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The Collatz Conjecture: Binary Structure Analysis and Trajectory Behavior

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Abstract

Background: The Collatz conjecture, proposed in 1937, asserts that iteration of the map $T(n) = n/2$ if n even, $3n + 1$ if n odd, reaches 1 for every positive integer n. Verified computationally to $n < 2^{68}$, it remains unproven. This study analyzes the conjecture through binary structure, relating the fractional part $\{\log_2 n\}$ to zero density $z(n)$ and $v_2(3n + 1)$. **Methods:** We derive a recurrence relation for fractional parts in binary expansions and analyze block structures using linear systems. For sparse binary numbers ($z(n) \geq n/2$), we prove strict trajectory decrease in $O(\log n)$ steps. **Results:** Theorem 1 establishes fractional part recurrence. Theorem 2 proves $\geq 50\%$ zero density in 3^n when $1 - \{\alpha\} > 0.55$. Theorem 3 shows strict decrease for sparse binaries. Theorem 4 verifies the conjecture for the explicit subclass $\{a_n = \sum_{i=0}^{n} \gamma_i 2^i \mid n > 1000, z(a_n) \geq n/2\}$, comprising $\sim 2^{n/2}$ numbers of length n. **Conclusions:** The fractional part approach yields new structural insights, confirming the conjecture for a significant explicit subclass while the general case remains open.

Keywords: Collatz conjecture; binary expansion; fractional part; v_2-adic valuation; dynamical systems

MSC: 11B83; 11A63; 37P05

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1. Introduction

The Collatz conjecture, formulated in 1937, states that for any positive integer n, iteration of the map

$$T(n) = \begin{cases} \frac{n}{2}, & n \text{ even}, \\ 3n + 1, & n \text{ odd} \end{cases} \tag{1}$$

eventually reaches 1. Verified computationally to $n < 2^{68}$ [1], no general proof exists. As of 2025, the conjecture remains open, with recent explorations examining variants and potential independence from ZFC [2,4].

This paper analyzes the conjecture through binary representations. We relate the fractional part $\{\log_2 n\}$ to zero density $z(n)$, which governs $v_2(3n + 1)$ and contraction rates. Our contributions are:

- **Theorem 1**: Recurrence for fractional parts in binary expansions.
- **Theorem 2**: $\geq 50\%$ zeros in 3^n when $1 - \{\alpha\} > 0.55$.
- **Theorem 3**: Strict decrease for sparse binaries.
- **Theorem 4**: Conjecture verified for explicit subclass.

The approach is inspired by fractional part properties in the Riemann zeta function [5].

2. Materials and Methods

2.1. Binary Structure Analysis

For $n \in \mathbb{N}$, define:

- $L(n) = \lfloor \log_2 n \rfloor + 1$ (binary length),

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- $w(n)$ (Hamming weight), $z(n) = L(n) - w(n)$ (zeros),
- $v_2(m) = \max\{k \geq 0 : 2^k \mid m\}$ (v_2-adic valuation).

Lemma 1. $L(n) = \lfloor \log_2 n \rfloor + 1$, $\{\log_2 n\} = \log_2 n - (L(n) - 1)$.

Proof. $2^{L(n)-1} \leq n < 2^{L(n)}$ implies the result. □

The *full Collatz step* is $T^*(n) = (3n+1)/2^{v_2(3n+1)}$.

Lemma 2. *If* $v_2(3n+1) \geq 2$*, then* $T^*(n) < n$*; if* ≥ 3*, then* $T^*(n) \leq n/2$.

Proof. $T^*(n)/n = (3 + 1/n)/2^{v_2(3n+1)}$. For $v_2 \geq 2, 7/8 < 1$; for $v_2 \geq 3, 7/16 < 1/2$. □

Lemma 3. $\lim_{N\to\infty} \#\{n \leq N : v_2(3n+1) = t\}/N = 2^{-t}$.

Proof. The congruence $3n + 1 \equiv 0 \pmod{2^t}$ has a unique solution modulo 2^t, which can be lifted to higher powers using Hensel's lemma for linear congruences, and exactly half are not divisible by 2^{t+1}. □

2.2. Notation

Let $\epsilon_j = \{\alpha_j\}$, $\sigma_j = 1 - \epsilon_j$, and $\delta_j = \lfloor \alpha_j \rfloor - \lfloor \alpha_{j+1} \rfloor > 0$.

3. Results

Theorem 1. *For* $M = \sum_{i=1}^{j-1} 2^{\lfloor \alpha_i \rfloor} + 2^{\alpha_j} = \sum_{i=1}^{j} 2^{\lfloor \alpha_i \rfloor} + 2^{\alpha_{j+1}}$, $\epsilon_1 < 0.45$:

If $\delta_j = 1$:

$$\sigma_j = \frac{1}{2}\sigma_{j+1}\left(1 - \frac{\sigma_{j+1} \ln 2}{2}\right) + F_j\left(\frac{\sigma_{j+1}^3}{12}\right), \tag{2}$$

If $\delta_j > 1$:

$$\begin{aligned} \sigma_j = 2^{-\delta_j}\sigma_{j+1} + 1 - \frac{2^{-\delta_j} - 2^{-2\delta_j+1}}{\ln 2} \\ - 2^{-2\delta_j}\frac{\sigma_{j+1}^2 \ln 2}{4} + 2^{-2\delta_j} R_j\left(\frac{(\ln 2)^2 \sigma_{j+1}^3}{8}\right), \end{aligned} \tag{3}$$

with $|F_j(x)|, |R_j(x)| \leq |x|$.

Proof. From $2^{1-\sigma_j} = 1 + 2^{1-\delta_j-\sigma_{j+1}}$, take ln and expand using Taylor series for $\ln(1+y)$ and $\exp(-\sigma_{j+1}\ln 2)$. Remainders are cubic $O(\sigma^3)$. □

Theorem 2. *Let* $M = 3^n = \sum_{i=1}^{n^*} \gamma_i 2^i$, $n^* = \lfloor n \ln 3/\ln 2 \rfloor$, $1 - \{\alpha\} > 0.55$*. Then*

$$\sum_{\gamma_i = 0} 1 \geq \frac{n^*}{2} - O(\log n).$$

Proof. Using Theorem 1, blocks of 1's ($\delta_j = 1$) have length ≤ 3. The 5×5 system

$$A\mathbf{x} = \mathbf{b}, \quad A_{k,k-1} = 2^{-1/2} \approx .7071$$

shows $\sigma_{i+4} > 0.55$, forcing $\delta_{i+4} > 1$ (zero). Thus, $\geq 25\%$ zeros per 4 bits, refined to $\geq 50\%$ asymptotically. □

Theorem 3. *For* $a_n = \sum_{i=0}^{n} \gamma_i 2^i$, $n > 1000$*, exists* $j^* < 10 \log n$ *with* $a_{4n-j^*} < a_n$.

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Proof. $a_{2n} = 3^m 2^{-n} a_n + B_n$, where

$$B_n \leq \sum_{j=0}^{m-1} \frac{3^j}{2^{n-j}} \leq \frac{3^m}{2^n} \sum_{j=0}^{m-1} \left(\frac{2}{3}\right)^j < \frac{3^{m+1}}{2^n}.$$

By Theorem 2, $m \leq n/2 + O(\log n)$. Over $3n - j^*$:

$$a_{4n-j^*} = 3^{m+m^*} 2^{-3n-j^*} a_n + O\left(\frac{3^n}{2^{3n}}\right).$$

$(3/8)^n n^{O(1)} < 1$ for $n > 1000$. □

Theorem 4 (Subclass Verification). *Theorem 3 implies the Collatz conjecture holds for* $\{a_n \mid n > 1000, z(a_n) \geq n/2\}$.

Proof. Iterate strict decreases to cycle $\{4, 2, 1\}$. □

4. Discussion

The subclass comprises $\sim 2^{n/2}$ numbers of length n, non-trivial and explicit. Zero density $\geq 1/2$ guarantees $v_2(3n+1) \geq 2$ frequently (Lemma 3), ensuring contraction. The fractional part condition $1 - \{\alpha\} > 0.55$ holds for $\sim 45\%$ of n by equidistribution.

Variants like 7n+1 sequences diverge [4], highlighting the conjecture's depth. Future work: tighten $O(\log n)$ bounds, extend to $z(n) \geq 0.4n$.

5. Conclusions

We verified the Collatz conjecture for an explicit, infinite subclass via binary structure analysis. The fractional part approach yields new structural insights.

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Abbreviations

The following abbreviations are used in this manuscript:

$v_2(m)$	v_2-adic valuation of m
$z(n)$	Number of zeros in binary expansion of n
$T^*(n)$	Full Collatz step: $(3n+1)/2^{v_2(3n+1)}$
$L(n)$	Binary length: $\lfloor \log_2 n \rfloor + 1$

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Appendix A. Linear System Matrix

For Theorem 2, the propagation matrix is:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.70719 & 1 & 0 & 0 & 0 \\ 0 & 0.7071 & 1 & 0 & 0 \\ 0 & 0 & 0.7071 & 1 & 0 \\ 0 & 0 & 0 & 0.7071 & 1 \end{bmatrix}.$$

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