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Article

Positive Solutions for Fractional Boundary Value Problems with Fractional Conditions Using Induction and Convolution of Lower-Order Problems

Jeffrey W. Lyons

Department of Mathematical Sciences, The Citadel, 171 Moultrie Street, Charleston, SC 29409 USA; jlyons3citadel.edu

Abstract: This paper examines the conditions for the existence and nonexistence of positive solutions to a class of nonlinear Riemann-Liouville fractional boundary value problems of order $\alpha + 2n$, where $\alpha \in (m-1, m]$ with $m \geq 3$ and $m, n \in \mathbb{N}$. The problem's nonlinearity is continuous and depends on a positive parameter. We derive constraints on this parameter that dictate whether positive solutions can be found. Our approach involves constructing a Green's function by combining the Green's functions of a lower-order fractional boundary value problem and a right-focal boundary value problem. Leveraging the properties of this Green's function, we apply Krasnosel'skii's Fixed Point Theorem to establish our results. Several examples are presented to illustrate the existence and nonexistence regions.

Keywords: positive solutions; nonexistence; convolution; induction; right focal; fractional derivative

MSC: 26A33, 34A08

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1. Introduction

Let $m, n \in \mathbb{N}$, $m \geq 3$, with $\alpha \in (m-1, m]$ and $\beta \in [1, m-1]$. Consider the following Riemann-Liouville fractional boundary value problem

$$D_{0^+}^{\alpha+2n} u(t) + (-1)^n \lambda g(t) f(u) = 0, \quad 0 < t < 1, \tag{1}$$

subject to the right-focal inspired fractional boundary conditions

$$\begin{gathered} u^{(i)}(0) = 0, \quad i = 0, 1, \ldots, m-2, \quad D_{0^+}^{\beta} u(1) = 0, \\ D_{0^+}^{\alpha+2l} u(0) = D_{0^+}^{\alpha+2l+1} u(1) = 0, \quad l = 0, 1, \ldots, n-1. \end{gathered} \tag{2}$$

We assume that $f : [0, \infty) \to [0, \infty)$ and $g : [0, 1] \to [0, \infty)$ are continuous functions with $g(t)$ satisfying the condition $\int_0^1 g(t)\, dt > 0$ and $\lambda > 0$ is a positive parameter. This paper is concerned with the existence and nonexistence of positive solutions to (1), (2).

We adopt the approach of Eloe et al. in [5] by constructing the Green's function associated with the given problem. This is done by convolving the Green's function $G_0(t, s)$ for a lower-order problem with the Green's function of a right-focal boundary value problem. We then use an inductive process to build the higher-order Green's function corresponding to (1), (2). Additionally, we present key properties of the lower-order Green's functions, as established in [11], and show that these properties extend to the higher-order Green's function, providing proofs where necessary. Finally, we apply this framework in an implementation of the Krasnosel'skii Fixed Point Theorem.

Our method involves the analysis of the operator defined by

$$Tu(t) = (-1)^n \lambda \int_0^1 G(t, s) g(s) f(u(s))\, ds,$$

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which is shown to have a fixed point under suitable conditions on the parameter λ. This fixed point is a positive solution to (1), (2).

This study builds upon the existing literature on fractional boundary value problems that utilize Krasnosel'skii's Fixed Point Theorem. Previous research has employed various fixed point theorems to establish the existence of positive solutions for similar problems, as seen in [1,2,6–9,11,13,15,16]. In this work, we leverage these findings to determine both the existence and nonexistence of positive solutions by deriving two distinct parameter constraints on λ formulated in terms of the liminf and limsup of the nonlinearity. This approach is fundamentally reliant on the properties of the Green's function, which plays a crucial role in proving the existence of positive solutions.

Section 2 introduces key definitions related to the Riemann-Liouville fractional derivative and offers directions for further study, along with a statement of Krasnosel'skii's Fixed Point Theorem. The following sections focus on constructing the Green's function and analyzing its properties. In Sections 5 and 6, we determine parameter intervals for λ that ensure the existence or nonexistence of positive solutions. Lastly, we provide examples to demonstrate the application of our main results.

2. Preliminaries and the Fixed Point Theorem

We begin by defining the Riemann-Liouville fractional integral which is used to define the Riemann-Liouville fractional derivative used in this work.

Definition 1. *Let $\nu > 0$. The Riemann-Liouville fractional integral of a function u of order ν, denoted $I_{0+}^{\nu}u$, is defined as*

$$I_{0+}^{\nu}u(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}u(s)ds,$$

provided the right-hand side exists.

Definition 2. *Let n denote a positive integer and assume $n-1 < \alpha \le n$. The Riemann-Liouville fractional derivative of order α of the function $u : [0,1] \to \mathbb{R}$, denoted $D_{0+}^{\alpha}u$, is defined as*

$$D_{0+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}u(s)ds = D^n I_{0+}^{n-\alpha}u(t),$$

provided the right-hand side exists.

For the interested reader, we cite [3,10,12,14] for further study of fractional calculus and fractional differential equations.

Now, we present Krasnosel'skii's Fixed Point Theorem.

Theorem 1 (Krasnosel'skii Fixed Point Theorem)**.** *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset X$ be a cone in $\mathcal{P}$. Assume that Ω_1, Ω_2 are open sets with $0 \in \Omega_1$, and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{P}$ be a completely continuous operator such that either*

1. $\|Tu\| \ge \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \le \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$; *or*
2. $\|Tu\| \le \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \ge \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$.

Then, T has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. The Green's Function

Now, we construct the Green's function used for (1), (2) by utilizing induction with a convolution of a lower-order problem and a right-focal problem. The procedure is similar to that found in [13].

First, the right-focal boundary value problem

$$-u'' = 0, \quad 0 < t < 1, \quad u(0) = 0, \quad u'(1) = 0$$

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has Green's function

$$G_{foc}(t,s) = \begin{cases} s, & 0 \le s < t \le 1, \\ t, & 0 \le t < s \le 1. \end{cases}$$

Let $G_0(t,s)$ be the Green's function for

$$-D_{0^+}^{\alpha} u = 0, \quad 0 < t < 1, \quad u^{(i)}(0) = 0,\ i = 0, 1, \ldots, m-2, \quad D_{0^+}^{\beta} u(1) = 0,$$

which is given by ([4])

$$G_0(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1-\beta} - (t-s)^{\alpha-1}, & 0 \le s < t \le 1, \\ \\ t^{\alpha-1}(1-s)^{\alpha-1-\beta}, & 0 \le t \le s < 1. \end{cases}$$

For $k = 1, \ldots, n-1$, recursively define $G_k(t,s)$ by

$$G_k(t,s) = -\int_0^1 G_{k-1}(t,r) G_{foc}(r,s) dr.$$

Then,

$$G_n(t,s) = -\int_0^1 G_{n-1}(t,r) G_{foc}(r,s) dr,$$

is the Green's function for

$$-D_{0^+}^{\alpha+2n} u(t) = 0, \quad 0 < t < 1,$$

with boundary conditions (2), and $G_{n-1}(t,s)$ is the Green's function for

$$-D_{0^+}^{\alpha+2(n-1)} u(t) = 0, \quad 0 < t < 1,$$

with boundary conditions

$$u^{(i)}(0) = 0,\ i = 0, 1, \ldots, m-2, \quad D_{0^+}^{\beta} u(1) = 0,$$

$$D_{0^+}^{\alpha+2l} u(0) = D_{0^+}^{\alpha+2l+1} u(1) = 0,\ l = 0, 1, \ldots, n-2.$$

To see this, for the base case $k = 1$, consider the linear differential equation

$$D_{0^+}^{\alpha+2} u(t) + h(t) = 0, \quad 0 < t < 1,$$

satisfying the boundary conditions

$$u^{(i)}(0) = 0,\ i = 0, 1, \ldots, m-2, \quad D_{0^+}^{\beta} u(1) = 0,$$

$$D_{0^+}^{\alpha} u(0) = 0, \quad D_{0^+}^{\alpha+1} u(1) = 0.$$

Make the change of variable

$$v(t) = D_{0^+}^{\alpha+2-2} u(t).$$

Then,

$$D^2 v(t) = D^2 D_{0^+}^{\alpha+2-2} u(t) = D_{0^+}^{\alpha+2} u(t) = -h(t),$$

and since $v(t) = D_{0^+}^{\alpha} u(t)$,

$$v(0) = D_{0^+}^{\alpha} u(0) = 0 \quad \text{and} \quad v'(1) = D_{0^+}^{\alpha+1} u(1) = 0.$$

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Thus, v satisfies the right-focal boundary value problem

$$v'' + h(t) = 0, \quad 0 < t < 1,$$

$$v(0) = 0, \quad v'(1) = 0.$$

Also, u now satisfies a lower order boundary value problem,

$$D^{\alpha}_{0+}u(t) = v(t), \quad 0 < t < 1,$$

$$u^{(i)}(0) = 0, \ i = 0, 1, \dots, m-2, \quad D^{\beta}_{0+}u(1) = 0.$$

So,

$$\begin{aligned} u(t) &= \int_0^1 G_0(t,s)(-v(s))ds \\ &= \int_0^1 G_0(t,s)\left(-\int_0^1 G_{foc}(s,r)h(r)ds\right)dr \\ &= \int_0^1 \left(\int_0^1 -G_0(t,s)G_{foc}(s,r)ds\right)h(r)dr. \end{aligned}$$

Therefore,

$$u(t) = \int_0^1 G_1(t,s)h(s)ds,$$

where

$$G_1(t,s) = -\int_0^1 G_0(t,r)G_{foc}(r,s)dr.$$

For the inductive step, the argument is similar. Assume that $k = n-1$ is true, and consider the linear differential equation

$$D^{\alpha+2n}_{0^+}u(t) + k(t) = 0, \quad 0 < t < 1,$$

satisfying boundary conditions (2).

Make the change of variables

$$v(t) = D^{\alpha+2(n-1)}_{0^+}u(t)$$

so that

$$D^2v(t) = D^{\alpha+2n}_{0^+} = -k(t)$$

and

$$v(0) = D^{\alpha+2(n-1)}_{0^+}u(0) = 0 \quad \text{and} \quad v'(1) = D^{\alpha+2(n-1)+1}_{0^+}v(1) = 0.$$

Similar to before, $v(t)$ satisfies the right-focal boundary value problem

$$v'' + k(t) = 0, \quad 0 < t < 1,$$

$$v(0) = 0, \quad v'(1) = 0$$

while $u(t)$ satisfies the lower order problem

$$D^{\alpha+2(n-1)}_{0^+}u(t) = v(t), \quad 0 < t < 1,$$

$$u(0) = 0, \quad D^{\beta}_{0^+}u(1) = 0,$$

$$D^{\alpha+2l}_{0^+}u(0) = D^{\alpha+2l+1}_{0^+}u(1) = 0, \ l = 0, 1, \dots, n-2.$$

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By induction,

$$\begin{aligned} u(t) &= \int_0^1 G_{n-1}(t,s)(-v(s))ds \\ &= \int_0^1 \left(-\int_0^1 G_{n-1}(t,s)G_{foc}(s,r)ds \right) k(r)dr \\ &= \int_0^1 G_n(t,s)k(s)ds. \end{aligned}$$

Therefore,

$$u(t) = \int_0^1 G_n(t,s)k(s)ds,$$

where

$$G_n(t,s) = -\int_0^1 G_{n-1}(t,r)G_{foc}(r,s)dr.$$

So, the unique solution to

$$D_{0^+}^{\alpha+2n}u(t) + k(t) = 0, \quad 0 < t < 1,$$

satisfying boundary conditions (2) is given by

$$u(t) = \int_0^1 G_n(t,s)k(s)ds.$$

4. Green's Function Properties

We now discuss properties for $G_n(t,s)$ that are inherited from $G_0(t,s)$ and $G_{foc}(t,s)$. The results of the first lemma regarding $G_{foc}(t,s)$ are well-known and easily verifiable.

Lemma 1. *For $(t,s) \in [0,1] \times [0,1]$, $G_{foc}(t,s) \in C^{(1)}$ and $G_{foc}(t,s) \geq 0$.*

The following lemma regarding $G_0(t,s)$ is Lemma 3.1 proved in [11].

Lemma 2. *The following are true.*

(1) For $(t,s) \in [0,1] \times [0,1)$, $G_0(t,s) \in C^{(1)}$.
(2) For $(t,s) \in (0,1) \times (0,1)$, $G_0(t,s) > 0$ and $\frac{\partial}{\partial t}G_0(t,s) > 0$.
(3) For $(t,s) \in [0,1] \times [0,1)$, $t^{\alpha-1}G_0(1,s) \leq G_0(t,s) \leq G_0(1,s)$.

Finally, we prove inherited properties for $G_n(t,s)$ from Lemma 2.

Lemma 3. *The following are true.*

(1) For $(t,s) \in [0,1] \times [0,1)$, $G_n(t,s) \in C^{(1)}$.
(2) For $(t,s) \in (0,1) \times (0,1)$, $(-1)^n G_n(t,s) > 0$ and $(-1)^n \frac{\partial}{\partial t}G_n(t,s) > 0$.
(3) For $(t,s) \in [0,1] \times [0,1)$,

$$(-1)^n t^{\alpha-1}G_n(1,s) \leq (-1)^n G_n(t,s) \leq (-1)^n G_n(1,s).$$

Proof. We proceed inductively for each part.

For (1) with $(t,s) \in [0,1] \times [0,1)$, we have the base case $k = 1$

$$G_1(t,s) = -\int_0^1 G_0(t,r)G_{foc}(r,s)ds$$

so by Lemmas 1 and 2, $G_1(t,s) \in C^{(1)}$.

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Now, assume that $k = n - 1$ is true. Then,

$$G_n(t,s) = -\int_0^1 G_{n-1}(t,r)G_{foc}(r,s)ds$$

so by induction and Lemma 1, $G_n(t,s) \in C^{(1)}$.

For (2) with $(t,s) \in (0,1) \times (0,1)$ and using Lemmas 1 and 2, we have the base case $k = 1$

$$(-1)^1 G_1(t,s) = -\left(-\int_0^1 G_0(t,r)G_{foc}(r,s)dr\right) > 0$$

and

$$(-1)^1 \frac{\partial}{\partial t} G_1(t,s) = -\left(-\int_0^1 \frac{\partial}{\partial t} G_0(t,r)G_{foc}(r,s)dr\right) > 0.$$

Now, assume that $k = n - 1$ is true. Then, by induction and Lemma 1

$$\begin{aligned}(-1)^n G_n(t,s) &= (-1)^n \left(-\int_0^1 G_{n-1}(t,r)G_{foc}(r,s)dr\right) \\ &= (-1)^2 \left(\int_0^1 (-1)^{n-1} G_{n-1}(t,r)G_{foc}(r,s)dr\right) \\ &> 0,\end{aligned}$$

and

$$\begin{aligned}(-1)^n \frac{\partial}{\partial t} G_n(t,s) &= (-1)^n \left(-\int_0^1 \frac{\partial}{\partial t} G_{n-1}(t,r)G_{foc}(r,s)dr\right) \\ &= (-1)^2 \left(\int_0^1 (-1)^{n-1} \frac{\partial}{\partial t} G_{n-1}(t,r)G_{foc}(r,s)dr\right) \\ &> 0.\end{aligned}$$

For (3) with $(t,s) \in [0,1] \times [0,1)$ and using Lemma 2 (3), we have the base case $k = 1$

$$\begin{aligned}(-1)^1 t^{\alpha-1} G_1(1,s) &= -t^{\alpha-1}\left(-\int_0^1 G_0(1,r)G_{foc}(r,s)dr\right) \\ &= -\left(\int_0^1 -t^{\alpha-1} G_0(1,r)G_{foc}(r,s)dr\right) \\ &\le -\left(\int_0^1 -G_0(t,r)G_{foc}(r,s)dr\right) \\ &= -\left(-\int_0^1 G_0(t,r)G_{foc}(r,s)dr\right) \\ &= (-1)^1 G_1(t,s),\end{aligned}$$

and

$$\begin{aligned}(-1)^1 G_1(t,s) &= -\left(-\int_0^1 G_0(t,r)G_{foc}(r,s)dr\right) \\ &= \int_0^1 G_0(t,r)G_{foc}(r,s)dr \\ &\le \int_0^1 G_0(1,r)G_{foc}(r,s)dr \\ &= -\left(-\int_0^1 G_0(1,r)G_{foc}(r,s)dr\right) \\ &= (-1)^1 G_1(1,s).\end{aligned}$$

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Now, assume that $k = n-1$ is true. Then,

$$
\begin{aligned}
(-1)^n t^{\alpha-1} G_n(1,s) &= (-1)^n t^{\alpha-1}\left(-\int_0^1 G_{n-1}(1,r)G_{foc}(r,s)dr\right) \\
&= (-1)^2\left(\int_0^1 (-1)^{n-1} t^{\alpha-1} G_{n-1}(1,r)G_{foc}(r,s)dr\right) \\
&\le (-1)^2\left(\int_0^1 (-1)^{n-1} G_{n-1}(t,r)G_{foc}(r,s)dr\right) \\
&= (-1)^n\left(-\int_0^1 G_{n-1}(t,r)G_{foc}(r,s)dr\right) \\
&= (-1)^n G_n(t,s),
\end{aligned}
$$

and

$$
\begin{aligned}
(-1)^n G_n(t,s) &= (-1)^n\left(-\int_0^1 G_{n-1}(t,r)G_{foc}(r,s)dr\right) \\
&= (-1)^2\left(\int_0^1 (-1)^{n-1} G_{n-1}(t,r)G_{foc}(r,s)dr\right) \\
&\le (-1)^2\left(\int_0^1 (-1)^{n-1} G_{n-1}(1,r)G_{foc}(r,s)dr\right) \\
&= (-1)^n\left(-\int_0^1 G_{n-1}(1,r)G_{foc}(r,s)dr\right) \\
&= (-1)^n G_n(1,s).
\end{aligned}
$$

□

5. Existence of Solutions

We are now in position to demonstrate the existence of positive solutions to (1), (2) based upon the parameter λ using the Krasnosel'skii Fixed Point Theorem and our constructed Green's function and properties.

Define the constants

$$A_{G_n} = \int_0^1 (-1)^n s^{\alpha-1} G_n(1,s)g(s)ds, \quad B_{G_n} = \int_0^1 (-1)^n G_n(1,s)g(s)ds,$$

$$F_0 = \limsup_{u\to 0^+} \frac{f(u)}{u}, \quad f_0 = \liminf_{u\to 0^+} \frac{f(u)}{u},$$

$$F_\infty = \limsup_{u\to\infty} \frac{f(u)}{u}, \quad f_\infty = \liminf_{u\to\infty} \frac{f(u)}{u}.$$

Let $\mathcal{B} = C[0,1]$ be a Banach space with norm

$$\|u\| = \max_{t\in[0,1]} |u(t)|.$$

Define the cone

$$
\begin{aligned}
\mathcal{P} = \{u \in \mathcal{B} : u(0) = 0,\ u(t) \text{ is nondecreasing, and} \\
t^{\alpha-1}u(1) \le u(t) \le u(1) \text{ on } [0,1]\Big\}.
\end{aligned}
$$

Define the operator $T : \mathcal{P} \to \mathcal{B}$ by

$$Tu(t) = (-1)^n \lambda \int_0^1 G_n(t,s)g(s)f(u(s))ds.$$

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Lemma 4. *The operator $T : \mathcal{P} \to \mathcal{P}$ is completely continuous.*

Proof. Let $u \in \mathcal{P}$. Then, by definition,

$$Tu(0) = (-1)^n \lambda \int_0^1 G_n(0,s)g(s)f(u(s))ds = 0.$$

Also, for $t \in (0,1)$ and by Lemma 3 (2),

$$\frac{\partial}{\partial t}[Tu(t)] = (-1)^n \lambda \int_0^1 \frac{\partial}{\partial t} G_n(t,s)g(s)f(u(s))ds > 0$$

which implies that $Tu(t)$ is nondecreasing.

Next, for $t \in [0,1]$ and by Lemma 3,

$$\begin{aligned} t^{\alpha-1}Tu(1) &= t^{\alpha-1}(-1)^n \lambda \int_0^1 G_n(1,s)g(s)f(u(s))ds \\ &\le (-1)^n \lambda \int_0^1 G_n(t,s)g(s)f(u(s))ds \\ &= Tu(t), \end{aligned}$$

and

$$\begin{aligned} Tu(t) &= (-1)^n \lambda \int_0^1 G_n(t,s)g(s)f(u(s))ds \\ &\le (-1)^n \lambda \int_0^1 G_n(1,s)g(s)f(u(s))ds \\ &= Tu(1). \end{aligned}$$

Therefore, $Tu \in \mathcal{P}$. A standard application of the Arzela-Ascoli Theorem yields that T is completely continuous. □

Theorem 2. *If*

$$\frac{1}{A_{G_n} f_\infty} < \lambda < \frac{1}{B_{G_n} F_0},$$

then (1), (2) *has at least one positive solution.*

Proof. Since $F_0 \lambda B_{G_n} < 1$, there exists an $\epsilon > 0$ such that

$$(F_0 + \epsilon)\lambda B_{G_n} \le 1.$$

Also since

$$F_0 = \limsup_{u \to 0^+} \frac{f(u)}{u},$$

there exists an $H_1 > 0$ such that

$$f(u) \le (F_0 + \epsilon)u \quad \text{for} \quad u \in (0, H_1].$$

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Define $\Omega_1 = \{u \in \mathcal{B} : \|u\| < H_1\}$. If $u \in \mathcal{P} \cap \partial\Omega_1$, then $\|u\| = H_1$, and

$$\begin{aligned}
|(Tu)(1)| &= (-1)^n \lambda \int_0^1 G_n(1,s) g(s) f(u(s)) ds \\
&\le (-1)^n \lambda \int_0^1 G_n(1,s) g(s) (F_0 + \epsilon) u(s) ds \\
&\le (F_0 + \epsilon) u(1) \lambda \int_0^1 (-1)^n G_n(1,s) g(s) ds \\
&\le (F_0 + \epsilon) \|u\| \lambda B_{G_n} \\
&\le \|u\|.
\end{aligned}$$

Since $Tu \in \mathcal{P}$, $\|Tu\| \le \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_1$.

Next, since $f_\infty \lambda > \dfrac{1}{A_{G_n}}$, there exists a $c \in (0,1)$ and an $\epsilon > 0$ such that

$$(f_\infty - \epsilon)\lambda > \left((-1)^n \int_c^1 s^{\alpha-1} G_n(1,s) g(s) ds \right)^{-1}.$$

Since

$$f_\infty = \liminf_{u \to \infty} \frac{f(u)}{u},$$

there exists an $H_3 > 0$ such that

$$f(u) \ge (f_\infty - \epsilon) u \quad \text{for} \quad u \in [H_3, \infty).$$

Define

$$H_2 = \max\left\{ \frac{H_3}{c^{\alpha-1}}, 2H_1 \right\},$$

and define $\Omega_2 = \{u \in \mathcal{B} : \|u\| < H_2\}$.

Let $u \in \mathcal{P} \cap \partial\Omega_2$. Then, $\|u\| = H_2$. Notice for $t \in [c,1]$,

$$u(t) \ge t^{\alpha-1} u(1) \ge c^{\alpha-1} H_2 \ge c^{\alpha-1} \frac{H_3}{c^{\alpha-1}} = H_3.$$

Therefore,

$$\begin{aligned}
|(Tu)(1)| &\ge (-1)^n \lambda \int_c^1 G_n(1,s) g(s) f(u(s)) ds \\
&\ge \lambda \int_c^1 (-1)^n G_n(1,s) g(s) (f_\infty - \epsilon) u(s) ds \\
&\ge \lambda (f_\infty - \epsilon) u(1) (-1)^n \int_c^1 s^{\alpha-1} G_n(1,s) g(s) ds \\
&\ge \|u\|.
\end{aligned}$$

Hence, $\|Tu\| \ge \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_2$. Notice since $H_1 < H_2$ we have $\overline{\Omega}_1 \subset \Omega_2$. Thus, by Theorem 1 (1), T has a fixed point $u \in \mathcal{P}$. By the definition of T, this fixed point is a positive solution of (1), (2). □

Theorem 3. *If*

$$\frac{1}{A_{G_n} f_0} < \lambda < \frac{1}{B_{G_n} F_\infty},$$

then (1), (2) *has at least one positive solution.*

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Proof. Since $f_0 \lambda A_{G_n} > 1$, there exists an $\epsilon > 0$ such that

$$(f_0 - \epsilon)\lambda A_{G_n} \geq 1.$$

Then, since

$$f_0 = \liminf_{u \to 0^+} \frac{f(u)}{u},$$

there exists an $H_1 > 0$ such that

$$f(u) \geq (f_0 - \epsilon)u, \quad t \in (0, H_1].$$

Define $\Omega_1 = \{u \in \mathcal{B} : \|u\| < H_1\}$. If $u \in \mathcal{P} \cap \partial\Omega_1$, then $u(t) \leq H_1$ for $t \in [0,1]$. So,

$$\begin{aligned}
|(Tu)(1)| &= (-1)^n \lambda \int_0^1 G_n(1,s)g(s)f(u(s))ds \\
&\geq (-1)^n \lambda \int_0^1 G_n(1,s)g(s)(f_0 - \epsilon)u(s)ds \\
&\geq \lambda(f_0 - \epsilon)u(1) \int_0^1 (-1)^n s^{\alpha-1} G_n(1,s)g(s)ds \\
&\geq \lambda(f_0 - \epsilon)\|u\| A_{G_n} \\
&\geq \|u\|.
\end{aligned}$$

Thus, $\|Tu\| \geq \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_1$.

Next, since $F_\infty B_{G_n} \lambda < 1$, there exists an $\epsilon \in (0,1)$ such that

$$((F_\infty + \epsilon)B_{G_n} + \epsilon)\lambda \leq 1.$$

Since

$$F_\infty = \limsup_{u \to \infty} \frac{f(u)}{u},$$

there exists an $H_3 > 0$ such that

$$f(u) \leq (F_\infty + \epsilon)u, \quad u \in [H_3, \infty).$$

Define

$$M = \max_{u \in [0, H_3]} f(u).$$

Now, there exists a $k \in (0,1)$ with

$$(-1)^n \int_0^k G_n(1,s)g(s)ds \leq \frac{\epsilon}{M}.$$

Let

$$H_2 = \max\left\{2H_1, \frac{H_3}{k^{\alpha-1}}, 1\right\},$$

and define $\Omega_2 = \{u \in \mathcal{B} : \|u\| < H_2\}$. Let $u \in \mathcal{P} \cap \partial\Omega_2$. Then, $\|u\| = H_2$ and so,

$$u(1) = H_2 \geq \frac{H_3}{k^{\alpha-1}} > H_3.$$

Now, $u(0) = 0$. So, by the Intermediate Value Theorem, there exists a $\gamma \in (0,1)$ with $u(\gamma) = H_3$. But, for $t \in [k,1]$, we have

$$u(t) \geq t^{\alpha-1}u(1) = t^{\alpha-1}H_2 \geq k^{\alpha-1}\frac{H_3}{k^{\alpha-1}} = H_3.$$

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So, $\gamma \in (0, k]$. Moreover, since $u(t)$ is nondecreasing, this implies

$$0 \le u(t) \le H_3, \quad t \in [0, \gamma)$$

and

$$u(t) \ge H_3, \quad t \in (\gamma, 1].$$

Therefore,

$$\begin{aligned}
|(Tu)(1)| &= (-1)^n \lambda \int_0^1 G_n(1,s)g(s)f(u(s))ds \\
&= \lambda\left((-1)^n \int_0^\gamma G_n(1,s)g(s)f(u(s))ds + (-1)^n \int_\gamma^1 G_n(1,s)g(s)f(u(s))ds\right) \\
&\le \lambda\left(M \int_0^\gamma (-1)^n G_n(1,s)g(s)ds + (-1)^n \int_\gamma^1 G_n(1,s)g(s)(F_\infty + \epsilon)u(s)ds\right) \\
&\le \lambda\left(M\frac{\epsilon}{M} + (F_\infty + \epsilon)u(1) \int_\gamma^1 (-1)^n G_n(1,s)g(s)ds\right) \\
&\le \lambda(\epsilon + (F_\infty + \epsilon)\|u\|\, B_{G_n}) \\
&\le \lambda(\epsilon\|u\| + (F_\infty + \epsilon)\|u\|\, B_{G_n}) \\
&= \lambda\|u\|(\epsilon + (F_\infty + \epsilon)B_{G_n}) \\
&\le \|u\|
\end{aligned}$$

Thus, $\|Tu\| \le \|u\|$ for $u \in \mathcal{P} \cap \partial\Omega_2$. Notice that since $H_1 < H_2$ we have $\overline{\Omega}_1 \subset \Omega_2$. Thus, by Theorem 1 (2), T has a fixed point $u \in \mathcal{P}$. By the definition of T, this fixed point is a positive solution of (1), (2). □

6. Nonexistence Results

Now, we provide two nonexistence of positive solutions results based upon the size of the parameter λ. First, we need the following Lemma.

Lemma 5. *Suppose $D_{0^+}^{\alpha+2n}u \in C[0,1]$. If $(-1)^n(-D_{0^+}^{\alpha+2n}u(t)) \ge 0$ for all $t \in [0,1]$ and $u(t)$ satisfies (2), then*

(1) $u'(t) \ge 0, \quad 0 \le t \le 1$, *and*
(2) $t^{\alpha-1}u(1) \le u(t) \le u(1), \quad 0 \le t \le 1.$

Proof. Let $0 \le t \le 1$.

For (1), by Lemma 3 (2),

$$\begin{aligned}
u'(t) &= \int_0^1 \frac{\partial}{\partial t}G_n(t,s)(-D_{0^+}^{\alpha+2n}u(s))ds \\
&= \int_0^1 (-1)^n \frac{\partial}{\partial t}G_n(t,s)(-1)^n(-D_{0^+}^{\alpha+2n}u(s))ds \\
&> 0.
\end{aligned}$$

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For (2), by Lemma 3 (3),

$$\begin{aligned}
t^{\alpha-1}u(1) &= t^{\alpha-1}\int_0^1 G_n(1,s)(-D_{0^+}^{\alpha+2n}u(s))ds \\
&= \int_0^1 (-1)^n t^{\alpha-1} G_n(1,s)(-1)^n(-D_{0^+}^{\alpha+2n}u(s))ds \\
&\leq \int_0^1 (-1)^n G_n(t,s)(-1)^n(-D_{0^+}^{\alpha+2n}u(s))ds \\
&= \int_0^1 G_n(t,s)(-D_{0^+}^{\alpha+2n}u(s))ds \\
&= u(t),
\end{aligned}$$

and

$$\begin{aligned}
u(t) &= \int_0^1 G_n(t,s)(-D_{0^+}^{\alpha+2n}u(s))ds \\
&= \int_0^1 (-1)^n G_n(t,s)(-1)^n(-D_{0^+}^{\alpha+2n}u(s))ds \\
&\leq \int_0^1 (-1)^n G_n(1,s)(-1)^n(-D_{0^+}^{\alpha+2n}u(s))ds \\
&= \int_0^1 G_n(1,s)(-D_{0^+}^{\alpha+2n}u(s))ds \\
&= u(1).
\end{aligned}$$

□

Theorem 4. *If*

$$\lambda < \frac{u}{B_{G_n} f(u)}$$

for all $u \in (0,\infty)$*, then no positive solution exists to* (1), (2).

Proof. For contradiction, suppose that $u(t)$ is a positive solution to (1), (2). Then, $(-1)^n(-D_{0^+}^{\alpha+2n}u(t)) = \lambda g(t) f(u(t)) \geq 0$. So by Lemma 5,

$$\begin{aligned}
u(1) &= (-1)^n \lambda \int_0^1 G_n(1,s) g(s) f(u(s))ds \\
&< (-1)^n (B_{G_n})^{-1} \int_0^1 G_n(1,s) g(s) u(s)ds \\
&\leq u(1)(B_{G_n})^{-1} \int_0^1 (-1)^n G_n(1,s) g(s)ds \\
&= u(1),
\end{aligned}$$

a contradiction. □

Theorem 5. *If*

$$\lambda > \frac{u}{A_{G_n} f(u)}$$

for all $u \in (0,\infty)$*, then no positive solution exists to* (1), (2).

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Proof. For contradiction, suppose that $u(t)$ is a positive solution to (1), (2). Then, $(-1)^n(-D_{0^+}^{\alpha+2n}u(t)) = \lambda g(t)f(u(t)) \geq 0$. So by Lemma 5,

$$\begin{aligned} u(1) &= (-1)^n\lambda\int_0^1 G_n(1,s)g(s)f(u(s))ds \\ &> (-1)^n(A_{G_n})^{-1}\int_0^1 G_n(1,s)g(s)u(s)ds \\ &\geq u(1)(A_{G_n})^{-1}\int_0^1 (-1)^n s^{\alpha-1}G_n(1,s)g(s)ds \\ &= u(1), \end{aligned}$$

a contradiction. □

7. An Example

Finally, we calculate approximate bounds of the parameter λ for the existence and nonexistence of positive solutions for specific example. We use Theorems 2, 4, and 5. Examples constructed using Theorems 3, 4, and 5 are found similarly.

Set $n = 2$, $m = 3$, $\alpha = 2.5$, $\beta = 1.5$, and $g(t) = t$. We note that that $g(t) \geq 0$ is continuous for $0 \leq t \leq 1$ and $\int_0^1 g(t)dt > 0$. Now, we have that

$$\begin{aligned} G_0(1,s) &= \frac{1}{\Gamma(2.5)}\begin{cases} 1^{1.5}(1-s)^0 - (1-s)^{1.5}, & 0 \leq s < t \leq 1, \\ 1^{1.5}(1-s)^0, & 0 \leq t \leq s < 1 \end{cases} \\ &= \frac{1-(1-s)^{1.5}}{\Gamma(2.5)}, \end{aligned}$$

and we compute

$$\begin{aligned} A_{G_2} &= \int_0^1 (-1)^2 s^{1.5} G_2(1,s)(s)ds \\ &= \int_0^1 \left[-\int_0^1 G_1(1,r_1)G_{foc}(r_1,s)dr_1\right] s^{2.5}ds \\ &= \int_0^1 \left[-\int_0^1 \left(\int_0^1 -G_0(1,r_2)G_{foc}(r_2,r_1)dr_2\right) G_{foc}(r_1,s)dr_1\right] s^{2.5}ds \\ &\approx 0.03071, \end{aligned}$$

and

$$\begin{aligned} B_{G_2} &= \int_0^1 (-1)^2 G_2(1,s)(s)ds \\ &= \int_0^1 \left[-\int_0^1 G_1(1,r_1)G_{foc}(r_1,s)dr_1\right] sds \\ &= \int_0^1 \left[-\int_0^1 \left(\int_0^1 -G_0(1,r_2)G_{foc}(r_2,r_1)dr_2\right) G_{foc}(r_1,s)dr_1\right] sds \\ &\approx 0.04749. \end{aligned}$$

Now that we have A_{G_2} and B_{G_2}, applying the Theorems is much simpler as they are based on the liminfs and limsups of choice of $f(u)$.

Example 1. *We demonstrate an example for Theorems 2, 4, and 5. Set $f(u) = u\ln(u+1) + 2u$. We note that $f(u) \geq 0$ is continuous for $u \geq 0$. Thus, the fractional boundary value problem is*

$$D_{0^+}^{6.5}u(t) + \lambda t(u\ln(u+1) + 2u) = 0, \quad 0 < t < 1, \tag{3}$$

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$$u(0) = u'(0) = 0, \quad D^{1.5}_{0^+}(1) = 0 \tag{4}$$
$$D^{2.5}_{0^+}u(0) = D^{3.5}_{0^+}(1) = 0, \quad D^{4.5}_{0^+}(0) = D^{5.5}_{0^+}(1) = 0.$$

We compute the liminfs and limsups for $f(u)/u = \ln(u+1) + 2$.

$$f_\infty = \liminf_{u\to\infty}(\ln(u+1)+2) = \infty, \qquad F_0 = \limsup_{u\to 0^+}(\ln(u+1)+2) = 2$$
$$f_0 = \liminf_{u\to 0^+}(\ln(u+1)+2) = 2, \qquad F_\infty = \limsup_{u\to\infty}(\ln(u+1)+2) = \infty.$$

Then, we have

$$\frac{1}{A_{G_2} f_\infty} \approx \frac{1}{0.03031\cdot\infty} = 0$$

and

$$\frac{1}{B_{G_2} F_0} \approx \frac{1}{0.04749\cdot 2} \approx 10.52853.$$

Next, for $u \in (0,\infty)$, *we investigate*

$$\frac{u}{B_{G_2} f(u)} = \frac{1}{B_{G_2}(\ln(u+1)+2)}.$$

We calculate

$$\inf_{u\in(0,\infty)} \frac{1}{B_{G_2}(\ln(u+1)+2)} = \frac{1}{B_{G_2}} \inf_{u\in(0,\infty)} \frac{1}{\ln(u+1)+2} \approx \frac{1}{0.04749}(0) = 0.$$

Finally, for $u \in (0,\infty)$, *we investigate*

$$\frac{u}{A_{G_2} f(u)} = \frac{1}{A_{G_2}(\ln(u+1)+2)}.$$

We calculate

$$\sup_{u\in(0,\infty)} \frac{1}{A_{G_2}(\ln(u+1)+2)} = \frac{1}{A_{G_2}} \sup_{u\in(0,\infty)} \frac{1}{\ln(u+1)+2} \approx \frac{1}{0.030307}\left(\frac{1}{2}\right) \approx 16.49784.$$

Therefore, by Theorems 2 and 5, if $0 < \lambda < 16.49$, *then* (3), (4) *has at least one positive solution, and if* $\lambda > 16.49$, *then* (3), (4) *does not have a positive solution. We note that Theorem 4 did not yield a meaningful result here which was expected as a solution exists for small positive* λ.

Remark 1. *Lastly, we note that to find a meaning* λ *range for both nonexistence results and either existence results simultaneously with* $g(t) = t$, *we could choose rational function* $f(u)$ *with a quadratic numerator and linear denominator. Thus,* $f(u)/u$ *is a rational function with a linear numerator and denominator leading to finite values for each liminf and limsup.*

8. Conclusions

In this article, we studied Riemann-Liouville fractional differential equations with order $\alpha + 2n$ with $n \in \mathbb{N}$ that includes a parameter λ. The two-point boundary conditions are influenced by standard right-focal conditions. We established the Green's function for the boundary value problem by utilizing a convolution of a lower-order problem and standard right-focal problem by making a change of variables. Then, we inductively defined the Green's function for the higher order problem.

Next, we inductively proved many properties inherited by the Green's function from the lower-order problems. These properties permitted an application of the Krasnosel'skii Fixed Point Theorem to establish the existence of positive solutions based upon the size of λ. We also established the

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nonexistence of positive solutions based upon choice of λ via contradiction. Finally, we discussed a specific example and proved existence and nonexistence based on the choice of λ.

Future research may be to use the approach in this work to establish existence and nonexistence of positive solutions for other types of boundary conditions. Another avenue could be considering a singularity at $f(0)$.

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