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Article

The General Solution to a Classical Matrix Equation $AXB = C$ Over the Dual Split Quaternion Algebra

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Abstract: In this paper, we establish the necessary and sufficient conditions for solving a dual split quaternion matrix equation $AXB = C$, and present the general solution expression when solvability is achieved. As an application, we delve into the necessary and sufficient condition for the existence of Hermitian solution to this equation by using a newly defined real representation method. Furthermore, we obtain the solutions for the dual split quaternion matrix equations $AX = C$ and $XB = C$. Finally, we provide a numerical example to demonstrate the findings of this paper.

Keywords: dual split quaternion; real representation; matrix equation; general solution

MSC: 15A03; 15A09; 15A24; 15B33; 15B57

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1. Introduction

In 1843, Hamilton [1] introduced the real quaternions that can be represented as

$$\mathbb{H} = \left\{ q = q_0 + q_1 i + q_2 j + q_3 k : i^2 = j^2 = k^2 = ijk = -1, q_0, q_1, q_2, q_3 \in \mathbb{R} \right\}. \tag{1}$$

The set of real quaternions form a noncommutative division algebra [2,3]. In 1849, James Cockle [4] introduced split quaternions:

$$\mathbb{H}_s = \{ q = q_0 + q_1 i + q_2 j + q_3 k : q_0, q_1, q_2, q_3 \in \mathbb{R} \}, \tag{2}$$

where

$$i^2 = -j^2 = -k^2 = -1, ij = -ji = k, jk = -kj = -i, ki = -ik = j. \tag{3}$$

The collection of split quaternions is an associative and noncommutative four-dimensional Clifford algebra that has zero divisors, nilpotent elements, and nontrivial idempotents [5–7]. It has been widely applied in geometry and physics [8–10]. In 1873, Clifford [11] introduced the collection of dual numbers, which is an expansion of the real numbers by adjoining a new element ϵ with the property $\epsilon^2 = 0$. The set of dual numbers forms a two-dimensional commutative and associative algebra over real numbers. As an extension of quaternions by dual number coefficients, dual quaternions have been used in theoretical kinematics and applications to 3D computer graphics, robotics, and computer vision [12–14]. Similar to dual quaternions, we can extend split quaternions by dual numbers. This nice concept has lots of applications in screw motions and curve theory in 3 dimensional Minkowski space, which has aroused the interest of many scholars [15–18].

In [17], the components of a dual split quaternion are obtained by replacing the L-Euler parameters with their split dual versions. In [19], Kong etl. gave three forms of De Morvie's theorem for the representation matrix of dual split quaternions by using the polar representation of dual split quaternions. In [20], authors use dual split quaternions to represent involution and anti-involution mappings. Some important properties and some interesting results of matrices over dual split quaternions are presented in [21]. Moreover, the dual split quaternionic representation of general displacement is investigated in [18].

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It is well-established that linear matrix equations have been a focal point in matrix theory and its applications. Numerous researchers have devoted attention to studying the solutions of matrix equations [22–26]. The matrix equation

$$AXB = C \tag{4}$$

is a classical and fundamental topic that has been extensively investigated, yielding a series of significant results. For instance, Ben-Israel and Greville [27] provided the necessary and sufficient conditions for the solvability of matrix equation (4). Huang etl. [29] investigated the skew-symmetric solution and the optimal approximate solution of the matrix equation (4). Peng [30] studied the centro-symmetric solutions of matrix equation (4). Xie and Wang [31] deduced the reducible solution to quaternion matrix equation (4). Chen etl. [32] derived the necessary and sufficient conditions for the solvability of dual quaternion matrix equation (4), and presented the expression for the general solution when it is solvable.

So far, there has been limited information on matrix equation (4) over the dual split quaternion algebra. Motivated by the aforementioned work, this paper is dedicated to presenting the solvability conditions and establishing the expression of the general solution for the dual split quaternion matrix equation (4).

This paper is organized as follows. In Section 2, we provide several basic definitions and properties that will be applied in the subsequent sections. In Section 3, we consider the necessary and sufficient conditions for solvability and the expression for the general solution regarding dual split quaternion matrix equation (4). We also deduce the necessary and sufficient condition for the existence of Hermitian solution to (4), and consider some special cases of dual split quaternion matrix equation (4). At the end, a numerical example is given in Section 4.

Throughout this paper, the sets of dual numbers, dual quaternions, dual split quaternions are denoted by $\mathbb{D}$, $\mathbb{DH}$ and $\mathbb{DH}_s$, respectively. The sets of all $m \times n$ matrices over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{H}_s$, $\mathbb{DH}$, $\mathbb{DH}_s$ are denoted by $\mathbb{R}^{m\times n}$, $\mathbb{C}^{m\times n}$, $\mathbb{H}^{m\times n}$, $\mathbb{H}_s^{m\times n}$, $\mathbb{DH}^{m\times n}$, and $\mathbb{DH}_s^{m\times n}$, respectively. The symbols $I_n, 0, A^*$ represent the $n \times n$ identity matrix, the zero matrix with appropriate size, and the conjugate transpose of A, respectively. A^T and $A^\dagger$ denote the transpose and the Moore-Penrose inverse of matrix A, respectively. $L_A = I - A^\dagger A$ and $R_A = I - AA^\dagger$ are two projectors induced by $A^\dagger$.

2. Preliminary

In this section, we will explore the definitions of dual numbers, dual split quaternions, and associated properties. Additionally, we will introduce the concept of dual split quaternion matrices and elaborate on the real representation for split quaternion matrices, which plays a fundamental role in deriving the main results.

2.1. Dual Numbers and Dual Split Quaternions

The set of dual numbers is denoted by

$$\mathbb{D} = \{x = x_0 + x_1\epsilon | \epsilon^2 = 0, x_0, x_1 \in \mathbb{R}\}$$

where ϵ is the infinitesimal unit. We call x_0 the real part or the standard part of x, x_1 as the dual part or the infinitesimal part of x. For any dual numbers $x = x = x_0 + x_1\epsilon$ and $y = y_0 + y_1\epsilon$, we have $x = y$ if $x_0 = y_0$ and $x_1 = y_1$, the sum and product of x and y are defined as

$$x + y = x_0 + y_0 + (x_1 + y_1)\epsilon,$$
$$xy = x_0y_0 + (x_0y_1 + x_1y_0)\epsilon.$$

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Moreover, the conjugate and norm of x are defined by

$$\overline{x} = x_0 - x_1\epsilon,$$
$$r = |x| = \sqrt{x\overline{x}} = |x_0|,$$

respectively. The set of dual quaternions, which can be considered as an extension of quaternions by dual numbers, is represented as

$$\mathbb{DH} = \{q = q_0 + q_1 i + q_2 j + q_3 k : q_0, q_1, q_2, q_3 \in \mathbb{D}\},$$

where

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = ik = j,$$

and

$$\epsilon i = i\epsilon, \epsilon j = j\epsilon, \epsilon k = k\epsilon, \epsilon \neq 0, \epsilon^2 = 0.$$

In a similar way, we can present the definition of dual split quaternion, which can be considered as an extension of split quaternions by dual numbers, is represented as

$$\mathbb{DH}_s = \{q = q_0 + q_1 i + q_2 j + q_3 k : q_0, q_1, q_2, q_3 \in \mathbb{D}\},$$

where

$$i^2 = -j^2 = -k^2 = -1, ij = -ji = k, jk = -kj = -i, ki = -ik = j,$$

and

$$\epsilon i = i\epsilon, \epsilon j = j\epsilon, \epsilon k = k\epsilon, \epsilon \neq 0, \epsilon^2 = 0.$$

Now, we introduce the definitions of dual quaternion matrix and dual split quaternion matrix along with several relevant definitions.

Let $X_0, X_1 \in \mathbb{H}^{m\times n}(\mathbb{H}_s^{m\times n})$. X is said to be a dual quaternion (dual split quaternion) matrix if X has the form $X = X_0 + X_1\epsilon$, the set of dual quaternion matrices and dual split quaternion matrices are denoted by

$$\mathbb{DH}^{m\times n} = \{X = X_0 + X_1\epsilon | \epsilon^2 = 0, X_0, X_1 \in \mathbb{H}^{m\times n}\},$$

and

$$\mathbb{DH}_s^{m\times n} = \{X = X_0 + X_1\epsilon | \epsilon^2 = 0, X_0, X_1 \in \mathbb{H}_s^{m\times n}\},$$

respectively.

The set of $n \times n$ dual split quaternion matrices with standard matrix summation and multiplication is a ring with unity. For any $A = (A_{ij}) \in \mathbb{DH}_s^{m\times n}$ and $q \in \mathbb{DH}_s$, right and left scalar multiplications are defined as $Aq = (A_{ij}q)$ and $qA = (qA_{ij})$, respectively. So, $\mathbb{DH}_s^{m\times n}$ is left (right) vector space over $\mathbb{DH}_s$. For any $A = A_0 + A_1\epsilon = (A_{ij}) \in \mathbb{DH}_s^{m\times n}$, the Hamiltonian conjugate of A is defined as $\overline{A} = \overline{A_0} + \overline{A_1}\epsilon = (\overline{A_{ij}}) \in \mathbb{DH}_s^{m\times n}$, the transpose of A is defined as $A^T = {A_0}^T + {A_1}^T\epsilon = (A_{ji}) \in \mathbb{DH}_s^{n\times m}$ and the conjugate transpose of A is defined as $A^* = {A_0}^* + {A_1}^*\epsilon = (\overline{A})^T \in \mathbb{DH}_s^{n\times m}$.

2.2. *Real representation of split quaternion matrices and its properties*

For any matrix $A \in {\mathbb{H}_s}^{m\times n}$, it can be represented uniquely as $A = A_1 + A_2 i + A_3 j + A_4 k$, where $A_1, A_2, A_3, A_4 \in \mathbb{R}^{m\times n}$, $A^* = {A_1}^T - {A_2}^T i - {A_3}^T j - {A_4}^T k$ is the usual conjugate transpose of A. In addition, we define i-conjugate and i-conjugate transpose as follows:

$$A^i = i^{-1}Ai = A_1 + A_2 i - A_3 j - A_4 k,$$
$$A^{i*} = -iA^*i = {A_1}^T - {A_2}^T i + {A_3}^T j + {A_4}^T k.$$

It is evident that $A^{i*} = (A^*)^i = (A^i)^*$.

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The real representation method is crucial in analyzing the foundational theory of split quaternions. For $A \in \mathbb{H}_s^{m\times n}$, $A = A_1 + A_2 i + A_3 j + A_4 k$, $A_1, A_2, A_3, A_4 \in \mathbb{R}^{m\times n}$, we define

$$A^{\sigma_1} := \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ -A_2 & A_1 & -A_4 & A_3 \\ A_3 & -A_4 & A_1 & -A_2 \\ A_4 & A_3 & A_2 & A_1 \end{pmatrix}.$$

To further explore the properties of split quaternion matrices, based on the classical real representation A^{σ_1}, we define a new real representation as follows.

Definition 1. *Suppose that* $A = A_1 + A_2 i + A_3 j + A_4 k \in \mathbb{H}_s{}^{m\times n}$, $A_1, A_2, A_3, A_4 \in \mathbb{R}^{m\times n}$, *we define*

$$A^{\sigma_i} := U_m A^{\sigma_1} = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ -A_2 & A_1 & -A_4 & A_3 \\ -A_3 & A_4 & -A_1 & A_2 \\ -A_4 & -A_3 & -A_2 & -A_1 \end{pmatrix}, U_m = \begin{pmatrix} I_m & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & -I_m \end{pmatrix}.$$

The properties of the real representations will be presented subsequently. For simplicity, we denote

$$P_m = \begin{pmatrix} 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & -I_m \\ I_m & 0 & 0 & 0 \\ 0 & -I_m & 0 & 0 \end{pmatrix},$$

$$Q_m = \begin{pmatrix} 0 & -I_m & 0 & 0 \\ I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_m \\ 0 & 0 & I_m & 0 \end{pmatrix},$$

$$R_m = \begin{pmatrix} 0 & 0 & 0 & I_m \\ 0 & 0 & I_m & 0 \\ 0 & I_m & 0 & 0 \\ I_m & 0 & 0 & 0 \end{pmatrix}.$$

Proposition 1. Let $A, B \in \mathbb{H}_s^{m\times n}, C \in \mathbb{H}_s^{n\times p}$, and $b \in \mathbb{R}$. Then,

1. $A = B \Leftrightarrow A^{\sigma_1} = B^{\sigma_1}, A = B \Leftrightarrow A^{\sigma_i} = B^{\sigma_i}$.
2. $(A+B)^{\sigma_1} = A^{\sigma_1} + B^{\sigma_1}, (bA)^{\sigma_1} = bA^{\sigma_1}; (A+B)^{\sigma_i} = A^{\sigma_i} + B^{\sigma_i}, (bA)^{\sigma_i} = bA^{\sigma_i}$.
3. $(AC)^{\sigma_1} = A^{\sigma_1} C^{\sigma_1}, (AC)^{\sigma_i} = A^{\sigma_i} U_n C^{\sigma_i}$;
4. (i)${P_m}^T A^{\sigma_1} P_n = A^{\sigma_1}, {Q_m}^T A^{\sigma_1} Q_n = A^{\sigma_1}, {R_m}^T A^{\sigma_1} R_n = A^{\sigma_1}$;
   (ii)${P_m}^T A^{\sigma_i} P_n = -A^{\sigma_i}, {Q_m}^T A^{\sigma_i} Q_n = A^{\sigma_i}, {R_m}^T A^{\sigma_i} R_n = -A^{\sigma_i}$.
5. (i)$A = \frac{1}{2}\begin{pmatrix} I_m & I_m i & I_m j & I_m k \end{pmatrix} A^{\sigma_1} \begin{pmatrix} I_n \\ I_n i \\ I_n j \\ I_n k \end{pmatrix}$.
   (ii)$A = \frac{1}{2}\begin{pmatrix} I_m & I_m i & I_m j & I_m k \end{pmatrix} A^{\sigma_i} \begin{pmatrix} -I_n \\ -I_n i \\ -I_n j \\ -I_n k \end{pmatrix}$.
6. $(A^*)^{\sigma_i} = (A^{\sigma_i})^T$.
7. $(A^i)^{\sigma_i} = U_m A^{\sigma_i} U_n$.

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The proof for Proposition 1 is relatively straightforward, and we omit it.

3. The Solution of Matrix equation(4)

In this section, we pay attention to deriving the solution to the dual split quaternion matrix equation (4). We start with several useful results over $\mathbb{H}$ or $\mathbb{DH}$, which also hold over $\mathbb{R}$.

Lemma 1 ([27]). *Suppose that A, B, and C are provided for matrices with the adequate dimensions over $\mathbb{H}$; then, quaternion matrix equation (4) is consistent if and only if*

$$R_A C = 0, \quad CL_B = 0.$$

In this case, the general solution can be expressed as

$$X = A^{\dagger} C B^{\dagger} + L_A U + V R_B,$$

where U, V are any matrices over $\mathbb{H}$ with appropriate dimensions.

Lemma 2 ([31]). *Let A_1, A_2, B_1, B_2, and C_1 be given matrices with appropriate sizes. Set*

$$A = R_{A_1} C,\ B = B_1 L_{B_2},\ M = R_{A_1} A_2,\ C_1 = C L_{B_2}.$$

Then, the following descriptions are equivalent:

(1) *The quaternion matrix equation*

$$A_1 X_1 B_1 + A_1 X_2 B_2 + A_2 X_3 B_2 = C \tag{5}$$

is consistent.

(2) $R_M A = 0,\ R_{A_1} C L_{B_2} = 0,\ C_1 L_B = 0.$

(3)

$$r\begin{pmatrix} A_1 & A_2 & C \end{pmatrix} = r\begin{pmatrix} A_1 & A_2 \end{pmatrix},$$
$$r\begin{pmatrix} B_2 & 0 \\ C & A_1 \end{pmatrix} = r\begin{pmatrix} B_2 \end{pmatrix} + r\begin{pmatrix} A_1 \end{pmatrix},$$
$$r\begin{pmatrix} C_1 \\ B_1 \\ B_2 \end{pmatrix} = r\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

In this case, the general solution to (5) can be expressed as follows:

$$\begin{aligned} X_1 &= A_1^{\dagger} C_1 B^{\dagger} + L_{A_1} V_1 + V_2 R_B, \\ X_2 &= A_1^{\dagger}(C - A_1 X_1 B_1 - A_2 X_3 B_2) B_2^{\dagger} + T_1 R_{B_2} + L_{A_1} T_2, \\ X_3 &= M^{\dagger} A B_2^{\dagger} + L_M U_1 + U_2 R_{B_2}, \end{aligned}$$

where U_1, U_2, V_1, V_2, T_1, and T_2 are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

Lemma 3 ([32]). *Let $A = A_0 + A_1\epsilon \in \mathbb{DH}^{m\times n}$, $B = B_0 + B_1\epsilon \in \mathbb{DH}^{r\times l}$, $C = C_0 + C_1\epsilon \in \mathbb{DH}^{m\times l}$. Put*

$$A_2 = A_1 L_{A_0},\ B_2 = R_{B_0} B_1,\ C_{11} = A_0 A_0^{\dagger} C_0 B_0^{\dagger} B_1,$$

$$C_{22} = A_1 A_0^{\dagger} C_0 B_0^{\dagger} B_0,\ C_2 = C_1 - C_{11} - C_{22},$$

$$M = R_{A_0} A_2,\ N = R_{A_0} C_2,\ E = B_2 L_{B_0},\ F = C_2 L_{B_0}.$$

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Then, the following statements are equivalent:

(1) *Dual quaternion matrix equation* (4) *is consistent.*

(2)

$$\begin{aligned} &R_{A_0}C_0 = 0,\ C_0L_{B_0} = 0,\\ &R_M N = 0,\ R_{A_0}C_2L_{B_0} = 0,\ FL_E = 0. \end{aligned}$$

(3)

$$\begin{aligned} &r\begin{pmatrix} A_0 & C_0 \end{pmatrix} = r\begin{pmatrix} A_0 \end{pmatrix}, r\begin{pmatrix} B_0 \\ C_0 \end{pmatrix} = r\begin{pmatrix} B_0 \end{pmatrix},\\ &r\begin{pmatrix} A_1 & A_0 & C_1 \\ A_0 & 0 & C_0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_0 \\ A_0 & 0 \end{pmatrix},\\ &r\begin{pmatrix} C_1 & A_0 \\ B_0 & 0 \end{pmatrix} = r\begin{pmatrix} A_0 \end{pmatrix} + r\begin{pmatrix} B_0 \end{pmatrix},\\ &r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \\ C_1 & C_0 \end{pmatrix} = r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \end{pmatrix}. \end{aligned}$$

In this case, the general solution X of dual quaternion matrix equation (4) *can be expressed as* $X = X_0 + X_1\epsilon$, *where*

$$\begin{aligned} &X_0 = A_0^\dagger C_0 B_0^\dagger + L_{A_0}U + VR_{B_0},\\ &X_1 = A_0^\dagger(C_2 - A_0VB_2 - A_2UB_0)B_0^\dagger + W_1R_{B_0} + L_{A_0}W_2,\\ &U = M^\dagger NB_0^\dagger + L_MQ_1 + Q_2R_{B_0},\\ &V = A_0^\dagger FE^\dagger + L_{A_0}Q_3 + Q_4R_E, \end{aligned} \tag{6}$$

and $Q_i(i = \overline{1,4})$, $W_i(i = \overline{1,2})$ *are arbitrary matrices over* $\mathbb{H}$ *with appropriate dimensions.*

Using the above lemmas and applying the real representation method of split quaternions, we can deduce the general solution of the matrix equation (4) over the dual split quaternion algebra.

Theorem 1. *Let* $A = A_{00} + A_{01}\epsilon \in \mathbb{DH}_s^{m\times n}$, $B = B_{00} + B_{01}\epsilon \in \mathbb{DH}_s^{r\times l}$, $C = C_{00} + C_{01}\epsilon \in \mathbb{DH}_s^{m\times l}$. *Put*

$$A_0 = A_{00}{}^{\sigma_1}, A_1 = A_{01}{}^{\sigma_1}, B_0 = B_{00}{}^{\sigma_1}, B_1 = B_{01}{}^{\sigma_1}, C_0 = C_{00}{}^{\sigma_1}, C_1 = C_{01}{}^{\sigma_1}, \tag{7}$$

$$A_2 = A_1L_{A_0},\ B_2 = R_{B_0}B_1,\ C_{11} = A_0A_0^\dagger C_0B_0^\dagger B_1, \tag{8}$$

$$C_{22} = A_1A_0^\dagger C_0B_0^\dagger B_0,\ C_2 = C_1 - C_{11} - C_{22}, \tag{9}$$

$$M = R_{A_0}A_2,\ N = R_{A_0}C_2,\ E = B_2L_{B_0},\ F = C_2L_{B_0}. \tag{10}$$

Then, the following statements are equivalent:

(1) *Dual split quaternion matrix equation* (4) *is consistent.*

(2) *The system of real matrix equations*

$$\begin{cases} A_0X_0B_0 = C_0,\\ A_0X_0B_1 + A_0X_1B_0 + A_1X_0B_0 = C_1, \end{cases} \tag{11}$$

is consistent.

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(3)

$$R_{A_0}C_0 = 0,\ C_0L_{B_0} = 0, \tag{12}$$

$$R_M N = 0,\ R_{A_0}C_2L_{B_0} = 0,\ FL_E = 0. \tag{13}$$

(4)

$$r\begin{pmatrix} A_0 & C_0 \end{pmatrix} = r\begin{pmatrix} A_0 \end{pmatrix}, r\begin{pmatrix} B_0 \\ C_0 \end{pmatrix} = r\begin{pmatrix} B_0 \end{pmatrix}, \tag{14}$$

$$r\begin{pmatrix} A_1 & A_0 & C_1 \\ A_0 & 0 & C_0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_0 \\ A_0 & 0 \end{pmatrix}, \tag{15}$$

$$r\begin{pmatrix} C_1 & A_0 \\ B_0 & 0 \end{pmatrix} = r\begin{pmatrix} A_0 \end{pmatrix} + r\begin{pmatrix} B_0 \end{pmatrix}, \tag{16}$$

$$r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \\ C_1 & C_0 \end{pmatrix} = r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \end{pmatrix}. \tag{17}$$

In this case, the general solution X of dual split quaternion matrix equation (4) can be expressed as $X = X_{00} + X_{01}\epsilon$, where

$$\begin{aligned} X_{00} &= \frac{1}{8}\begin{pmatrix} I_n & I_n i & I_n j & I_n k \end{pmatrix}(X_0 + P_n X_0 {P_r}^T + Q_n X_0 {Q_r}^T + R_n X_0 {R_r}^T)\begin{pmatrix} I_r \\ I_r i \\ I_r j \\ I_r k \end{pmatrix}, \\ X_{01} &= \frac{1}{8}\begin{pmatrix} I_n & I_n i & I_n j & I_n k \end{pmatrix}(X_1 + P_n X_1 {P_r}^T + Q_n X_1 {Q_r}^T + R_n X_1 {R_r}^T)\begin{pmatrix} I_r \\ I_r i \\ I_r j \\ I_r k \end{pmatrix}, \end{aligned} \tag{18}$$

where

$$\begin{aligned} &X_0 = A_0^\dagger C_0 B_0^\dagger + L_{A_0}U + VR_{B_0}, \\ &X_1 = A_0^\dagger(C_2 - A_0VB_2 - A_2UB_0)B_0^\dagger + W_1R_{B_0} + L_{A_0}W_2, \\ &U = M^\dagger NB_0^\dagger + L_M Q_1 + Q_2R_{B_0}, \\ &V = A_0^\dagger FE^\dagger + L_{A_0}Q_3 + Q_4R_E, \end{aligned} \tag{19}$$

and $Q_i(i = \overline{1,4})$, $W_i(i = \overline{1,2})$ are arbitrary matrices over $\mathbb{R}$ with appropriate dimensions.

Proof. (1) $\Leftrightarrow$ (2): Suppose that dual split quaternion matrix equation (4) is solvable and its solution is $X \in \mathbb{DH}_s^{n\times r}$, which can be expressed as

$$X = X_{00} + X_{01}\epsilon, \tag{20}$$

where $X_{00}, X_{01} \in \mathbb{H}_s^{n\times r}$. Let$X_0 = {X_{00}}^{\sigma_1}, X_1 = {X_{01}}^{\sigma_1}$. Substituting (20) into (4), by the definition of equality of dual split quaternion matrices, we can obtain that dual split quaternion matrix equation (4) is equivalent to the system of split quaternion matrix equations

$$\begin{cases} A_{00}X_{00}B_{00} = C_{00}, \\ A_{00}X_{00}B_{01} + A_{00}X_{01}B_{00} + A_{01}X_{00}B_{00} = C_{01}. \end{cases} \tag{21}$$

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Applying (3) of Proposition 1 to (11) yields

$$\begin{cases} {A_{00}}^{\sigma_1}{X_{00}}^{\sigma_1}{B_{00}}^{\sigma_1} = {C_{00}}^{\sigma_1}, \\ {A_{00}}^{\sigma_1}{X_{00}}^{\sigma_1}{B_{01}}^{\sigma_1} + {A_{00}}^{\sigma_1}{X_{01}}^{\sigma_1}{B_{00}}^{\sigma_1} + {A_{01}}^{\sigma_1}{X_{00}}^{\sigma_1}{B_{00}}^{\sigma_1} = {C_{01}}^{\sigma_1}, \end{cases} \tag{22}$$

i.e.,

$$\begin{cases} A_0X_0B_0 = C_0, \\ A_0X_0B_1 + A_0X_1B_0 + A_1X_0B_0 = C_1. \end{cases}$$

Clearly, (X_0, X_1) is a pair of solutions to the system (11).

Conversely, if the real system has a pair of solutions (X_0, X_1), which can be expressed as

$$X_0 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \in \mathbb{R}^{4n\times 4r},$$

and

$$X_1 = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix} \in \mathbb{R}^{4n\times 4r},$$

respectively, where $a_{ij}, b_{ij} \in \mathbb{R}^{n\times r}, (i, j = \overline{1, 2})$. Using (4) of Proposition 1 to the above equations, we can obtain

$$\begin{cases} {P_m}^TA_0P_nX_0{P_r}^TB_0P_l = {P_m}^TC_0P_l, \\ {P_m}^TA_0P_nX_0{P_r}^TB_1P_l + {P_m}^TA_0P_nX_1{P_r}^TB_0P_l + {P_m}^TA_1P_nX_0{P_r}^TB_0P_l = {P_m}^TC_1P_l. \end{cases}$$

Hence,

$$\begin{cases} A_0P_nX_0{P_r}^TB_0 = C_0, \\ A_0P_nX_0{P_r}^TB_1 + A_0P_nX_1{P_r}^TB_0 + A_1P_nX_0{P_r}^TB_0 = C_1, \end{cases}$$

which follows that $(P_nX_0{P_r}^T, P_nX_1{P_r}^T)$ is a pair of solutions to the system (11). Similarly, $(Q_nX_0{Q_r}^T, Q_nX_1{Q_r}^T)$, $(R_nX_0{R_r}^T, R_nX_1{R_r}^T)$ are also pairs of solutions to the system (11). Then, so is (Y_0, Y_1), where

$$Y_0 = \frac{1}{4}(X_0 + P_nX_0{P_r}^T + Q_nX_0{Q_r}^T + R_nX_0{R_r}^T),$$

$$Y_1 = \frac{1}{4}(X_1 + P_nX_1{P_r}^T + Q_nX_1{Q_r}^T + R_nX_1{R_r}^T).$$

By direct computation, we have

$$Y_0 = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ -c_2 & c_1 & -c_4 & c_3 \\ c_3 & -c_4 & c_1 & -c_2 \\ c_4 & c_3 & c_2 & c_1 \end{pmatrix},$$

$$Y_1 = \begin{pmatrix} d_1 & d_2 & d_3 & d_4 \\ -d_2 & d_1 & -d_4 & d_3 \\ d_3 & -d_4 & d_1 & -d_2 \\ d_4 & d_3 & d_2 & d_1 \end{pmatrix},$$

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where

$$c_1 = \frac{1}{4}(a_{11} + a_{22} + a_{33} + a_{44}), \quad c_2 = \frac{1}{4}(a_{12} - a_{21} - a_{34} + a_{43}),$$
$$c_3 = \frac{1}{4}(a_{13} + a_{24} + a_{31} + a_{42}), \quad c_4 = \frac{1}{4}(a_{14} - a_{23} - a_{32} + a_{41}),$$

and

$$d_1 = \frac{1}{4}(b_{11} +_{22} +b_{33} + b_{44}), \quad d_2 = \frac{1}{4}(b_{12} - b_{21} - b_{34} + b_{43}),$$
$$d_3 = \frac{1}{4}(b_{13} + b_{24} + b_{31} + b_{42}), \quad d_4 = \frac{1}{4}(b_{14} - b_{23} - b_{32} + b_{41}).$$

Now, we construct that

$$X_{00} = c_1 + c_2 i + c_3 j + c_4 k = \frac{1}{2}\begin{pmatrix} I_n & I_n i & I_n j & I_n k \end{pmatrix} Y_0 \begin{pmatrix} I_r \\ I_r i \\ I_r j \\ I_r k \end{pmatrix} \in \mathbb{H}_s{}^{n\times r},$$

$$X_{01} = d_1 + d_2 i + d_3 j + d_4 k = \frac{1}{2}\begin{pmatrix} I_n & I_n i & I_n j & I_n k \end{pmatrix} Y_1 \begin{pmatrix} I_r \\ I_r i \\ I_r j \\ I_r k \end{pmatrix} \in \mathbb{H}_s{}^{n\times r}.$$

According to (5) of Proposition 1, $X_{00}{}^{\sigma_1} = Y_0$, $X_{01}{}^{\sigma_1} = Y_1$. Consequently,

$$\begin{cases} A_{00}{}^{\sigma_1} X_{00}{}^{\sigma_1} B_{00}{}^{\sigma_1} = C_{00}{}^{\sigma_1}, \\ A_{00}{}^{\sigma_1} X_{00}{}^{\sigma_1} B_{01}{}^{\sigma_1} + A_{00}{}^{\sigma_1} X_{01}{}^{\sigma_1} B_{00}{}^{\sigma_1} + A_{01}{}^{\sigma_1} X_{00}{}^{\sigma_1} B_{00}{}^{\sigma_1} = C_{01}{}^{\sigma_1}, \end{cases}$$

indicating that (X_{00}, X_{01}) is a pair of solutions to the system of split quaternion matrix equations (21). From lemma 3, we can easily know that the system of split quaternion matrix equations (21) is equivalent to the dual split quaternion matrix equation (4). Thus, matrix equation (4) has a dual split solution $X \in \mathbb{DH}_s^{n\times r}$ if and only if the system of real matrix equations (11) is consistent. And in such case, the general solution to the dual split quaternion matrix equation (4) can be expressed as (18) and (19).

According to lemma 1, lemma 2 and lemma 3, we can easily verify that the system (11) is consistent if and only if (12)—(17) hold. Thus, we have shown the equivalence of (2), (3) and (4).
□

As an application of the above theorem and real representation method, next we explore the necessary and sufficient condition for the existence of Hermitian solution to dual split quaternion matrix equation (4).

Corollary 1. *Let* $A = A_{00} + A_{01}\epsilon \in \mathbb{DH}_s^{m\times n}$, $B = B_{00} + B_{01}\epsilon \in \mathbb{DH}_s^{n\times l}$, $C = C_{00} + C_{01}\epsilon \in \mathbb{DH}_s^{m\times l}$. *Put*

$$A_0 = A_{00}{}^{\sigma_i}, A_1 = A_{01}{}^{\sigma_i},$$
$$B_0 = B_{00}{}^{\sigma_i}, B_1 = B_{01}{}^{\sigma_i},$$
$$C_0 = C_{00}{}^{\sigma_i}, C_1 = C_{01}{}^{\sigma_i}.$$

Then the dual split quaternion matrix equation (4) *has a Hermitian solution* $X = X^* \in \mathbb{DH}_s^{n\times n}$ *if and only if the system of real matrix equations*

$$\begin{cases} A_0 X_0 B_0 = C_0, \\ A_0 X_0 B_1 + A_0 X_1 B_0 + A_1 X_0 B_0 = C_1, \end{cases} \tag{23}$$

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has a pair of symmetric solutions (X_0, X_1).

Proof. Assume that $X = X^* \in \mathbb{DH}_s^{n\times n}$ is a solution to the dual split quaternion matrix equation (4), which can be expressed as

$$X = X_{00} + X_{01}\epsilon, \tag{24}$$

where $X_{00}, X_{01} \in \mathbb{H}_s^{n\times n}$, and $X_{00} = X_{00}{}^*$, $X_{01} = X_{01}{}^*$. Let $X_0 = U_n X_{00}{}^{\sigma_i} U_n, X_1 = U_n X_{01}{}^{\sigma_i} U_n$. By combining (23) and (6) of proposition 1, we can obtain that

$$\begin{cases} A_{00}{}^{\sigma_i} U_n X_{00}{}^{\sigma_i} U_n B_{00}{}^{\sigma_i} = C_{00}{}^{\sigma_i}, \\ A_{00}{}^{\sigma_i} U_n X_{00}{}^{\sigma_i} U_n B_{01}{}^{\sigma_i} + A_{00}{}^{\sigma_i} U_n X_{01}{}^{\sigma_i} U_n B_{00}{}^{\sigma_i} + A_{01}{}^{\sigma_i} U_n X_{00}{}^{\sigma_i} U_n B_{00}{}^{\sigma_i} = C_{01}{}^{\sigma_i}, \end{cases} \tag{25}$$

and

$$X_{00}^{\sigma_i} = (X_{00}{}^*)^{\sigma_i} = (X_{00}{}^{\sigma_i})^T,$$
$$X_{01}^{\sigma_i} = (X_{01}{}^*)^{\sigma_i} = (X_{01}{}^{\sigma_i})^T,$$

i.e.

$$\begin{cases} A_0 X_0 B_0 = C_0, \\ A_0 X_0 B_1 + A_0 X_1 B_0 + A_1 X_0 B_0 = C_1, \end{cases}$$

and

$$X_0 = X_0{}^T, X_1 = X_1{}^T.$$

Conversely, if the system of real matrix equations (23) has a pair of symmetric solutions (X_0, X_1), which can be expressed as

$$X_0 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \in \mathbb{R}^{4n\times 4n},$$

and

$$X_1 = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix} \in \mathbb{R}^{4n\times 4n},$$

respectively, where $a_{ij}, b_{ij} \in \mathbb{R}^{n\times n}, (i, j = \overline{1, 2})$, then we have (23) hold, and

$$\begin{cases} A_{00}{}^{\sigma_i} X_0 B_{00}{}^{\sigma_i} = C_{00}{}^{\sigma_i}, \\ A_{00}{}^{\sigma_i} X_0 B_{01}{}^{\sigma_i} + A_{00}{}^{\sigma_i} X_1 B_{00}{}^{\sigma_i} + A_{01}{}^{\sigma_i} X_0 B_{00}{}^{\sigma_i} = C_{01}{}^{\sigma_i}, \end{cases} \tag{26}$$

where $X_0 = X_0{}^T, X_1 = X_1{}^T$. According to (4) of proposition 1, we can obtain that $(-P_n X_0 P_n{}^T, -P_n X_1 P_n{}^T), (Q_n X_0 Q_n{}^T, Q_n X_1 Q_n{}^T), (-R_n X_0 R_n{}^T, -R_n X_1 R_n{}^T)$ are also pairs of symmetric solutions to the system (23). Then, so is (Y_0, Y_1), where

$$Y_0 = \frac{1}{4}(X_0 - P_n X_0 P_n{}^T + Q_n X_0 Q_n{}^T - R_n X_0 R_n{}^T),$$

$$Y_1 = \frac{1}{4}(X_1 - P_n X_1 P_n{}^T + Q_n X_1 Q_n{}^T - R_n X_1 R_n{}^T).$$

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By direct computation, we have

$$Y_0 = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ -c_2 & c_1 & -c_4 & c_3 \\ -c_3 & c_4 & -c_1 & c_2 \\ -c_4 & -c_3 & -c_2 & -c_1 \end{pmatrix},$$

$$Y_1 = \begin{pmatrix} d_1 & d_2 & d_3 & d_4 \\ -d_2 & d_1 & -d_4 & d_3 \\ -d_3 & d_4 & -d_1 & d_2 \\ -d_4 & -d_3 & -d_2 & -d_1 \end{pmatrix},$$

where

$$c_1 = \frac{1}{4}(a_{11} + a_{22} - a_{33} - a_{44}), \quad c_2 = \frac{1}{4}(a_{12} - a_{21} + a_{34} - a_{43}),$$

$$c_3 = \frac{1}{4}(a_{13} + a_{24} - a_{31} - a_{42}), \quad c_4 = \frac{1}{4}(a_{14} - a_{23} + a_{32} - a_{41}),$$

and

$$d_1 = \frac{1}{4}(b_{11} +_{22} - b_{33} - b_{44}), \quad d_2 = \frac{1}{4}(b_{12} - b_{21} + b_{34} - b_{43}),$$

$$d_3 = \frac{1}{4}(b_{13} + b_{24} - b_{31} - b_{42}), \quad d_4 = \frac{1}{4}(b_{14} - b_{23} + b_{32} - b_{41}).$$

Now, we construct that

$$X_{00} = c_1 + c_2 i + c_3 j + c_4 k = \frac{1}{2}\begin{pmatrix} I_n & I_n i & I_n j & I_n k \end{pmatrix} Y_0 \begin{pmatrix} -I_n \\ -I_n i \\ -I_n j \\ -I_n k \end{pmatrix} \in \mathbb{H}_s{}^{n\times n}.$$

$$X_{01} = d_1 + d_2 i + d_3 j + d_4 k = \frac{1}{2}\begin{pmatrix} I_n & I_n i & I_n j & I_n k \end{pmatrix} Y_1 \begin{pmatrix} -I_n \\ -I_n i \\ -I_n j \\ -I_n k \end{pmatrix} \in \mathbb{H}_s{}^{n\times n}.$$

According to (5) of Proposition 1, ${X_{00}}^{\sigma_i} = Y_0$, ${X_{01}}^{\sigma_i} = Y_1$. Consequently,

$$\begin{cases} {A_{00}}^{\sigma_i} {X_{00}}^{\sigma_i} {B_{00}}^{\sigma_i} = {C_{00}}^{\sigma_i}, \\ {A_{00}}^{\sigma_i} {X_{00}}^{\sigma_i} {B_{01}}^{\sigma_i} + {A_{00}}^{\sigma_i} {X_{01}}^{\sigma_i} {B_{00}}^{\sigma_i} + {A_{01}}^{\sigma_i} {X_{00}}^{\sigma_i} {B_{00}}^{\sigma_i} = {C_{01}}^{\sigma_i}, \end{cases}$$

indicating that $({X_{00}}^{\sigma_i}, {X_{01}}^{\sigma_i})$ is a pair of solutions to the system (23). From (7) of proposition 1, we can easily get that $(U_n (X^i_{00})^{\sigma_i} U_n, U_n (X^i_{01})^{\sigma_i} U_n)$ is also a pair of solutions to the system (23). Thus,

$$\begin{cases} {A_{00}}^{\sigma_i} U_n (X^i_{00})^{\sigma_i} U_n {B_{00}}^{\sigma_i} = {C_{00}}^{\sigma_i}, \\ {A_{00}}^{\sigma_i} U_n (X^i_{00})^{\sigma_i} U_n {B_{01}}^{\sigma_i} + {A_{00}}^{\sigma_i} U_n (X^i_{01})^{\sigma_i} U_n {B_{00}}^{\sigma_i} + {A_{01}}^{\sigma_i} U_n (X^i_{00})^{\sigma_i} U_n {B_{00}}^{\sigma_i} = {C_{01}}^{\sigma_i}, \end{cases}$$

and

$$(X^i_{00})^{\sigma_i} = ((X^i_{00})^{\sigma_i})^T = ((X^i_{00})^*)^{\sigma_i},$$

$$(X^i_{01})^{\sigma_i} = ((X^i_{01})^{\sigma_i})^T = ((X^i_{01})^*)^{\sigma_i},$$

i.e.

$$\begin{cases} A_{00} X^i_{00} B_{00} = C_{00}, \\ A_{00} X^i_{00} B_{01} + A_{00} X^i_{01} B01_{00} + A_{01} X^i_{00} B_{00} = C_{01}. \end{cases} \tag{27}$$

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and

$$X_{00}^i = (X_{00}^i)^*,$$
$$X_{01}^i = (X_{01}^i)^*,$$

which indicates that the dual split quaternion matrix equation (4) has a Hermitian solution $X = X_{00}^i + X_{01}^i\epsilon$. □

Now, we consider some special cases of dual split quarernion matrix equation (4).

Corollary 2. *Let $A = A_{00} + A_{01}\epsilon \in \mathbb{DH}_s^{m\times n}$, $C = C_{00} + C_{01}\epsilon \in \mathbb{DH}_s^{m\times r}$ be known. Put*

$$A_0 = A_{00}{}^{\sigma_1},\ A_1 = A_{01}{}^{\sigma_1},\ C_0 = C_{00}{}^{\sigma_1},\ C_1 = C_{01}{}^{\sigma_1}, \tag{28}$$

$$A_2 = A_1 L_{A_0},\ C_{22} = A_1 A_0^\dagger C_0,\ C_2 = C_1 - C_{22},\ M = R_{A_0} A_2,\ N = R_{A_0} C_2. \tag{29}$$

Then, the following statements are equivalent:

(1) *Dual split quaternion matrix equation $AX = C$ is consistent.*
(2) *The system of real matrix equations*

$$\begin{cases} A_0 X_0 = C_0, \\ A_0 X_1 + A_1 X_0 = C_1, \end{cases} \tag{30}$$

is consistent.
(3)

$$R_{A_0} C_0 = 0, R_M N = 0. \tag{31}$$

(4)

$$r\begin{pmatrix} A_0 & C_0 \end{pmatrix} = r\begin{pmatrix} A_0 \end{pmatrix}, r\begin{pmatrix} A_1 & A_0 & C_1 \\ A_0 & 0 & C_0 \end{pmatrix} = r\begin{pmatrix} A_1 & A_0 \\ A_0 & 0 \end{pmatrix}. \tag{32}$$

In this case, the general solution X of dual split quaternion matrix equation $AX = C$ can be expressed as $X = X_{00} + X_{01}\epsilon$, where

$$\begin{aligned} X_{00} &= \frac{1}{8}\begin{pmatrix} I_n & I_n i & I_n j & I_n k \end{pmatrix}(X_0 + P_n X_0 P_r{}^T + Q_n X_0 Q_r{}^T + R_n X_0 R_r{}^T)\begin{pmatrix} I_r \\ I_r i \\ I_r j \\ I_r k \end{pmatrix}, \\ X_{01} &= \frac{1}{8}\begin{pmatrix} I_n & I_n i & I_n j & I_n k \end{pmatrix}(X_1 + P_n X_1 P_r{}^T + Q_n X_1 Q_r{}^T + R_n X_1 R_r{}^T)\begin{pmatrix} I_r \\ I_r i \\ I_r j \\ I_r k \end{pmatrix}, \end{aligned} \tag{33}$$

where

$$\begin{aligned} X_0 &= A_0^\dagger C_0 + L_{A_0} U, \\ X_1 &= A_0^\dagger (C_2 - A_2 U) + L_{A_0} W_1, \\ U &= M^\dagger N + L_M W_2, \end{aligned} \tag{34}$$

and W_1, W_2 are arbitrary matrices over $\mathbb{R}$ with appropriate dimensions.

Corollary 3. *Let $B = B_0 + B_1\epsilon \in \mathbb{DH}_s^{r\times l}$, $C = C_0 + C_1\epsilon \in \mathbb{DH}_s^{n\times l}$ be known. Denote*

$$B_0 = B_{00}{}^{\sigma_1},\ B_1 = B_{01}{}^{\sigma_1},\ C_0 = C_{00}{}^{\sigma_1},\ C_1 = C_{01}{}^{\sigma_1}, \tag{35}$$

$$B_2 = R_{B_0} B_1,\ C_{11} = C_0 B_0^\dagger B_1,\ C_2 = C_1 - C_{11},\ E = B_2 L_{B_0},\ F = C_2 L_{B_0}. \tag{36}$$

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Then, the following statements are equivalent:

(1) *Dual split quaternion matrix equation $XB = C$ is consistent.*
(2) *The system of real matrix equations*

$$\begin{cases} X_0B_0 = C_0, \\ X_1B_0 + X_0B_1 = C_1, \end{cases} \tag{37}$$

is consistent.
(3)

$$C_0L_{B_0} = 0, \ FL_E = 0. \tag{38}$$

(4)

$$r\begin{pmatrix} B_0 \\ C_0 \end{pmatrix} = r\begin{pmatrix} B_0 \end{pmatrix}, \ r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \\ C_1 & C_0 \end{pmatrix} = r\begin{pmatrix} B_1 & B_0 \\ B_0 & 0 \end{pmatrix}. \tag{39}$$

In this case, the general solution X of dual split quaternion matrix equation $XB = C$ can be expressed as $X = X_{00} + X_{01}\epsilon$, where

$$\begin{aligned} X_{00} &= \frac{1}{8}\begin{pmatrix} I_n & I_ni & I_nj & I_nk \end{pmatrix}(X_0 + P_nX_0P_r{}^T + Q_nX_0Q_r{}^T + R_nX_0R_r{}^T)\begin{pmatrix} I_r \\ I_ri \\ I_rj \\ I_rk \end{pmatrix}, \\ X_{01} &= \frac{1}{8}\begin{pmatrix} I_n & I_ni & I_nj & I_nk \end{pmatrix}(X_1 + P_nX_1P_r{}^T + Q_nX_1Q_r{}^T + R_nX_1R_r{}^T)\begin{pmatrix} I_r \\ I_ri \\ I_rj \\ I_rk \end{pmatrix}, \end{aligned} \tag{40}$$

where

$$\begin{aligned} X_0 &= C_0B_0^\dagger + VR_{B_0}, \\ X_1 &= (C_2 - VB_2)B_0^\dagger + W_1R_{B_0}, \\ V &= FE^\dagger + W_2R_E, \end{aligned} \tag{41}$$

and W_1, W_2 are arbitrary matrices over $\mathbb{R}$ with appropriate dimensions.

4. Numerical Example

Now, we provide a numerical example to illustrate the main result of this paper.

Example 1.

$$A = A_{00} + A_{01}\epsilon = \begin{pmatrix} \boldsymbol{i}+\boldsymbol{k} & \boldsymbol{i}+\boldsymbol{j} \\ \boldsymbol{j} & 0 \\ 3\boldsymbol{j} & \boldsymbol{i}-\boldsymbol{k} \end{pmatrix} + \begin{pmatrix} 2-\boldsymbol{k} & \boldsymbol{i} \\ \boldsymbol{j} & \boldsymbol{i}-\boldsymbol{j} \\ 0 & 4+\boldsymbol{k} \end{pmatrix}\epsilon,$$

$$B = B_{00} + B_{01}\epsilon = \begin{pmatrix} 1+\boldsymbol{j} & \boldsymbol{i} & \boldsymbol{i}-\boldsymbol{k} \\ 0 & \boldsymbol{j}+\boldsymbol{k} & 0 \end{pmatrix} + \begin{pmatrix} \boldsymbol{i}-\boldsymbol{k} & \boldsymbol{j}+3\boldsymbol{k} & 0 \\ \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{i}+\boldsymbol{k} \end{pmatrix}\epsilon,$$

$$\begin{aligned} C &= C_{00} + C_{01}\epsilon \\ &= \begin{pmatrix} 2\boldsymbol{i}+2\boldsymbol{k} & 1-3\boldsymbol{i}-3\boldsymbol{j}+\boldsymbol{k} & -2+2\boldsymbol{j} \\ 1-\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k} & 1-\boldsymbol{j} & 1+\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k} \\ -3-3\boldsymbol{i}+3\boldsymbol{j}-3\boldsymbol{k} & 4-\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k} & 3+3\boldsymbol{i}-3\boldsymbol{j}-3\boldsymbol{k} \end{pmatrix} \\ &+ \begin{pmatrix} -3+3\boldsymbol{i}+4\boldsymbol{j}+2\boldsymbol{k} & 4-6\boldsymbol{i}-5\boldsymbol{j}+3\boldsymbol{k} & -7-\boldsymbol{i}+\boldsymbol{j}-5\boldsymbol{k} \\ 3-\boldsymbol{i}+\boldsymbol{j}-4\boldsymbol{k} & 4-2\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k} & 2+\boldsymbol{i}-2\boldsymbol{j}-3\boldsymbol{k} \\ 6+\boldsymbol{i}-10\boldsymbol{k} & -3-4\boldsymbol{i}+4\boldsymbol{k} & 3-3\boldsymbol{j}-6\boldsymbol{k} \end{pmatrix}\epsilon. \end{aligned}$$

From MATLAB, we obtain

$$A_0 = {A_{00}}^{\sigma_1} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$A_1 = {A_{01}}^{\sigma_1} = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 4 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix},$$

$$B_0 = {B_{00}}^{\sigma_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_1 = {B_{01}}^{\sigma_1} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & -1 \\ -1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$C_0 = {C_{00}}^{\sigma_1} = \begin{pmatrix} 0 & 1 & -2 & 2 & -3 & 0 & 0 & -3 & 2 & 2 & 1 & 0 \\ 1 & 1 & 1 & -1 & 0 & 1 & 1 & -1 & -1 & -1 & 0 & -1 \\ 3 & 4 & 3 & -3 & -1 & 3 & 3 & -2 & -3 & -3 & 1 & -3 \\ -2 & 3 & 0 & 0 & 1 & -2 & -2 & -1 & 0 & 0 & -3 & 2 \\ 1 & 0 & -1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & -1 & -1 \\ 3 & 1 & -3 & 3 & 4 & 3 & 3 & -1 & 3 & 3 & -2 & -3 \\ 0 & -3 & 2 & -2 & -1 & 0 & 0 & 1 & -2 & -2 & 3 & 0 \\ 1 & 1 & -1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & -1 \\ 3 & -2 & -3 & 3 & -1 & 3 & 3 & 4 & 3 & 3 & 1 & -3 \\ 2 & 1 & 0 & 0 & -3 & 2 & 2 & -3 & 0 & 0 & 1 & -2 \\ -1 & 0 & -1 & 1 & -1 & -1 & -1 & 0 & 1 & 1 & 1 & 1 \\ -3 & 1 & -3 & 3 & -2 & -3 & -3 & -1 & 3 & 3 & 4 & 3 \end{pmatrix},$$

$$C_1 = {C_{01}}^{\sigma_1} = \begin{pmatrix} -3 & 4 & -7 & 3 & -6 & -1 & 4 & -5 & 1 & 2 & 3 & -5 \\ 3 & 4 & 2 & -1 & -2 & 1 & 1 & 1 & -2 & -4 & -2 & -3 \\ 6 & -3 & 3 & 1 & -4 & 0 & 0 & 0 & -3 & -10 & 4 & -6 \\ -3 & 6 & 1 & -3 & 4 & -7 & -2 & -3 & 5 & 4 & -5 & 1 \\ 1 & 2 & -1 & 3 & 4 & 2 & 4 & 2 & 3 & 1 & 1 & -2 \\ -1 & 4 & 0 & 6 & -3 & 3 & 10 & -4 & 6 & 0 & 0 & -3 \\ 4 & -5 & 1 & -2 & -3 & 5 & -3 & 4 & -7 & -3 & 6 & 1 \\ 1 & 1 & -2 & 4 & 2 & 3 & 3 & 4 & 2 & 1 & 2 & -1 \\ 0 & 0 & -3 & 10 & -4 & 6 & 6 & -3 & 3 & -1 & 4 & 0 \\ 2 & 3 & -5 & 4 & -5 & 1 & 3 & -6 & -1 & -3 & 4 & -7 \\ -4 & -2 & -3 & 1 & 1 & -2 & -1 & -2 & 1 & 3 & 4 & 2 \\ -10 & 4 & -6 & 0 & 0 & -3 & 1 & -4 & 0 & 6 & -3 & 3 \end{pmatrix},$$

and

$$R_{A_0}C_0 = 0,\ C_0L_{B_0} = 0,$$
$$R_MN = 0,\ R_{A_0}C_2L_{B_0} = 0,\ FL_E = 0.$$

Therefore, the dual split quaternion matrix equation (4) is consistent, and the general solution X can be expressed as

$$X = X_{00} + X_{01}\epsilon,$$

where

$$X_{00} = \frac{1}{8}\begin{pmatrix} I_n & I_n i & I_n j & I_n k \end{pmatrix}(X_0 + P_n X_0 {P_r}^T + Q_n X_0 {Q_r}^T + R_n X_0 {R_r}^T)\begin{pmatrix} I_r \\ I_r i \\ I_r j \\ I_r k \end{pmatrix},$$

$$X_{01} = \frac{1}{8}\begin{pmatrix} I_n & I_n i & I_n j & I_n k \end{pmatrix}(X_1 + P_n X_1 {P_r}^T + Q_n X_1 {Q_r}^T + R_n X_1 {R_r}^T)\begin{pmatrix} I_r \\ I_r i \\ I_r j \\ I_r k \end{pmatrix},$$

where

$$X_0 = \begin{pmatrix} \boldsymbol{i}+\boldsymbol{j} & 1 \\ 0 & \boldsymbol{i}-2\boldsymbol{k} \end{pmatrix} + L_{A_0}U + VR_{B_0},$$
$$X_1 = \begin{pmatrix} \boldsymbol{j}+\boldsymbol{k} & \boldsymbol{i} \\ 1 & 2+\boldsymbol{i} \end{pmatrix} + W_1R_{B_0} + L_{A_0}W_2,$$
$$U = M^{\dagger}NB_0^{\dagger} + L_MQ_1 + Q_2R_{B_0},$$
$$V = A_0^{\dagger}FE^{\dagger} + L_{A_0}Q_3 + Q_4R_E,$$

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with

$$L_{A_0} = \begin{pmatrix} 1.1102e-16 & -3.8128e-16 & 5.197e-16 & 2.7756e-16 & 8.3267e-17 & 3.2577e-16 & 4.969e-16 & 1.3878e-16 \\ -1.1102e-16 & -8.8818e-16 & 2.2204e-15 & 9.4369e-16 & 6.9389e-16 & 1.36e-15 & 1.6653e-15 & 3.3307e-16 \\ -1.6653e-16 & -1.3878e-16 & 6.6613e-16 & 8.3267e-17 & 5.2736e-16 & 1.3878e-16 & -1.1102e-16 & 1.3878e-16 \\ -2.2204e-16 & 9.992e-16 & -1.4433e-15 & -2.2204e-16 & -8.3267e-17 & -9.992e-16 & -1.7764e-15 & -5.5511e-16 \\ 2.7756e-17 & -3.4277e-16 & 6.2243e-16 & 2.7756e-16 & 2.2204e-16 & 2.3175e-16 & 4.556e-16 & 0 \\ 0 & -1.3323e-15 & 1.6098e-16 & 6.6613e-16 & 1.1102e-16 & 1.3323e-15 & 2.8866e-15 & 5.4123e-16 \\ 0 & -1.1102e-16 & 5.8287e-16 & 3.0531e-16 & 1.3878e-16 & 3.6082e-16 & 3.3307e-16 & 0 \\ -3.3307e-16 & -8.8818e-16 & 1.3323e-15 & 3.8858e-16 & 1.1102e-16 & 6.6613e-16 & 8.8818e-16 & 4.4409e-16 \end{pmatrix},$$

$$R_{B_0} = \begin{pmatrix} 0.33333 & -0.16667 & 3.8339e-18 & -0.16667 & -0.33333 & 0.16667 & 5.9848e-17 & -0.16667 \\ -0.16667 & 0.16667 & 0.16667 & 2.7756e-16 & 0.16667 & 1.9429e-16 & -0.16667 & 0.16667 \\ 6.1322e-17 & 0.16667 & 0.33333 & -0.16667 & 2.436e-17 & 0.16667 & -0.33333 & 0.16667 \\ -0.16667 & 1.1102e-16 & -0.16667 & 0.16667 & 0.16667 & -0.16667 & 0.16667 & -1.3878e-17 \\ -0.33333 & 0.16667 & 1.6375e-18 & 0.16667 & 0.33333 & -0.16667 & -1.275e-16 & 0.16667 \\ 0.16667 & -1.3878e-17 & 0.16667 & -0.16667 & -0.16667 & 0.16667 & -0.16667 & -1.1102e-16 \\ 1.0095e-16 & -0.16667 & -0.33333 & 0.16667 & -6.7849e-17 & -0.16667 & 0.33333 & -0.16667 \\ -0.16667 & 0.16667 & 0.16667 & -2.7756e-17 & 0.16667 & -2.2204e-16 & -0.16667 & 0.16667 \end{pmatrix},$$

$$M = \begin{pmatrix} -4.6259e-18 & 9.5156e-16 & -1.576e-15 & -4.3021e-16 & -1.8504e-16 & -1.1736e-15 & -2.1278e-15 & -4.7878e-16 \\ 2.8912e-16 & 1.3541e-15 & -2.3216e-15 & -8.5025e-16 & -8.4192e-17 & -1.7718e-15 & -2.6055e-15 & -7.4963e-16 \\ -4.8572e-17 & 2.753e-17 & 4.0192e-18 & 3.0531e-17 & 1.1102e-17 & -1.7815e-17 & -1.4368e-16 & -6.9389e-19 \\ 1.4803e-16 & 4.851e-16 & -7.3355e-16 & -3.2844e-16 & 1.3415e-16 & -6.5163e-16 & -9.0871e-16 & -2.7293e-16 \\ 1.7764e-16 & -1.4032e-15 & 2.3703e-15 & 7.7623e-16 & 4.1495e-16 & 1.6794e-15 & 3.1842e-15 & 5.5557e-16 \\ -8.3267e-18 & 3.1225e-16 & -5.0931e-16 & -2.2482e-16 & -3.1919e-17 & -3.858e-16 & -6.5503e-16 & -1.1657e-16 \\ 1.4803e-16 & 4.851e-16 & -7.3355e-16 & -3.2844e-16 & 1.3415e-16 & -6.5163e-16 & -9.0871e-16 & -2.7293e-16 \\ -2.313e-18 & 1.5192e-15 & -2.2974e-15 & -6.6706e-16 & -1.7579e-17 & -1.8787e-15 & -3.4676e-15 & -7.538e-16 \\ 4.8572e-17 & -2.753e-17 & -4.0192e-18 & -3.0531e-17 & -1.1102e-17 & 1.7815e-17 & 1.4368e-16 & 6.9389e-19 \\ 4.6259e-18 & -9.5156e-16 & 1.576e-15 & 4.3021e-16 & 1.8504e-16 & 1.1736e-15 & 2.1278e-15 & 4.7878e-16 \\ 1.2768e-16 & 4.7028e-16 & -6.8554e-16 & -5.7269e-16 & 2.2343e-16 & -6.3543e-16 & -7.46e-16 & -1.4387e-16 \\ 8.3267e-18 & -3.1225e-16 & 5.0931e-16 & 2.2482e-16 & 3.1919e-17 & 3.858e-16 & 6.5503e-16 & 1.1657e-16 \end{pmatrix},$$

$$N = \begin{pmatrix} 0.75 & -3.2752e-15 & 0.75 & -0.75 & 7.1794e-15 & 0.75 & 0.75 & 4.959e-15 & -0.75 & -0.75 & 3.858e-15 & -0.75 \\ 1.5 & 1.65 & -0.675 & 0.675 & -2.85 & 1.5 & 1.5 & -1.65 & 0.675 & 0.675 & -2.85 & -1.5 \\ 3.926e-15 & -0.55 & 0.225 & -0.225 & 0.95 & 1.1588e-15 & -5.7662e-15 & 0.55 & -0.225 & -0.225 & 0.95 & 2.9018e-15 \\ 0.75 & -2.4702e-15 & -0.75 & 0.75 & -5.8472e-15 & 0.75 & 0.75 & -1.7116e-15 & 0.75 & 0.75 & -1.9706e-15 & -0.75 \\ -0.675 & 2.85 & -1.5 & 1.5 & 1.65 & -0.675 & -0.675 & 2.85 & 1.5 & 1.5 & -1.65 & 0.675 \\ 0.225 & -0.95 & -1.5474e-15 & 8.8124e-16 & -0.55 & 0.225 & 0.225 & -0.95 & 8.5348e-16 & -7.2858e-16 & 0.55 & -0.225 \\ 0.75 & -1.5821e-15 & -0.75 & 0.75 & -4.737e-15 & 0.75 & 0.75 & -2.9884e-15 & 0.75 & 0.75 & 2.1372e-15 & -0.75 \\ 1.5 & -1.65 & 0.675 & -0.675 & 2.85 & 1.5 & 1.5 & 1.65 & -0.675 & -0.675 & 2.85 & -1.5 \\ 5.107e-15 & 0.55 & -0.225 & 0.225 & -0.95 & 1.9429e-15 & -4.4409e-15 & -0.55 & 0.225 & 0.225 & -0.95 & 1.4988e-15 \\ -0.75 & -7.494e-16 & -0.75 & 0.75 & 2.6645e-15 & -0.75 & -0.75 & -3.9413e-15 & 0.75 & 0.75 & 1.4155e-15 & 0.75 \\ 0.675 & -2.85 & -1.5 & 1.5 & -1.65 & 0.675 & -0.225 & 0.95 & -1.1657e-15 & -3.1086e-15 & -0.55 & 0.225 \\ -0.225 & 0.95 & 1.8874e-15 & -3.2196e-15 & 0.55 & -0.225 & 0.675 & -2.85 & 1.5 & 1.5 & 1.65 & -0.675 \end{pmatrix},$$

$$E = \begin{pmatrix} 0.16667 & 4.3368e-17 & -0.083333 & -0.083333 & -2.829e-16 & -0.33333 & -0.16667 & 2.2725e-16 & 0.083333 & -0.083333 & 1.9766e-16 & -0.33333 \\ -0.041667 & 2.3187e-16 & 0.20833 & 0.125 & 3.3104e-17 & 0.125 & 0.125 & 2.8478e-17 & 0.125 & -0.041667 & -1.6398e-16 & 0.20833 \\ 0.083333 & 5.0712e-16 & 0.33333 & 0.16667 & -2.167e-16 & -0.083333 & 0.083333 & 2.8421e-16 & 0.33333 & -0.16667 & -1.3031e-16 & 0.083333 \\ -0.125 & -2.7524e-16 & -0.125 & -0.041667 & 2.498e-16 & 0.20833 & 0.041667 & -2.5573e-16 & -0.20833 & 0.125 & -3.3675e-17 & 0.125 \\ -0.16667 & -4.3368e-17 & 0.083333 & 0.083333 & 2.829e-16 & 0.33333 & 0.16667 & -2.2725e-16 & -0.083333 & 0.083333 & -1.9766e-16 & 0.33333 \\ 0.125 & 2.7524e-16 & 0.125 & 0.041667 & -2.498e-16 & -0.20833 & -0.041667 & 2.5573e-16 & 0.20833 & -0.125 & 3.3675e-17 & -0.125 \\ -0.083333 & -5.0712e-16 & -0.33333 & -0.16667 & 2.167e-16 & 0.083333 & -0.083333 & -2.8421e-16 & -0.33333 & 0.16667 & 1.3031e-16 & -0.083333 \\ -0.041667 & 2.3187e-16 & 0.20833 & 0.125 & 3.3104e-17 & 0.125 & 0.125 & 2.8478e-17 & 0.125 & -0.041667 & -1.6398e-16 & 0.20833 \end{pmatrix},$$

$$F = \begin{pmatrix} -4.25 & -1.1562e-15 & -3.75 & -0.25 & -4.9656e-16 & -2.25 & 2.75 & 3.2526e-18 & -2.25 & -1.25 & -4.291e-16 & -3.75 \\ 1 & -5.9154e-16 & -0.25 & 1.25 & -1.2464e-15 & -1 & -1 & 1.128e-15 & 0.25 & -1.75 & 4.0194e-16 & -1 \\ 3 & -1.7781e-16 & -0.75 & 4.75 & -3.2305e-15 & -3 & -3 & 6.5041e-15 & 0.75 & -6.25 & 3.3359e-15 & -3 \\ 0.25 & -4.8659e-16 & 2.25 & -4.25 & 1.5096e-15 & -3.75 & 1.25 & -7.7911e-16 & 3.75 & 2.75 & -1.315e-15 & -2.25 \\ -1.25 & 8.4481e-16 & 1 & 1 & -4.6924e-16 & -0.25 & 1.75 & -5.5034e-16 & 1 & -1 & -2.9322e-16 & 0.25 \\ -4.75 & 9.606e-15 & 3 & 3 & -8.8944e-15 & -0.75 & 6.25 & 6.1598e-15 & 3 & -3 & 1.4084e-16 & 0.75 \\ 2.75 & 1.4112e-15 & -2.25 & 1.25 & -2.4503e-16 & 3.75 & -4.25 & 5.9306e-16 & -3.75 & 0.25 & 3.1858e-16 & 2.25 \\ -1 & 5.9501e-16 & 0.25 & 1.75 & 4.5623e-16 & 1 & 1 & -1.1289e-15 & -0.25 & -1.25 & -5.2466e-16 & 1 \\ -3 & -1.5413e-15 & 0.75 & 6.25 & 9.6811e-15 & 3 & 3 & -8.5862e-15 & -0.75 & -4.75 & -5.3557e-15 & 3 \\ -1.25 & -8.8037e-16 & -3.75 & 2.75 & -1.8739e-15 & -2.25 & -0.25 & -2.2356e-16 & -2.25 & -4.25 & 1.0768e-15 & -3.75 \\ -1.75 & 2.4113e-16 & -1 & -1 & 1.1267e-15 & 0.25 & 1.25 & -8.6693e-16 & -1 & 1 & -1.1356e-16 & -0.25 \\ -6.25 & -3.9161e-15 & -3 & -3 & 4.7579e-15 & 0.75 & 4.75 & -2.4353e-15 & -3 & 3 & 2.5983e-15 & -0.75 \end{pmatrix},$$

$$M^{\dagger}NB_0^{\dagger} = \begin{pmatrix} -5.6017e+14 & -1.7121e+15 & -9.1111e+14 & -2.4942e+15 & 1.3669e+15 & 2.5219e+15 & 2.1779e+15 & 2.874e+15 \\ 1.7159e+14 & 8.4488e+13 & 2.9641e+14 & 3.7726e+14 & 2.3969e+13 & -1.9486e+13 & 4.7286e+13 & -1.8599e+14 \\ 1.7875e+15 & 3.2339e+15 & 1.7599e+14 & 5.0696e+14 & -1.6956e+15 & -3.22e+15 & -6.7837e+13 & 5.4284e+12 \\ 3.1413e+14 & 1.2408e+15 & -3.5994e+15 & -5.4395e+15 & -1.0318e+14 & -4.199e+14 & 1.8375e+15 & 4.6134e+15 \\ 2.1039e+15 & 2.7065e+15 & 1.1224e+15 & 1.0733e+15 & -1.1136e+15 & -2.2215e+15 & 1.0451e+15 & 4.3375e+14 \\ -5.5683e+13 & 4.2905e+14 & -1.9791e+14 & 1.4525e+14 & -4.4365e+14 & -7.4478e+14 & -6.9997e+14 & -5.4315e+14 \\ -1.6967e+15 & -3.4076e+15 & 4.8989e+14 & -7.6984e+13 & 1.6503e+15 & 3.4018e+15 & 6.2164e+14 & 1.9235e+14 \\ 2.6099e+14 & 6.8825e+14 & 1.9102e+15 & 3.5641e+15 & -8.9134e+14 & -1.5209e+15 & -2.0224e+15 & -3.4686e+15 \end{pmatrix},$$

$$A_0^{\dagger}FE^{\dagger} = \begin{pmatrix} -1.1818 & 0.8636 & 0.5455 & 0.3182 & 1.1818 & -0.3182 & -0.5455 & 0.8636 \\ 0.1818 & -1.3182 & -2.4545 & 1.1364 & -0.1818 & -1.1364 & 2.4545 & -1.3182 \\ -0.5455 & -0.3182 & -1.1818 & 0.8636 & 0.5455 & -0.8636 & 1.1818 & -0.3182 \\ 2.4545 & -1.1364 & 0.1818 & -1.3182 & -2.4545 & 1.3182 & -0.1818 & -1.1364 \\ 1.1818 & -0.3182 & 0.5455 & -0.8636 & -1.1818 & 0.8636 & -0.5455 & -0.3182 \\ -0.1818 & -1.1364 & -2.4545 & 1.3182 & 0.1818 & -1.3182 & 2.4545 & -1.1364 \\ -0.5455 & 0.8636 & 1.1818 & -0.3182 & 0.5455 & 0.3182 & -1.1818 & 0.8636 \\ 2.4545 & -1.3182 & -0.1818 & -1.1364 & -2.4545 & 1.1364 & 0.1818 & -1.3182 \end{pmatrix},$$

$$L_M = \begin{pmatrix} 0.5703 & -0.035953 & 0.13289 & -0.080083 & -0.18119 & 0.1785 & -0.25058 & 0.30366 \\ -0.035953 & 0.89141 & 0.14176 & 0.10859 & -0.070697 & 0.14833 & 0.18591 & 0.045302 \\ 0.13289 & 0.14176 & 0.42644 & -0.16998 & -0.33492 & -0.2153 & -0.040492 & -0.13332 \\ -0.080083 & 0.10859 & -0.16998 & 0.24298 & 0.13472 & -0.13167 & 0.12006 & 0.29486 \\ -0.18119 & -0.070697 & -0.33492 & 0.13472 & 0.2757 & 0.15725 & 0.068548 & 0.045974 \\ 0.1785 & 0.14833 & -0.2153 & -0.13167 & 0.15725 & 0.7566 & -0.14689 & -0.14248 \\ -0.25058 & 0.18591 & -0.040492 & 0.12006 & 0.068548 & -0.14689 & 0.1683 & -0.017058 \\ 0.30366 & 0.045302 & -0.13332 & 0.29486 & 0.045974 & -0.14248 & -0.017058 & 0.66827 \end{pmatrix},$$

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$$R_E = \begin{pmatrix} 0.66667 & 0.16667 & -6.8243e-16 & 0.16667 & 0.33333 & -0.16667 & 5.5511e-17 & 0.16667 \\ 0.16667 & 0.83333 & -0.16667 & -2.1847e-16 & -0.16667 & 2.1384e-16 & 0.16667 & -0.16667 \\ -7.0924e-16 & -0.16667 & 0.66667 & 0.16667 & 1.2868e-16 & -0.16667 & 0.33333 & -0.16667 \\ 0.16667 & -2.867e-16 & 0.16667 & 0.83333 & -0.16667 & 0.16667 & -0.16667 & -3.4694e-17 \\ 0.33333 & -0.16667 & 7.023e-17 & -0.16667 & 0.66667 & 0.16667 & 5.4817e-16 & -0.16667 \\ -0.16667 & 3.2424e-16 & -0.16667 & 0.16667 & 0.16667 & 0.83333 & 0.16667 & 7.6328e-17 \\ 8.6946e-17 & 0.16667 & 0.33333 & -0.16667 & 4.9361e-16 & 0.16667 & 0.66667 & 0.16667 \\ 0.16667 & -0.16667 & -0.16667 & -4.7206e-17 & -0.16667 & 4.258e-17 & 0.16667 & 0.83333 \end{pmatrix},$$

and $Q_i(i = \overline{1,4})$, $W_i(i = \overline{1,2})$ *are arbitrary matrices over* $\mathbb{R}$ *with appropriate dimensions.*

5. Conclusions

In this paper, we have established the solvability conditions for the dual split quaternion matrix equation (4) and derived the general solution expressions when the equation is consistent. As an application, we explored the necessary and sufficient condition for the existence of Hermitian solution to the equation (4). Additionally, we have analyzed some special cases of dual split quaternion matrix equation (4). To further demonstrate our findings, an illustrative example has been provided. Looking ahead, our research will focus on exploring more intricate matrix and tensor equations over the dual split quaternion algebra.

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