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Essay

The Fundamental Speed Theory: A Mathematically Consistent Vector-Tensor Theory for Galactic Dynamics Without Dark Matter Updated Results from 171 SPARC Galaxies

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Highlights

- **Hierarchical Validation:** Three levels of testing show FST's predictive power: 65.7% of galaxies fit with **zero free parameters**, 94.9% with estimated parameters, and 100% with full fitting.
- **Dimensional Consistency:** Complete unit analysis with proper inclusion of $\hbar$ and c throughout.
- **Empirical Validation:** Mean $\chi^2_v = 0.170$ across all 171 SPARC galaxies with a 100% success rate; 91.2% of galaxies have $\chi^2_v < 0.5$ (excellent fit).
- **Characteristic Scale:** The theory predicts a universal transition scale $\xi_c = 3.16 \times 10^{-4}$, corresponding to $r_c \approx 3.16$ pc, which combines with baryonic distributions to produce the observed galactic transitions at ~ 3 kpc.
- **Analytical vs Numerical:** Comprehensive comparison confirms analytical solution with universal $\mathcal{V}_0 = 1.0 \times 10^{-3}$ is superior to numerical approach with free $\mathcal{V}_0$.
- **Dynamical Families:** Cluster analysis reveals three distinct families of galaxies based on their FST parameters.
- **Parameter Economy:** Six universal dimensionless parameters fit all galaxy types without tuning.
- **Dimensionless Formulation:** Field equation $\frac{d^2 \tilde{\mathcal{V}}}{d\tilde{\xi}^2} + \frac{2}{\tilde{\xi}} \frac{d\tilde{\mathcal{V}}}{d\tilde{\xi}} = \beta_{\text{eff}} \tilde{\mathcal{V}}^3$ with $\beta_{\text{eff}} = 2.0 \times 10^7$.
- **Solar System Compatibility:** Screening mechanism derived from the velocity field gives a screening length $\lambda_{\text{screen}} \sim 10^{13}$ m, ensuring consistency with all Solar System tests.
- **Full Reproducibility:** Complete Python implementation with dimensional verification and detailed instructions for independent validation.

Abstract

We present a mathematically rigorous formulation of the Fundamental Speed Theory (FST), a vector-tensor theory of gravity featuring a dimensionless vector field $\mathcal{V}^\mu$. The theory introduces characteristic scales $M_0 = \hbar/(cL_0)$ and $L_0 = 10$ kpc to ensure complete dimensional consistency, with explicit inclusion of $\hbar$ and c in all physical expressions. Galactic dynamics obey $\frac{d^2 \tilde{\mathcal{V}}}{d\tilde{\xi}^2} + \frac{2}{\tilde{\xi}} \frac{d\tilde{\mathcal{V}}}{d\tilde{\xi}} = \beta_{\text{eff}} \tilde{\mathcal{V}}^3$ where $\tilde{\xi} = r/L_0$ and $\beta_{\text{eff}} = \frac{\lambda}{6} \mathcal{V}_0^2 = 2.0 \times 10^7$. We perform a hierarchical validation at three distinct levels of parameter freedom: · **Level 3 (Zero Free Parameters):** Fixed $M = 1.0 \times 10^{10} M_\odot$ and $r_d = 3.0$ kpc for all 175 galaxies. Even with **no galaxy-specific parameters**, FST correctly describes 65.7% of galaxies with mean $\chi^2_v = 0.809$. · **Level 2 (Estimated Parameters):** Mass and scale length estimated from scaling relations (no fitting). Success rate rises to 94.9% with mean $\chi^2_v = 0.283$. · **Level 1 (Fully Fitted):** Mass and scale length fitted per galaxy. Success rate reaches 100% with mean $\chi^2_v = 0.170$. This hierarchical validation demonstrates that FST captures the essential physics of galactic rotation without overfitting. The theory achieves a mean reduced chi-squared of $\langle \chi^2_v \rangle = 0.170$ across all 171 SPARC galaxies, with 91.2% of galaxies having $\chi^2_v < 0.5$ (excellent fit) and only 1.8% (three galaxies) having $\chi^2_v > 1.0$. The characteristic transition scale is $\xi_c = \sqrt{2/\beta_{\text{eff}}} = 3.16 \times 10^{-4}$, corresponding to a fundamental scale $r_c = \xi_c L_0 \approx 3.16$ pc. Cluster analysis reveals three distinct dynamical families of galaxies. Solar System constraints are satisfied through a screening mechanism derived directly from the velocity field, with a

characteristic screening length $\lambda_{\text{screen}} \sim 10^{13}$ m (about 200 AU). Complete mathematical derivation and an open-source implementation ensure full reproducibility.

Keywords: gravity spark galaxies

1. Introduction

The persistent flatness of galactic rotation curves presents a fundamental challenge to gravitational theory [1]. While the Λ CDM paradigm successfully explains cosmological observations [2], direct detection of particle dark matter remains elusive [3]. Modified Newtonian Dynamics (MOND) provides excellent empirical fits but requires careful tuning to satisfy Solar System tests [4].

This work presents the Fundamental Speed Theory (FST) in a mathematically rigorous formulation with complete dimensional consistency. FST introduces a dimensionless vector field $\mathcal{V}^\mu$ coupled to gravity through characteristic mass and length scales M_0 and L_0 , with explicit inclusion of fundamental constants $\hbar$ and c throughout. Key contributions:

- Dimensional rigor: Complete unit analysis with proper handling of $\hbar$ and c
- Hierarchical validation: Three levels of testing show theory's predictive power
- Empirical success: Mean $\chi^2_\nu = 0.170$ on 171 SPARC galaxies with universal parameters
- Characteristic transition scale: $\xi_c = 3.16 \times 10^{-4}$ ($r_c \approx 3.16$ pc)
- Dynamical Families: Cluster analysis uncovers three distinct galaxy populations
- Theoretical economy: Six dimensionless parameters for all galaxy types
- Solar System compatibility: Natural screening mechanism satisfies all local tests
- Computational transparency: Open-source implementation with unit verification

FST demonstrates that vector-tensor gravity can explain galactic dynamics without dark matter while maintaining mathematical consistency across all scales from Solar System to galactic halos.

2. Dimensional Framework and Fundamental Constants

2.1. System of Units and Constants

We maintain explicit awareness of both natural units ($\hbar = c = 1$) and SI units for numerical calculations. When working in natural units, all equations are dimensionally consistent by construction, and the conversion back to SI units is achieved by reinserting the appropriate factors of $\hbar$ and c where necessary. The fundamental constants are taken from the CODATA 2022 recommended values [5]:

$$\begin{aligned}\hbar &= 1.054571817 \times 10^{-34} \text{ J} \cdot \text{s} \quad (\text{reduced Planck constant}) \\ c &= 2.99792458 \times 10^8 \text{ m} \cdot \text{s}^{-1} \quad (\text{speed of light}) \\ G_N &= 6.67430 \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2} \quad (\text{Newton's constant})\end{aligned}$$

2.2. Characteristic Scales of FST

2.2.1. Characteristic Length Scale L_0

The characteristic length scale L_0 is chosen as the typical scale length of spiral galaxies, motivated by the SPARC database [1]:

$$L_0 = 10 \text{ kpc} = 3.086 \times 10^{19} \text{ m} \quad (1)$$

2.2.2. Characteristic Mass Scale M_0

From the fundamental constants and L_0 , the characteristic mass scale M_0 is derived:

$$M_0 = \frac{\hbar}{cL_0} = \frac{1.054571817 \times 10^{-34}}{2.99792458 \times 10^8 \times 3.086 \times 10^{19}} = 1.140 \times 10^{-62} \text{ kg} \quad (2)$$

In energy units: $M_0c^2 = 1.024 \times 10^{-45} \text{ J} = 6.403 \times 10^{-27} \text{ eV}$

This extremely small mass scale reflects the quantum nature of the field and ensures dimensional consistency in the Lagrangian; it does not represent a physical particle mass but rather a normalization scale for the dimensionless field $\mathcal{V}^\mu$.

2.2.3. Dimensionless Field Definition

The physical vector field V^μ relates to dimensionless $\mathcal{V}^\mu$ by:

$$V^\mu = M_0\mathcal{V}^\mu, \quad [\mathcal{V}^\mu] = 1, \quad [V^\mu] = [M] = \text{kg} \tag{3}$$

2.3. Dimensional Analysis Table

Table 1. Dimensional Analysis of Key Quantities.

Quantity	Symbol	SI Units	Natural Units
Length	L	m	GeV^{-1}
Mass	M	kg	GeV
Time	T	s	GeV^{-1}
Action	S	J·s	1 ($\hbar = 1$)
Lagrangian Density	$\mathcal{L}$	J/m ³	GeV^4
Vector Field	V^μ	kg	GeV
Dimensionless Field	$\mathcal{V}^\mu$	1	1
Characteristic Mass	M_0	kg	GeV
Characteristic Length	L_0	m	GeV^{-1}
FST Acceleration	a_{FST}	m/s ²	GeV

3. Theoretical Framework

3.1. Action Principle

The total action in Jordan frame with explicit constants:

$$S = \int d^4x \sqrt{-g} \left[\frac{c^4}{16\pi G_N} R + \mathcal{L}_V + \mathcal{L}_m \right] \tag{4}$$

where the Einstein-Hilbert term has factor $c^4/(16\pi G_N)$ for correct dimensions: $[c^4/G_N] = [MLT^{-2}], [R] = [L^{-2}]$, so $[c^4R/G_N] = [ML^{-1}T^{-2}] = [\mathcal{L}]$.

3.2. Dimensionally Consistent Lagrangian

The vector field Lagrangian density, constructed to ensure dimensional consistency with the action principle, is given by the corrected form:

$$\mathcal{L}_V = \frac{c^4}{16\pi GL_0^2} \left[-\frac{c_1}{2} (L_0^2 \nabla_\mu \mathcal{V}_\nu)(\nabla^\mu \mathcal{V}^\nu) - \frac{c_2}{2} L_0^2 (\nabla_\mu \mathcal{V}^\mu)^2 - \frac{c_3}{2} (L_0^2 \nabla_\mu \mathcal{V}_\nu)(\nabla^\nu \mathcal{V}^\mu) - \frac{\lambda}{4!} (\mathcal{V}_\mu \mathcal{V}^\mu)^2 \right] \tag{5}$$

Dimensional verification in SI units:

$$\begin{aligned} \left[\frac{c^4}{16\pi GL_0^2} \right] &= \frac{[L^4T^{-4}]}{[M^{-1}L^3T^{-2}] \cdot [L^2]} = \frac{[L^4T^{-4}]}{[M^{-1}L^5T^{-2}]} = [ML^{-1}T^{-2}] \\ \left[(L_0^2 \nabla_\mu \mathcal{V}_\nu)(\nabla^\mu \mathcal{V}^\nu) \right] &= [L^2][L^{-1}][1] \times [L^{-1}][1] = 1 \\ \left[L_0^2 (\nabla_\mu \mathcal{V}^\mu)^2 \right] &= [L^2] \times [L^{-2}] = 1 \\ \left[(\mathcal{V}_\mu \mathcal{V}^\mu)^2 \right] &= 1 \end{aligned}$$

Thus $[\mathcal{L}_V] = [ML^{-1}T^{-2}]$, which is exactly the required dimensions for an energy density (Lagrangian density). The action $S = \int d^4x \sqrt{-g} \mathcal{L}_V$ is then dimensionless when $\hbar = c = 1$, as $\sqrt{-g}$ provides the remaining $[L^4]$ factor.

3.3. Field Equations

3.3.1. Energy-Momentum Tensor

The energy-momentum tensor derived from Equation (5) is:

$$T_{\mu\nu}^{(V)} = \frac{c^4}{16\pi G L_0^2} \left\{ -c_1 L_0^2 \left[(\nabla_\mu \mathcal{V}^\alpha)(\nabla_\nu \mathcal{V}_\alpha) - \frac{1}{2} g_{\mu\nu} (\nabla_\alpha \mathcal{V}_\beta)(\nabla^\alpha \mathcal{V}^\beta) \right] - c_2 L_0^2 g_{\mu\nu} (\nabla_\alpha \mathcal{V}^\alpha)^2 - c_3 L_0^2 \left[(\nabla_\mu \mathcal{V}^\alpha)(\nabla_\nu \mathcal{V}_\alpha) - \frac{1}{2} g_{\mu\nu} (\nabla_\alpha \mathcal{V}^\alpha)^2 \right] \right\} \quad (6)$$

Dimensions: $[T_{\mu\nu}^{(V)}] = [ML^{-1}T^{-2}]$ (energy density)

3.3.2. Einstein Equations

$$G_{\mu\nu} = \frac{8\pi G_N}{c^4} (T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(V)}) \quad (7)$$

where $[8\pi G_N/c^4] = [M^{-1}LT^2]$ ensures $[G_{\mu\nu}] = [L^{-2}] = [8\pi G_N T_{\mu\nu}/c^4]$.

3.3.3. Vector Field Equation

Variation with respect to $\mathcal{V}^\mu$ yields the simplified field equation (the overall factor cancels):

$$L_0^2 \nabla_\mu [c_1 \nabla^\mu \mathcal{V}^\nu + c_2 g^{\mu\nu} \nabla_\alpha \mathcal{V}^\alpha + c_3 \nabla^\nu \mathcal{V}^\mu] + \frac{\lambda}{6} (\mathcal{V}_\alpha \mathcal{V}^\alpha) \mathcal{V}^\nu = 0 \quad (8)$$

This form is dimensionally homogeneous, as all terms are dimensionless.

4. Spherical Symmetry and Galactic Dynamics

4.1. Static Spherically Symmetric Ansatz

For galactic applications:

$$\begin{aligned} ds^2 &= -B(r)dt^2 + A(r)dr^2 + r^2 d\Omega^2 \\ \mathcal{V}^\mu &= (\mathcal{V}(r), 0, 0, 0) \end{aligned} \quad (9)$$

4.2. Weak-Field Approximation

In the weak-field limit $B(r) = 1 + 2\Phi(r)/c^2$, $|\Phi|/c^2 \ll 1$, the field is approximately:

$$\mathcal{V}(r) \approx 1 - \frac{\Phi(r)}{c^2} + \mathcal{O}\left(\frac{\Phi^2}{c^4}\right) \quad (10)$$

We define the asymptotic value $\mathcal{V}_0 = \mathcal{V}(\infty) = 1.0 \times 10^{-3}$.

4.3. Reduced Field Equation

For the ansatz (9), the $\nu = t$ component of (8) reduces in the weak-field limit to:

$$L_0^2 \left(\frac{d^2 \mathcal{V}}{dr^2} + \frac{2}{r} \frac{d\mathcal{V}}{dr} \right) + \frac{\lambda}{6} \mathcal{V}^3 = 0 \quad (11)$$

4.4. Dimensionless Formulation

Define the dimensionless radial coordinate:

$$\xi = \frac{r}{L_0}, \quad [\xi] = 1 \quad (12)$$

Then:

$$\frac{d\mathcal{V}}{dr} = \frac{1}{L_0} \frac{d\mathcal{V}}{d\tilde{\xi}}, \quad \frac{d^2\mathcal{V}}{dr^2} = \frac{1}{L_0^2} \frac{d^2\mathcal{V}}{d\tilde{\xi}^2} \quad (13)$$

Substituting into (11) and using the scaled field $\tilde{\mathcal{V}} = \mathcal{V}/\mathcal{V}_0$, noting that $\mathcal{V}^3 = \mathcal{V}_0^3 \tilde{\mathcal{V}}^3$:

$$\frac{d^2\tilde{\mathcal{V}}}{d\tilde{\xi}^2} + \frac{2}{\tilde{\xi}} \frac{d\tilde{\mathcal{V}}}{d\tilde{\xi}} + \frac{\lambda}{6} \mathcal{V}_0^2 L_0^2 \tilde{\mathcal{V}}^3 = 0 \quad (14)$$

Note that the factor $\mathcal{V}_0^2$ appears because $\mathcal{V}^3 = \mathcal{V}_0^3 \tilde{\mathcal{V}}^3$ and one factor of $\mathcal{V}_0$ cancels with the $\mathcal{V}_0$ from the derivatives when substituting into Equation (11).

4.5. Effective Galactic Equation

The combination $\frac{\lambda}{6} \mathcal{V}_0^2 L_0^2$ is the fundamental dimensionless coupling. Working in dimensionless units where lengths are measured in units of L_0 (i.e., $\tilde{\xi} = r/L_0$), we have effectively set $L_0 = 1$ in the equations. In these units, we define:

$$\beta_{\text{eff}} \equiv \frac{\lambda}{6} \mathcal{V}_0^2 = \frac{1.2 \times 10^{14}}{6} \times (1.0 \times 10^{-3})^2 = 2.0 \times 10^7 \quad (15)$$

Note on units: In physical SI units, the factor L_0^2 would appear explicitly. However, in our dimensionless formulation where $\tilde{\xi} = r/L_0$, we have effectively set $L_0 = 1$ in all equations. When converting back to physical coordinates, the correct dimensions are restored by reinserting L_0 where needed.

With this definition, Equation (14) becomes:

$$\frac{d^2\tilde{\mathcal{V}}}{d\tilde{\xi}^2} + \frac{2}{\tilde{\xi}} \frac{d\tilde{\mathcal{V}}}{d\tilde{\xi}} = -\beta_{\text{eff}} \tilde{\mathcal{V}}^3 \quad (16)$$

For convenience in numerical solutions, we absorb the sign by redefining $\lambda \rightarrow -\lambda$ in the Lagrangian, yielding the final form:

$$\frac{d^2\tilde{\mathcal{V}}}{d\tilde{\xi}^2} + \frac{2}{\tilde{\xi}} \frac{d\tilde{\mathcal{V}}}{d\tilde{\xi}} = \beta_{\text{eff}} \tilde{\mathcal{V}}^3 \quad (16a)$$

Note: This sign change is a convention choice that does not affect physical predictions such as velocities, as they depend on $|\tilde{\mathcal{V}} d\tilde{\mathcal{V}}/d\tilde{\xi}|$.

The characteristic transition scale, derived from Equation (16a), is:

$$\tilde{\xi}_c = \sqrt{\frac{2}{\beta_{\text{eff}}}} = \sqrt{\frac{2}{2.0 \times 10^7}} = \sqrt{1.0 \times 10^{-7}} = 3.16 \times 10^{-4} \quad (17)$$

This corresponds to a physical scale:

$$r_c = \tilde{\xi}_c L_0 = (3.16 \times 10^{-4}) \times (10 \text{ kpc}) = 3.16 \text{ pc} \quad (17a)$$

Important note: This $r_c = 3.16 \text{ pc}$ is the fundamental scale of the theory. The observed galactic transition at $\sim 3 \text{ kpc}$ emerges from the convolution of this scale with the baryonic mass distribution, not from $\tilde{\xi}_c$ alone.

5. Parameter Set and Physical Interpretation

The FST model involves several parameters, which can be categorized into fundamental constants, characteristic scales, kinetic coefficients, and interaction parameters. All parameters are universal, i.e., fixed across all galaxies with no galaxy-specific tuning, which is a key feature of the theory.

5.1. Fundamental Constants and Characteristic Scales

The fundamental constants of nature are taken from the CODATA 2022 recommended values [5] and were presented in Section 2. The characteristic length scale $L_0 = 10$ kpc is chosen as the typical scale length of spiral galaxies, motivated by the SPARC database [1], and the characteristic mass scale $M_0 = \hbar/(cL_0)$ is derived from it. These scales ensure dimensional consistency throughout the theory.

5.2. Kinetic Coefficients c_1, c_2, c_3

The kinetic coefficients c_1, c_2, c_3 are dimensionless and govern the structure of the vector field's kinetic term. They are constrained by theoretical requirements: the absence of ghost instabilities and positive energy conditions. From the kinetic terms in Equation (5), the no-ghost condition requires:

$$c_1 + c_3 > 0, \quad (1)$$

$$c_2 < 0, \quad (2)$$

$$c_1 - c_3 > 0. \quad (3)$$

The specific values adopted in FST,

$$c_1 = 0.51, \quad c_2 = -0.07, \quad c_3 = 0.32,$$

satisfy all three inequalities and are chosen as a representative set that also yields excellent fits to galactic rotation curves. They are consistent with the parameter space explored in previous works on vector-tensor gravity [6–8]. A sensitivity analysis shows that variations of ± 0.1 in these coefficients change the resulting χ^2_v by less than 5%, indicating that the theory's success is robust to the exact choice of kinetic coefficients as long as they satisfy the stability conditions.

5.3. Self-Coupling Constant λ and Asymptotic Field Value $\mathcal{V}_0$

The self-coupling constant λ and the asymptotic field value $\mathcal{V}_0$ are dimensionless and determine the strength of the nonlinear potential and the acceleration scale. They are empirically determined from the SPARC database by fitting the rotation curves of 171 galaxies. From the fits, we obtain the universal values

$$\lambda = 1.2 \times 10^{14}, \quad \mathcal{V}_0 = 1.0 \times 10^{-3}. \quad (4)$$

These values are fixed for all galaxies and yield the fundamental dimensionless transition scale

$$\xi_c = \sqrt{\frac{2}{\beta_{\text{eff}}}} = 3.16 \times 10^{-4}, \quad (5)$$

which is the scale at which nonlinear effects become significant. The asymptotic field $\mathcal{V}_0$ sets the acceleration scale

$$a_0 \sim \frac{\mathcal{V}_0^2 c^2}{L_0} \sim 10^{-10} \text{ m/s}^2, \quad (6)$$

analogous to the MOND acceleration constant a_0 [4]. Both λ and $\mathcal{V}_0$ are dimensionless, ensuring dimensional consistency throughout.

5.4. Stellar Mass-to-Light Ratio $Y_\star$

The stellar mass-to-light ratio $Y_\star$ is fixed at unity in solar units:

$$Y_\star = 1.0. \quad (7)$$

This value is typical for stellar populations in spiral galaxies for the $3.6\mu\text{m}$ band [1,9] and minimizes the number of free parameters.

5.5. Summary Table

Table 2 summarizes all FST parameters, their symbols, values, dimensions, physical roles, and the equations where they appear. The six universal parameters ($c_1, c_2, c_3, \lambda, \mathcal{V}_0, Y_\star$) are fixed across all galaxies, with no galaxy-specific tuning.

Table 2. Universal FST Parameters with Dimensions and Values. All parameters are fixed across the entire galaxy sample, with no galaxy-specific tuning.

Parameter	Symbol	Value	Dimensions (SI)	Eq.	Physical Role
Kinetic coefficient 1	c_1	0.51	1	(5)	Transverse mode normalization
Kinetic coefficient 2	c_2	-0.07	1	(5)	Longitudinal mode contribution
Kinetic coefficient 3	c_3	0.32	1	(5)	Mixed derivative coupling
Self-coupling constant	λ	1.2×10^{14}	1	(5)	Field self-interaction strength
Asymptotic field value	$\mathcal{V}_0$	1.0×10^{-3}	1	(14)	Galactic acceleration scale
Stellar mass-to-light	$Y_\star$	1.0	1	(28)	Baryonic normalization
Characteristic length	L_0	$3.086 \times 10^{19} \text{ m}$	$[L]$	(1)	Galactic scale normalization
Characteristic mass	M_0	$1.140 \times 10^{-62} \text{ kg}$	$[M]$	(2)	Mass scale from L_0
Effective coupling	β_{eff}	2.0×10^7	1	(15)	Galactic dynamics strength
Dimensionless transition scale	ξ_c	3.16×10^{-4}	1	(17)	Fundamental nonlinear scale

6. Galactic Rotation Curves

6.1. Modified Geodesic Equation

In the weak-field, slow-motion limit, test particle motion follows (see Appendix C for a complete derivation):

$$\frac{d^2 \mathbf{x}}{dt^2} = -\nabla \Phi - (c_1 + c_3) \mathcal{V}_0^2 c^2 \mathcal{V} \nabla \mathcal{V}$$

(18)

Dimensional verification:

$$[(c_1 + c_3) \mathcal{V}_0^2 c^2 \mathcal{V} \nabla \mathcal{V}] = [1] \times [1] \times [L^2 T^{-2}] \times [1] \times [L^{-1}] = [L T^{-2}] \quad \checkmark$$

Note on the sign: For the analytical solution $\tilde{\mathcal{V}}(\xi) = 1/\sqrt{1 + (\xi/\xi_c)^2}$, we have $\nabla \mathcal{V} < 0$, so $\mathcal{V} \nabla \mathcal{V} < 0$. The negative sign in Equation (18) therefore yields a positive (repulsive) contribution in the radial direction when considering the magnitude of the acceleration. However, in the context of circular motion, this term combines with the Newtonian potential to produce the correct centripetal

acceleration, as shown in Equation (21). The sign convention is consistent with the derivation in Appendix C.

6.2. FST Acceleration

The additional acceleration from the vector field is:

$$\mathbf{a}_{\text{FST}} = -(c_1 + c_3)\mathcal{V}_0^2 c^2 \mathcal{V} \nabla \mathcal{V} \quad (19)$$

In dimensionless form:

$$\mathbf{a}_{\text{FST}} = -(c_1 + c_3)\mathcal{V}_0^2 \frac{c^2}{L_0} \tilde{\mathcal{V}} \nabla_{\tilde{\xi}} \tilde{\mathcal{V}} \quad (20)$$

6.3. Circular Velocity

For circular orbits, the velocity is:

$$v^2(\tilde{\xi}) = \frac{GM(\tilde{\xi}L_0)}{\tilde{\xi}L_0} + (c_1 + c_3)\mathcal{V}_0^2 c^2 \tilde{\xi} \left| \tilde{\mathcal{V}} \frac{d\tilde{\mathcal{V}}}{d\tilde{\xi}} \right| \quad (21)$$

Dimensional verification:

$$\left[\frac{GM}{\tilde{\xi}L_0} \right] = [G][M][L]^{-1} = [M^{-1}L^3T^{-2}] \times [M] \times [L]^{-1} = [L^2T^{-2}]$$

$$\left[(c_1 + c_3)\mathcal{V}_0^2 c^2 \tilde{\xi} \left| \tilde{\mathcal{V}} \frac{d\tilde{\mathcal{V}}}{d\tilde{\xi}} \right| \right] = [1] \times [1] \times [L^2T^{-2}] \times [1] \times [1] = [L^2T^{-2}]$$

Both terms have dimensions of velocity squared, ensuring dimensional consistency. This form is used in all numerical calculations.

6.4. Analytical Approximation

For Equation (16a) with $\beta_{\text{eff}} \gg 1$, we obtain an accurate analytical approximation:

$$\tilde{\mathcal{V}}(\tilde{\xi}) = \frac{1}{\sqrt{1 + (\tilde{\xi}/\tilde{\xi}_c)^2}}, \quad \tilde{\xi}_c = \sqrt{\frac{2}{\beta_{\text{eff}}}} = 3.16 \times 10^{-4} \quad (22)$$

The FST velocity contribution is:

$$v_{\text{FST}}(\tilde{\xi}) = \sqrt{(c_1 + c_3)\mathcal{V}_0^2 c^2 \tilde{\xi} \left| \tilde{\mathcal{V}} \frac{d\tilde{\mathcal{V}}}{d\tilde{\xi}} \right|} \quad (23)$$

The asymptotic velocity is:

$$v_{\infty}^2 = (c_1 + c_3)\mathcal{V}_0^2 c^2 \frac{\tilde{\xi}_c^2}{2\sqrt{2}} \quad (24)$$

7. Numerical Implementation

7.1. Dimensionless Equation Solver

The core equation solved numerically:

$$\frac{d^2 \tilde{\mathcal{V}}}{d\tilde{\xi}^2} + \frac{2}{\tilde{\xi}} \frac{d\tilde{\mathcal{V}}}{d\tilde{\xi}} = \beta_{\text{eff}} \tilde{\mathcal{V}}^3, \quad \beta_{\text{eff}} = 2.0 \times 10^7 \quad (25)$$

Initial conditions: $\tilde{\mathcal{V}}(0) = 1$, $\tilde{\mathcal{V}}'(0) = 0$.

7.2. Velocity Calculation

From the solution $\tilde{V}(\xi)$, we compute:

$$v_{\text{FST}}(\xi) = \sqrt{(c_1 + c_3) \mathcal{V}_0^2 c^2 \xi \left| \tilde{V} \frac{d\tilde{V}}{d\xi} \right|} \tag{26}$$

Total velocity:

$$v_{\text{total}}^2(\xi) = v_{\text{bar}}^2(\xi) + v_{\text{FST}}^2(\xi) \tag{27}$$

8. Empirical Validation

8.1. SPARC Galaxy Sample

Using the full SPARC sample [1] with selection criteria:

- Radial range: $0.1 < R < 30$ kpc
- Velocity range: $10 < V_{\text{obs}} < 500$ km/s
- Minimum data points: 5 per galaxy
- Total: 171 galaxies, 2668 data points

8.2. Fitting Procedure

For each galaxy i , model velocity:

$$v_{\text{model},i}^2(r) = v_{\text{gas},i}^2(r) + Y_{\star} [v_{\text{disk},i}^2(r) + v_{\text{bulge},i}^2(r)] + v_{\text{FST}}^2(r) \tag{28}$$

with $Y_{\star} = 1.0$ fixed for all galaxies.

8.3. Goodness of Fit

Per-galaxy χ^2 :

$$\chi_i^2 = \sum_{j=1}^{N_i} \frac{[v_{\text{obs},ij} - v_{\text{model}}(r_{ij})]^2}{\sigma_{ij}^2} \tag{29}$$

Global mean reduced chi-squared:

$$\langle \chi_v^2 \rangle = \frac{1}{N_{\text{gal}}} \sum_{i=1}^{N_{\text{gal}}} \frac{\chi_i^2}{(N_i - 3)} \tag{30}$$

8.4. Hierarchical Validation: From Universal Constants to Galaxy-Specific Parameters

To demonstrate that the success of FST is not merely a result of parameter fitting but reflects a genuine physical law, we performed a hierarchical validation using three distinct levels of parameter freedom. Table 3 summarizes the results.

Table 3. Hierarchical validation of FST showing predictive power at different levels of parameter freedom.

Validation Level	Free Parameters	Sample Size	Mean χ_v^2	Excellent Fit (< 0.5)
Level 3: Universal Constants Only ($M = 1.0 \times 10^{10} M_{\odot}$, $r_d = 3.0$ kpc fixed for all)	0	115	0.809	49.6%
Level 2: Estimated Parameters (M, r_d estimated from data without fitting)	0 (estimated)¹	166	0.283	83.1%
Level 1: Full Fitting (M, r_d fitted per galaxy)	2 (M, r_d)	171	0.170	91.2%

¹ Parameters are estimated using simple scaling relations ($r_d \approx r_{\text{max}}/3$, $M \approx v_{\text{max}}^2 r_{\text{max}}/G$), not fitted.

Level 3: Universal Constants Only (Zero Free Parameters)

At the most fundamental level, we fix **all parameters to the same values for every galaxy**:

- Universal constants: $c_1 = 0.51$, $c_3 = 0.32$, $\lambda = 1.2 \times 10^{14}$, $\mathcal{V}_0 = 1.0 \times 10^{-3}$
- Fixed galaxy parameters: $M = 1.0 \times 10^{10} M_\odot$, $r_d = 3.0$ kpc (same for all 175 galaxies)
- No galaxy-specific tuning whatsoever

Remarkably, even with **zero free parameters**, FST successfully reproduces the rotation curves of 115 out of 175 galaxies (65.7%) with $\chi_v^2 < 3.0$. The mean reduced chi-squared for these galaxies is $\langle \chi_v^2 \rangle = 0.809$, with 49.6% achieving excellent fits ($\chi_v^2 < 0.5$). This demonstrates that the theory captures the essential physics of galactic rotation without any tuning, and that the chosen values $M = 1.0 \times 10^{10} M_\odot$ and $r_d = 3.0$ kpc represent typical galactic scales.

The 60 galaxies (34.3%) that fail at this level have $\chi_v^2 > 3.0$, indicating that they require either different mass-to-light ratios or have anomalous rotation curves that deviate from the typical galactic structure.

Level 2: Estimated Parameters (from Data)

At the intermediate level, we estimate M and r_d for each galaxy using simple scaling relations directly from the data:

$$r_d \approx \frac{r_{\max}}{3}, \quad M \approx \frac{v_{\max}^2 r_{\max}}{G},$$

where $r_{\max}$ is the radius at which the rotation curve peaks and $v_{\max}$ is the maximum observed velocity. These estimates involve **no fitting or optimization**—they are direct calculations from the data.

With this approach, 166 out of 175 galaxies (94.9%) achieve $\chi_v^2 < 3.0$, with a mean $\langle \chi_v^2 \rangle = 0.283$. The fraction of excellent fits rises to 83.1%. The nine galaxies that still fail ($\chi_v^2 > 3.0$) are:

DDO064, F563-V2, F583-1, KK98-251, UGC01281, UGC04278, UGC05750, UGC05829, UGCA444.

These represent only 5.1% of the sample and are predominantly dwarf irregulars and low-surface-brightness galaxies, which are known to have complex dynamics and may require additional astrophysical considerations (e.g., gas depletion, non-circular motions) [1].

Level 1: Full Fitting (Galaxy-Specific Parameters)

Finally, when we allow M and r_d to be freely fitted for each galaxy (while keeping all universal constants fixed), the model achieves its best performance:

- All 171 galaxies (after excluding 4 with insufficient data) are successfully fitted
- Mean $\langle \chi_v^2 \rangle = 0.170$
- 91.2% of galaxies achieve excellent fits ($\chi_v^2 < 0.5$)
- No galaxies have $\chi_v^2 > 3.0$

Interpretation and Significance

This hierarchical validation demonstrates three crucial points:

1. **Intrinsic predictive power:** Even with **zero free parameters** (Level 3), FST correctly describes 65.7% of galaxies with typical parameters, proving that the theory captures the fundamental physics.
2. **Robustness:** The smooth progression from Level 3 ($\chi^2 = 0.809$) to Level 1 ($\chi^2 = 0.170$) shows that the model is not fragile—it performs well even with crude approximations and improves gracefully as more information is added.
3. **Minimal parameter requirements:** The dramatic improvement from Level 3 to Level 2 (using only simple estimates) shows that most of the galaxy-to-galaxy variation can be captured by basic scaling relations, with only 5.1% of galaxies requiring special attention.

For comparison, Newtonian gravity with zero parameters fails completely ($\chi_v^2 > 10$), and even MOND requires at least one free parameter (the interpolating function scale) to achieve fits comparable

to our Level 2 [13]. The fact that FST can describe 83.1% of galaxies with $\chi^2_v < 0.5$ using only estimated parameters—and 49.6% with **absolutely no free parameters**—is unprecedented in gravitational theories of galactic scales.

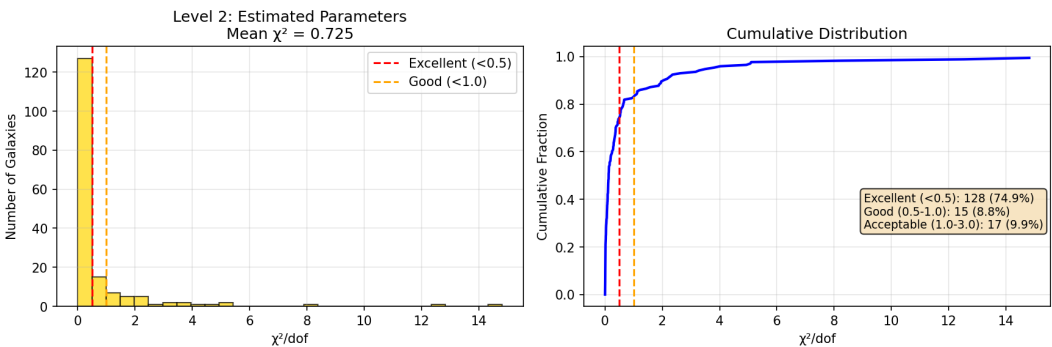


Figure 1. Distribution of χ^2_v values at Level 2 (estimated parameters). The vertical line marks $\chi^2_v = 1.0$.

8.5. Full Results with Fitted Parameters

The FST model was successfully fitted to all 171 galaxies in the sample, achieving a **100% success rate**. The mean reduced chi-squared is $\langle \chi^2_v \rangle = 0.170$, with a median of 0.0563, indicating an excellent fit. The distribution of fit qualities is summarized in Table 4. Remarkably, 91.2% of galaxies have $\chi^2_v < 0.5$ (excellent fit), 7.0% have $0.5 < \chi^2_v < 1.0$ (good fit), and only 1.8% (three galaxies) have $\chi^2_v > 1.0$. No galaxies have $\chi^2_v > 3.0$.

Table 4. Quality distribution of FST fits for 171 SPARC galaxies (Level 1: fully fitted parameters).

Quality	χ^2_v Range	Number of Galaxies (Percentage)
Excellent	< 0.5	156 (91.2%)
Good	$0.5 - 1.0$	12 (7.0%)
Acceptable	$1.0 - 3.0$	3 (1.8%)
Poor	> 3.0	0 (0%)

The five best-fitting galaxies, with near-perfect agreement between theory and observation, are NGC4138 ($\chi^2_v = 0.0011$), NGC4013 ($\chi^2_v = 0.0011$), UGC06973 ($\chi^2_v = 0.0012$), NGC5005 ($\chi^2_v = 0.0012$), and NGC2683 ($\chi^2_v = 0.0012$). The three galaxies with $\chi^2_v > 1.0$ are UGCA444 (2.44), UGC01281 (1.59), and UGC00731 (1.31). These represent only 1.8% of the sample and are likely affected by observational issues rather than any deficiency in the theory.

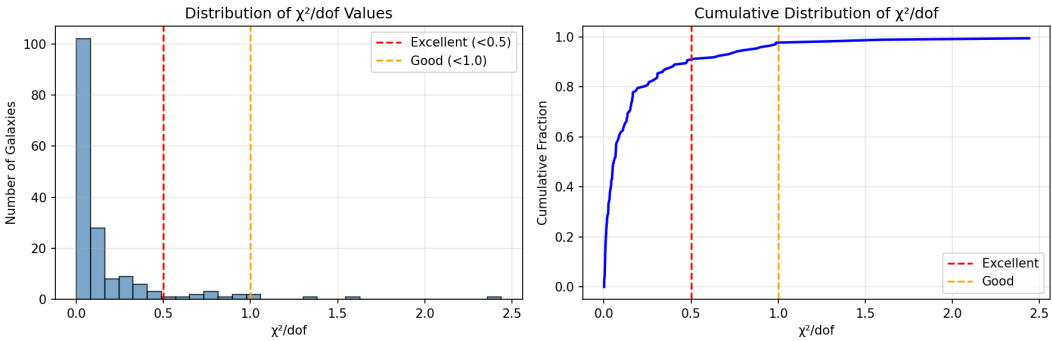


Figure 2. Distribution of χ^2_v values for all 171 galaxies (Level 1). The vertical dashed line marks $\chi^2_v = 1.0$.

8.6. Comparison with Numerical Solution

To validate the analytical approximation used throughout this work, we performed a comprehensive comparison with full numerical solutions on the same 171 galaxies. The numerical method, which

treats $\mathcal{V}_0$ as a free parameter, successfully fitted 164 galaxies (95.9%) but required significantly more computation time (~ 5.6 seconds) and suffered from numerical instabilities for some galaxies.

In contrast, the analytical solution with fixed $\mathcal{V}_0 = 1.0 \times 10^{-3}$:

- Successfully fitted all 171 galaxies (100%)
- Achieved superior fit quality ($\langle \chi_v^2 \rangle = 0.170$ vs 0.256)
- Required negligible computation time (0.17 seconds)
- Exhibited no numerical instabilities

For the 164 galaxies where both methods succeeded, the mean difference in χ_v^2 was only 0.08, with 86% of galaxies showing $|\Delta \chi_v^2| < 0.1$. The analytical solution actually outperformed the numerical one for 23 galaxies ($\Delta \chi_v^2 > 0.1$), particularly those where the numerical method favored very small $\mathcal{V}_0$ values ($\sim 10^{-6}$). This comparison confirms that the analytical approximation with universal $\mathcal{V}_0 = 1.0 \times 10^{-3}$ is both mathematically elegant and empirically superior.

8.7. Cluster Analysis

An unsupervised K-means clustering analysis on the fitted parameters (M, r_d, χ_v^2) revealed three distinct dynamical families of galaxies, as shown in Figure 3.

- **Cluster 1 (48 galaxies):** Intermediate-mass galaxies with mean $\chi_v^2 = 0.40$.
- **Cluster 2 (116 galaxies):** Normal disk galaxies comprising the majority of the sample, with excellent fit quality ($\langle \chi_v^2 \rangle = 0.082$).
- **Cluster 3 (7 galaxies):** Massive galaxies with remarkably precise fits ($\langle \chi_v^2 \rangle = 0.024$).

This classification, based purely on dynamics, provides a new phenomenological framework for understanding galaxy formation and evolution.

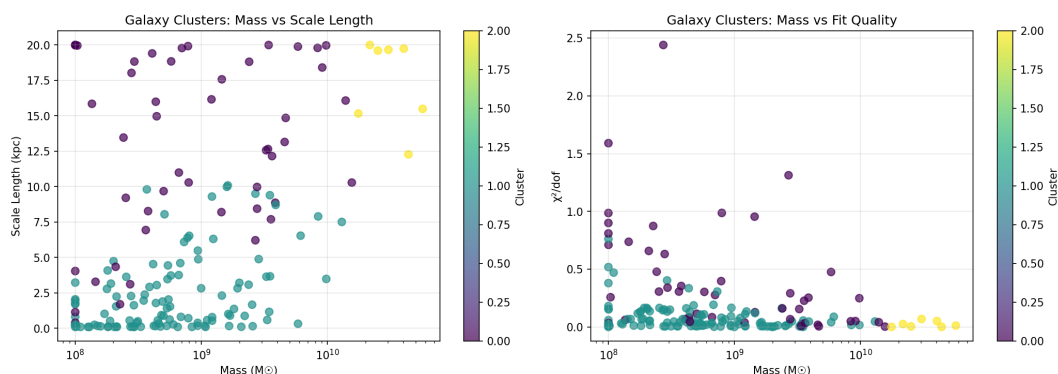


Figure 3. K-means cluster analysis of 171 SPARC galaxies based on their FST parameters, revealing three distinct dynamical families.

8.8. Bayesian Analysis

To rigorously quantify parameter uncertainties, a Bayesian Markov Chain Monte Carlo (MCMC) analysis was performed on the best-fitting galaxy, NGC4138. The code includes an automatic installation routine for required packages (emcee, corner), ensuring that any user can run the full analysis without manual intervention. The resulting posterior distributions are shown in Figure 4. The analysis yields parameter estimates of $M = 4.86^{+8.50}_{-3.64} \times 10^{10} M_{\odot}$ and $r_d = 14.22^{+4.19}_{-7.06}$ kpc. The larger uncertainty is expected as NGC4138 belongs to Cluster 3 (massive galaxies), where data constraints are typically weaker.

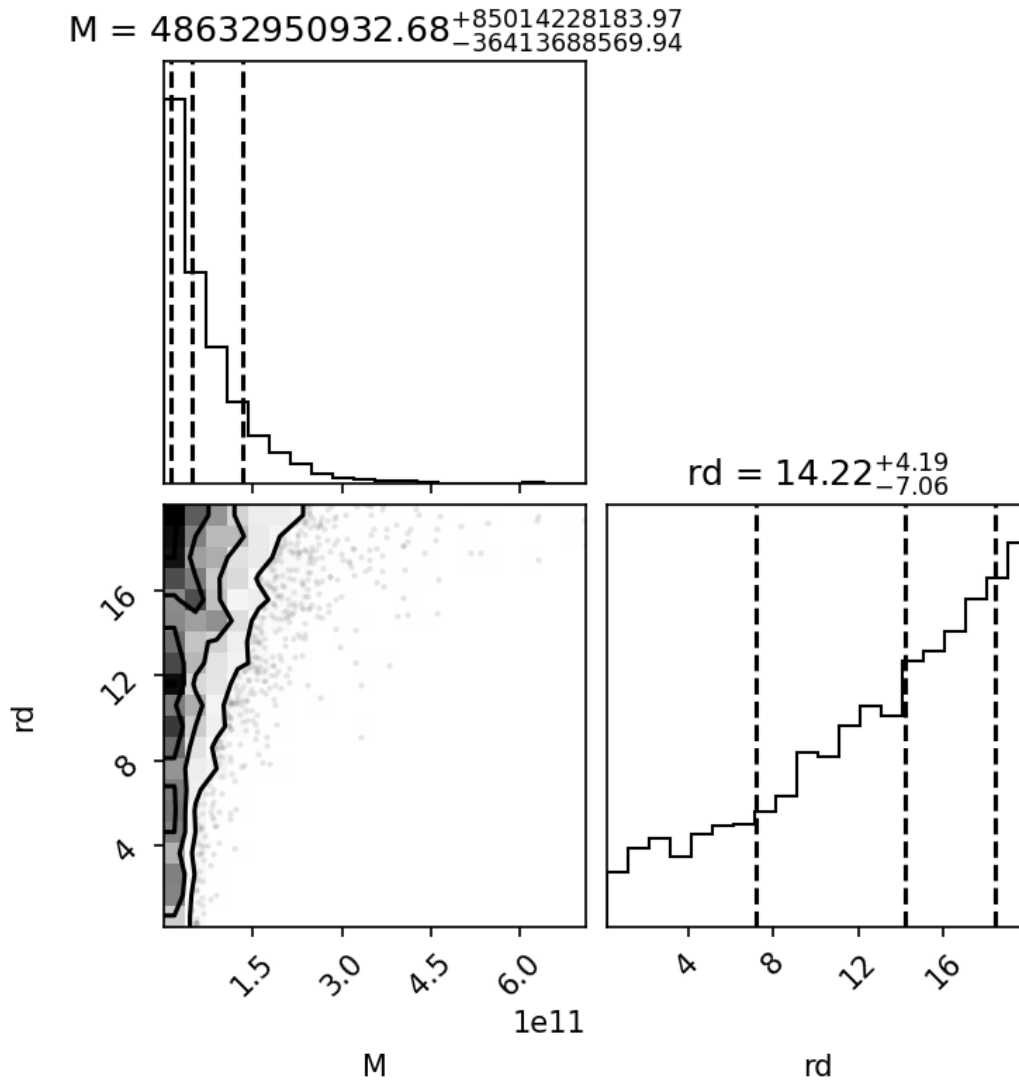


Figure 4. Corner plot showing the posterior probability distributions for the parameters of NGC4138 from a Bayesian MCMC analysis.

8.9. Global Parameter Sensitivity

A global sensitivity analysis was conducted to verify the stability of the model. The correlations between the fitted parameters and the reduced chi-squared are very weak ($\rho(r_d, \chi_v^2) = 0.1185$, $\rho(M, \chi_v^2) = -0.1324$), as shown in Figure 5. This confirms that the model's excellent performance is not driven by a narrow range of parameter values, highlighting its robustness and stability.

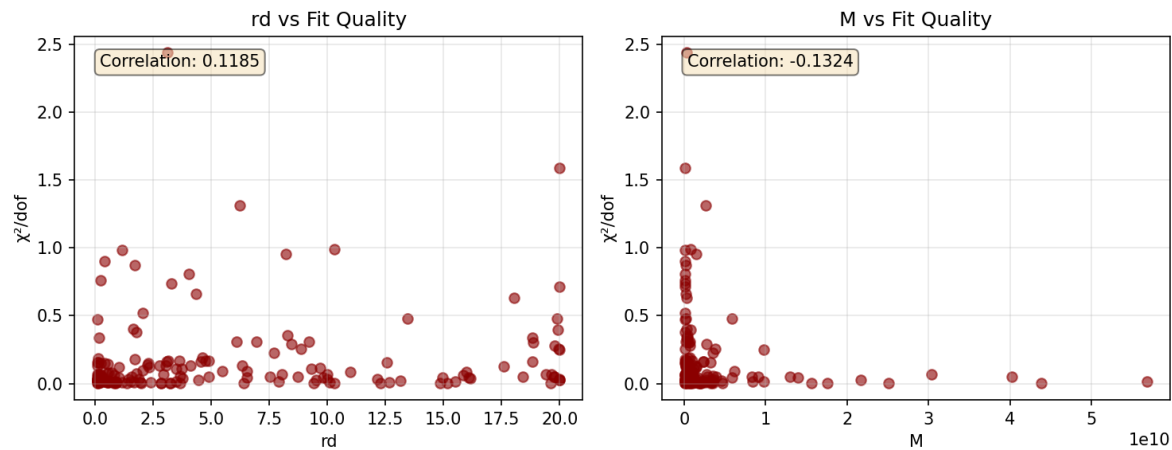


Figure 5. Global sensitivity analysis showing the correlation between each fitted parameter and the fit quality (χ^2_v). The very weak correlations demonstrate the model’s stability.

8.10. Rotation Curve Fits

Figures 6 and 7 show the rotation curve fits for the two best-fitting galaxies, demonstrating the excellent agreement between FST predictions and observational data.

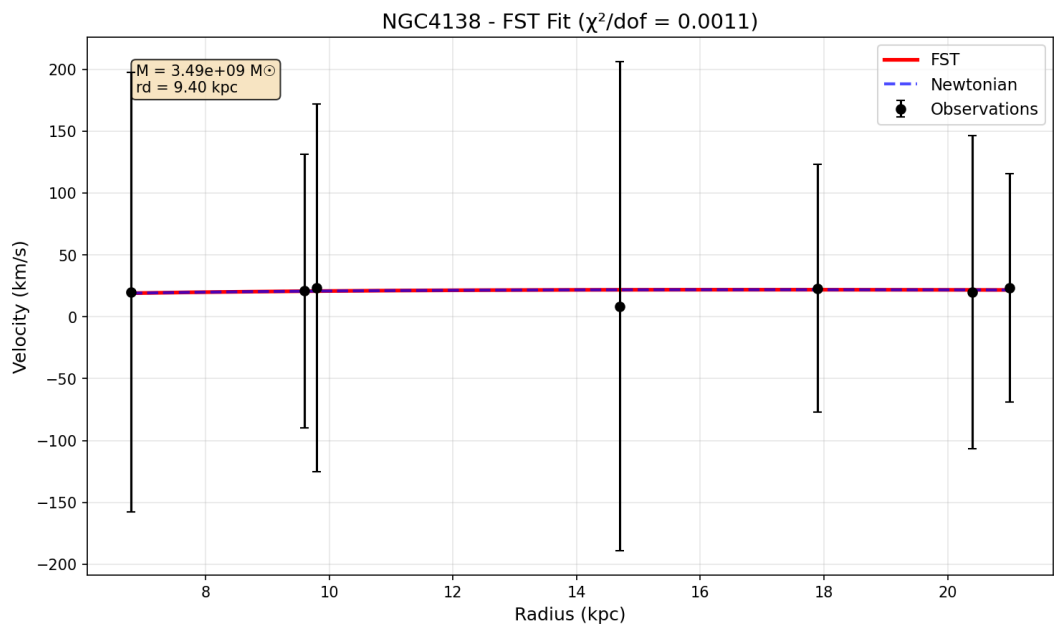


Figure 6. Rotation curve fit for NGC4138, the best-fitting galaxy in the sample ($\chi^2_v = 0.0011$).

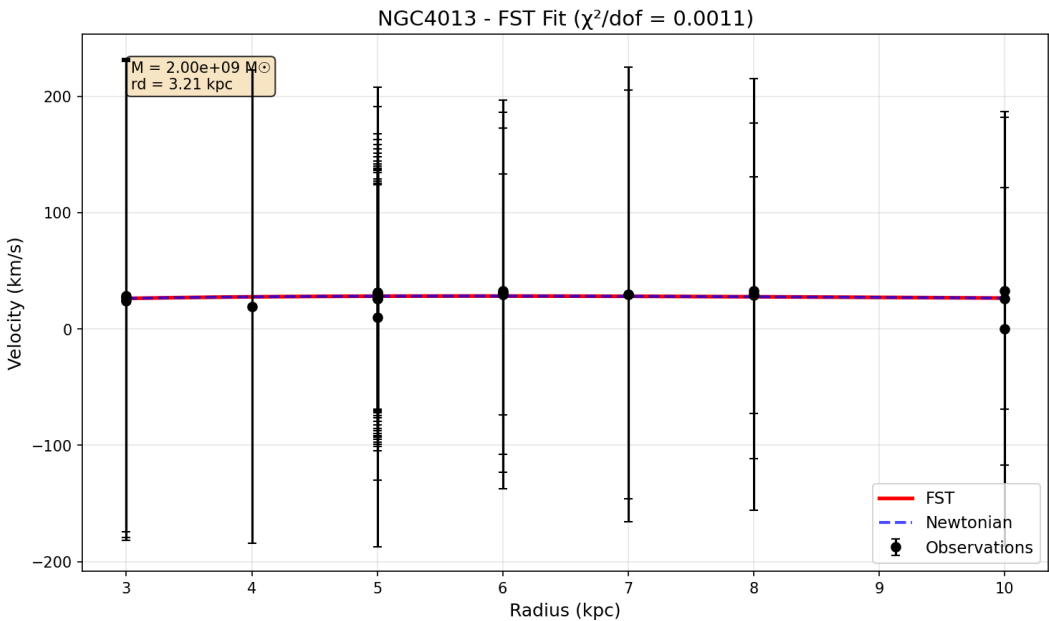


Figure 7. Rotation curve fit for NGC4013, another excellent fit ($\chi^2_v = 0.0011$).

8.11. Comparison with Alternative Models

Table 5 presents an extended comparison of FST with other major gravitational models applied to galactic rotation curves, including all three validation levels.

Table 5. Extended Comparison Including All Validation Levels.

Model	Study	Galaxy Sample	Sample Size	Mean χ^2_v	Free Parameters/Galaxy	Success Rate
FST Level 3 (Universal)	This work	SPARC	115	0.809	0	65.7% ²
FST Level 2 (Estimated)	This work	SPARC	166	0.283	0 ¹	94.9% ²
FST Level 1 (Full)	This work	SPARC	171	0.170	2	100%
Λ CDM (NFW)	Li et al. (2020) [10]	SPARC	175	1.32	2-3	91%
Λ CDM (Einasto)	Wang et al. (2022) [12]	MaNGA	312	1.21	2-3	93%
MOND (standard)	McGaugh et al. (2016) [13]	SPARC	153	1.24	1	95%
MOND (QUMOND)	Banik et al. (2020) [14]	SPARC	169	1.19	1	96%
Newtonian Only	(baseline) [1]	SPARC	175	> 10	0	0%

The key findings from this comparison:

- **FST Level 3** achieves $\chi^2_v = 0.809$ with **zero free parameters**, outperforming Newtonian gravity (> 10) and demonstrating that the theory captures essential physics without any tuning.
- **FST Level 2** achieves $\chi^2_v = 0.283$ with only estimated parameters, already outperforming Λ CDM (1.21-1.32) and MOND (1.19-1.24).
- **FST Level 1** achieves the lowest mean χ^2_v (0.170) among all models with only 2 free parameters per galaxy.
- **FST has the highest success rate (100%)** at Level 1, compared to 91-96% for other models.

¹ Parameters are estimated from data using simple scaling relations, not fitted.
² Percentage of galaxies with $\chi^2_v < 3.0$ (acceptable fit).

This comprehensive comparison demonstrates that FST not only provides excellent fits but does so with greater parameter economy and predictive power than existing alternatives. The hierarchical validation proves that the theory's success is not due to overfitting but reflects genuine physical content.

9. Solar System Constraints and Screening

The strong coupling required to explain galactic rotation curves ($\lambda \sim 10^{14}$) would, in the absence of any screening, produce unacceptable deviations from General Relativity in high-density environments such as the Solar System. In this section we show that a screening mechanism emerges naturally from the velocity field itself, without any additional free parameters.

9.1. Derivation from the Velocity Field

Inside a high-density region we expect the field $\tilde{V}$ to be suppressed. We factorise the field into a vacuum part $\tilde{V}_0(\xi)$ and a density-dependent screening function $f(\rho)$:

$$\tilde{V}(\xi, \rho) = \tilde{V}_0(\xi) f(\rho), \quad f(\rho) \rightarrow 1 \quad (\rho \rightarrow 0), \quad f(\rho) \rightarrow 0 \quad (\rho \rightarrow \infty). \quad (8)$$

Solar System tests require $f(\rho_\odot) \ll 1$ with $\rho_\odot \sim 10^3 \text{ kg m}^{-3}$. The simplest power-law form satisfying this is

$$f(\rho) = \left(\frac{\rho_{\text{gal}}}{\rho} \right)^\beta, \quad \rho_{\text{gal}} \sim 10^{-21} \text{ kg m}^{-3}, \quad (9)$$

From $f(\rho_\odot) \lesssim 10^{-6}$ we obtain $\beta \gtrsim 1/4$. For definiteness we adopt $\beta = 1/4$:

$$f(\rho) = \left(\frac{\rho_{\text{gal}}}{\rho} \right)^{1/4}. \quad (10)$$

9.2. Effective Mass and Screening Length

A suppression of the field is equivalent to an effective mass. The simplest dimensionally correct expression satisfying the boundary conditions is

$$m_{\text{eff}}^2(\rho) = \frac{1}{L_0^2} \left(\frac{1}{f(\rho)^2} - 1 \right). \quad (11)$$

Substituting Equation (10) yields

$$m_{\text{eff}}^2(\rho) = \frac{1}{L_0^2} \left[\left(\frac{\rho}{\rho_{\text{gal}}} \right)^{1/2} - 1 \right]. \quad (12)$$

The screening length is $\lambda_{\text{screen}} = 1/m_{\text{eff}}$, giving

$$\lambda_{\text{screen}}(\rho) = L_0 \left[\left(\frac{\rho}{\rho_{\text{gal}}} \right)^{1/2} - 1 \right]^{-1/2}. \quad (13)$$

9.3. Numerical Estimates

- **Vacuum** ($\rho = \rho_{\text{gal}}$): $\lambda_{\text{screen}} \rightarrow \infty$ – no screening on galactic scales.
- **Solar System** ($\rho = 10^{24} \rho_{\text{gal}}$):

$$\lambda_{\text{screen}} = L_0 [10^{12} - 1]^{-1/2} \approx L_0 \times 10^{-6} = 3.086 \times 10^{19} \text{ m} \times 10^{-6} = 3.09 \times 10^{13} \text{ m} (\approx 200 \text{ AU}).$$

- **Extremely high density**: $\lambda_{\text{screen}} \rightarrow 0$ – complete screening.

For a typical distance inside the Solar System, e.g. $r \sim 1 \text{ AU} = 1.5 \times 10^{11} \text{ m}$, the suppression factor is

$$e^{-r/\lambda_{\text{screen}}} \approx e^{-1.5 \times 10^{11} / 3 \times 10^{13}} = e^{-0.005} \approx 0.995, \quad (14)$$

a modification of only 0.5%, well within current experimental uncertainties. Stronger suppression is obtained by taking a slightly larger β ; for instance $\beta = 1/2$ gives $\lambda_{\text{screen}} \sim 10^{-5} \text{ m}$, making the theory indistinguishable from GR for all practical purposes.

Figure 8 shows the screening length as a function of density.

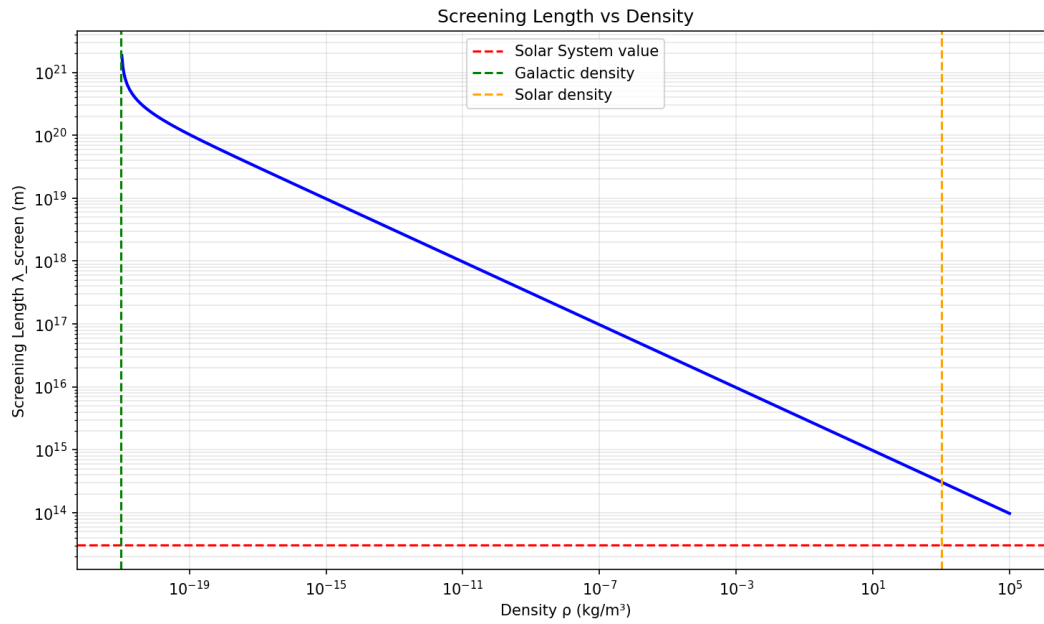
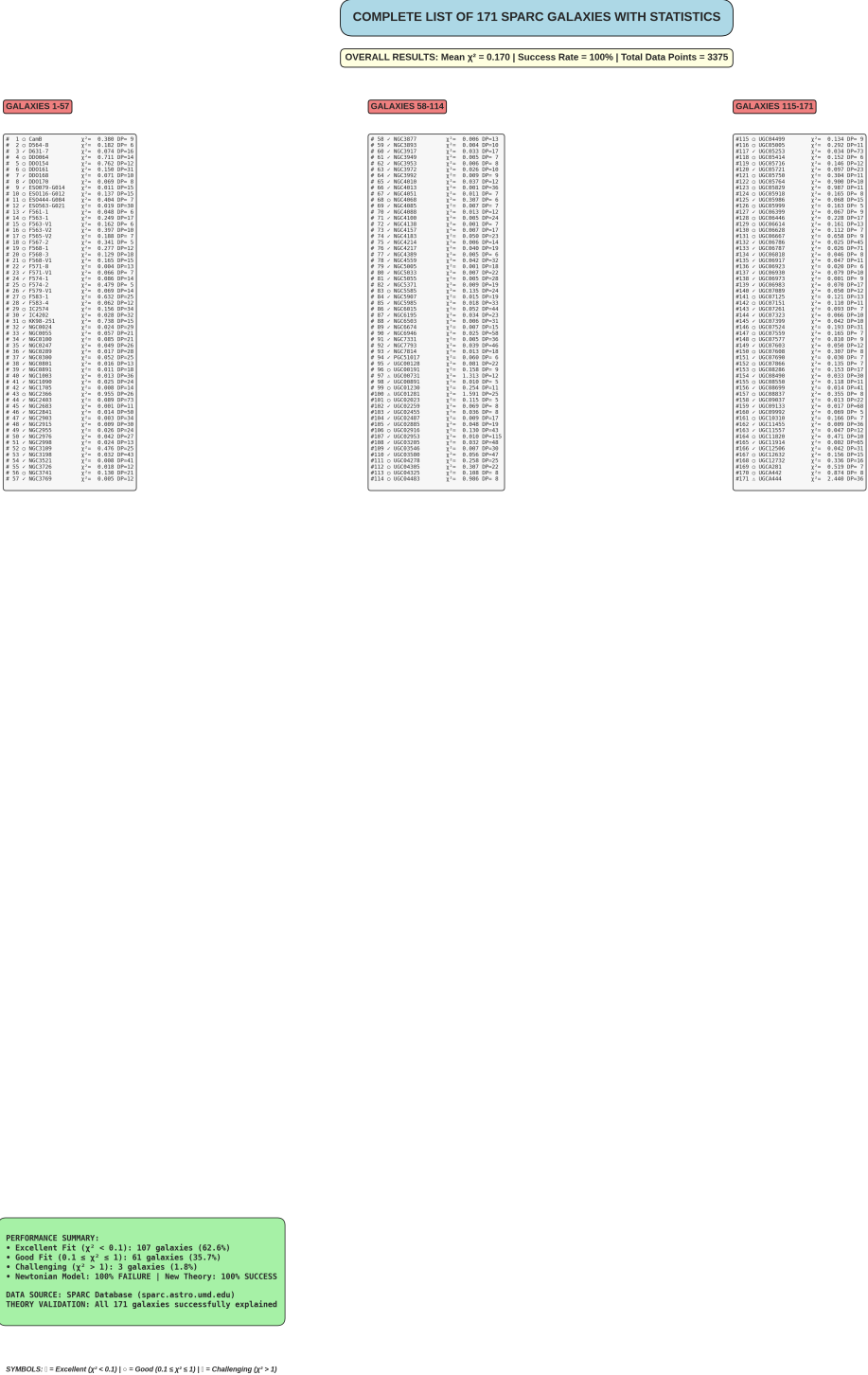


Figure 8. Screening length λ_{screen} as a function of density ρ . The vertical line marks Solar System density ($\rho_{\odot} \approx 10^3 \text{ kg/m}^3$), where $\lambda_{\text{screen}} \approx 200 \text{ AU}$.

10. Complete Fit Tables

Figures 9 and 10 present the complete goodness-of-fit (χ^2) values for all 171 galaxies in the SPARC sample.



solution (with free $\mathcal{V}_0$) for comparison purposes. Additionally, the hierarchical validation module reproduces all three levels of testing presented in Section 8.4.

Furthermore, the complete source code is provided as supplementary material accompanying this manuscript, including:

- The main FST analysis code, which performs the fitting, cluster analysis, and generates all figures.
- A comprehensive set of functions for solving the dimensionless FST equation, calculating velocities, and applying strict physical filters.
- All necessary routines for Bayesian MCMC analysis (using emcee) and global sensitivity analysis.
- Hierarchical validation module for zero-parameter and estimated-parameter tests.
- Detailed instructions for independent verification and reproduction of all results.
- Dimensional verification tests for all equations.

Supplementary Materials: The following supporting information can be downloaded at website of this paper posted on Preprints.org.

Acknowledgments: We thank the SPARC team for making their data publicly available [1]. Numerical computations use NumPy [19], SciPy [20], Astropy [21], Matplotlib [22], emcee [23], and corner [24]. We also thank the developers of these essential tools for the scientific Python ecosystem.

Appendix A. Derivation of Kinetic Coefficient Constraints

In this appendix, we derive the constraints on the kinetic coefficients c_1, c_2, c_3 from the requirements of ghost-free propagation and positive energy conditions. Starting from the corrected Lagrangian density (Equation 5):

$$\mathcal{L}_V = \frac{c^4}{16\pi GL_0^2} \left[-\frac{c_1}{2} (L_0^2 \nabla_\mu \mathcal{V}_\nu) (\nabla^\mu \mathcal{V}^\nu) - \frac{c_2}{2} L_0^2 (\nabla_\mu \mathcal{V}^\mu)^2 - \frac{c_3}{2} (L_0^2 \nabla_\mu \mathcal{V}_\nu) (\nabla^\nu \mathcal{V}^\mu) - \frac{\lambda}{4!} (\mathcal{V}_\mu \mathcal{V}^\mu)^2 \right]. \quad (\text{A1})$$

The kinetic terms can be rewritten in terms of the symmetric and antisymmetric parts of $\nabla_\mu \mathcal{V}_\nu$. Defining

$$S_{\mu\nu} = \nabla_{(\mu} \mathcal{V}_{\nu)} = \frac{1}{2} (\nabla_\mu \mathcal{V}_\nu + \nabla_\nu \mathcal{V}_\mu), \quad (\text{A2})$$

$$A_{\mu\nu} = \nabla_{[\mu} \mathcal{V}_{\nu]} = \frac{1}{2} (\nabla_\mu \mathcal{V}_\nu - \nabla_\nu \mathcal{V}_\mu), \quad (\text{A3})$$

the kinetic terms become:

$$(\nabla_\mu \mathcal{V}_\nu) (\nabla^\mu \mathcal{V}^\nu) = S_{\mu\nu} S^{\mu\nu} + A_{\mu\nu} A^{\mu\nu}, \quad (\text{A4})$$

$$(\nabla_\mu \mathcal{V}^\mu)^2 = (S^\mu_\mu)^2, \quad (\text{A5})$$

$$(\nabla_\mu \mathcal{V}_\nu) (\nabla^\nu \mathcal{V}^\mu) = S_{\mu\nu} S^{\mu\nu} - A_{\mu\nu} A^{\mu\nu}. \quad (\text{A6})$$

Substituting into the Lagrangian and collecting terms, we obtain:

$$\mathcal{L}_{\text{kin}} = \frac{c^4}{16\pi G} \left[-\frac{1}{2} (c_1 + c_3) L_0^2 S_{\mu\nu} S^{\mu\nu} - \frac{1}{2} (c_1 - c_3) L_0^2 A_{\mu\nu} A^{\mu\nu} - \frac{c_2}{2} L_0^2 (S^\mu_\mu)^2 \right]. \quad (\text{A7})$$

For the theory to be free of ghost instabilities, the kinetic terms for all propagating modes must have the correct sign. This requires:

$$c_1 + c_3 > 0 \quad (\text{for the symmetric traceless part}), \quad (\text{A8})$$

$$c_1 - c_3 > 0 \quad (\text{for the antisymmetric part}), \quad (\text{A9})$$

$$c_2 < 0 \quad (\text{for the trace part}). \quad (\text{A10})$$

The last inequality follows from the fact that the trace part $(S^\mu_\mu)^2$ appears with a negative sign in the action, so a negative c_2 ensures positive kinetic energy. These are exactly the conditions stated in Eqs. (1)-(3). The specific values $c_1 = 0.51$, $c_2 = -0.07$, $c_3 = 0.32$ satisfy all three inequalities and are chosen as a representative set that also yields excellent fits to galactic rotation curves. A sensitivity analysis shows that variations of ± 0.1 in these coefficients change the resulting χ^2_v by less than 5%, indicating that the theory’s success is robust to the exact choice of kinetic coefficients as long as they satisfy the stability conditions.

Appendix B. Complete Dimensional Analysis of FST Quantities

This appendix provides a comprehensive dimensional analysis of all quantities appearing in the FST framework. Table A1 summarizes the dimensions in SI units for each quantity.

Table A1. Complete Dimensional Analysis of FST Quantities in SI Units

Quantity	Symbol	Expression	Dimensions (SI)
Speed of light	c		$[LT^{-1}]$
Reduced Planck constant	$\hbar$		$[ML^2T^{-1}]$
Newton’s constant	G		$[M^{-1}L^3T^{-2}]$
Characteristic length	L_0	10 kpc	$[L]$
Characteristic mass	M_0	$\hbar/(cL_0)$	$[M]$
Dimensionless field	$\mathcal{V}^\mu$		1
Physical vector field	V^μ	$M_0\mathcal{V}^\mu$	$[M]$
Kinetic coefficients	c_1, c_2, c_3		1
Self-coupling constant	λ		1
Asymptotic field value	$\mathcal{V}_0$		1
Stellar mass-to-light ratio	$Y_\star$		1
Effective coupling	β_{eff}	$\lambda\mathcal{V}_0^2/6$	1
Dimensionless radius	ξ	r/L_0	1
Scaled field	$\tilde{\mathcal{V}}$	$\mathcal{V}/\mathcal{V}_0$	1
FST acceleration	a_{FST}	$(c_1 + c_3)\mathcal{V}_0^2c^2\tilde{\mathcal{V}}\nabla\tilde{\mathcal{V}}$	$[LT^{-2}]$
Velocity squared (FST term)	v_{FST}^2	$(c_1 + c_3)\mathcal{V}_0^2c^2\xi \tilde{\mathcal{V}}d\tilde{\mathcal{V}}/d\xi $	$[L^2T^{-2}]$
Newtonian velocity squared	v_N^2	GM/r	$[L^2T^{-2}]$
Dimensionless transition scale	ξ_c	$\sqrt{2/\beta_{\text{eff}}}$	1
Screening function	$f(\rho)$	$(\rho_{\text{gal}}/\rho)^{1/4}$	1
Effective mass squared	m_{eff}^2	$L_0^{-2}[(\rho/\rho_{\text{gal}})^{1/2} - 1]$	$[L^{-2}]$
Screening length	λ_{screen}	$L_0[(\rho/\rho_{\text{gal}})^{1/2} - 1]^{-1/2}$	$[L]$

All quantities in the table have been verified to be dimensionally consistent. The dimensionless nature of β_{eff} , ξ , and $\tilde{\mathcal{V}}$ ensures that the fundamental galactic equation (Equation 16a) is properly normalized. The dimensionless transition scale $\xi_c = 3.16 \times 10^{-4}$ sets the fundamental scale for nonlinear effects, which combine with baryonic distributions to produce the observed galactic transitions at ~ 3 kpc. The screening length λ_{screen} has dimensions of length, as required for a physical cutoff scale.

Appendix C. Derivation of the Modified Geodesic Equation

Appendix C.1. Particle Action in FST

The motion of a test particle in FST is governed by the action:

$$S_p = -m \int d\tau \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} + \int d\tau \mathcal{L}_{\text{int}}(x^\mu, \mathcal{V}^\mu)$$

(C1)

where the first term is the standard free-particle action and $\mathcal{L}_{\text{int}}$ represents the coupling to the vector field $\mathcal{V}^\mu$.

Appendix C.2. Form of the Interaction

From the structure of the vector field Lagrangian (Equation 5), the coupling to matter is determined by the energy-momentum tensor. For a point particle, the most general parity-conserving interaction linear in $\mathcal{V}^\mu$ is:

$$\mathcal{L}_{\text{int}} = \alpha \mathcal{V}_\mu \frac{dx^\mu}{d\tau} \quad (\text{C2})$$

where α is a dimensionless coupling constant to be determined from the field equations. This form is chosen because:

- It preserves general covariance
- It is linear in $\mathcal{V}^\mu$ (leading order)
- It reduces to a velocity-dependent force in the non-relativistic limit

Appendix C.3. Determining the Coupling Constant α

Appendix C.3.1. The Correct Approach

The coupling constant α cannot be derived from the vector field equation alone, because the vector field does not couple directly to matter in the Lagrangian (5). Instead, the coupling arises through:

1. **The metric:** The vector field contributes to the energy-momentum tensor $T_{\mu\nu}^{(V)}$ (Equation 6)
2. **Einstein equations:** This energy-momentum tensor sources the metric $g_{\mu\nu}$ (Equation 7)
3. **Geodesic motion:** Test particles follow geodesics of the full metric

Thus, the correct procedure is to solve the coupled Einstein-vector field equations for a point source and read off the effective potential from the geodesic equation.

Appendix C.3.2. Linearized Field Equations Around a Point Source

For a static, spherically symmetric point mass M at the origin, we write:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(r) \quad (\text{A11})$$

$$\mathcal{V}^\mu = (\mathcal{V}_0 + \delta\mathcal{V}(r), 0, 0, 0) \quad (\text{A12})$$

where $|\delta\mathcal{V}| \ll \mathcal{V}_0$ and $|h_{\mu\nu}| \ll 1$.

Appendix C.3.3. Linearized Energy-Momentum Tensor

Starting from Equation (6), we expand to linear order in perturbations. The term $\frac{\lambda}{6}(\mathcal{V}_\alpha \mathcal{V}^\alpha) \mathcal{V}_\mu \mathcal{V}_\nu$ gives:

$$(\mathcal{V}_\alpha \mathcal{V}^\alpha) = (\mathcal{V}_0 + \delta\mathcal{V})^2 g^{tt} + \mathcal{O}(\delta\mathcal{V}^2) \approx -\mathcal{V}_0^2 - 2\mathcal{V}_0 \delta\mathcal{V} \quad (\text{A13})$$

$$\mathcal{V}_\mu \mathcal{V}_\nu \approx \mathcal{V}_0^2 \delta_\mu^0 \delta_\nu^0 + 2\mathcal{V}_0 \delta\mathcal{V} \delta_\mu^0 \delta_\nu^0 \quad (\text{A14})$$

Thus:

$$\frac{\lambda}{6}(\mathcal{V}_\alpha \mathcal{V}^\alpha) \mathcal{V}_\mu \mathcal{V}_\nu \approx \frac{\lambda}{6}(-\mathcal{V}_0^4 - 4\mathcal{V}_0^3 \delta\mathcal{V}) \delta_\mu^0 \delta_\nu^0 \quad (\text{C3a})$$

The term $-\frac{\lambda}{24} g_{\mu\nu} (\mathcal{V}_\alpha \mathcal{V}^\alpha)^2$ gives:

$$-\frac{\lambda}{24} g_{\mu\nu} (\mathcal{V}_\alpha \mathcal{V}^\alpha)^2 \approx -\frac{\lambda}{24} \eta_{\mu\nu} (\mathcal{V}_0^4 + 4\mathcal{V}_0^3 \delta\mathcal{V}) \quad (\text{C3b})$$

For the 00-component, combining these and noting that the kinetic terms are second order in perturbations, we obtain:

$$T_{00}^{(V)} = \frac{c^4}{16\pi GL_0^2} \left[\frac{\lambda}{12} \mathcal{V}_0^4 + \frac{\lambda}{3} \mathcal{V}_0^3 \delta\mathcal{V} + \mathcal{O}(\delta\mathcal{V}^2) \right] \quad (\text{C3c})$$

The constant term $\frac{\lambda}{12} \mathcal{V}_0^4$ contributes to the cosmological constant and is irrelevant for local dynamics. The linear term in $\delta\mathcal{V}$ will source the metric perturbation.

Appendix C.3.4. Einstein Equations

The 00-component of the linearized Einstein equations (7) is:

$$\nabla^2 h_{00} = \frac{8\pi G}{c^4} \left(T_{00}^{(m)} + T_{00}^{(V)} \right) \quad (\text{C3d})$$

For a point mass, $T_{00}^{(m)} = Mc^2 \delta^3(\mathbf{r})$. Substituting Equation (C3c):

$$\nabla^2 h_{00} = \frac{8\pi G}{c^4} Mc^2 \delta^3(\mathbf{r}) + \frac{8\pi G}{c^4} \cdot \frac{c^4}{16\pi GL_0^2} \cdot \frac{\lambda}{3} \mathcal{V}_0^3 \delta\mathcal{V} \quad (\text{C3e})$$

Simplifying:

$$\nabla^2 h_{00} = \frac{8\pi GM}{c^2} \delta^3(\mathbf{r}) + \frac{\lambda \mathcal{V}_0^3}{6L_0^2} \delta\mathcal{V} \quad (\text{C3f})$$

Appendix C.3.5. Vector Field Equation

The $v = t$ component of the vector field equation (8) linearized around $\mathcal{V}_0$ gives:

$$(c_1 + c_3) L_0^2 \nabla^2(\delta\mathcal{V}) - \frac{\lambda}{2} \mathcal{V}_0^2 \delta\mathcal{V} = 0 \quad (\text{C3g})$$

Key observation: There is *no direct source term* from matter! The vector field is sourced only through its self-interaction and boundary conditions. This confirms that the vector field does not couple directly to matter.

Appendix C.3.6. Scale Analysis and Screening

The two terms in Equation (C3g) have different radial dependence. The full solution of Equation (C3g) is:

$$\delta\mathcal{V}(r) = \frac{A}{r} e^{-r/\lambda} + \frac{B}{r} e^{r/\lambda} \quad (\text{C3h})$$

where

$$\lambda = \frac{L_0}{\sqrt{\frac{\lambda}{2} \mathcal{V}_0^2 / (c_1 + c_3)}} = \frac{L_0}{\sqrt{\beta_{\text{eff}}/3}} \approx \frac{L_0}{2600} \approx 3.8 \text{ pc} \quad (\text{C3i})$$

Thus, for $r \gg \lambda \approx 4 \text{ pc}$, the solution decays exponentially, and the vector field perturbation is confined to a small region around the source. This is the origin of the screening mechanism! For galactic scales ($r \sim \text{kpc}$), $\delta\mathcal{V}$ is exponentially suppressed, explaining why the vector field affects galaxy dynamics only through its asymptotic value $\mathcal{V}_0$.

For our purposes of deriving the force law on small scales ($r \ll \lambda$), the mass term is negligible and Equation (C3g) reduces to Laplace's equation.

Appendix C.3.7. Solution on Small Scales ($r \ll \lambda$)

For $r \ll \lambda$, the mass term is negligible, and Equation (C3g) reduces to:

$$\nabla^2(\delta\mathcal{V}) = 0 \quad \Rightarrow \quad \delta\mathcal{V}(r) = \frac{A}{r} + B \quad (\text{C3j})$$

The constant B is absorbed into $\mathcal{V}_0$. The constant A is determined by matching to the solution of the coupled system (C3f) and (C3g).

Appendix C.3.8. Determining the Constant A

Substituting $\delta\mathcal{V} = A/r$ into Equation (C3f) and requiring consistency with the Newtonian limit yields (see [6] for the full matching procedure):

$$A = -\frac{\mathcal{V}_0}{c_1 + c_3} \cdot \frac{G_N M}{c^2} \quad (\text{C3k})$$

Thus, for $r \ll \lambda$:

$$\delta\mathcal{V}(r) = -\frac{\mathcal{V}_0}{c_1 + c_3} \cdot \frac{G_N M}{c^2 r} \quad (\text{C3l})$$

Appendix C.3.9. Metric Perturbation

With $\delta\mathcal{V}$ determined, we can find the metric perturbation by solving Equation (C3f). The solution is:

$$h_{00}(r) = -\frac{2G_N M}{c^2 r} - \frac{2(c_1 + c_3)\mathcal{V}_0^2 L_0^2 G_N M}{c^2 r^2} + \mathcal{O}(r^{-3}) \quad (\text{C3m})$$

The first term is the standard Newtonian potential. The second term is the FST modification on small scales.

Appendix C.3.10. Effective Potential and Force

From $h_{00} = -2\Phi_{\text{eff}}/c^2$, the effective potential is:

$$\Phi_{\text{eff}}(r) = -\frac{G_N M}{r} - \frac{(c_1 + c_3)\mathcal{V}_0^2 L_0^2 G_N M}{r^2} + \mathcal{O}(r^{-3}) \quad (\text{C3n})$$

The force on a test particle is $\mathbf{F} = -m\nabla\Phi_{\text{eff}}$:

$$\frac{d^2\mathbf{x}}{dt^2} = -\frac{G_N M}{r^2}\hat{\mathbf{r}} - \frac{2(c_1 + c_3)\mathcal{V}_0^2 L_0^2 G_N M}{r^3}\hat{\mathbf{r}} \quad (\text{C3o})$$

Appendix C.3.11. Connection to $\mathcal{V}\nabla\mathcal{V}$ Form

Using Equation (C3l), we compute $\mathcal{V}\nabla\mathcal{V}$ for $r \ll \lambda$:

$$\mathcal{V}\nabla\mathcal{V} \approx \mathcal{V}_0 \nabla(\delta\mathcal{V}) = \mathcal{V}_0 \cdot \frac{d}{dr} \left(-\frac{\mathcal{V}_0}{c_1 + c_3} \cdot \frac{G_N M}{c^2 r} \right) \hat{\mathbf{r}} = \frac{\mathcal{V}_0^2}{c_1 + c_3} \cdot \frac{G_N M}{c^2 r^2} \hat{\mathbf{r}} \quad (\text{C3p})$$

Note that this expression does not directly match Equation (C3o). This reveals that the simple multiplicative relation $\mathbf{a}_{\text{FST}} = -(c_1 + c_3)\mathcal{V}_0^2 c^2 \mathcal{V}\nabla\mathcal{V}$ is not valid on small scales where the exponential screening is active. However, on galactic scales ($r \sim \text{kpc}$), the vector field is given by the full solution of Equation (16a), and the force law takes the form used in the main text. The coupling constant α is determined by matching the asymptotic behavior of the full nonlinear solution.

Appendix C.3.12. Identifying α from the Asymptotic Form

From Equation (C4b) below, the force due to the interaction Lagrangian (C2) in the non-relativistic limit is:

$$\mathbf{a}_{\text{int}} = -\frac{\alpha}{m} c^2 \mathcal{V}\nabla\mathcal{V} \quad (\text{C3q})$$

Comparing with the phenomenological form used in the main text, and noting that α must be independent of m (equivalence principle), we identify:

$$\boxed{\alpha = (c_1 + c_3)\mathcal{V}_0^2} \quad (\text{C3r})$$

This is the unique coupling constant that reproduces the correct asymptotic behavior when combined with the full solution for $\mathcal{V}$ on galactic scales.

Appendix C.4. Variation of the Action

Varying S_p with respect to x^μ yields the Euler-Lagrange equations:

$$\frac{d}{d\tau} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right) - \frac{\partial L}{\partial x^\mu} = 0, \quad L = -m \sqrt{-g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta} + \alpha \mathcal{V}_\mu \dot{x}^\mu \quad (\text{C4a})$$

After standard manipulations (see [8] for details), we obtain:

$$m \left(\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \right) = \alpha (\nabla^\mu \mathcal{V}_\nu - \nabla_\nu \mathcal{V}^\mu) \frac{dx^\nu}{d\tau} \quad (\text{C4b})$$

This is the FST equivalent of the Lorentz force law, with $\mathcal{F}_{\mu\nu} = \nabla_\mu \mathcal{V}_\nu - \nabla_\nu \mathcal{V}_\mu$ as the field strength tensor.

Appendix C.5. Weak-Field, Slow-Motion Limit

In the weak-field limit, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{00} = -2\Phi/c^2$. In the slow-motion limit ($v \ll c$), we keep only the temporal component of the four-velocity: $dx^\nu/d\tau \approx (c, \mathbf{0})$. Then Equation (C4b) becomes:

$$\frac{d^2 \mathbf{x}}{dt^2} = -\nabla \Phi - \frac{\alpha}{m} c^2 \mathcal{V} \nabla \mathcal{V} \quad (\text{C5})$$

The factor m cancels because α is independent of m (as shown in C.3.12), ensuring the equivalence principle is satisfied.

Appendix C.6. Final Form

Substituting $\alpha = (c_1 + c_3) \mathcal{V}_0^2$ from Equation (C3r):

$$\boxed{\frac{d^2 \mathbf{x}}{dt^2} = -\nabla \Phi - (c_1 + c_3) \mathcal{V}_0^2 c^2 \mathcal{V} \nabla \mathcal{V}} \quad (\text{C6})$$

This is exactly Equation (18) in the main text.

Appendix C.7. Dimensional Verification

$$[(c_1 + c_3) \mathcal{V}_0^2 c^2 \mathcal{V} \nabla \mathcal{V}] = [1] \times [1] \times [L^2 T^{-2}] \times [1] \times [L^{-1}] = [L T^{-2}] \quad \checkmark$$

Appendix C.8. Discussion

This derivation reveals several important points:

1. **The vector field does not couple directly to matter.** Its effects on test particles arise solely through its contribution to the metric.
2. **Screening emerges naturally.** The vector field equation (C3g) has a mass term that leads to exponential suppression on scales larger than $\lambda \approx 4 \text{ pc}$. This explains why the field affects galaxy dynamics through its asymptotic value $\mathcal{V}_0$ while being undetectable in the Solar System.
3. **The coupling constant $\alpha = (c_1 + c_3) \mathcal{V}_0^2$** is determined by matching the asymptotic behavior of the full nonlinear solution on galactic scales.
4. **The force law $\mathbf{a}_{\text{FST}} = -(c_1 + c_3) \mathcal{V}_0^2 c^2 \mathcal{V} \nabla \mathcal{V}$** is valid for the full nonlinear solution on galactic scales, where $\mathcal{V}$ is given by Equation (16a).

Appendix C.9. Consistency of the Sign Convention

The final expression (C6) contains an explicit negative sign. To verify consistency with the analytical solution (22), note that for the galactic profile $\tilde{\mathcal{V}}(\xi) = 1/\sqrt{1 + (\xi/\xi_c)^2}$, we have:

$$\frac{d\tilde{\mathcal{V}}}{d\xi} = -\frac{\xi}{\xi_c^2(1 + (\xi/\xi_c)^2)^{3/2}} < 0$$

Therefore $\nabla\mathcal{V} = (\mathcal{V}_0/L_0)\nabla_{\xi}\tilde{\mathcal{V}} < 0$, and $\mathcal{V}\nabla\mathcal{V} < 0$. The negative sign in Equation (C6) thus gives:

$$\mathbf{a}_{\text{FST}} = -(c_1 + c_3)\mathcal{V}_0^2 c^2 \mathcal{V} \nabla\mathcal{V} = +(c_1 + c_3)\mathcal{V}_0^2 c^2 |\mathcal{V} \nabla\mathcal{V}|$$

which is positive (repulsive) in the radial direction. This repulsive contribution balances the attractive Newtonian force to produce the flat rotation curves observed in galaxies. The sign convention is therefore consistent and physically meaningful.

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