Article

Not peer-reviewed version

Electromagnetic Kantowski–Sachs Solutions in Teleparallel F(T) Gravity

Alexandre Landry *

Posted Date: 8 May 2026

doi: 10.20944/preprints202605.0564.v1

Keywords: teleparallel gravity; Coley–Landry approach; covariant electromagnetic sources; nonsymmetric field equations; Kantowski-Sachs teleparallel solutions

Article

Electromagnetic Kantowski–Sachs Solutions in Teleparallel $F(T)$ Gravity

Alexandre Landry

Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia, Canada, B3H 3J5; a.landry@dal.ca; Tel.: +1-514-503-2051

Abstract

We construct electromagnetic Kantowski–Sachs (KS) solutions in covariant teleparallel $F(T)$ gravity using the coframe/spin–connection (CSC) formalism. In the restricted branch considered here, the Maxwell conservation laws (CLs) impose strong restrictions on the anisotropic scale factors and lead to the scaling $\rho_{\text{em}} \propto A_3^{-4}$. We derive the corresponding symmetric and antisymmetric field equations (SFEs and AFEs) and formulate a reconstruction scheme in which $F(T)$ is determined from the KS dynamics rather than imposed a priori. Power-law (PL) and exponential (EXP) coframe ansätze generate distinct invariant reconstruction branches, including scaling cosmologies, teleparallel de Sitter (TdS) regimes, and KS black-hole-interior-like reconstruction branches. The resulting models are organized using the Coley–Landry invariant classification and analyzed through leading-order stability conditions $F_T > 0$, $F_{TT} > 0$.

Keywords: teleparallel gravity; Coley–Landry approach; covariant electromagnetic sources; non-symmetric field equations; Kantowski-Sachs teleparallel solutions

---

1. Introduction

Teleparallel $F(T)$ gravity provides a conceptually distinct and geometrically rich reformulation of gravitation, in which the gravitational interaction is encoded in spacetime torsion rather than curvature [1–4]. In this framework, gravity can be interpreted as a gauge theory of translations, described by a coframe $h^a{}_\mu$ and a spin-connection $\omega^a{}_{b\mu}$ defining a curvature-free but torsion-full geometry [5–8]. This viewpoint places gravitation on a similar conceptual footing as other fundamental interactions, notably electromagnetism, which is governed by gauge principles.

A major development in modern teleparallel gravity is the fully covariant formulation based on independent coframe/spin-connection (CSC) pairs, ensuring local Lorentz invariance and a consistent separation between inertial and gravitational effects [2,9,10]. This formulation resolves long-standing ambiguities related to frame dependence and provides a robust geometric foundation for modified teleparallel theories such as $F(T)$ gravity. In this context, the choice of coframe becomes an intrinsic geometric ingredient, rather than a mere computational tool.

These considerations naturally motivate the use of invariant classification methods. In particular, the Coley–Landry program extends the Cartan–Karlhede (CK) algorithm to teleparallel geometries, allowing a systematic classification of spacetimes at the level of the coframe and spin-connection [10–19]. This approach encodes spacetime symmetries through Lie–derivative conditions and provides a powerful framework for constructing physically inequivalent solutions beyond purely metric-based analyses. Recent developments have demonstrated its effectiveness in the study of spherically symmetric and cosmological teleparallel spacetimes [14–17].

Among anisotropic cosmological models, the Kantowski–Sachs (KS) spacetime plays a central role. Originally introduced in [20], KS geometries describe homogeneous but anisotropic universes with a four-dimensional isometry group, and arise naturally both in early-universe cosmology and in the interior region of Schwarzschild black holes (BHs) [21–23]. Their dynamical properties have

 doi:10.20944/preprints202605.0564.v1

been extensively studied in general relativity (GR) and modified gravity theories, including $f(R)$, $f(T)$, $f(Q)$ and related frameworks [24–29]. In teleparallel gravity, KS spacetimes provide a minimal yet non-trivial setting where torsion dynamics can significantly deviate from GR while remaining analytically tractable [16,30].

Most previous investigations in teleparallel $F(T)$ gravity have focused on scalar fields or perfect fluids, primarily motivated by dark energy, inflation, and late-time cosmology [3,4]. These studies have led to a wide class of viable cosmological models, including quintessence, phantom, and quintom scenarios. However, electromagnetic fields represent a fundamentally different class of sources. Their intrinsically anisotropic energy–momentum tensor is [30,31]:

$$\Theta^a{}_b = \mathrm{Diag}(\rho_{\mathrm{em}}, -\rho_{\mathrm{em}}, \rho_{\mathrm{em}}, \rho_{\mathrm{em}}), \tag{1}$$

where $\rho_{em} = -P_r = P_t$ in the orthonormal KS coframe. This structure naturally matches anisotropic KS geometries and provides a direct probe of anisotropic gravitational dynamics.

From a physical perspective, electromagnetic fields in anisotropic cosmologies play a crucial role in several contexts. They are central to primordial magnetogenesis scenarios, contribute to anisotropic inflation, and may leave observable imprints in the cosmic microwave background through statistical anisotropies. In strong-gravity regimes, such as BH interiors, electromagnetic fields can significantly modify the causal structure and stability properties of KS regions. These effects become even more intricate in teleparallel gravity, where torsion induces additional couplings that can mimic effective interactions between geometry and gauge fields [3,10].

In the teleparallel framework, electromagnetic fields must satisfy both the gravitational field equations (FEs) and the covariant Maxwell system [32–34]:

$$\nabla_\mu F^{\mu\nu} = J^\nu, \quad \nabla_{[\alpha} F_{\mu\nu]} = 0, \tag{2}$$

together with current conservation $\nabla_\nu J^\nu = 0$. These constraints strongly restrict the admissible coframe dynamics. Within the restricted Maxwell-compatible branch considered below, the electromagnetic energy density scales as

$$\rho_{\mathrm{em}} \propto A_3^{-4}(t),$$

reflecting flux conservation in the angular sector of the anisotropically evolving KS background [16,31]. This scaling establishes a direct coupling between the electromagnetic sector and the torsion scalar, leading to a highly constrained dynamical system.

The main goal of this work is to develop a unified and systematic framework for constructing exact electromagnetic KS solutions in covariant $F(T)$ gravity. Our approach combines three key ingredients: (i) the covariant CSC formalism, (ii) the Coley–Landry invariant classification, and (iii) the full set of Maxwell constraints. Unlike many previous studies relying on predefined $F(T)$ ansatze, we adopt a reconstruction strategy in which the functional form of $F(T)$ is determined directly from the FEs. The novelty of the present work is threefold. First, we treat the Maxwell sector covariantly in time-dependent KS geometry rather than imposing an effective fluid by hand. Second, we use the Maxwell conservation laws (CLs) to constrain the KS scale factors and reconstruct admissible $F(T)$ functions. Third, we classify the resulting Power-law (PL), exponential (EXP), and Teleparallel de Sitter (TdS) branches within the Coley–Landry invariant framework [19]. Throughout the paper, KS geometries are interpreted in two complementary ways: as homogeneous anisotropic cosmologies and as effective descriptions of BH-interior regions. This dual interpretation motivates the analysis of both anisotropic cosmological expansion and compact-object-like reconstruction sectors.

The structure of the paper is as follows. In Section 2, we derive the full set of teleparallel FEs with electromagnetic sources in KS geometry. In Section 3, we construct exact solutions using PL coframe ansatze, while Section 4 is devoted to EXP ansatze and TdS regimes. In Section 5, we extend the analysis to more general electromagnetic configurations, including BH solutions and teleparallel analogues of Reissner–Nordstrom–de Sitter (RN–dS) spacetimes. Finally, in Section 6, we summarize the main results and discuss their physical implications.

2. Teleparallel Field Equations, Maxwell Sector, and Kantowski–Sachs Geometry

2.1. Teleparallel $F(T)$ Gravity and General Conservation Laws

In the covariant formulation of teleparallel gravity, the gravitational dynamics is described by a generalized $F(T)$ action including a minimally coupled electromagnetic sector. The action is given by [1,3,10,15,16]:

$$S_{F(T)} = \int d^4x \left[\frac{h}{2\kappa} F(T) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right], \tag{3}$$

where $h = \det(h^a{}_\mu)$ is the coframe determinant, T is the torsion scalar, and $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the electromagnetic field tensor.

Varying the action with respect to the coframe $h^a{}_\mu$ yields the general FEs in covariant $F(T)$ gravity [2,7]:

$$\begin{aligned} \kappa \Theta^\mu{}_a &= h^{-1} F_T \partial_\nu \left(h S^{\mu\nu}{}_a \right) + F_{TT} S^{\mu\nu}{}_a \partial_\mu T + \frac{F}{2} h^\mu{}_a \\ &\quad - F_T T^b{}_{a\nu} S^{\mu\nu}{}_b - F_T \omega^b{}_{a\nu} S^{\mu\nu}{}_b, \end{aligned} \tag{4}$$

where $F_T = dF/dT$ and $F_{TT} = d^2F/dT^2$, $S^{\mu\nu}{}_a$ is the superpotential, $T^b{}_{a\nu}$ is the torsion tensor, and $\omega^b{}_{a\nu}$ is the spin-connection.

These equations can be decomposed into symmetric and antisymmetric parts [7,10]:

$$\kappa \Theta_{(ab)} = F_T \overset{\circ}{G}_{ab} + F_{TT} S^\mu{}_{(ab)} \partial_\mu T + \frac{g_{ab}}{2} [F - T F_T], \tag{5}$$

$$0 = F_{TT} S^\mu{}_{[ab]} \partial_\mu T. \tag{6}$$

The antisymmetric FEs impose strong constraints on the admissible CSC pairs and on the time dependence of the torsion scalar. In particular, nonlinear $F(T)$ branches with $F_{TT} \neq 0$ require compatibility between $\partial_t T$ and the antisymmetric superpotential sector.

From the matter sector, the canonical energy–momentum tensor is defined through the variation of the matter Lagrangian $\mathcal{L}_{\text{Source}}$ as [5,6]:

$$\Theta^\mu{}_a = \frac{1}{h} \frac{\delta \mathcal{L}_{\text{Source}}}{\delta h^a{}_\mu}, \tag{7}$$

and satisfies the covariant CL:

$$\overset{\circ}{\nabla}_\nu \Theta^{\mu\nu} = 0. \tag{8}$$

This relation corresponds to the standard conservation of energy–momentum in GR, now arising from diffeomorphism invariance within the teleparallel framework. Decomposing into symmetric and antisymmetric components, one obtains:

$$\Theta_{[ab]} = 0, \qquad \Theta_{(ab)} = T_{ab}, \tag{9}$$

where T_{ab} denotes the physical symmetric energy–momentum tensor.

This structure assumes minimal coupling between matter and the metric, implying a vanishing hypermomentum. In the general case, a non-zero hypermomentum leads to modified CLs. From Eqs. (5)–(6), one can rewrite the SFEs as an effective gravitational balance equation [6,10]:

$$0 = \kappa \Theta_{ab} - F_T \overset{\circ}{G}_{ab} - F_{TT} S^\mu{}_{ab} \partial_\mu T - \frac{g_{ab}}{2} [F - T F_T]. \tag{10}$$

 doi:10.20944/preprints202605.0564.v1

Equation (10) is therefore not an additional CL, but rather a convenient rewriting of the symmetric FEs in terms of an effective gravitational balance. It illustrates that, in teleparallel gravity, the effective balance equations are directly linked to the torsional structure of spacetime and to the functional form of $F(T)$. For minimally coupled electromagnetic matter with vanishing hypermomentum, the usual metric-compatible CL in Eq. (8) is recovered, ensuring consistency with GR solutions.

Remark. This formulation emphasizes the dynamical origin of the CLs within the covariant teleparallel framework. In teleparallel $F(T)$ gravity, the presence of F_T and F_{TT} terms can be interpreted as inducing effective couplings between torsion and matter, potentially modifying the energy dynamics, especially in anisotropic electromagnetic configurations.

2.2. Teleparallel Kantowski–Sachs Coframe/Spin-Connection Pair from Lie Algebra

In the covariant formulation of teleparallel gravity, admissible geometries are determined by symmetry requirements imposed simultaneously on the coframe and the spin-connection. These requirements are naturally expressed through Lie derivatives along the Killing vectors (KVs) generating the spacetime isometry group. A teleparallel geometry must satisfy [10–13]:

$$\mathcal{L}_X h^a = \lambda^a{}_b h^b, \qquad \mathcal{L}_X \omega^a{}_{bc} = 0, \tag{11}$$

where h^a is the orthonormal coframe, $\mathcal{L}_X$ denotes the Lie derivative along a KV X, and $\lambda^a{}_b$ generates local Lorentz transformations $\Lambda^a{}_b$.

In addition, a proper teleparallel geometry must satisfy the vanishing curvature condition, ensuring that gravitation is entirely encoded in torsion rather than curvature [2,5,10]:

$$R^a{}_{b\mu\nu} = \partial_\mu \omega^a{}_{b\nu} - \partial_\nu \omega^a{}_{b\mu} + \omega^a{}_{e\mu}\omega^e{}_{b\nu} - \omega^a{}_{e\nu}\omega^e{}_{b\mu} = 0 \;\;\Rightarrow\;\; \omega^a{}_{b\mu} = \Lambda^a{}_c \partial_\mu \Lambda^c{}_b. \tag{12}$$

A natural orthonormal coframe adapted to time-dependent KS geometries in coordinates (t, r, θ, ϕ) is given by [14–17]:

$$h^a{}_\mu = \mathrm{Diag}[1,\ A_2(t),\ A_3(t),\ A_3(t)\sin\theta]. \tag{13}$$

Without loss of generality, we fix $A_1(t) = 1$ through a time reparametrization, a standard procedure in homogeneous cosmologies [21,22].

An alternative parametrization is obtained by imposing $A_3(t) = t$,

$$h^a{}_\mu = \mathrm{Diag}[A_1(t),\ A_2(t),\ t,\ t\sin\theta]. \tag{14}$$

This branch is useful for comparison with areal-time descriptions and KS BH-interior-like geometries. In the remainder of Secs. 3–5, however, we use the cosmological-time gauge $A_1 = 1$, which is better suited to reconstruction in homogeneous cosmology [14–17].

Although Eqs. (13)–(14) can be rewritten using tetrads $e^a{}_\mu$ in orthonormal gauges, the coframe formulation $h^a{}_\mu$ is preferable in covariant teleparallel gravity. Indeed, the theory is invariant under local Lorentz transformations, and the CSC pair provides a consistent separation between inertial and gravitational effects [2,7,10].

In contrast, tetrad-only formulations may lead to frame-dependent results or ambiguities in the presence of non-trivial spin-connections [2,9]. The covariant approach resolves these issues and ensures the physical consistency of solutions.

The non-vanishing components of the spin-connection $\omega^a{}_{bc}$ compatible with KS symmetry can be parametrized as $\omega^a{}_{bc} = \omega^a{}_{bc}(\psi(t), \chi(t))$, where ψ and χ are arbitrary functions [14–17].

Imposing the antisymmetric part of the $F(T)$ FEs (cf. Eq. (6)) yields the constraints:

$$\psi = 0, \qquad \chi = \frac{\pi}{2} \quad (\text{or } \tfrac{3\pi}{2}),$$

which correspond to proper covariant teleparallel frames [2,10].

 doi:10.20944/preprints202605.0564.v1

Having fixed the KS coframe branch, we now specify the spin-connection required for covariance and local Lorentz invariance. The resulting non-zero components of the spin-connection are:

$$\omega^2{}_{34} = -\omega^2{}_{43} = \delta, \qquad \omega^3{}_{44} = -\frac{\cos\theta}{A_3 \sin\theta}, \tag{15}$$

with $\delta = \pm 1$ corresponding to equivalent discrete teleparallel branches, reflecting a discrete symmetry of the connection [15,16].

The coframe–spin-connection pair defined by Eqs. (13)–(15) satisfies the symmetry conditions, the zero-curvature condition, and the AFEs, thereby defining a fully consistent covariant teleparallel KS geometry.

Moreover, this construction is equivalent to that obtained via the CK invariant classification extended to teleparallel gravity [11–14]. This approach classifies spacetimes at the level of the coframe and spin-connection rather than solely through the metric, allowing the identification of physically inequivalent geometries that would otherwise appear identical in purely metric-based analyses. This further highlights the central role of the Coley–Landry method in constructing and interpreting teleparallel solutions [15,16].

2.3. Electromagnetic Source Conservation Laws

The electromagnetic CLs for an equation (13) KS spacetime and an energy-momentum defined by equation (1) are:

$$0 = \partial_t \rho_{em} + 4\partial_t(\ln A_3(t))\,\rho_{em}, \tag{16}$$

where $\rho_{em} = \frac{1}{2}\left(E^2 + B^2\right)$. The $E(t)$ and $B(t)$ terms denote the physical radial electric and magnetic amplitudes measured in the orthonormal coframe. We simplify equation (16) as:

$$0 = \partial_t \ln\left[\rho_{em}(t)\, A_3^4(t)\right] \quad \Rightarrow \quad \rho_{em}(t) = \frac{\rho_{em\,0}}{A_3^4(t)}. \tag{17}$$

The electromagnetic sector must also satisfy the covariant Maxwell equations defined as:

$$\nabla_\mu F^{\mu\nu} = J^\nu \quad \text{and} \quad \nabla_\alpha F_{\mu\nu} + \nabla_\nu F_{\alpha\mu} + \nabla_\mu F_{\nu\alpha} = 0. \tag{18}$$

These constraints distinguish the time-dependent KS electromagnetic sector from the static SS electromagnetic solutions studied previously [15,18,32–34]. We add the conserved 4-current:

$$\nabla_\nu J^\nu = 0, \tag{19}$$

for a completely conserved electromagnetic solution. In terms of KS spacetime, we find that equations (18)–(19) are summarized as:

$$\begin{aligned} A_2 A_3^2 \rho_{\text{elec}} &= \partial_t(A_2 A_3^2 E), \quad A_3^2 J^r = \partial_t(A_3^2 E), \\ 0 &= \partial_t(A_3^2 B), \quad 0 = \partial_t(A_2 A_3^2 \rho_{\text{elec}}), \end{aligned} \tag{20}$$

$$\begin{aligned} \rho_{elec}(t) &= \frac{\rho_{elec\,0}}{A_2(t)\, A_3^2(t)}, \quad E(t) = \frac{(\rho_{elec\,0}\, t + E_0)}{A_2(t)\, A_3^2(t)} \quad B(t) = \frac{B_0}{A_3^2(t)} \\ J^r(t) &= \frac{1}{A_2^2(t)\, A_3^2(t)}\left[\rho_{elec\,0}\left(A_2(t) - t\partial_t A_2(t)\right) - E_0 \partial_t A_2(t)\right]. \end{aligned} \tag{21}$$

The product $A_2(t)A_3^2(t)$ is proportional to the comoving spatial volume element of the KS geometry. Within the restricted Maxwell-compatible branch, substituting equations (21) into equation (17) yields

$$\rho_{em}(t) = \frac{1}{2A_3^4(t)}\left(\frac{(\rho_{elec\,0}\,t + E_0)^2}{A_2^2(t)} + B_0^2\right) = \frac{\rho_{em\,0}}{A_3^4(t)}. \tag{22}$$

If one imposes the pure angular flux scaling $\rho_{\rm em} = \rho_{\rm em,0}A_3^{-4}$ together with the Maxwell solutions (21), then Eq. (22) defines a restricted Maxwell-compatible branch. In this branch one obtains

$$A_2(t) = \pm\frac{\rho_{\rm elec,0}t + E_0}{\sqrt{2\rho_{\rm em,0} - B_0^2}} = A_{20}(t + \tilde{E}_0), \tag{23}$$

where

$$A_{20} = \pm\frac{\rho_{\rm elec,0}}{\sqrt{2\rho_{\rm em,0} - B_0^2}}, \qquad \tilde{E}_0 = \frac{E_0}{\rho_{\rm elec,0}}, \qquad 2\rho_{\rm em,0} > B_0^2, \tag{24}$$

with $\rho_{\rm elec,0} \neq 0$. The condition $2\rho_{\rm em,0} > B_0^2$ ensures that the restricted branch remains real-valued. This branch corresponds to a special current-free configuration compatible with the pure angular flux scaling. It should not be interpreted as the most general Maxwell solution in KS geometry. More general current profiles may relax this constraint and lead to different $A_2(t)$ evolutions.

Using equation (14) ansatz formulation, equation (16) will be:

$$0 = \partial_t\rho_{em} + \frac{4}{t}\rho_{em}, \quad \Rightarrow \quad \rho_{em} = \frac{\rho_{em\,0}}{t^4}. \tag{25}$$

Equations (20)–(21) solutions will be summarized by:

$$\begin{aligned}\rho_{elec}(t) =& \frac{\rho_{elec\,0}}{A_2(t)\,t^2} \quad E(t) = \frac{1}{A_2(t)\,t^2}\left[\rho_{elec\,0}\int dt'\,A_1(t') + E_0\right], \quad B(t) = \frac{B_0}{t^2},\\ J^r(t) =& \frac{1}{A_1A_2^2t^2}\frac{d}{dt}\left[\frac{\rho_{\rm elec,0}\int^t A_1(t')dt' + E_0}{A_2(t)}\right].\end{aligned} \tag{26}$$

In the $A_3 = t$ branch, the current component $J^r(t)$ generally does not vanish unless additional constraints are imposed on $A_1(t)$, $A_2(t)$, and the electric source constants. In the remainder of the paper, we mainly use the $A_1 = 1$ cosmological-time branch, while Eq. (26) is retained for comparison with the areal-time parametrization.

2.4. Symmetric and Unified Field Equations

With the KS-compatible CSC pair fixed, the torsion scalar and the independent symmetric FEs can be written entirely in terms of the two scale factors $A_2(t)$ and $A_3(t)$. These equations form the dynamical core of the reconstruction procedure. The torsion scalar and the symmetric FE components for the $\chi = \frac{\pi}{2}$ $(\delta = +1)$ case in equation (5) are [15,16]:

$$T = 2(\ln(A_3))'\Big((\ln(A_3))' + 2(\ln(A_2))'\Big) - \frac{2}{A_3^2}, \tag{27}$$

$$\kappa\rho_{em} = -\tfrac{1}{2}[F - TF_T] + F_T\left[\frac{1}{A_3^2} + 2(\ln A_2)'(\ln A_3)' + (\ln A_3)'^2\right], \tag{28}$$

$$-\kappa\rho_{em} = \tfrac{1}{2}[F - TF_T] - 2\partial_t(F_T)(\ln A_3)' - F_T\left[\frac{1}{A_3^2} + \frac{2A_3''}{A_3} + (\ln A_3)'^2\right], \tag{29}$$

$$\kappa\rho_{em} = \tfrac{1}{2}[F - TF_T] - \partial_t(F_T)\Big[(\ln A_3)' + (\ln A_2)'\Big] - F_T\left[\frac{A_2''}{A_2} + \frac{A_3''}{A_3} + (\ln A_2)'(\ln A_3)'\right], \tag{30}$$

 doi:10.20944/preprints202605.0564.v1

where F_T is not assumed to be constant. Compared with some KS $F(T)$-gravity formulations in the literature [26,30], the present covariant CSC formulation leads to modified component expressions. For the $\delta = -1$ branch, only minor sign changes occur in selected terms of Eqs. (28)–(30), corresponding to sign changes in selected connection-dependent terms. Apart from these sign changes, the general structure of Eqs. (28)–(30) remains unchanged.

Combining Eqs. (28)–(29), and assuming $A_3' \neq 0$, eliminates the $[F - TF_T]$ and $\rho_{\rm em}$ terms and yields

$$\partial_t(\ln F_T) = \left[(\ln A_2)' - \frac{A_3''}{A_3'}\right]. \tag{31}$$

Equation (31) assumes $F_T(T) \neq 0$ and excludes degenerate TEGR-like branches with constant FT. Equation (31) plays the role of the master reconstruction equation for the KS sector and represents the KS analogue of the reconstruction equation used in the static SS case, with the radial derivative replaced by the time derivative. Equation (31) is valid on branches where $F_T \neq 0$, $A_3' \neq 0$, and $\partial_t \ln F_T$ is well defined; branches violating these conditions must be treated separately. Eqs. (27), (30) and (31) together provide the master teleparallel FEs system used in Secs. 3–5.

3. Exact Power-Law Solutions in Kantowski–Sachs Teleparallel Gravity

In this section, we derive exact solutions of the FEs (27)–(30) using a PL ansatz within the covariant teleparallel framework. We adopt the Coley–Landry approach based on invariant classification and the CSC pair [15–17], ensuring the consistency of the teleparallel geometry.

3.1. Power-Law Ansatz and Torsion Scalar

We consider the KS coframe ansatz with

$$A_1 = 1, \quad A_2(t) = b_0 t^b, \quad A_3(t) = c_0 t^c. \tag{32}$$

We restrict to $t > 0$, so that the PL branches are real and the logarithmic derivatives are well defined. The exponent b controls the radial scale factor, while c controls the angular two-sphere. Their difference measures the anisotropic shear of the KS geometry.

The logarithmic derivatives are

$$H_2 = \frac{\dot{A}_2}{A_2} = \frac{b}{t}, \qquad H_3 = \frac{\dot{A}_3}{A_3} = \frac{c}{t}, \qquad \sigma^2 \propto (H_2 - H_3)^2. \tag{33}$$

The torsion scalar becomes

$$T = \frac{2c(c+2b)}{t^2} - \frac{2}{c_0^2 t^{2c}}. \tag{34}$$

The sign of T depends on the balance between the kinetic torsion contribution and the angular-curvature term.

Three integrable cases emerge:

- $c = 1$: $T \sim t^{-2}$ (scaling regime). This case is especially useful because both the angular-curvature term and the kinetic torsion contribution scale as t^{-2}, allowing a direct algebraic inversion $t = t(T)$.
- $c = -2b$: $T \sim t^{-2c}$ (pure angular-torsional regime). In this branch, the kinetic contribution $2c(c+2b)t^{-2}$ vanishes and the torsion scalar is controlled entirely by the angular-curvature term.
- $c \neq \{1, -2b\}$ (general or intermediate regimes):

$$0 = 2c(c+2b)\, t^{-2} - \frac{2}{c_0^2}\, t^{-2c} - T. \tag{35}$$

Each branch leads to a specific local inversion $t = t(T)$, whenever $\dot{T} \neq 0$, and therefore to a corresponding reconstruction of teleparallel $F(T)$. The cases $c = 1$ and $c = -2b$ are the analytically tractable branches used below; the intermediate branch generally requires implicit reconstruction or a case-by-case inversion of $T(t)$. Most developments and results concern the $c = 1$ branch.

3.2. Coley–Landry Reconstruction Method

Following [13,15,16], we express all physical quantities in terms of the invariant scalar T. After substituting the PL ansatz into the reduced system (27), (30), and (31), all time-dependent terms can be expressed as powers of T on locally invertible branches. The resulting equation for $F(T)$ takes the Euler form

$$T^2 F_{TT} + \gamma_1 T F_T + \gamma_0 F = \kappa\rho(T), \tag{36}$$

where γ_1 and γ_0 are branch-dependent constants determined by the KS exponents (b, c) and by the electromagnetic sector under consideration. The coefficients γ_1 and γ_0 are fixed only after the invariant branch and electromagnetic source scaling have been selected. The source term $\rho(T)$ is fixed by the Maxwell branch under consideration; hence the electromagnetic sector determines the particular solution, while the homogeneous part encodes the vacuum teleparallel branch. The homogeneous solution is

$$F_h(T) = C_1 T^{m_1} + C_2 T^{m_2}, \tag{37}$$

with

$$m_{1,2} = \frac{1}{2}\left[1 - \gamma_1 \pm \sqrt{(\gamma_1 - 1)^2 - 4\gamma_0}\right]. \tag{38}$$

The constants C_1 and C_2 parametrize the homogeneous teleparallel sector, while the coefficient λ appearing below is fixed by the electromagnetic source normalization. When the discriminant $(\gamma_1 - 1)^2 - 4\gamma_0$ is negative, the homogeneous sector corresponds to logarithmic oscillatory modes in T, which require a separate physical interpretation. The reduction to Eq. (36) assumes that $T(t)$ is locally invertible on the branch considered and sufficiently differentiable. Points satisfying $\dot{T} = 0$ correspond to critical invariant branches and must be treated separately.

3.3. Radial Electric Field

For the Maxwell-compatible electric branch, the relevant electric density scales as

$$\rho_E(t) \sim t^{-4b},$$

which corresponds to the effective orthonormal scaling $E_r \sim A_2^{-2}$. Here E_r denotes the orthonormal radial electric amplitude.

In the scaling case $c = 1$, one finds

$$\rho_E(T) \sim T^{2b}. \tag{39}$$

The reconstructed function is therefore

$$F(T) = C_1 T^{m_1} + C_2 T^{m_2} + \lambda T^{2b}. \tag{40}$$

This demonstrates that the electric field directly imprints the anisotropy parameter b into the gravitational action.

3.4. Radial Magnetic Field

For a radial magnetic field,

$$B_r \propto \frac{1}{A_3^2}, \quad \rho_B \sim t^{-4c}. \tag{41}$$

For $c = 1$, we obtain

$$\rho_B(T) \sim T^2, \tag{42}$$

leading to

$$F(T) = C_1 T^{m_1} + C_2 T^{m_2} + \lambda T^2. \tag{43}$$

This universal quadratic behavior highlights the strong coupling between torsion and magnetic energy.

3.5. Transverse Electromagnetic Field

For transverse fields,

$$\rho_\perp \sim \frac{1}{A_2^2 A_3^2} \sim t^{-2(b+c)}. \tag{44}$$

For $c = 1$,

$$\rho_\perp(T) \sim T^{b+1}, \tag{45}$$

and thus

$$F(T) = C_1 T^{m_1} + C_2 T^{m_2} + \lambda T^{b+1}. \tag{46}$$

The different powers of T in Eqs. (40), (43), and (46) show that the Maxwell sector does not merely source the geometry; it selects distinct invariant reconstruction classes depending on whether the dominant contribution is electric, magnetic, or transverse. The corresponding PL branches are summarized in Table 1.

Table 1. Electromagnetic reconstruction branches in the PL KS sector.

EM branch	Density scaling for $c = 1$	Reconstructed term
Radial electric	$\rho_E(T) \sim T^{2b}$	T^{2b}
Radial magnetic	$\rho_B(T) \sim T^2$	T^2
Transverse	$\rho_\perp(T) \sim T^{b+1}$	T^{b+1}

Table 1 refers to the scaling branch $c = 1$, for which $T \sim t^{-2}$. Other values of c lead to different implicit reconstruction branches. The table highlights how different electromagnetic sectors select distinct invariant powers in the reconstructed teleparallel action.

3.6. Cosmological Solutions

The scale factors describe anisotropic cosmologies:

- $b = c$: isotropic expansion rates. Strictly speaking, the spatial topology remains KS rather than flat FLRW.
- $b > c$: anisotropic shear-dominated phase,
- $c < 0$: contracting radial direction (BH-interior-like behavior).

The torsion scalar behaves as

$$T \sim t^{-2} \quad \text{or} \quad T \sim t^{-2c}, \tag{47}$$

leading to PL cosmological expansion driven by torsion.

The mean expansion rate and shear are

$$H = \frac{1}{3}(H_2 + 2H_3) = \frac{b+2c}{3t}, \qquad \sigma^2 = \frac{1}{3}(H_2 - H_3)^2.$$

Thus, the isotropic limit corresponds to $b = c$, while $b \neq c$ describes a shear-dominated anisotropic phase. For constant PL exponents (b, c), the dimensionless ratio σ^2/H^2 remains constant.

The averaged volume is:

$$V(t) = A_2 A_3^2 = b_0 c_0^2 t^{b+2c}, \qquad \Theta = 3H = \frac{b+2c}{t}. \tag{48}$$

 doi:10.20944/preprints202605.0564.v1

For $t > 0$, accelerated KS volume expansion occurs when the effective mean scale factor exponent satisfies

$$\frac{b+2c}{3} > 1.$$

The universe expands on average when

$$b + 2c > 0,$$

while $b + 2c < 0$ corresponds to average contraction.

3.7. Stability Analysis

Stability requires

$$F_T > 0, \quad F_{TT} > 0, \tag{49}$$

to avoid ghosts and classical instabilities [3,10]. The conditions $F_T > 0$ and $F_{TT} > 0$ are necessary leading-order conditions. A complete stability analysis would require coupled perturbations of the two KS scale factors and the electromagnetic sector.

For $F(T) = T^n$, we find

$$F_T = nT^{n-1}, \quad F_{TT} = n(n-1)T^{n-2}. \tag{50}$$

For non-integer n, the sign of T must be fixed on the branch considered. Equivalently, one may work with $|T|^n$ or with shifted variables when T crosses zero.

Thus:

- $n > 1$: potentially stable, provided the branch satisfies $F_T > 0$ and $F_{TT} > 0$,
- $0 < n < 1$: typically unstable because F_{TT} changes sign or becomes singular depending on the sign branch of T,
- $n < 0$: generally pathological or associated with ghost-like effective torsional sectors.

Linear perturbations δT satisfy

$$\ddot{\delta T} + M_{\text{eff}}^2 \delta T = 0, \tag{51}$$

with

$$M_{\text{eff}}^2 \sim \frac{F_T}{F_{TT}}, \tag{52}$$

provided $F_{TT} \neq 0$. This expression should be interpreted as a leading-order scalar-torsion diagnostic, not as the full perturbative spectrum of the anisotropic KS system.

For quadratic gravity $F(T) = T + \alpha T^2$, one finds stable oscillatory modes if $\alpha > 0$, and then

$$F_T = 1 + 2\alpha T, \qquad F_{TT} = 2\alpha.$$

Thus $\alpha > 0$ ensures $F_{TT} > 0$, while $F_T > 0$ additionally requires $1 + 2\alpha T > 0$ on the physical branch considered.

3.8. Discussion

Therefore, the electromagnetic source does not merely act as an external matter sector; through the Maxwell CLs it selects admissible torsion scalings and hence restricts the reconstructed $F(T)$ branches.

In summary, the PL sector provides a direct map between anisotropic KS expansion and reconstructed teleparallel actions. Electric sources encode the radial anisotropy exponent b, magnetic sources select the universal quadratic correction in the scaling branch, and transverse sources generate mixed radial-angular powers. This provides the time-dependent KS counterpart of the static SS reconstruction program, with cosmological time replacing the radial coordinate and Maxwell CLs replacing radial flux conservation.

 doi:10.20944/preprints202605.0564.v1

4. Exponential Solutions and Teleparallel de Sitter Regime

In this section, we construct exact solutions of the FEs (27)–(30) using an EXP ansatz within the covariant teleparallel framework. The analysis follows the Coley–Landry invariant approach based on the CSC pair [10,15,16], allowing a consistent and frame-independent reconstruction of $F(T)$.

4.1. Exponential Ansatz and Torsion Structure

We consider the KS metric with

$$A_1 = 1, \quad A_2(t) = b_0 e^{bt}, \quad A_3(t) = c_0 e^{ct}. \tag{53}$$

The logarithmic derivatives are constant:

$$\frac{\dot{A}_2}{A_2} = b, \quad \frac{\dot{A}_3}{A_3} = c. \tag{54}$$

The torsion scalar becomes

$$T = 2c(c + 2b) - \frac{2}{c_0^2} e^{-2ct}. \tag{55}$$

Defining

$$T_0 = 2c(c + 2b), \quad X \equiv T_0 - T, \tag{56}$$

we obtain

$$X = \frac{2}{c_0^2} e^{-2ct} > 0, \tag{57}$$

for $c_0 \neq 0$.

This relation is exactly invertible:

$$t = -\frac{1}{2c} \ln\left(\frac{c_0^2 X}{2}\right). \tag{58}$$

The case $c = 0$ must be treated separately, since the invariant X becomes constant and the inversion $t = t(X)$ degenerates. Thus, all physical quantities can be expressed as functions of the invariant $X = T_0 - T$.

The mean expansion and shear are

$$H = \frac{b + 2c}{3}, \qquad \sigma^2 = \frac{1}{3}(b - c)^2,$$

so the EXP branch describes anisotropic EXP expansion whenever $b + 2c > 0$ and $b \neq c$. The isotropic EXP limit is recovered for $b = c$, whereas $b \neq c$ gives constant nonzero shear.

For $c > 0$, $X \to 0$ as $t \to \infty$, and the solution approaches the TdS point $T = T_0$ [3,19]. The limit $X \to 0$ therefore corresponds to the invariant TdS fixed point. For $c < 0$, X grows and the same ansatz describes a contracting angular sector, relevant to KS interior-like regimes.

4.2. Coley–Landry Reduction

Following [15,16], the FEs reduce to an Euler-type equation in X:

$$X^2 F_{TT} + \Gamma_1 X F_T + \Gamma_0 F = \kappa X^\alpha, \tag{59}$$

with constant coefficients within a fixed invariant branch.

The homogeneous solution reads

$$F_h(T) = C_1 X^{m_1} + C_2 X^{m_2}, \tag{60}$$

 doi:10.20944/preprints202605.0564.v1

with

$$m_{1,2} = \frac{1}{2}\left[1 - \Gamma_1 \pm \sqrt{(\Gamma_1 - 1)^2 - 4\Gamma_0}\right]. \tag{61}$$

The discriminant $(\Gamma_1 - 1)^2 - 4\Gamma_0$ determines whether the invariant modes are real shifted powers of X or logarithmic oscillatory modes. Here Γ_1 and Γ_0 are branch-dependent coefficients determined by the reduced KS FEs after all time dependence has been expressed through $X = T_0 - T$. As in the PL sector, the homogeneous solution describes the vacuum teleparallel invariant branch, while the Maxwell density scaling determines the particular reconstructed term.

4.3. Radial Electric Field

Maxwell equations yield

$$E_r \propto \frac{1}{A_2^2} \Rightarrow \rho_E \sim e^{-4bt}. \tag{62}$$

Using the inversion relation, we find

$$\rho_E(X) \sim X^{2b/c}. \tag{63}$$

The reconstructed function is therefore

$$F(T) = C_1 X^{m_1} + C_2 X^{m_2} + \lambda X^{2b/c}. \tag{64}$$

This demonstrates that the electric field encodes the anisotropic expansion rate b/c into the gravitational Lagrangian.

4.4. Radial Magnetic Field

For a radial magnetic field,

$$B_r \propto \frac{1}{A_3^2}, \quad \rho_B \sim e^{-4ct}. \tag{65}$$

Thus,

$$\rho_B(X) \sim X^2, \tag{66}$$

leading to

$$F(T) = C_1 X^{m_1} + C_2 X^{m_2} + \lambda X^2. \tag{67}$$

This quadratic behavior is universal and independent of the anisotropy parameters.

4.5. Transverse Electromagnetic Field

For transverse fields,

$$\rho_\perp \sim \frac{1}{A_2^2 A_3^2} \sim e^{-2(b+c)t}, \tag{68}$$

which implies

$$\rho_\perp(X) \sim X^{(b+c)/c}. \tag{69}$$

Thus,

$$F(T) = C_1 X^{m_1} + C_2 X^{m_2} + \lambda X^{(b+c)/c}. \tag{70}$$

The resulting EXP reconstruction branches are summarized in Table 2. This table assumes $c \neq 0$, so that the invariant map $t \leftrightarrow X$ is well defined. Table 2 illustrates how the electromagnetic sector determines the shifted invariant reconstruction powers in the EXP branch.

 doi:10.20944/preprints202605.0564.v1

Table 2. Electromagnetic reconstruction branches in the EXP KS sector.

EM branch	Density scaling	Reconstructed term
Radial electric	$\rho_E(X) \sim X^{2b/c}$	$X^{2b/c}$
Radial magnetic	$\rho_B(X) \sim X^2$	X^2
Transverse	$\rho_\perp(X) \sim X^{(b+c)/c}$	$X^{(b+c)/c}$

4.6. Teleparallel de Sitter Solutions

At late times ($t \to \infty$), we have

$$T \to T_0 = \text{const.}, \tag{71}$$

which corresponds to a TdS phase [3,19].

The existence condition for such solutions is

$$F(T_0) - 2T_0 F_T(T_0) = 0, \tag{72}$$

assuming $F_T(T_0) \neq 0$. This condition constrains the coefficients of the reconstructed $F(T)$ evaluated at the constant-torsion point T_0. For constant torsion, the FEs reduce to an effective GR-like branch with an induced cosmological contribution. This relation should be understood as the reduced TdS existence condition for the KS branch considered here, rather than as a universal condition for all $F(T)$ cosmologies [19].

4.7. Cosmological Behavior

The EXP ansatz describes anisotropic inflationary regimes:

- $b = c$: isotropic expansion rates in the KS topology,
- $b > c$: anisotropic inflation,
- $c < 0$: contracting radial direction (BH-interior-like regime).

For $c < 0$, the angular sector contracts and the solution is better interpreted as a KS interior-like branch rather than a late-time cosmology.

The invariant variable evolves as

$$X \sim e^{-2ct}, \tag{73}$$

showing exponential relaxation toward the dS attractor. The averaged volume evolves as

$$V(t) = A_2 A_3^2 = b_0 c_0^2 e^{(b+2c)t},$$

so $b + 2c > 0$ gives exponentially expanding average volume, while $b + 2c < 0$ gives contraction, with isotropic expansion rates recovered in the limit $b = c > 0$.

4.8. Stability Analysis

Stability requires [3,10,19]

$$F_T > 0, \quad F_{TT} > 0. \tag{74}$$

For the isolated monomial $F(T) = \alpha X^n$, one obtains

$$F_T = -\alpha n X^{n-1}, \qquad F_{TT} = \alpha n(n-1) X^{n-2}.$$

For $X > 0$ and $n > 1$, the conditions $F_T > 0$ and $F_{TT} > 0$ cannot generally be satisfied simultaneously by the monomial alone. Thus:

- $n = 1$: marginal,
- $n < 1$: unstable.

Perturbations around the dS point,

$$T = T_0 + \delta T, \tag{75}$$

 doi:10.20944/preprints202605.0564.v1

lead to

$$\ddot{\delta T} + M_{\text{eff}}^2 \delta T = 0, \tag{76}$$

with

$$M_{\text{eff}}^2 = \frac{F_T(T_0)}{2T_0 F_{TT}(T_0)}, \tag{77}$$

whenever $T_0 F_{TT}(T_0) \neq 0$. This expression should be interpreted as a leading-order scalar-torsion diagnostic, not as the full perturbative spectrum of the anisotropic KS system.

For the shifted model

$$F(T) = T + \alpha X^2,$$

one finds

$$F_T = 1 - 2\alpha X, \qquad F_{TT} = 2\alpha.$$

Hence a viable branch requires

$$\alpha > 0, \qquad 1 - 2\alpha X > 0.$$

Equivalently, the viable shifted-quadratic branch satisfies

$$0 < X < \frac{1}{2\alpha}.$$

Since $X \to 0$ for $c > 0$, the condition $1 - 2\alpha X > 0$ is automatically satisfied sufficiently close to the TdS attractor for finite positive α.

For shifted models $F(T) = T + \alpha X^n$, the TEGR term contributes positively to F_T, while the nonlinear correction controls F_{TT}. This illustrates why shifted TdS models must be analyzed as complete functions rather than as isolated monomial corrections.

4.9. Discussion

The EXP ansatz naturally leads to a shifted functional dependence $F(T) = f(T_0 - T)$, rather than a pure PL in T. This reflects the presence of a TdS attractor and provides a novel mechanism for anisotropic inflation in $F(T)$ gravity. Different electromagnetic configurations generate distinct scaling exponents, highlighting the predictive power of the Coley–Landry invariant approach. However, the physical viability of these branches depends on the full shifted model, the sign of F_T, the positivity of F_{TT}, and the behavior of anisotropic shear perturbations. Thus, the EXP sector provides the natural KS setting for studying anisotropic TdS attractors sourced by electromagnetic fields.

5. General Electromagnetic Sources and Kantowski–Sachs Reconstruction Branches

In this section, we extend the PL and EXP reconstruction schemes to more general effective electromagnetic scalings. The goal is not to construct a complete global BH spacetime, but rather to identify KS invariant branches that can be interpreted as BH-interior-like or RN–dS-like local reduced sectors within covariant $F(T)$ gravity. The reconstruction remains local in the invariant variable T or $X = T_0 - T$, and its physical viability must be checked branch by branch.

5.1. General Electromagnetic Sources

Beyond the standard radial electric and magnetic fields, one can consider more general electromagnetic configurations described by the invariant

$$\mathcal{I} = F_{\mu\nu} F^{\mu\nu}. \tag{78}$$

The parametrization

$$\rho_{\text{EM}} = \rho_0 A_2^{-p} A_3^{-q}$$

 doi:10.20944/preprints202605.0564.v1

should be understood as an effective scaling ansatz. Specific Maxwell branches correspond to particular values of (p,q), whereas nonlinear electrodynamics or additional currents may generate more general effective exponents [32–34].

Using the PL and EXP solutions derived in Sections 3–4, this general scaling becomes:

- For the PL branch with $c=1$, one obtains
$$\rho_{\rm EM}(T)\sim T^{(pb+q)/2}.$$
- For the angular-torsional branch $c=-2b$, the exponent is modified according to the corresponding implicit map $t=t(T)$. In the EXP sector,
$$\rho_{\rm EM}(X)\sim X^{(pb+qc)/(2c)},\qquad c\neq 0.$$

This organizes a broad class of effective electromagnetic source scalings within a single reconstruction scheme, enabling systematic reconstruction of $F(T)$ as illustrated in Table 3.

Table 3. Effective electromagnetic scaling branches in KS geometry.

Source branch	Effective scaling	(p,q)
Radial electric	restricted branch or $A_2^{-2}A_3^{-4}$-type effective scaling	branch dependent
Radial magnetic	A_3^{-4}	$(0,4)$
Transverse EM	$A_2^{-2}A_3^{-2}$	$(2,2)$
General effective EM	$A_2^{-p}A_3^{-q}$	(p,q)

Table 3 shows that the standard electric, magnetic, and transverse sectors are recovered as special cases of the effective (p,q) parametrization.

5.2. General Reconstruction Algorithm

Following the Coley–Landry invariant method, the reconstruction algorithm can be summarized as follows. First, one selects a KS coframe branch (A_2,A_3) and computes the torsion scalar $T(t)$ [15,16]. Second, one verifies local invertibility of the invariant map, either $t=t(T)$ or $t=t(X)$. Third, the electromagnetic density is rewritten as $\rho_{\rm EM}(T)$ or $\rho_{\rm EM}(X)$. Finally, the reduced FEs are solved as Euler-type equations,

$$T^2F_{TT}+\gamma_1TF_T+\gamma_0F=\kappa\rho_{\rm EM}(T), \tag{79}$$
$$X^2F_{TT}+\Gamma_1XF_T+\Gamma_0F=\kappa\rho_{\rm EM}(X). \tag{80}$$

The branch coefficients γ_i and Γ_i are fixed only after a specific invariant branch and electromagnetic scaling are selected.

The general solution is
$$F(T)=C_1T^{m_1}+C_2T^{m_2}+F_{\rm part}(T), \tag{81}$$
or in the EXP case,
$$F(T)=C_1X^{m_1}+C_2X^{m_2}+F_{\rm part}(X). \tag{82}$$

This algorithm is valid only on invariant branches where $\dot T\neq 0$, or equivalently where the map $t\mapsto T$ or $t\mapsto X$ is locally invertible. Critical points must be treated separately.

5.3. Kantowski–Sachs Black Hole Solutions

The KS metric naturally describes homogeneous BH-interior-like sectors, rather than a complete exterior BH spacetime [21–23]. A global BH interpretation requires matching to an exterior region and analyzing horizons, causal structure, and geodesic completeness. Using the EXP ansatz, one obtains:

$$ds^2 = dt^2 - b_0^2 e^{2bt} dr^2 - c_0^2 e^{2ct} d\Omega^2. \tag{83}$$

For $c < 0$, the angular sector contracts, mimicking the contraction of the two-sphere inside a BH-interior-like region. The torsion scalar behaves as

$$T = T_0 - X, \quad X \sim e^{-2ct}. \tag{84}$$

Thus, for $c < 0$, the EXP branch provides a KS interior-like effective high-torsion sector. The associated reconstructed $F(T)$ models may be interpreted as local teleparallel analogues of charged BH-interior-like geometries, but not yet as complete BH solutions.

5.4. Teleparallel RN–de Sitter Solutions

We now identify a reduced teleparallel analogue of the RN–dS structure. In GR, the RN–dS lapse function is [23,32–34]:

$$A_1^2(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3} r^2. \tag{85}$$

In the reduced KS reconstruction, an analogous effective structure may be represented by the functional form:

$$F(T) = T + \alpha T^2 + \beta (T_0 - T)^n. \tag{86}$$

The terms correspond to:

- T: GR limit,
- T^2: nonlinear high-torsion correction, which may mimic charge-like scaling in suitable KS branches,
- $(T_0 - T)^n$: shifted TdS correction, associated with the effective cosmological sector.

Effective horizon-like structures would emerge after reconstruction of the corresponding exterior metric sector, where lapse-function zeros define causal horizons.

This suggests that suitable nonlinear torsion corrections can mimic RN–dS-like effective structures at the level of the reduced KS dynamics. A complete RN–dS interpretation would require the reconstruction of the corresponding exterior metric sector and a comparison of horizon invariants.

5.5. Other Classes of Solutions

Such extensions are natural from the invariant electromagnetic perspective [32–34]. The following classes should be regarded as reconstruction templates rather than fully analyzed physical solutions:

- **Nonlinear electrodynamics:**
$$\rho \sim \mathcal{I}^k \Rightarrow F(T) \sim T^{\gamma k}, \tag{87}$$
- **Magnetically dominated universes:**
$$F(T) = T + \alpha T^2 + \lambda T^m, \tag{88}$$
- **Mixed EM + scalar field systems:**
$$F(T) = T + V(\phi(T)) + \rho_{\mathrm{EM}}(T), \tag{89}$$
- **Exact dS attractors:**
$$F(T_0) - 2T_0 F_T(T_0) = 0. \tag{90}$$

Each class requires separate checks of the Maxwell equations, the AFEs, the positivity of F_T, the sign of F_{TT}, and the behavior of the anisotropic shear. These solutions extend the PL and EXP branches by allowing additional invariant powers generated by nonlinear electromagnetic or mixed matter sectors.

 doi:10.20944/preprints202605.0564.v1

5.6. Stability and Physical Viability

The necessary leading-order viability conditions are

$$F_T > 0, \qquad F_{TT} > 0,$$

but these are not sufficient for full stability in KS geometry. These conditions must be imposed together with the Maxwell equations and the AFEs, since a stable $F(T)$ branch is not necessarily compatible with the covariant CSC constraints. Since KS spacetimes contain independent radial and angular scale factors, a complete analysis must also control anisotropic shear perturbations and electromagnetic backreaction.

For generalized shifted models

$$F(T) = T + \alpha T^2 + \beta X^n,$$

one obtains

$$F_T = 1 + 2\alpha T - \beta n X^{n-1}, \qquad F_{TT} = 2\alpha + \beta n(n-1)X^{n-2}.$$

For $X > 0$, the shifted correction contributes with opposite signs to F_T and F_{TT}, so the sign of β must be chosen together with n and the allowed invariant range. Thus, viability depends on the signs and magnitudes of α, β, n, and on the invariant branch considered.

The effective scalar-torsion mass,

$$M^2_{\text{eff}} \sim \frac{F_T}{F_{TT}},$$

should again be interpreted as a leading-order diagnostic rather than the full perturbation spectrum. In particular, $M^2_{\text{eff}} > 0$ is necessary for scalar-torsion stability, but not sufficient for full KS stability.

5.7. Discussion

This section extends the PL and EXP reconstruction schemes to broader electromagnetic scaling branches. The main result is that effective EM sources select characteristic invariant powers in $F(T)$, allowing KS cosmological, interior-like, and RN–dS-like local reduced sectors to be treated within a unified Coley–Landry reconstruction framework. However, the BH-like and RN–dS-like interpretations remain local and branch-dependent unless supplemented by an exterior matching, horizon analysis, and full perturbative stability study. In this sense, the solutions constructed here should be viewed as teleparallel KS building blocks for more complete charged compact-object models [3,10]. This interpretation is consistent with the role of KS geometries as local models of anisotropic cosmology and BH-interior-like regions [21–23].

6. Discussion and Conclusions

In this work, we constructed exact solutions in covariant $F(T)$ teleparallel gravity with electromagnetic sources within the KS geometry, using the invariant Coley–Landry approach [10,15–17]. This framework ensures consistency at the level of the CSC pair, the symmetric and AFEs, and allows a systematic reconstruction of the gravitational action from invariant torsional dynamics.

Two main classes of solutions were obtained. PL configurations (Section 3) lead to $F(T)$ models expressed as combinations of T^n, while EXP solutions (Section 4) naturally generate shifted forms $F(T) = f(T_0 - T)$ and admit TdS regimes [19]. In both cases, electromagnetic sources (electric, magnetic, and transverse) strongly constrain the admissible functional dependence of $F(T)$, highlighting a direct link between matter content and torsion dynamics [3]. In particular, the Maxwell CLs act as dynamical selection rules for admissible torsion branches, strongly constraining the reconstructed teleparallel action.

We further showed that KS geometries admit local BH-interior-like and RN–dS-like reconstruction branches in covariant $F(T)$ gravity. In this context, nonlinear torsion corrections effectively reproduce charge-like and cosmological-constant-like contributions without relying on curvature invariants.

These solutions should however be interpreted as local KS reduced sectors unless supplemented by an exterior matching and global horizon analysis.

From a cosmological perspective, the EXP solutions describe anisotropic inflationary regimes with a dS attractor $T \to T_0$, satisfying the condition $F(T_0) - 2T_0 F_T(T_0) = 0$ [3,10]. Stability analysis shows that viable models require $F_T > 0$ and $F_{TT} > 0$, which are satisfied for broad classes such as $F(T) = T + \alpha T^2 + \beta(T_0 - T)^n$ with $\alpha > 0$ and $n > 1$.

More generally, the Coley–Landry invariant approach provides a natural geometric framework for organizing inequivalent teleparallel reconstruction branches directly at the level of the coframe and spin-connection [15,16,19,31]. Overall, our results demonstrate that electromagnetic fields act as effective reconstruction sources for $F(T)$ gravity, providing a unified mechanism to generate both cosmological and BH solutions within a torsion-based framework. The KS geometry therefore provides a unified setting in which anisotropic cosmological evolution and BH-interior-like dynamics can be analyzed within the same invariant teleparallel formalism.

Recent studies have also shown that covariant teleparallel reconstruction methods can generate weak-massive wormhole configurations supported by suitable $F(T)$ sectors, further illustrating the richness of torsion-based compact-object phenomenology [35]. Future work includes the study of quasi-normal modes and perturbative spectra of the teleparallel BH solutions, as well as gravitational lensing and observational signatures. Confrontation with cosmological data (e.g., Planck) could further constrain viable teleparallel $F(T)$ models. On the theoretical side, extending the invariant classification program and incorporating additional fields (scalar sectors or nonlinear electrodynamics) may provide further insight into the role of torsion, invariant geometry, and gauge structures in modified theories of gravity.

Funding: This research received no external funding.

Data Availability Statement: All data is included in this manuscript.

Acknowledgments: Thanks to A.A. Coley for useful and constructive comments.

Conflicts of Interest: The author declares no conflicts of interest.

References

1. R. Aldrovandi and J. G. Pereira, *Teleparallel Gravity: An Introduction* (Springer, Dordrecht, 2013).
2. M. Krššák and E. N. Saridakis, "The covariant formulation of $f(T)$ gravity," Class. Quantum Grav. **33**, 115009 (2016).
3. Y.-F. Cai, S. Capozziello, M. De Laurentis, and E. N. Saridakis, "$f(T)$ teleparallel gravity and cosmology," Rept. Prog. Phys. **79**, 106901 (2016).
4. S. Bahamonde et al., Phys. Rept. **775**, 1 (2021).
5. F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Ne'eman, "Metric-affine gauge theory of gravity: field equations, Noether identities, world spinors, and breaking of dilation invariance," Phys. Rept. **258**, 1 (1995).
6. Y. N. Obukhov, "Poincaré gauge gravity: selected topics," Int. J. Geom. Meth. Mod. Phys. **3**, 95 (2006).
7. M. Hohmann, L. Järv, and U. Ualikhanova, "Covariant formulation of scalar-torsion gravity," Phys. Rev. D **97**, 104011 (2018).
8. M. Krssak and J. G. Pereira, Eur. Phys. J. C **75**, 519 (2015).
9. A. Golovnev, T. Koivisto and M. Sandstad, Class. Quant. Grav. **34**, 145013 (2017).
10. M. Krššák, R. J. van den Hoogen, J. G. Pereira, C. G. Böhmer, and A. A. Coley, "Teleparallel theories of gravity: illuminating a fully invariant approach," Class. Quantum Grav. **36**, 183001 (2019).
11. A. A. Coley, Class. Quant. Grav. **26**, 195015 (2009).
12. A. A. Coley, G. Papadopoulos and N. Pelavas, Class. Quant. Grav. **28**, 125007 (2011).
13. Coley, A.A.; van den Hoogen, R.J.; McNutt, D.D. Symmetry and Equivalence in Teleparallel Gravity. *J. Math. Phys.* **2020**, *61*, 072503.
14. D. D. McNutt, A. A. Coley and R. J. van den Hoogen, J. Math. Phys. **64**, 032503 (2023).
15. A. A. Coley, A. Landry, R. J. van den Hoogen and D. D. McNutt, Eur. Phys. J. C **84**, 334 (2024).
16. A. Landry, Symmetry **16**, 953 (2024).

 doi:10.20944/preprints202605.0564.v1

17. van den Hoogen, R.J.; Forance, H. Teleparallel Geometry with Spherical Symmetry: The diagonal and proper frames. *J. Cosmol. Astrophys.* **2024**, *11*, 033.
18. A. Landry, Static spherically symmetric perfect fluid solutions in teleparallel $F(T)$ gravity. *Axioms* **2024**, *13*, 333.
19. Coley, A.A.; Landry, A.; van den Hoogen, R.J.; McNutt, D.D. Generalized Teleparallel de Sitter geometries. *Eur. Phys. J. C* **2023**, *83*, 977.
20. R. Kantowski and R. K. Sachs, J. Math. Phys. **7**, 443 (1966).
21. M. Ryan and L. Shepley, *Homogeneous Relativistic Cosmologies*, Princeton (1975).
22. G. F. R. Ellis and M. A. H. MacCallum, Commun. Math. Phys. **12**, 108 (1969).
23. S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge (1973).
24. C. G. Boehmer, A. Mussa and N. Tamanini, Class. Quant. Grav. **28**, 245020 (2011).
25. G. G. L. Nashed, Eur. Phys. J. C **81**, 324 (2021).
26. M. E. Rodrigues et al., Astrophys. Space Sci. **357**, 129 (2015).
27. G. Leon and A. A. Roque, JCAP **05**, 032 (2014).
28. N. Dimakis et al., Eur. Phys. J. C **83**, 794 (2023).
29. A. D. Millano et al., Phys. Rev. D **109**, 124044 (2024).
30. M. J. Amir and M. Yussouf, Int. J. Theor. Phys. **54**, 2798 (2015).
31. A. Landry, Electromagnetic Sources Teleparallel Robertson–Walker F(T)-Gravity Solutions, Mathematics, **14**(1), 48 (2026).
32. J. D. Jackson, *Classical Electrodynamics*, 3rd ed., Wiley, New York (1999).
33. L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, Pergamon Press, Oxford (1975).
34. F. W. Hehl and Y. N. Obukhov, *Foundations of Classical Electrodynamics*, Birkhäuser, Boston (2003).
35. A. Landry et al., Weak-Massive Wormholes in Covariant Teleparallel $F(T)$ Gravity, to appear soon in Int. J. of Geom. Meth. in Mod. Phys., arXiv:2508.06290 (2026).