(t),\ x(0) = x_0 \text{ and } x'(0) = \dot{x}_0, \tag{16}$$

may be approximated by means of the solution to the integer order linear ode

$$\ddot{x} + 2\varepsilon\omega_0^{2/\alpha}\dot{x} + (\varepsilon^2 + \kappa^2)\omega_0^{4/\alpha}x = f(t),\ x(0) = x_0 \text{ and } x'(0) = \dot{x}_0. \tag{17}$$

2.2. Second Approach.

We approximate the function $E_\alpha(t)$ by means of the solution to the following damped Duffing oscillator:

$$\ddot{x} + 2\varepsilon\dot{x} + \omega^2 x + \lambda x^3 = 0,\ x(0) = 1 \text{ and } x'(0) = 0. \tag{18}$$

The approximate solution to the Duffing oscillator (18) is given by

$$\hat{x}(t) = \exp(-\varepsilon t)(\cos(\psi t) + c\sin((t))),$$

where

$$\psi(t) = \frac{4\varepsilon t w_0 - 2w_1 e^{-2\varepsilon t} - w_2 e^{-4\varepsilon t} + 2w_1 + w_2}{4\varepsilon}.$$

$$w_2 = -\frac{w_1^2}{2w_0}, w_1 = \frac{3\lambda\left(c^2+1\right)}{8w_0}, w_0 = \sqrt{\omega^2 - \varepsilon^2}.$$

$$9c^5\lambda^2 + 18c^3\lambda^2 - 48c^3\lambda w_0^2 + 9c\lambda^2 - 48c\lambda w_0^2 - 128cw_0^4 + 128\varepsilon w_0^3 = 0.$$

$$c \approx \frac{128\varepsilon w_0^3\left(-9\lambda^2+48\lambda w_0^2+128w_0^4\right)^2}{-729\lambda^6-73728\lambda^2 w_0^6(4\varepsilon^2+3\lambda)+49152\lambda w_0^8(16\varepsilon^2+9\lambda)+11664\lambda^5 w_0^2-31104\lambda^4 w_0^4+2359296\lambda w_0^{10}+2097152w_0^{12}}.$$

The constants ε, ω and λ are chosen in order to minimize the distance between $E_\alpha(t)$ and $\hat{x}(t)$. For example, for $\alpha = 1.94$ we have

$E_\alpha(t) \approx \hat{x}(t)$ for the parameter values
$\varepsilon = 0.0493409, \omega = 0.987951, \lambda = 0.0850072$
with error $\mathcal{E}$ =0.0317695.

See Figure 3. The approximation reads

$$\hat{x}(t) = e^{-0.0493409t}\left(\begin{array}{c} 0.0484414\sin\left(\begin{array}{c} 0.986718t + 0.00269236e^{-0.197364t} - \\ 0.328152e^{-0.0986819t} + 0.325459 \end{array}\right) + \\ \\ \cos\left(\begin{array}{c} 0.986718t + 0.00269236e^{-0.197364t} - \\ 0.328152e^{-0.0986819t} + 0.325459 \end{array}\right) \end{array}\right).$$

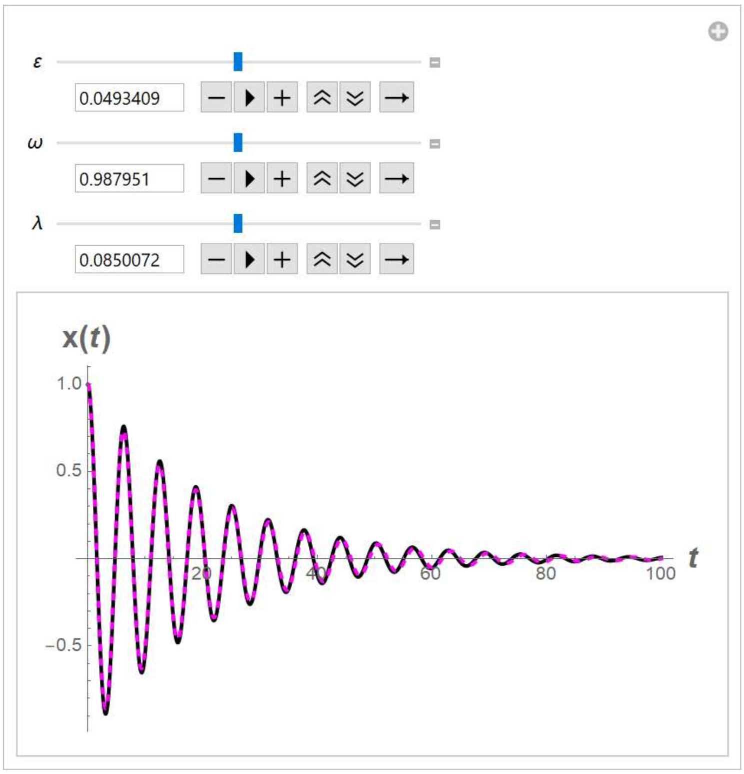

Figure 3

The optimal values for ε, ω and λ are presented in Tables 3 and 4.

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Table 3

α	ε	ω	λ	$\mathcal{E}$	α	ε	ω	λ	$\mathcal{E}$
1.1	0.846303	0.930862	0.884961	0.0171003	1.45	0.473497	1.05113	1.	0.0762138
1.11	0.819062	0.986339	0.922694	0.0656855	1.46	0.464217	1.04177	1.01563	0.0728971
1.12	0.78959	1.02964	0.952144	0.102819	1.47	0.456201	1.0339	1.02659	0.0699269
1.3	0.520815	1.1	1.	0.109876	1.48	0.449818	1.02813	1.02921	0.0677535
1.31	0.51581	1.10007	1.00064	0.106596	1.49	0.445441	1.02511	1.01982	0.0668268
1.32	0.511247	1.10012	1.00111	0.103345	1.5	0.443439	1.02546	0.994739	0.0675971
1.33	0.507419	1.10013	1.00127	0.100156	1.51	0.453873	1.04258	0.88869	0.0774791
1.34	0.504619	1.1001	1.00095	0.0970607	1.52	0.465004	1.06115	0.775027	0.0882171
1.35	0.503141	1.1	1.	0.0940907	1.53	0.47478	1.07861	0.665486	0.0985199
1.36	0.508144	1.10111	0.995711	0.0913459	1.54	0.48115	1.09241	0.571805	0.107096
1.37	0.51384	1.1018	0.991105	0.0887738	1.56	0.458282	1.07782	0.574683	0.103544
1.38	0.519303	1.10172	0.986658	0.0863892	1.57	0.42924	1.04957	0.670786	0.0914237
1.39	0.523612	1.10054	0.982846	0.0842073	1.58	0.397181	1.01792	0.781836	0.0775939
1.4	0.525842	1.09791	0.980146	0.082243	1.59	0.364353	0.985573	0.895638	0.0633542
1.41	0.518262	1.09002	0.983014	0.0808407	1.6	0.333002	0.955221	1.	0.0500041
1.42	0.508459	1.08087	0.986952	0.0796039	1.61	0.316897	0.94374	1.01775	0.0457454
1.43	0.497211	1.07099	0.991442	0.0784653	1.62	0.30388	0.936089	1.01791	0.0432498
1.44	0.485297	1.0609	0.995964	0.0773577	1.63	0.293315	0.931415	1.00455	0.0420912

Table 4

α	ε	ω	λ	$\mathcal{E}$	α	ε	ω	λ	$\mathcal{E}$
1.64	0.284567	0.928862	0.981714	0.0418436	1.82	0.136274	0.955633	0.371744	0.0371835
1.65	0.277001	0.927576	0.953453	0.042081	1.83	0.128556	0.958458	0.342611	0.036875
1.66	0.268002	0.922459	0.944079	0.0404954	1.84	0.120812	0.961242	0.314609	0.036445
1.67	0.25941	0.917961	0.932321	0.0390132	1.85	0.113054	0.964054	0.287458	0.0358902
1.68	0.251085	0.914287	0.917167	0.0376788	1.86	0.106934	0.94057	0.369439	0.0211623
1.69	0.242887	0.911643	0.897605	0.0365366	1.87	0.0995198	0.93741	0.368993	0.0168011
1.7	0.234676	0.910234	0.872621	0.0356312	1.88	0.0913099	0.947681	0.314275	0.0193282
1.71	0.225338	0.912182	0.830561	0.0353886	1.89	0.0828028	0.964489	0.233441	0.0252649
1.72	0.215951	0.915299	0.783716	0.0353761	1.9	0.074497	0.980938	0.154645	0.0311327
1.73	0.206619	0.919312	0.733735	0.0355429	1.91	0.0668909	0.990136	0.106044	0.0334531
1.74	0.197446	0.923949	0.682267	0.0358379	1.92	0.0604829	0.985188	0.115791	0.0287474
1.75	0.188538	0.928936	0.630959	0.0362103	1.93	0.0528581	0.987682	0.0945696	0.0275357
1.76	0.180756	0.93317	0.587923	0.0365159	1.94	0.0493409	0.987951	0.0850072	0.0317695
1.77	0.173256	0.937417	0.546729	0.0368201	1.95	0.0400411	0.990754	0.0641711	0.0279478
1.78	0.165952	0.941612	0.50741	0.0370955	1.96	0.0307866	0.994233	0.0411173	0.0220486
1.79	0.158756	0.94569	0.47	0.0373142	1.97	0.0239986	0.996001	0.027014	0.0206847
1.8	0.151584	0.949585	0.434531	0.0374487	1.98	0.0159786	0.998598	0.0109364	0.0157666
1.81	0.143954	0.952698	0.402291	0.0373737	1.99	0.00778619	0.998701	0.00630205	0.00811248

2.3. Third Approach.

We approximate the function $E_\alpha(t)$ by means of the solution to the following damped cubic-quintic Duffing oscillator:

$$\ddot{x} + 2\varepsilon\dot{x} + \omega^2 x + \lambda x^3 + \mu x^5 = 0,\ x(0) = 1 \text{ and } x'(0) = 0. \tag{19}$$

The approximate solution to the Duffing oscillator (19) is given by

$$\hat{X}(t) = \exp(-\varepsilon t)\left[\cos\psi(t) + c\sin\psi(t)\right], \tag{20}$$

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where

$$\psi(t) = \frac{1}{4\varepsilon}\left[4\varepsilon w_0 t - 2w_1 e^{-2\varepsilon t} - w_2 e^{-4\varepsilon t} + 2w_1 + w_2\right].$$

$$w_0 = \sqrt{\omega^2 - \varepsilon^2},\, w_1 = \frac{3}{8w_0}\left(c^2+1\right)\lambda,\, w_2 = \frac{\left(c^2+1\right)^2}{128w_0^3}\left(40\mu w_0^2 - 9\lambda^2\right).$$

$$c^5\left(40\mu w_0^2 - 9\lambda^2\right) + c^3\left(-18\lambda^2 + 48\lambda w_0^2 + 80\mu w_0^2\right) + c\left(-9\lambda^2 + 48\lambda w_0^2 + 40\mu w_0^2 + 128w_0^4\right) - 128\varepsilon w_0^3 = 0.$$

$$c \approx \frac{128\varepsilon w_0^3\left(-9\lambda^2+8w_0^2(6\lambda+5\mu)+128w_0^4\right)^2}{\left[\begin{array}{c} -729\lambda^6 + 8192w_0^8\left(32\varepsilon^2(3\lambda+5\mu) + 54\lambda^2 + 180\lambda\mu + 75\mu^2\right) - 512w_0^6\left(144\lambda^2\left(4\varepsilon^2+3\lambda\right) - 450\lambda\mu^2 - 125\mu^3\right) + \\ 1944\lambda^4 w_0^2(6\lambda+5\mu) - 1728\lambda^2 w_0^4\left(18\lambda^2 + 60\lambda\mu + 25\mu^2\right) + 393216w_0^{10}(6\lambda+5\mu) + 2097152w_0^{12} \end{array}\right]}. \tag{21}$$

For example, let $\alpha = 1.95$. We have

$E_\alpha(t) \approx \hat{X}(t)$ for the parameter values

$$\varepsilon = 0.0383727, \omega = 0.995282, \lambda = 0.00295046 \text{ and } \mu = 0.0645417 \tag{22}$$

with error $\mathcal{E}$ =0.0200355.

The approximation reads

$$\hat{X}(t) = e^{-0.0383727t}\left(\begin{array}{c} 0.0377686\sin\left(0.994542t - 0.132498e^{-0.153491t} - 0.0145166e^{-0.0767453t} + 0.147015\right) + \\ \cos\left(0.994542t - 0.132498e^{-0.153491t} - 0.0145166e^{-0.0767453t} + 0.147015\right) \end{array}\right). \tag{23}$$

See Figure 4.

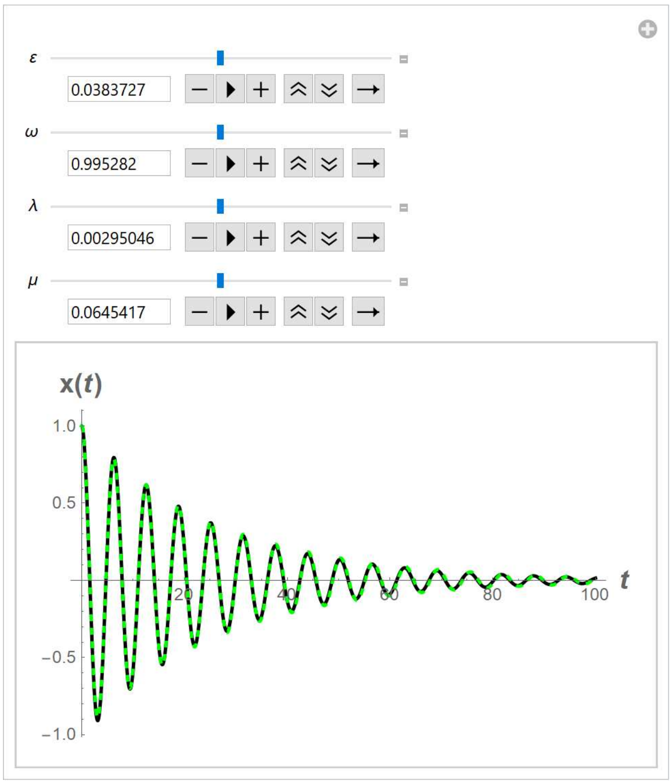

Figure 4

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The optimal values for ε, ω, λ and μ are presented in Tables 5–8.

Table 5

α	ε	ω	λ	μ	$\mathcal{E}$
1.05	1.17565	2.22404	15.4941	−11.1618	0.0477814
1.06	1.10125	1.94663	12.878	−8.69331	0.0436304
1.07	1.01882	1.61342	9.64906	−5.77276	0.0376546
1.08	0.940566	1.29437	6.36108	−2.878	0.0310302
1.09	0.87869	1.05475	3.56782	−0.486854	0.0249332
1.1	0.8454	0.934144	1.82306	0.922848	0.0205398
1.11	0.8529	0.93986	1.68056	0.873281	0.0190261
1.12	0.844896	0.952562	1.99981	0.474146	0.0203796
1.13	0.765094	0.861856	1.64031	0.835145	0.0188395
1.14	0.7598	0.856841	1.55488	0.751999	0.0195938
1.15	0.772489	0.904102	2.	0.0103973	0.0252311
1.16	0.713583	0.818608	1.64129	0.953723	0.0251636
1.17	0.680872	0.769385	1.34135	1.14769	0.0201201
1.18	0.659256	0.756388	1.40719	1.09896	0.0251916
1.19	0.524985	0.611582	0.865317	1.71172	0.00840014
1.2	0.522689	0.615157	0.883495	1.66922	0.00987507
1.21	0.528152	0.635468	0.945607	1.61112	0.00966996
1.22	0.475957	0.545394	0.725926	1.79614	0.015571
1.23	0.520775	0.639424	0.990248	1.60417	0.0103935
1.24	0.514867	0.637945	0.995238	1.59868	0.0105978
1.25	0.506777	0.632278	1.03835	1.69699	0.0130096
1.26	0.508517	0.651922	1.09638	1.62525	0.014026
1.27	0.511374	0.669358	1.11507	1.49976	0.0149413
1.28	0.508017	0.773248	0.100096	1.87625	0.0235894

Table 6

α	ε	ω	λ	μ	$\mathcal{E}$
1.29	0.498728	0.730784	0.658047	1.19529	0.0206834
1.3	0.538289	0.841076	0.0108725	1.99307	0.0255741
1.31	0.541734	0.831617	0.225146	1.80361	0.0260523
1.32	0.530351	0.839459	0.0906538	1.99627	0.0276624
1.33	0.520783	0.842107	0.0602407	2.	0.0262533
1.34	0.519176	0.835518	0.203658	1.83392	0.0272229
1.35	0.514045	0.845961	0.142618	1.91355	0.0279495
1.36	0.506794	0.84889	0.12195	1.93765	0.026471
1.37	0.498145	0.851198	0.0840845	1.98538	0.0271168
1.38	0.491564	0.854293	0.0747146	1.98609	0.027447
1.39	0.485332	0.856817	0.0743162	1.97601	0.0276241
1.4	0.478038	0.860825	0.0499686	1.98704	0.0281142
1.41	0.471071	0.86455	0.0383513	1.97258	0.0284721
1.42	0.463325	0.868035	0.0111147	1.98595	0.0289776
1.43	0.451257	0.859243	0.0028895	1.99311	0.0296543
1.44	0.440832	0.866644	0.00556457	1.99144	0.0316863
1.45	0.430451	0.866411	0.00717314	1.98093	0.033017
1.46	0.423856	0.870769	0.00997726	1.93857	0.0334097
1.47	0.406235	0.845041	$2.67e-6$	2.	0.0361714
1.48	0.396591	0.841827	0.	1.99077	0.0378698
1.49	0.387603	0.840605	0.0000168982	1.97843	0.039593
1.5	0.403126	0.891162	0.181392	1.42029	0.034919
1.51	0.434224	0.974812	0.476172	0.517205	0.0260887
1.52	0.471959	1.07288	0.816407	−0.529995	0.0153428

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Table 7

α	ε	ω	λ	μ	$\mathcal{E}$
1.53	0.507396	1.16672	1.13414	−1.52047	0.00492236
1.54	0.531597	1.23769	1.36143	−2.25338	−0.00293189
1.55	0.535626	1.2672	1.43032	−2.52788	−0.00597906
1.56	0.510548	1.23669	1.27285	−2.14315	−0.00197828
1.57	0.447424	1.12776	0.821074	−0.898334	0.0113113
1.58	0.337319	0.922392	0.00704273	1.4074	0.0361305
1.59	0.329015	0.935955	0.00605698	1.39103	0.0351725
1.6	0.321299	0.946724	0.0031614	1.32339	0.0338844
1.61	0.313321	0.953591	0.000525712	1.24439	0.0329386
1.62	0.304228	0.955486	0.000319626	1.19394	0.0330078
1.63	0.29317	0.951369	0.00471285	1.21193	0.0347647
1.64	0.282542	0.94767	0.00643761	1.23091	0.0363964
1.65	0.272447	0.944916	0.00621097	1.24234	0.0377003
1.66	0.262987	0.943635	0.00475002	1.23767	0.0384734
1.67	0.254263	0.944355	0.00277182	1.20837	0.0385132
1.68	0.246377	0.947608	0.000993464	1.14588	0.0376167
1.69	0.239431	0.953925	0.000132019	1.04168	0.0355812
1.7	0.233526	0.963835	0.000904563	0.887219	0.032204
1.71	0.226541	0.946524	0.114213	0.80477	0.0353159
1.72	0.218686	0.923961	0.227652	0.780667	0.0400929
1.73	0.210173	0.918206	0.228815	0.801244	0.0417115
1.74	0.201216	0.951524	0.00529895	0.852831	0.0353476
1.75	0.193623	0.958479	−0.242128	1.2301	0.0390437
1.76	0.184082	0.953293	−0.182603	1.1432	0.0401316

Table 8

α	ε	ω	λ	μ	$\mathcal{E}$
1.77	0.17396	0.948153	0.000107094	0.824036	0.038852
1.78	0.164627	0.955285	0.122232	0.504506	0.0354456
1.79	0.157451	0.986949	$1.64e-6$	0.416506	0.030153
1.8	0.150436	1.00039	−0.0366678	0.352533	0.0304374
1.81	0.143349	1.00226	−0.0244455	0.308749	0.0332365
1.82	0.135959	0.999292	0.	0.281316	0.0354879
1.83	0.128035	0.998245	0.	0.266394	0.0341293
1.84	0.119666	0.995605	0.00290616	0.253632	0.0317824
1.85	0.111489	0.99387	0.00733691	0.233966	0.0303116
1.86	0.104143	0.995524	0.0119107	0.198335	0.0315813
1.87	0.0982656	1.00304	0.0152459	0.137675	0.037456
1.88	0.0886001	0.993957	0.0534113	0.0984177	0.0338215
1.89	0.0827019	1.0007	0.000262891	0.1361	0.0329433
1.9	0.074831	0.998052	0.00923344	0.118255	0.0286814
1.91	0.0677163	0.998997	$2.64e-6$	0.110885	0.0292408
1.92	0.0605911	1.00195	0.0154867	0.0478608	0.0337178
1.93	0.0529403	1.00113	0.000349593	0.0618894	0.0291377
1.94	0.0449666	1.00184	0.0192709	0.00417361	0.0335329
1.95	0.0383727	0.995282	0.00295046	0.0645417	0.0200355
1.96	0.0311126	1.00263	0.00293461	0.00259494	0.0286641
1.97	0.0238845	1.00898	0.00547476	−0.0476078	0.032584
1.98	0.0151849	1.00024	0.00229705	0.000384475	0.0160657
1.99	0.00719774	0.996993	0.000967276	0.018135	0.00456411
2.	0.	1.	0.	0.	0.

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2.4. Fourth Approach.

We approximate the function $E_\alpha(t)$ by means of the solution to the following damped cubic-quintic-heptic Duffing oscillator:

$$\ddot{x} + 2\varepsilon\dot{x} + \omega^2 x + \lambda x^3 + \mu x^5 + \nu x^7 = 0,\, x(0) = 1 \text{ and } x'(0) = 0. \tag{24}$$

The approximate solution to the Duffing oscillator (24) is given by

$$\hat{X}(t) = \exp(-\varepsilon t)\left[\cos(\psi(t)) + c\sin(\psi(t))\right], \tag{25}$$

where

$$\psi(t) = \frac{1}{12\varepsilon}\left[12\varepsilon t w_0 - 6w_1 e^{-2\varepsilon t} - 3w_2 e^{-4\varepsilon t} - 2w_3 e^{-6\varepsilon t} + 6w_1 + 3w_2 + 2w_3\right].$$

$$w_0 = \sqrt{\omega^2 - \varepsilon^2},\; w_1 = \frac{3}{8w_0}\left(c^2\lambda + \lambda\right),\; w_2 = \frac{\left(c^2+1\right)^2\left(40\mu w_0^2 - 9\lambda^2\right)}{128 w_0^3},\, w_3 = \frac{\left(c^2+1\right)^3\left(27\lambda^3 - 120\lambda\mu w_0^2 + 280\nu w_0^4\right)}{1024 w_0^5}.$$

$$c^7\left(27\lambda^3 - 120\lambda\mu w_0^2 + 280\nu w_0^4\right) + c^5\left(81\lambda^3 - 72\lambda^2 w_0^2 - 360\lambda\mu w_0^2 + 320\mu w_0^4 + 840\nu w_0^4\right) +$$

$$c^3\left(81\lambda^3 - 144\lambda^2 w_0^2 - 360\lambda\mu w_0^2 + 384\lambda w_0^4 + 640\mu w_0^4 + 840\nu w_0^4\right) +$$

$$c\left(27\lambda^3 - 72\lambda^2 w_0^2 - 120\lambda\mu w_0^2 + 384\lambda w_0^4 + 320\mu w_0^4 + 280\nu w_0^4 + 1024 w_0^6\right) - 1024\varepsilon w_0^5 = 0.$$

$$c \approx \frac{1024\varepsilon w_0^5\left(27\lambda^3 + 8w_0^4(48\lambda + 40\mu + 35\nu) - 24\lambda w_0^2(3\lambda + 5\mu) + 1024 w_0^6\right)^2}{\left[\begin{array}{c} 19683\lambda^9 + 65536 w_0^{14}\left(128\varepsilon^2(48\lambda + 80\mu + 105\nu) + 3456\lambda^2 + 1440\lambda(4\mu + 7\nu) + 75(8\mu + 7\nu)^2\right) - \\ 512 w_0^{12}\left(\begin{array}{c} 147456\varepsilon^2\lambda(2\lambda + 5\mu) + 55296\lambda^3 + 552960\lambda^2\mu + 3600\lambda\left(64\mu^2 - 49\nu^2\right) - \\ 125(8\mu + 7\nu)^3 \end{array}\right) + \\ 4608\lambda w_0^{10}\left(\begin{array}{c} 18432\varepsilon^2\lambda^2 + 10368\lambda^3 - 75\lambda\left(192\mu^2 + 336\mu\nu + 49\nu^2\right) - \\ 125\mu(8\mu + 7\nu)^2 \end{array}\right) - \\ 52488\lambda^7 w_0^2(3\lambda + 5\mu) + 5832\lambda^5 w_0^4\left(216\lambda^2 + 15\lambda(24\mu + 7\nu) + 200\mu^2\right) - \\ 3456\lambda^3 w_0^6\left(756\lambda^3 + 945\lambda^2(4\mu + \nu) + 225\lambda\mu(12\mu + 7\nu) + 500\mu^3\right) + \\ 1728\lambda^2 w_0^8\left(\begin{array}{c} 3456\lambda^3 + 1800\lambda^2(8\mu + 7\nu) + 75\lambda\left(320\mu^2 + 224\mu\nu + 49\nu^2\right) + \\ 1000\mu^2(8\mu + 7\nu) \end{array}\right) + \\ 25165824 w_0^{16}(48\lambda + 40\mu + 35\nu) + 1073741824 w_0^{18}. \end{array}\right]} \tag{26}$$

The optimal values for ε, ω, λ, μ and are presented in Tables 9 and 10.

Table 9

α	ε	ω	λ	μ	ν	$\mathcal{E}$
1.07	1.01882	1.09443	9.64906	−5.77276	2.93378	0.0247321
1.08	0.940566	1.00712	6.36108	−2.878	2.47401	0.0161492
1.09	0.87869	0.999945	3.56782	−0.486854	0.159733	0.0149198
1.17	0.680872	0.9	1.34135	1.14769	0.181797	0.0292205
1.3	0.538289	0.9	0.0108725	1.99307	2.70761	0.0325142
1.31	0.541734	0.9	0.225146	1.80361	2.00866	0.0317996
1.32	0.530351	0.9	0.0906538	1.99627	2.39849	0.0322925
1.33	0.520783	0.9	0.0602407	2.	2.46711	0.0328764

Table 9. *Cont.*

α	ε	ω	λ	μ	ν	$\mathcal{E}$
1.34	0.519176	0.9	0.203658	1.83392	1.99522	0.0325506
1.35	0.514045	0.9	0.142618	1.91355	2.19971	0.0320122
1.36	0.506794	0.9	0.12195	1.93765	2.29168	0.0321516
1.37	0.498145	0.9	0.0840845	1.98538	2.43472	0.0324978
1.38	0.491564	0.9	0.0747146	1.98609	2.49019	0.0324374
1.39	0.485332	0.9	0.0743162	1.97601	2.51203	0.0322589
1.4	0.478038	0.9	0.0499686	1.98704	2.61934	0.0322147
1.41	0.471071	0.9	0.0383513	1.97258	2.67605	0.0320753
1.42	0.463325	0.9	0.0111147	1.98595	2.78815	0.0320548
1.43	0.451257	0.900006	0.0028895	1.99311	2.73632	0.0333135
1.5	0.403126	0.9	0.181392	1.42029	1.95961	0.0316524
1.58	0.337319	0.949223	0.00704273	1.4074	2.21637	0.0337745
1.59	0.329015	0.951034	0.00605698	1.39103	2.15559	0.0333744
1.6	0.321299	0.95331	0.0031614	1.32339	2.09747	0.0324901
1.61	0.313321	0.955398	0.000525712	1.24439	2.03919	0.0317012
1.62	0.304228	0.95653	0.000319626	1.19394	1.97955	0.0317214
1.63	0.29317	0.962873	0.00471285	1.21193	1.71253	0.0334456
1.69	0.239431	0.978995	0.000132019	1.04168	1.1126	0.0337634
1.7	0.233526	0.971481	0.000904563	0.887219	1.3045	0.0302745
1.71	0.226541	0.954866	0.114213	0.80477	1.04249	0.0308724

Table 10

α	ε	ω	λ	μ	ν	$\mathcal{E}$
1.72	0.218686	0.944851	0.227652	0.780667	0.641011	0.0314681
1.73	0.210173	0.945083	0.228815	0.801244	0.573331	0.0314194
1.74	0.201216	0.959201	0.00529895	0.852831	1.29452	0.0308476
1.75	0.193623	0.995526	-0.242128	1.2301	1.93631	0.0327612
1.76	0.184082	0.986233	-0.182603	1.1432	1.6993	0.0303013
1.77	0.17396	0.978521	0.000107094	0.824036	0.86835	0.0274559
1.78	0.164627	0.974772	0.122232	0.504506	0.324126	0.0304066
1.79	0.157451	0.991066	$1.6e-6$	0.416506	0.604265	0.0284246
1.8	0.150436	0.990566	-0.0366678	0.352533	0.76488	0.027914
1.81	0.143349	0.983884	-0.0244455	0.308749	0.747054	0.028919
1.82	0.135959	0.980325	0.	0.281316	0.639564	0.029515
1.83	0.128035	0.984661	0.	0.266394	0.550911	0.0292163
1.84	0.119666	0.991314	0.00290616	0.253632	0.42927	0.0280272
1.85	0.111489	0.994873	0.00733691	0.233966	0.338185	0.0268657
1.86	0.104143	0.992817	0.0119107	0.198335	0.305948	0.0268938
1.87	0.0982656	0.984911	0.0152459	0.137675	0.336537	0.0295332
1.88	0.0886001	0.991155	0.0534113	0.0984177	0.108439	0.0280921
1.89	0.0827019	0.990087	0.000262891	0.1361	0.279865	0.0272129
1.9	0.074831	0.991493	0.00923344	0.118255	0.204796	0.0255918
1.91	0.0677163	0.99291	$2.64e-6$	0.110885	0.200892	0.0252421
1.92	0.0605911	0.992438	0.0154867	0.0478608	0.126686	0.0248368
1.93	0.0529403	0.994779	0.000349593	0.0618894	0.137506	0.0227101
1.94	0.0449666	0.994965	0.0192709	0.00417361	0.0529319	0.0212404
1.95	0.0383727	0.996123	0.00295046	0.0645417	0.0804873	0.0195729
1.96	0.0311126	0.996804	0.00293461	0.00259494	0.0575215	0.0177635
1.97	0.0238845	0.997511	0.00547476	-0.0476078	0.0257993	0.0170734
1.98	0.0151849	0.999264	0.00229705	0.000384475	0.00997451	0.0130932
1.99	0.00719774	0.998838	0.000967276	0.018135	0.0129342	0.0290255

2.5. Fifth Approach. Polynomial approximation.

Given a continuous function $\varphi(t)$ on $[-1,1]$ we may approximate it by means of a linear combination of Chebyshev polynomials in the form

$$\varphi(t) \approx P(t) := \sum_{r=0}^{N} c_r T_r(t), \tag{27}$$

where the constnats c_r are determined from the linera system

$$\varphi(\xi_j) = P(\xi_j),\ j = 0,1,2,3,...,N, \tag{28}$$

where

$$\xi_0 = -1,\, \xi_N = 1,\, \xi_{\lfloor N/2 \rfloor} = 0 \text{ and } \xi_j = \cos\left(\frac{j+1/2}{N+1}\pi\right) \text{ for } 0 < j < N \text{ and } j \neq_{\lfloor N/2 \rfloor}\ . \tag{29}$$

For example, for $N = 5$ we have

$$\varphi(t) \approx P_\varphi(t) := \sum_{r=0}^{5} c_r T_r(t), \tag{30}$$

where

$$\begin{aligned}
c_0 &= \varphi(0).\\
c_1 &= -0.174599\varphi(-1) + 0.816497\varphi(-0.707107) - 4.78174\varphi(-0.258819) + 3.8637\varphi(0) + 0.378937\varphi(0.707107) - 0.102802\varphi(1).\\
c_2 &= -0.5\varphi(-1) + 2\varphi(-0.707107) - 3\varphi(0) + 2.\varphi(0.707107) - 0.5\varphi(1).\\
c_3 &= 1.0238\varphi(-1) - 5.27792\varphi(-0.707107) + 14.3452\varphi(-0.258819) - 11.5911\varphi(0) + 1.69161\varphi(0.707107) - 0.191593\varphi(1).\\
c_4 &= \varphi(-1) - 2\varphi(-0.707107) + 2\varphi(0) - 2\varphi(0.707107) + \varphi(1).\\
c_5 &= -1.3492\varphi(-1) + 4.46142\varphi(-0.707107) - 9.56347\varphi(-0.258819) + 7.72741\varphi(0) - 2.07055\varphi(0.707107) + 0.794395\varphi(1).
\end{aligned} \tag{31}$$

Let $f(t)$ be a continuous on $[a,b]$ function. Define

$$\varphi_f(t) = f\left(\frac{b-a}{2}t + \frac{b+a}{2}\right) \text{ for } -1 \le t \le 1. \tag{32}$$

Then

$$f(t) \approx P_{\varphi_f}\left(\frac{2t}{b-a} - \frac{a+b}{b-a}\right) \text{ for } a \le t \le b. \tag{33}$$

Let us apply this technique for approximatingbthe Mittag-Leffler functions.

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Example. We have the approximations

$$
\begin{aligned}
&E_{1.9}(t) \approx \left(-7.8941 \times 10^{-21}\right) t^{20} + \left(2.33785 \times 10^{-18}\right) t^{19} - \left(3.17959 \times 10^{-16}\right) t^{18} + \\
&\left(2.63053 \times 10^{-14}\right) t^{17} - \left(1.47815 \times 10^{-12}\right) t^{16} + \left(5.96593 \times 10^{-11}\right) t^{15} - \\
&\left(1.78388 \times 10^{-9}\right) t^{14} + \left(4.01915 \times 10^{-8}\right) t^{13} - \left(6.87859 \times 10^{-7}\right) t^{12} + \\
&\left(8.9635 \times 10^{-6}\right) t^{11} - \left(8.88756 \times 10^{-5}\right) t^{10} + \left(6.70204 \times 10^{-4}\right) t^{9} - \\
&\left(3.8524 \times 10^{-3}\right) t^{8} + \left(1.68792 \times 10^{-2}\right) t^{7} - \left(5.47837 \times 10^{-2}\right) t^{6} + \\
&\left(1.18961 \times 10^{-1}\right) t^{5} - \left(1.52666 \times 10^{-1}\right) t^{4} + \left(2.48293 \times 10^{-1}\right) t^{3} - \\
&\left(6.64595 \times 10^{-1}\right) t^{2} - \left(2.41517 \times 10^{-3}\right) t + 1.
\end{aligned}
\tag{34}
$$

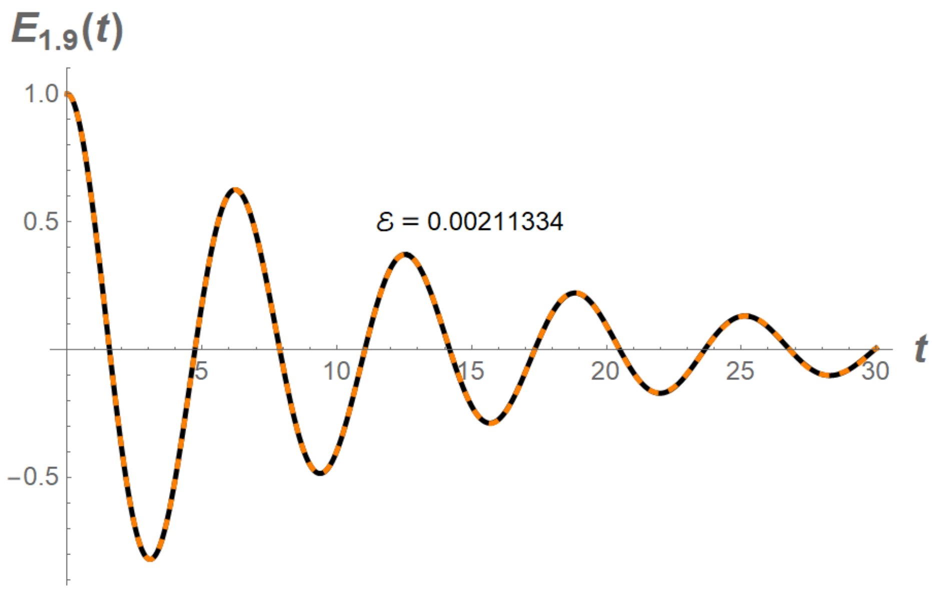

Figure 5

and

$$
\begin{aligned}
&E_{1.9,2}(t) \approx \left(-4.34152 \times 10^{-22}\right) t^{20} + \left(1.3397 \times 10^{-19}\right) t^{19} - \left(1.90429 \times 10^{-17}\right) t^{18} + \\
&\left(1.65206 \times 10^{-15}\right) t^{17} - \left(9.77053 \times 10^{-14}\right) t^{16} + \\
&\left(4.16738 \times 10^{-12}\right) t^{15} - \left(1.32279 \times 10^{-10}\right) t^{14} + \left(3.17957 \times 10^{-9}\right) t^{13} - \\
&\left(5.83735 \times 10^{-8}\right) t^{12} + \left(8.20778 \times 10^{-7}\right) t^{11} - \left(8.83439 \times 10^{-6}\right) t^{10} + \\
&\left(7.27272 \times 10^{-5}\right) t^{9} - \left(4.58522 \times 10^{-4}\right) t^{8} + \left(2.21304 \times 10^{-3}\right) t^{7} - \\
&\left(7.96561 \times 10^{-3}\right) t^{6} + \left(1.93654 \times 10^{-2}\right) t^{5} - \left(2.78186 \times 10^{-2}\right) t^{4} + \\
&\left(5.63977 \times 10^{-2}\right) t^{3} - \left(2.16129 \times 10^{-1}\right) t^{2} - \left(3.05442 \times 10^{-3}\right) t + 1.
\end{aligned}
\tag{35}
$$

See Figure 6.

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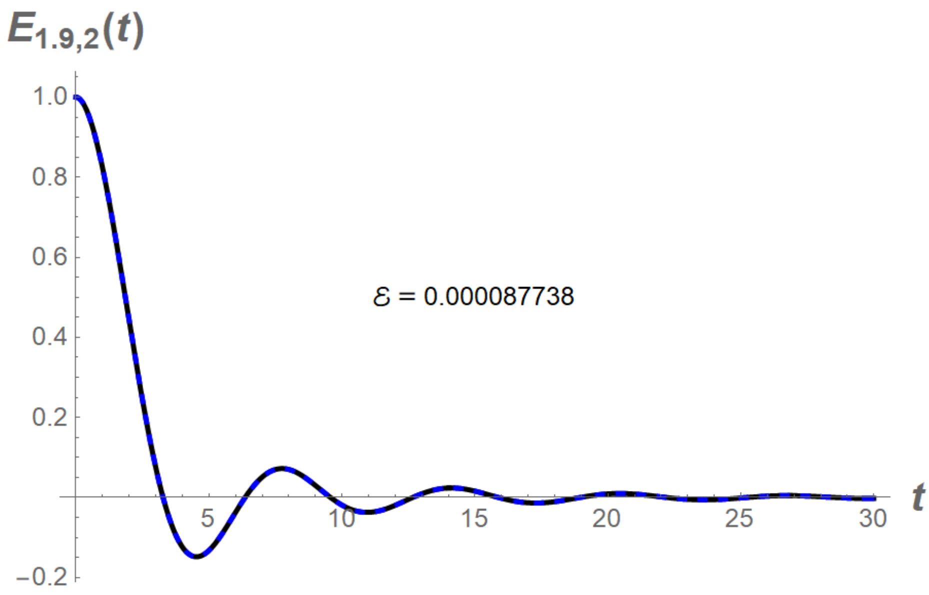

Figure 6

We obtained the following approximations.

2.6. Mittag Leffler function $E_\alpha(-t^\alpha)$.

For $0 \le t \le 10$ and $1.05 \le \alpha \le 2$:

$$
\begin{aligned}
&E_\alpha(-t^\alpha) \approx \\
&-3.31819(\alpha-2.63795)(\alpha-2.19825)(\alpha-0.462826)\left(\alpha^2-3.44556\alpha+3.00567\right)\left(\alpha^2-1.60872\alpha+1.11751\right)t+ \\
&9.4249(\alpha-2.63986)(\alpha-1.17758)(\alpha-0.593342)\left(\alpha^2-4.12231\alpha+4.25257\right)\left(\alpha^2-1.82145\alpha+1.20697\right)t^2+ \\
&-8.97968(\alpha-2.62418)(\alpha-2.22066)(\alpha-1.79894)(\alpha-1.09393)(\alpha-0.662815)\left(\alpha^2-1.92785\alpha+1.25783\right)t^3+ \\
&4.01742(\alpha-2.61737)(\alpha-2.20703)(\alpha-1.69479)(\alpha-1.06394)(\alpha-0.707285)\left(\alpha^2-1.99641\alpha+1.29223\right)t^4+ \\
&-0.950677(\alpha-2.60531)(\alpha-2.16609)(\alpha-1.63372)(\alpha-1.05013)(\alpha-0.73826)\left(\alpha^2-2.04488\alpha+1.31749\right)t^5+ \\
&0.122174(\alpha-2.58602)(\alpha-2.12322)(\alpha-1.59499)(\alpha-1.04277)(\alpha-0.760889)\left(\alpha^2-2.08064\alpha+1.33659\right)t^6+ \\
&-0.00804049(\alpha-2.56221)(\alpha-2.08505)(\alpha-1.56918)(\alpha-1.03844)(\alpha-0.777908)\left(\alpha^2-2.10766\alpha+1.35122\right)t^7+ \\
&0.000211896(\alpha-2.5367)(\alpha-2.0528)(\alpha-1.55133)(\alpha-1.03569)(\alpha-0.790949)\left(\alpha^2-2.12835\alpha+1.36247\right)t^8.
\end{aligned} \tag{36}
$$

For $10 \le t \le 20$ and $1.05 \le \alpha \le 2$:

$$
\begin{aligned}
&E_\alpha(-t^\alpha) \approx \\
&-243804.(\alpha-1.58723)(\alpha-1.54111)(\alpha-1.35855)(\alpha-1.23385)(\alpha-1.07874)(\alpha-1.01037)(\alpha-0.845726)+ \\
&110862.(\alpha-1.36324)(\alpha-1.2264)(\alpha-1.08101)(\alpha-1.00432)(\alpha-0.560557)\left(\alpha^2-3.1297\alpha+2.44928\right)t+ \\
&-19807.(\alpha-1.36665)(\alpha-1.22297)(\alpha-1.08162)(\alpha-1.00257)(\alpha-0.0214421)\left(\alpha^2-3.12999\alpha+2.45035\right)t^2+ \\
&1630.12(\alpha-1.36945)(\alpha-1.22108)(\alpha-1.08186)(\alpha-1.00176)(\alpha+1.3272)\left(\alpha^2-3.12995\alpha+2.45056\right)t^3+ \\
&-36.7302(\alpha-1.37183)(\alpha-1.21991)(\alpha-1.08198)(\alpha-1.0013)(\alpha+10.5852)\left(\alpha^2-3.12999\alpha+2.45065\right)t^4+ \\
&-3.94984(\alpha-8.28782)(\alpha-1.37386)(\alpha-1.21915)(\alpha-1.08204)(\alpha-1.00101)\left(\alpha^2-3.13029\alpha+2.45097\right)t^5+ \\
&0.342(\alpha-4.40187)(\alpha-1.37561)(\alpha-1.21863)(\alpha-1.08208)(\alpha-1.00082)\left(\alpha^2-3.13089\alpha+2.45166\right)t^6+ \\
&-0.010615(\alpha-3.44088)(\alpha-1.37712)(\alpha-1.21826)(\alpha-1.0821)(\alpha-1.00068)\left(\alpha^2-3.13176\alpha+2.4527\right)t^7+ \\
&0.00012203(\alpha-3.00628)(\alpha-1.37841)(\alpha-1.21798)(\alpha-1.08212)(\alpha-1.00058)\left(\alpha^2-3.13285\alpha+2.45406\right)t^8
\end{aligned} \tag{37}
$$

For $20 \le t \le 30$ and $1.05 \le \alpha \le 2$:

$$
\begin{aligned}
&E_\alpha(-t^\alpha) \approx \\
&1.66711\times 10^8(\alpha-1.48737)(\alpha-1.3924)(\alpha-1.21666)(\alpha-1.08216)(\alpha-1)\left(\alpha^2-3.58508\alpha+3.21328\right)+ \\
&-5.47563\times 10^7(\alpha-1.48737)(\alpha-1.3924)(\alpha-1.21666)(\alpha-1.08216)(\alpha-1)\left(\alpha^2-3.58375\alpha+3.21101\right)t+ \\
&7.82832\times 10^6(\alpha-1.48737)(\alpha-1.3924)(\alpha-1.21666)(\alpha-1.08216)(\alpha-1)\left(\alpha^2-3.58244\alpha+3.20875\right)t^2+ \\
&-636231.(\alpha-1.48737)(\alpha-1.3924)(\alpha-1.21666)(\alpha-1.08216)(\alpha-1)\left(\alpha^2-3.58113\alpha+3.20652\right)t^3+ \\
&32149.(\alpha-1.48737)(\alpha-1.3924)(\alpha-1.21666)(\alpha-1.08216)(\alpha-1)\left(\alpha^2-3.57984\alpha+3.20431\right)t^4+ \\
&-1034.23(\alpha-1.48738)(\alpha-1.3924)(\alpha-1.21666)(\alpha-1.08216)(\alpha-1)\left(\alpha^2-3.57856\alpha+3.2021\right)t^5+ \\
&20.6856(\alpha-1.48738)(\alpha-1.3924)(\alpha-1.21666)(\alpha-1.08216)(\alpha-1)\left(\alpha^2-3.57728\alpha+3.19991\right)t^6+ \\
&-0.235189(\alpha-1.48738)(\alpha-1.3924)(\alpha-1.21666)(\alpha-1.08216)(\alpha-1)\left(\alpha^2-3.57601\alpha+3.19773\right)t^7+ \\
&0.00116388(\alpha-1.48738)(\alpha-1.3924)(\alpha-1.21666)(\alpha-1.08216)(\alpha-1)\left(\alpha^2-3.57474\alpha+3.19554\right)t^8.
\end{aligned} \tag{38}
$$

$$\mathcal{E} = \text{ Error } = 0.0340668(3.59621-\alpha)(\alpha-0.997587)(\alpha-0.316375)\left(\alpha^2-4.49614\alpha+5.09479\right)\left(\alpha^2-3.41703\alpha+3.00412\right) \tag{39}$$

2.7. *Mittag Leffler function* $E_{\alpha,2}\left(-t^{\alpha}\right)$.

For $0 \leq t \leq 10$ and $1.05 \leq \alpha \leq 2$:

$$\begin{gathered} E_{\alpha,2}\left(-t^{\alpha}\right) \approx \\ -0.278939(\alpha-2.31677)(\alpha-2.12563)(\alpha+0.0893733)\left(\alpha^2-3.59157\alpha+3.32862\right)\left(\alpha^2-0.889475\alpha+1.39508\right)t \\ 0.826976(\alpha-2.37246)(\alpha-1.12366)(\alpha-0.129251)\left(\alpha^2-4.13191\alpha+4.35456\right)\left(\alpha^2-1.22713\alpha+1.33209\right)t^2 \\ -0.78555(\alpha-1.84908)(\alpha-1.05487)(\alpha-0.245395)\left(\alpha^2-4.49297\alpha+5.05579\right)\left(\alpha^2-1.3987\alpha+1.31131\right)t^3 \\ 0.341291(\alpha-2.27376)(\alpha-2.23048)(\alpha-1.69824)(\alpha-1.03336)(\alpha-0.315277)\left(\alpha^2-1.50041\alpha+1.3013\right)t^4 \\ -0.0772279(\alpha-2.35263)(\alpha-2.10666)(\alpha-1.63099)(\alpha-1.02433)(\alpha-0.359298)\left(\alpha^2-1.56281\alpha+1.29204\right)t^5 \\ 0.00940743(\alpha-2.36727)(\alpha-2.03824)(\alpha-1.59441)(\alpha-1.01982)(\alpha-0.386893)\left(\alpha^2-1.60025\alpha+1.28059\right)t^6 \\ -0.000584189(\alpha-2.36542)(\alpha-1.98829)(\alpha-1.57288)(\alpha-1.01729)(\alpha-0.403095)\left(\alpha^2-1.62059\alpha+1.26611\right)t^7 \\ 0.0000145004(\alpha-2.35663)(\alpha-1.95046)(\alpha-1.55959)(\alpha-1.01573)(\alpha-0.41085)\left(\alpha^2-1.62866\alpha+1.2485\right)t^8. \end{gathered} \tag{40}$$

For $10 \leq t \leq 20$ and $1.05 \leq \alpha \leq 2$:

$$\begin{gathered} E_{\alpha,2}\left(-t^{\alpha}\right) \approx \\ 60943.9(\alpha-2.06184)(\alpha-1.38175)(\alpha-1.19821)(\alpha-1.09657)(\alpha-0.995747)\left(\alpha^2-3.11686\alpha+2.43632\right)+ \\ -36159.(\alpha-2.04536)(\alpha-1.37969)(\alpha-1.20652)(\alpha-1.08925)(\alpha-0.997841)\left(\alpha^2-3.11793\alpha+2.4373\right)t+ \\ 9273.74(\alpha-2.02865)(\alpha-1.37991)(\alpha-1.20961)(\alpha-1.08675)(\alpha-0.998573)\left(\alpha^2-3.11906\alpha+2.43826\right)t^2+ \\ -1342.04(\alpha-2.01202)(\alpha-1.38072)(\alpha-1.21123)(\alpha-1.08549)(\alpha-0.998941)\left(\alpha^2-3.1206\alpha+2.43983\right)t^3+ \\ 119.803(\alpha-1.99573)(\alpha-1.38168)(\alpha-1.21224)(\alpha-1.08474)(\alpha-0.999161)\left(\alpha^2-3.12255\alpha+2.44205\right)t^4+ \\ -6.75367(\alpha-1.98001)(\alpha-1.38264)(\alpha-1.21292)(\alpha-1.08425)(\alpha-0.999304)\left(\alpha^2-3.12483\alpha+2.44481\right)t^5+ \\ 0.234774(\alpha-1.96503)(\alpha-1.38354)(\alpha-1.2134)(\alpha-1.08391)(\alpha-0.999404)\left(\alpha^2-3.12734\alpha+2.44795\right)t^6+ \\ -0.00460174(\alpha-1.95089)(\alpha-1.38434)(\alpha-1.21377)(\alpha-1.08366)(\alpha-0.999477)\left(\alpha^2-3.12998\alpha+2.45135\right)t^7+ \\ 0.0000389484(\alpha-1.93764)(\alpha-1.38506)(\alpha-1.21404)(\alpha-1.08348)(\alpha-0.999532)\left(\alpha^2-3.13267\alpha+2.45489\right)t^8 \end{gathered} \tag{41}$$

For $20 \leq t \leq 30$ and $1.05 \leq \alpha \leq 2$:

$$\begin{gathered} E_{\alpha,2}\left(-t^{\alpha}\right) \approx \\ 1.26462\times10^6(\alpha-1.48703)(\alpha-1.39288)(\alpha-1.21602)(\alpha-1.08268)(\alpha-0.999813)\left(\alpha^2-3.36036\alpha+2.82904\right)+ \\ -351313.(\alpha-1.48714)(\alpha-1.39271)(\alpha-1.21619)(\alpha-1.08251)(\alpha-0.99988)\left(\alpha^2-3.32038\alpha+2.76052\right)t+ \\ 40872.9(\alpha-1.48716)(\alpha-1.39267)(\alpha-1.21618)(\alpha-1.0825)(\alpha-0.999888)\left(\alpha^2-3.26096\alpha+2.65869\right)t^2+ \\ -2547.21(\alpha-1.68286)(\alpha-1.48833)(\alpha-1.4796)(\alpha-1.39284)(\alpha-1.21598)(\alpha-1.08259)(\alpha-0.999864)t^3+ \\ 88.908(\alpha-1.69704)(\alpha-1.48734)(\alpha-1.39215)(\alpha-1.282)(\alpha-1.21343)(\alpha-1.08319)(\alpha-0.999727)t^4+ \\ -1.56062(\alpha-1.70588)(\alpha-1.48733)(\alpha-1.39236)(\alpha-1.21738)(\alpha-1.08116)(\alpha-1.00051)(\alpha-0.765272)t^5+ \\ 0.00488049(\alpha-1.71229)(\alpha-1.48734)(\alpha-1.3924)(\alpha-1.21694)(\alpha-1.08188)(\alpha-1.00011)(\alpha+5.24412)t^6+ \\ 0.000247386(\alpha-3.46421)(\alpha-1.7173)(\alpha-1.48734)(\alpha-1.39241)(\alpha-1.21682)(\alpha-1.08201)(\alpha-1.00006)t^7+ \\ -2.74e-6(\alpha-2.57819)(\alpha-1.72137)(\alpha-1.48735)(\alpha-1.39241)(\alpha-1.21677)(\alpha-1.08206)(\alpha-1.00004)t^8. \end{gathered} \tag{42}$$

$$\mathcal{E} = \text{Error} = 0.0340671(3.59618-\alpha)(\alpha-0.997587)(\alpha-0.316376)\left(\alpha^2-4.49615\alpha+5.0948\right)\left(\alpha^2-3.41703\alpha+3.00412\right). \tag{43}$$

3. Physics Applications.

Suppose we are given that

$$\mathcal{D}_t^{\alpha}x(t)+x(t)+F(t,x,\dot{x})=0,\, x(0)=x_0 \text{ and } x'(0)=\dot{x}_0,,0\leq t\leq T. \tag{44}$$

We approximate the solution to (44) by means of the solution to the following integer order oscillator :

$$\ddot{x}+2\varepsilon\dot{x}\;+\omega^2x+\lambda x^3+\mu x^5+\nu x^7+F(t,x,\dot{x})=0,\, x(0)=x_0 \text{ and } x'(0)=\dot{x}_0, 0\leq t\leq T. \tag{45}$$

The error of this approximation is defined as

$$\mathcal{E}(x,T)=\max_{0\leq t\leq T}\left|\frac{1}{\Gamma(2-\alpha)}\int_0^t\frac{x''(s)ds}{(t-s)^{\alpha-1}}+x(t)+F(t,x(t),\dot{x}(t))\right|. \tag{46}$$

We will refer to this method as *Integerization Principle*. The oscillator (45) may be easily solved with the aid of the Runge-Kutta methods.

3.1. *The fractional Pendulum*

Let

$$\mathcal{D}_t^{\alpha}x(t)+\sin(x(t))=f(t),\, x(0)=x_0 \text{ and } x'(0)=\dot{x}_0,,0\leq t\leq T. \tag{47}$$

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We solve the i.v.p.

$$\ddot{x}+2\varepsilon\dot{x}+\omega^2\left(x-1/6x^3+1/120x^5\right)+\lambda x^3+\mu x^5+\nu x^7=f(t),\,x(0)=x_0 \text{ and } x'(0)=\dot{x}_0, 0\le t\le T. \tag{48}$$

The parameter values ε, ω, λ, μ and ν are obtained from Tables 8 and 9.

Example. Let

$$\mathcal{D}_t^{1.9}x(t)+\sin(x(t))=0.2\cos(0.3t),\,x(0)=0 \text{ and } x'(0)=0, 0\le t\le 30. \tag{49}$$

From Table 9 we get

$$\varepsilon=0.074831,\ \omega=0.991493,\ \lambda=0.00923344,\ \mu=0.118255,\ \nu=0.204796. \tag{50}$$

We replace the i.v.p. (49) with the i.v.p.

$$\begin{aligned}&\ddot{x}+0.149662\dot{x}+0.983059x-0.15461x^3+0.126447x^5+0.204796x^7=0.2\cos(0.3t).\\&x(0)=0,\,x'(0)=0,\ \ 0\le t\le 30.\end{aligned} \tag{51}$$

See Figure 7.

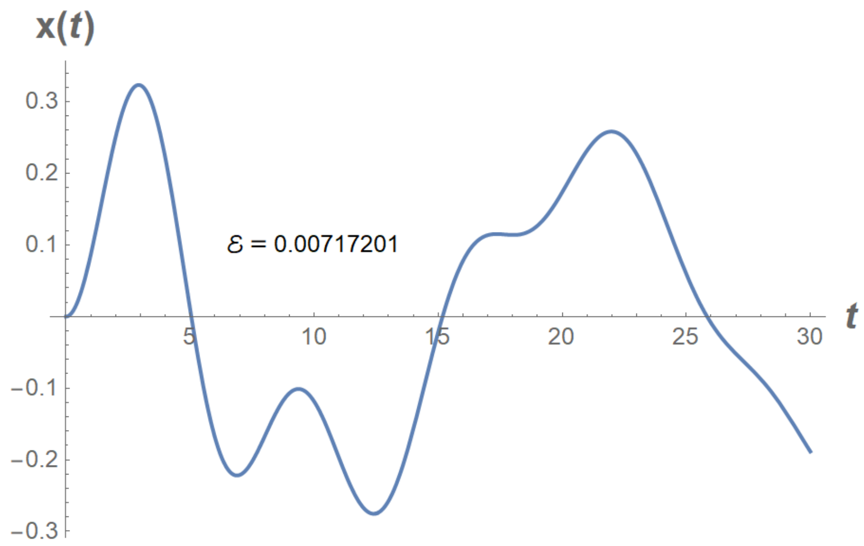

Figure 7

3.2. Fractional Van der Pol oscillator.

Let

$$\mathcal{D}_t^{1.85}x(t)-0.25(1-x(t)^2)x'(t)+x(t)=0,\,x(0)=0.5 \text{ and } x'(0)=0,\ 0\le t\le 30. \tag{52}$$

Parameter values:

$$\varepsilon=0.111489,\ \omega=0.994873,\ \lambda=0.00733691,\ \mu=0.233966,\ \nu=0.338185. \tag{53}$$

We solve the following Duffing-Van der Pol oscillator :

$$\begin{aligned}&x''(t)-0.25\left(1-x(t)^2\right)x'(t)+0.989773x(t)+0.222979x'(t)+0.00733691x(t)^3+0.233966x(t)^5+0.338185x(t)^7\\&=0.\\&x(0)=0.5\wedge x'(0)=0, 0\le t\le 30.\end{aligned} \tag{54}$$

See Figure 8.

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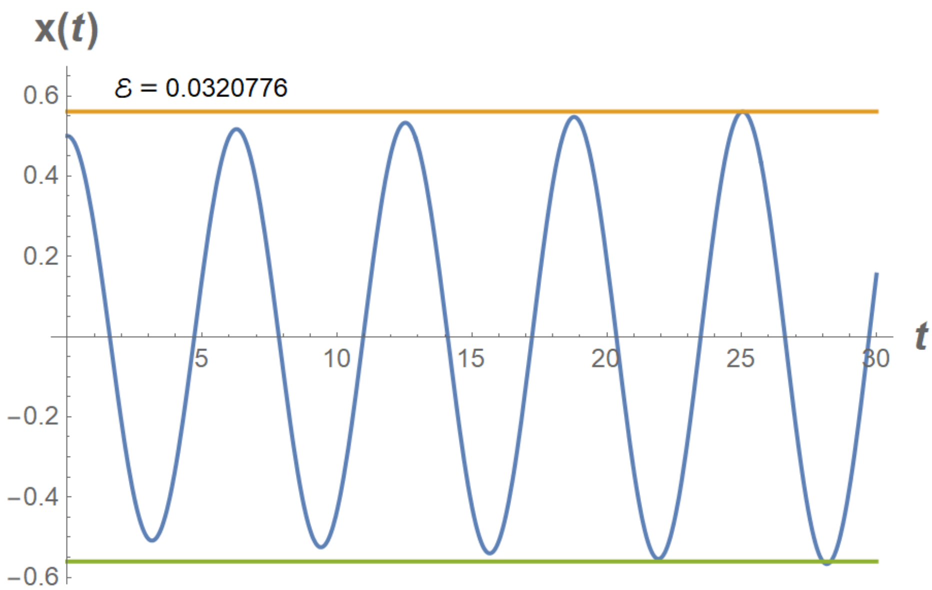

Figure 8

Let now

$$\mathcal{D}_t^{1.95}x(t) - 0.25(1 - x(t)^2)x'(t) + x(t) = 0.1\cos(3t), x(0) = 0 \text{ and } x'(0) = 0,\ 0 \le t \le 30. \tag{55}$$

Parameter values:

$$\varepsilon = 0.0383727,\ \omega = 0.996123,\ \lambda = 0.00295046,\ \mu = 0.0645417,\ \nu = 0.0804873. \tag{56}$$

We solve the following oscillator :

$$\begin{aligned} &x''(t) - 0.25\left(1 - x^2\right)\ \dot{x} + 0.0767453\dot{x} + 0.992262x(+0.00295046x^3 + 0.0645417x^5 + 0.0804873x^7 = 0.1\cos(3t) \\ &x(0) = 0,\ x'(0) = 0,\ 0 \le t \le 30 \end{aligned} \tag{57}$$

See Figure 9.

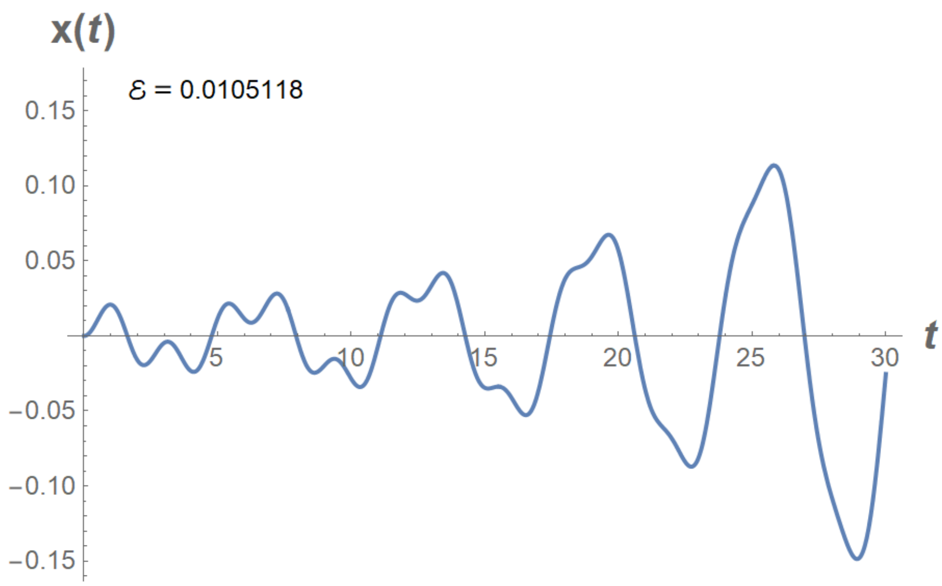

Figure 9

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3.3. Fractional linear oscillator.

Suppose we are given that

$$\mathcal{D}_t^{\alpha} x(t) + x(t) = f(t), x(0) = x_0 \text{ and } x'(0) = \dot{x}_0, , 0 \leq t \leq T. \tag{58}$$

We replace this oscillator with the oscillator

$$\ddot{x} + 2\varepsilon\dot{x} + \omega^2 x + \lambda x^3 + \mu x^5 + \nu x^7 = f(t), \;\; x(0) = x_0 \text{ and } x'(0) = \dot{x}_0, , 0 \leq t \leq T. \tag{59}$$

For example, let

$$\mathcal{D}_t^{1.75} x(t) + x(t) = 0.2\cos t, x(0) = 0 \text{ and } x'(0) = 0, 0 \leq t \leq 40. \tag{60}$$

Parameter values:

$$\varepsilon = 0.193623, \;\omega = 0.995526, \;\lambda = -0.242128, \;\mu = 1.2301, \;\nu = 1.93631. \tag{61}$$

We solve the following oscillator :

$$\begin{aligned} &\ddot{x} + 0.387247\dot{x} + 0.991072x - 0.242128x^3 + 1.2301x^5 + 1.93631x^7 = 0.2\cos(t) \\ &x(0) = 0, x'(0) = 0, 0 \leq t \leq 40 \end{aligned} \tag{62}$$

See Figure 10.

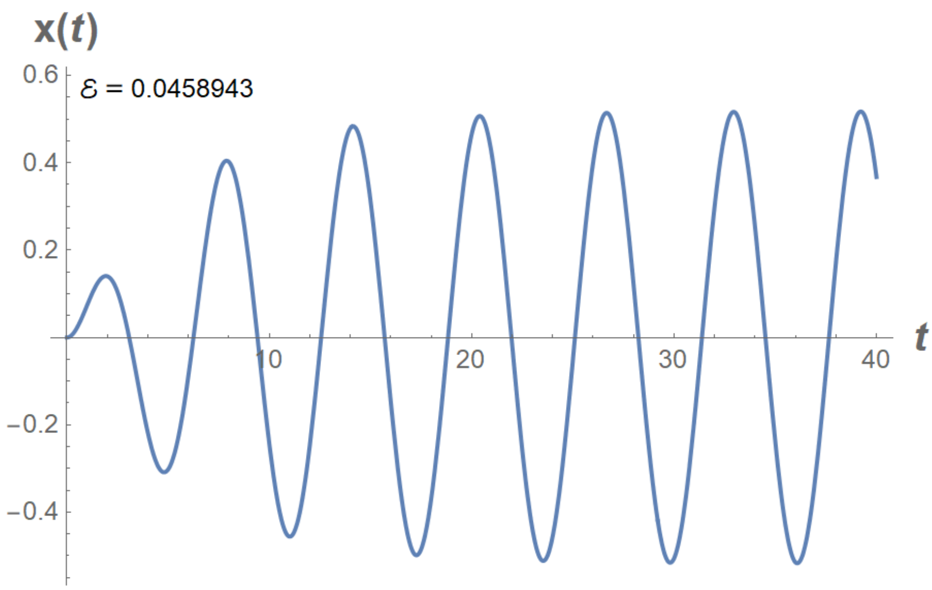

Figure 10

3.4. Fractional Duffing-Mathieu oscillator.

Let us consider the i.v.p.

$$\mathcal{D}_t^{1.9} x(t) + (1 - \cos(0.4t)) + x(t) + x^3(t) = 0, \;\; x(0) = 1 \text{ and } x'(0) = 0, 0 \leq t \leq 30. \tag{63}$$

Parameter values:

$$\varepsilon = 0.074831, \;\omega = 0.991493, \;\lambda = 0.00923344, \;\mu = 0.118255, \;\nu = 0.204796. \tag{64}$$

We solve the following oscillator :

$$\ddot{x} - x\cos(0.4t) + 0.149662\dot{x} + +0.983059x + 1.00923x^3 + 0.118255x^5 + 0.204796x^7 = 0. \quad (65)$$
$$x(0) = 1 \text{ and } x'(0) = 0, 0 \leq t \leq 30.$$

See Figure 11.

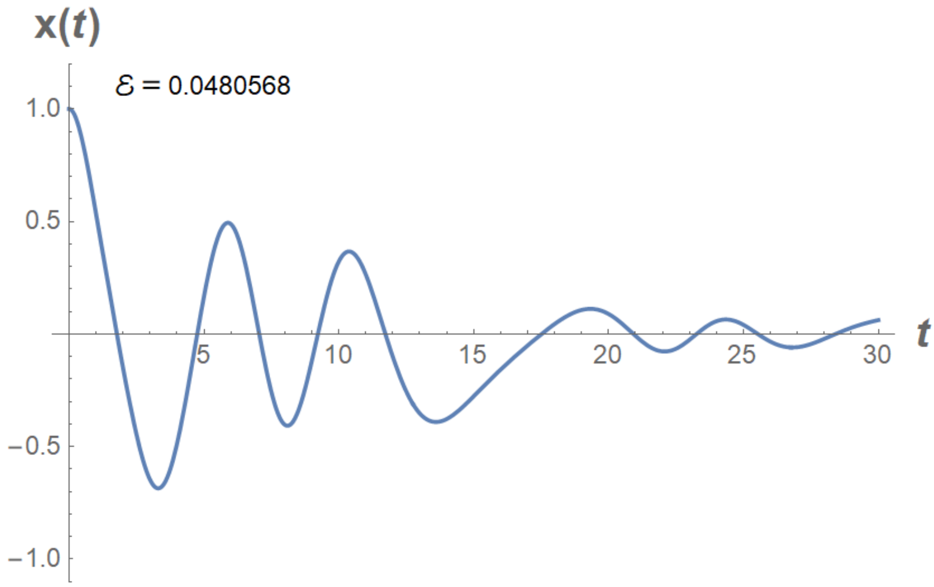

Figure 11

4. Conclusions.

We gave approximate expression for the Mittag-Leffler functions in terms of elementary functions making use of several appeoaches. This allowed us to establish a method we called integerization principle. We demosntrated in concrete examples the effectiveness of the proposed method. We also may solve other fractional nonlinear oescillators using the integerization principle.

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