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Article

Conditional Regularity of the Three-Dimensional Navier–Stokes Equations via High–High Triadic Absorption

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Abstract

We develop a structural framework for the regularity problem of the three-dimensional incompressible Navier–Stokes equations based on the Fourier-space geometry of triadic interactions. The central idea is that the nonlinear energy transfer can be decomposed into dyadic shell contributions and further classified into Low–Low, Low–High, and High–High channels. Within this decomposition, the potentially dangerous same-scale amplification mechanism is localized to the High–High channel, while the Low–Low and Low–High contributions are shown to be controllable through weighted paraproduct estimates. To quantify the High–High mechanism, we introduce a triadic-family decomposition together with shell-level observables including High–High family transfer, family coherence, coherent time sets, residence times, and shell defect quantities. Using these observables, we formulate a shellwise High–High absorption condition stating that the High–High transfer is dominated by the corresponding viscous scale up to a Sobolev-summable remainder. Under this condition, we prove a conditional regularity theorem: for strong solutions in $H^s(\mathbb{T}^3)$ with $s > 5/2$, the Sobolev norm remains bounded on any finite time interval, and hence finite-time blow-up does not occur. We then investigate why such an absorption condition is structurally natural. On the Fourier side, we analyze triadic geometry, helical sign structure, and phase dynamics, and show that persistent High–High amplification requires strong coherence and sufficiently long residence in coherent time sets. This provides an integral mechanism suppressing cumulative High–High transfer. On the PDE side, we introduce a relaxation formulation of the Navier–Stokes equations with an independent stress variable and establish a triple-dissipation structure consisting of viscous dissipation, defect-stress relaxation, and stress diffusion. A relative-entropy argument is then used to show that this enhanced dissipative structure is stably transferred to the Navier–Stokes limit. The result is not an unconditional resolution of the global regularity problem. Rather, it provides a precise reduction: the continuation problem for strong solutions is reduced to a shellwise High–High absorption condition with explicit geometric, temporal, and dissipative interpretations. In this sense, the regularity problem is reformulated in terms of the internal structure of the energy cascade, rather than solely by global norm criteria.

Keywords: three-dimensional Navier–Stokes equations; conditional regularity; triadic interaction; High–High absorption; dyadic shell decomposition; Littlewood–Paley analysis; helical decomposition; phase coherence; residence time; relaxation system; relative entropy

1. Introduction

The Navier–Stokes equations occupy a central position in fluid mechanics as the fundamental continuum model for viscous incompressible flows. Derived from the conservation of mass and momentum together with Newton's constitutive law for viscous stress, they have been extraordinarily successful across a wide range of physical and engineering applications, including aerodynamics, geophysical flows, oceanic circulation, combustion, process engineering, and

astrophysical fluid dynamics [2–4]. At the same time, turbulence theory has shown that the apparent simplicity of the equations conceals a remarkably rich multiscale nonlinear structure, most notably in the transfer of kinetic energy across wave numbers [14–18]. In this sense, the Navier–Stokes equations lie simultaneously at the foundations of continuum mechanics and at the heart of modern turbulence theory.

Despite this broad success, the mathematical structure of the three-dimensional incompressible Navier–Stokes equations still contains one of the most fundamental unresolved questions in analysis: whether a smooth solution issued from smooth initial data can develop a finite-time singularity, or instead remains smooth for all time. This problem was formulated by Fefferman as one of the Clay Millennium Problems [1]. Since the pioneering work of Leray and Hopf on global weak solutions [5,6], a large body of research has established conditional regularity criteria and partial regularity results. Among the classical milestones are the Beale–Kato–Majda blow-up criterion [7], the Prodi–Serrin integrability conditions [8,9], the partial regularity theory of Caffarelli, Kohn, and Nirenberg [10], and the geometric viewpoint based on vorticity direction due to Constantin and Fefferman [11]. More recent developments include analyticity and decay estimates [13], averaged-model blow-up constructions illuminating the role of nonlinear structure [12], and a broad functional-analytic framework for Navier–Stokes well-posedness and continuation theory [40,43,46]. Yet the full large-data global regularity problem for the three-dimensional Navier–Stokes equations remains open.

From the viewpoint of turbulence, however, it is natural to suspect that the decisive issue is not merely the size of a global norm, but rather the internal structure of the nonlinear energy transfer. In Fourier variables, the nonlinear term is represented as a superposition of triadic interactions, that is, interactions among three wave vectors satisfying the resonance relation $k + p + q = 0$. These triads are the basic carriers of energy transfer in the cascade process and therefore provide the most natural language for identifying which nonlinear channels are potentially dangerous. Extensive work in turbulence theory and numerical analysis has clarified the scale-locality and geometry of triadic transfer [16,21,22,42], the relevance of shell-type reductions [23], and the role of coherent structures, intermittency, and extreme gradients at high Reynolds number [19,20,24,27–29]. This accumulated knowledge strongly suggests that any mathematically meaningful reformulation of the regularity problem should be compatible with the triadic structure of the energy cascade.

The present work adopts precisely this viewpoint. Rather than approaching the regularity problem solely through global function-space criteria, we ask a more structural question:

Which Fourier-space interaction channel can actually produce dangerous high-frequency amplification, and under what condition is that channel neutralized by dissipation?

Our central claim is that the regularity problem can be reformulated as a control problem for the High–High triadic transfer. More precisely, after decomposing the nonlinear interaction into dyadic shells and regrouping the transfer into triadic families, we show that the Low–Low and Low–High contributions can be estimated by paraproduct-type bounds and treated as controllable perturbative terms in the Sobolev energy balance, whereas the only interaction channel that remains potentially capable of producing dangerous same-scale high-frequency amplification is the High–High channel. In this way, the regularity problem is reduced to a single structural issue: whether the shellwise High–High transfer is dominated by viscous dissipation up to a summable remainder. This motivates the introduction of the High–High Absorption Condition, which constitutes the central hypothesis of the present paper. The role of the weighted paraproduct estimates is therefore crucial: they ensure that the Low–Low and Low–High channels are handled inside the proof, so that the only external condition in the final continuation theorem is the High–High absorption property itself. This is exactly the logic developed in the revised shellwise closure framework and in the weighted Sobolev argument underlying the conditional regularity chapter and Appendix B.

It is important to state this point with full precision. This paper does not claim an unconditional proof of global regularity for the three-dimensional Navier–Stokes equations. Nor does it claim that every possible aspect of the turbulence cascade has been rigorously reduced to a closed theorem independent of all structural hypotheses. The main theorem is conditional. What it proves is the

following: if the High–High transfer in each dyadic shell is absorbed by the shellwise viscous dissipation up to a remainder that is summable in the Sobolev-weighted shell energy, then the Sobolev norm of the strong solution cannot blow up in finite time. In particular, once the Low–Low and Low–High channels are controlled by the internal estimates of the paper, the sole structural condition that remains at the level of the main continuation theorem is the High–High Absorption Condition. This logical distinction is essential. It prevents the reader from confusing the present result with an unconditional solution of the Millennium problem, while at the same time preserving the true conceptual reduction achieved here: namely, that the strong-solution continuation problem is reduced to the control of the High–High interaction channel.

The mathematical mechanism behind this reduction may be summarized as follows. First, the Navier–Stokes nonlinearity is decomposed in Fourier space by means of a dyadic shell decomposition and Littlewood–Paley projections [45]. The shellwise transfer is then split into Low–Low, Low–High, and High–High contributions. Second, the Low–Low and Low–High contributions are estimated by Bernstein, Hölder, Young, and paraproduct inequalities [40,45,46], yielding weighted bounds of lower order relative to the dissipative part of the Sobolev energy inequality. In particular, the low-frequency factor acts as a smooth coefficient in the paraproduct structure, so these nonlocal interactions do not create a dangerous same-scale resonance and can be absorbed into the dissipation after Sobolev-weighted summation. This is the analytic reason why these channels do not appear as independent external assumptions in the main theorem. Third, the remaining High–High contribution is isolated at the shell level and quantified through a family-based interaction observable. The resulting shellwise estimate leads, under the High–High Absorption Condition, to a closed differential inequality of Grönwall type [44] for the Sobolev energy. The closure of the weighted shell estimate and the resulting boundedness of the H^s -norm form the core of the conditional regularity theorem. This proof strategy is already visible in the revised conditional manuscript, especially in the Sobolev closure chapter and Appendix B, where the HL/LH channel is explicitly handled by weighted paraproduct estimates and the High–High channel alone remains as the essential condition.

The second major theme of this paper is that the High–High Absorption Condition is not introduced as a purely ad hoc analytic hypothesis. Rather, we argue that it has a natural dynamical origin. This part of the paper is motivated by two earlier lines of development. The first is a network master equation approach to fluid dynamics, in which interacting discrete states are constructed so as to satisfy conservation laws and the second law of thermodynamics at the structural level, and whose continuum limit yields Navier–Stokes-type equations [36]. The second is a stress relaxation formulation of the Navier–Stokes equations, in which the viscous stress is treated as an independent dynamic variable approaching its Newtonian equilibrium through a relaxation law, thereby generating an additional triple-dissipation structure [37]. In the present work these two themes are not auxiliary digressions. They serve to explain why a shellwise absorption mechanism should naturally arise: the relaxation dynamics produces defect-stress damping at high wave numbers, while a relative entropy argument transfers the corresponding stability structure from the relaxation system to the Navier–Stokes limit. In this sense, the paper has a deliberately double architecture:

$$\text{High–High Absorption} \Rightarrow \text{conditional regularity,}$$

together with

$$\text{relaxation damping + phase decorrelation} \Rightarrow \text{naturalization of High–High Absorption.}$$

This double structure is one of the distinguishing features of the present work and is already clearly visible in the longer structural manuscript, where the master-equation and relaxation chapters are placed not as isolated background material but as dynamical support for the absorption hypothesis itself.

A further key idea is that dangerous High–High transfer is not determined solely by amplitudes, but also by triadic phase coherence. Energy transfer in a triad depends not only on shell magnitudes but also on the relative phase of the interacting Fourier modes. Strong transfer requires a sufficient degree of coherent alignment, and such coherence becomes progressively more fragile at high wave numbers due to rapid phase variation. This motivates the introduction of triadic families, family coherence observables, coherent time sets, and residence-time budgets. In the later chapters we show how these quantities lead to sufficient conditions for the integral form of the High–High Absorption Condition. Thus the paper is not limited to the formal statement “assume absorption.” It also develops a detailed phase-dynamical scenario explaining why prolonged coherent High–High amplification should be difficult to sustain in the high-wave-number regime. This phase-residence viewpoint is already emphasized in the longer manuscript and remains one of the principal mechanisms by which the absorption condition is rendered physically and mathematically plausible.

From this perspective, the main contribution of the paper may be summarized in four steps.

First, the nonlinear term of the three-dimensional incompressible Navier–Stokes equations is reorganized into triadic families indexed by dyadic shells, thereby converting the triadic geometry into observable shell-level quantities.

Second, the Low–Low and Low–High channels are treated by weighted paraproduct estimates, so that they enter the Sobolev energy budget as controllable perturbative terms rather than as independent dangerous mechanisms [45]. This step is essential because it justifies, within the proof itself, why the final continuation criterion does not require additional external assumptions on those channels.

Third, the remaining High–High transfer is isolated and expressed through a High–High Absorption Condition, which compares the shellwise nonlinear transfer against the shellwise viscous dissipation plus a summable remainder. Under this condition, the weighted shell energy closes and yields bounded Sobolev norms on every finite time interval via a Grönwall argument [44].

Fourth, the paper develops a dynamical explanation of this condition through relaxation damping, phase decorrelation, coherent-set residence estimates, and relative entropy transfer from the relaxation system to the Navier–Stokes limit [36–38].

Taken together, these steps imply that the strong-solution continuation problem is reduced to the control of a single shellwise structural quantity: the High–High transfer. The logical meaning of this statement is precise. It does not mean that the entire regularity problem has been unconditionally solved. It means that, once the analytically controllable channels have been treated inside the proof, the sole unresolved channel at the level of the continuation theorem is the High–High channel.

The present analysis is also naturally connected to classical turbulence phenomenology. In Kolmogorov’s picture, the inertial range supports a roughly scale-independent energy flux [18], while dissipation dominates in the far ultraviolet. In dyadic shell variables, this picture suggests that same-scale transfer and viscous dissipation should compete most directly near the dissipation onset. The present High–High Absorption framework is fully consistent with this interpretation, but it goes beyond phenomenology by translating it into a mathematically explicit shellwise condition. In this way, the paper seeks to bridge the gap between the turbulence-cascade viewpoint of Frisch, Pope, McComb, Alexakis, Biferale, and others [14–17,21–23], and the PDE-regularity viewpoint of Leray, Hopf, Beale–Kato–Majda, Prodi, Serrin, Constantin–Fefferman, and related works [5–11,40,43,46].

For the convenience of the reader, we emphasize once more the principal logical distinction that governs the entire paper:

- the statements concerning the master equation, relaxation dynamics, relative entropy, coherence, and residence time are intended to explain why the High–High Absorption Condition is natural and how it may arise dynamically;
- the conditional regularity theorem itself requires only the shellwise absorption condition together with the internally proved weighted-control properties of the Low–Low and Low–High channels.

This distinction is central to avoiding misunderstanding. The dynamical chapters provide motivation and structural support; they do not enlarge the list of external assumptions in the main theorem.

We now state, informally, the main result of the paper.

Informal Main Theorem.

Let u be a local strong solution of the three-dimensional incompressible Navier–Stokes equations on the torus, and let $s > 5/2$. Suppose that the shellwise High–High transfer satisfies the High–High Absorption Condition with a remainder term summable in the Sobolev-weighted shell energy. Then the H^s -norm of u remains bounded on every finite time interval. Consequently, no finite-time blow-up can occur along the interval of existence.

The fully quantitative statement is given later in the paper after the shellwise transfer, the weighted energies, and the remainder class have been precisely defined. The role of the present introduction is to make clear why this theorem is not merely another conditional criterion expressed in a different notation. Its novelty lies in the fact that the criterion is formulated at the location where the dangerous nonlinear transfer is generated. Instead of imposing a condition on a global norm without structural interpretation, the paper identifies the potentially dangerous transfer channel, quantifies it shell by shell, and states regularity in terms of the absorption of that transfer.

The organization of the paper is as follows. In Chapter 2 we fix the Fourier, Leray-projection, and dyadic shell framework, and recall the triadic interaction structure of the Navier–Stokes nonlinearity [42,45]. In Chapter 3 we introduce the scale classification of triads and identify the High–High channel as the only potentially dangerous same-scale amplification mechanism at the shell level. In Chapters 4 and 5 we refine this viewpoint by introducing triadic families, coherence indicators, and family-level transfer observables. In Chapter 6 the High–High Absorption Condition is formulated in pointwise and integral forms, together with remainder summability. In Chapter 7 the weighted Sobolev energy is closed. A decisive ingredient is the weighted paraproduct estimate controlling the Low–Low and Low–High terms, so that the only external structural hypothesis in the theorem is the High–High absorption property itself [45]. In Chapters 8 through 13 we explain why this condition is dynamically natural, by means of the network master equation [36], the relaxation system and triple dissipation structure [37], the phase-residence analysis of coherent triadic families, and the relative entropy transfer of stability. Chapter 14 discusses the regularity implications of this framework, and the appendices collect the detailed shellwise, paraproduct, Bernstein, Hölder, Young, Littlewood–Paley, and turbulence-scaling estimates used in the main text [39,40,44,45].

In summary, the present work proposes a structural reformulation of the three-dimensional Navier–Stokes regularity problem. Its guiding principle is that the continuation problem for strong solutions should be read through the internal architecture of the energy cascade. Once the perturbative channels are controlled analytically, the issue reduces to one question only: Is the High–High transfer absorbed by dissipation, shell by shell?

The purpose of the paper is to show that, under a precise quantitative formulation of that condition, finite-time blow-up is excluded; and further, that this condition is not arbitrary, but emerges naturally from relaxation damping and phase-decorrelation mechanisms associated with the multiscale triadic dynamics.

2. The Structural Origin of Fluid Equations from a Master-Equation Perspective

2.1. Purpose and Positioning of This Chapter

The purpose of this chapter is to demonstrate that the theoretical framework employed in this study is not constructed in an ad hoc manner by taking the Navier–Stokes equations as a starting point but rather emerges naturally from the structure of more fundamental interacting systems. As stated in Chapter 1, the central claim of this work is that the possibility of finite-time breakdown of strong solutions to the three-dimensional Navier–Stokes equations can ultimately be reduced to the absorptive nature of High–High triadic interactions. However, such a High–High absorption

condition should not be introduced merely as a formal assumption. Instead, it is essential to clarify the structural background from which such a condition can naturally arise.

To this end, we revisit, based on our previous works [36,37], the concept of a network master equation, in which fluid motion is described as a discrete interacting system. The fundamental standpoint adopted here is not to assume the fluid equations *a priori* as partial differential equations, but to construct a system of finitely interacting local states governed by the following design principles:

- conservation of mass, momentum, and total energy,
- compliance with the second law of thermodynamics, i.e., non-decreasing total entropy,
- existence of a continuum limit under appropriate geometric and scaling assumptions.

This perspective is consistent with the hydrodynamic limit from the Boltzmann equation to fluid equations [30–35] and is closely aligned with structure-based formulations in nonequilibrium thermodynamics, such as the GENERIC framework [38].

It is important to emphasize that the network master equation is not employed as a direct tool for proving the main theorem. The theorem itself is established in later chapters through shellwise triadic analysis and the High–High Absorption Condition. Rather, the network master equation and its subsequent relaxation extension provide the dynamical foundation that renders the absorption condition natural. The role of this chapter is therefore not to provide external background, but to place the later notions of absorption, dissipation, and defect stress within a coherent structural framework.

We proceed by introducing node states and an abstract master equation, followed by an analysis of conservation and entropy structures. We then present a finite-volume realization and describe how continuum equations emerge. Subsequently, we explain the natural introduction of a relaxation system as a continuation of the continuum limit and finally clarify the connection to triadic interaction structures in Fourier space.

2.2. Node States and Abstract Interaction Network

Consider a network consisting of a finite number of nodes, $G = (V, E)$, where V denotes the set of nodes and E the set of edges. Each node $i \in V$ represents either a finite-volume cell in physical space or a coarse-grained local state. The state variable at node i is defined as $U_i = (\rho_i, m_i, e_i)$, where ρ_i is the density, m_i the momentum, and e_i the internal energy. The corresponding velocity is given by $u_i = \frac{m_i}{\rho_i}$. This choice of variables provides a minimal representation of fluid-mechanical conservation laws. The total energy at node i is expressed as

$$E_i = e_i + \frac{|m_i|^2}{2\rho_i}.$$

Let $N(i)$ denote the set of nodes adjacent to i . The most general form of an interaction-type master equation is then

$$\frac{dU_i}{dt} = \sum_{j \in N(i)} (A_{ij}(U_i, U_j) + D_{ij}(U_i, U_j)). \quad (1)$$

Here, A_{ij} represents reversible exchange fluxes, while D_{ij} represents dissipative fluxes. This decomposition is fundamental: reversible transport corresponds to conservative dynamics, whereas dissipative transport captures irreversible effects such as viscosity and diffusion.

2.3. Algebraic Conditions for Conservation

For the master equation (1) to be admissible in a fluid-mechanical sense, internal interactions must not create or destroy mass, momentum, or total energy. This is ensured by imposing antisymmetric across each edge (i, j) :

$$A_{ij}(U_i, U_j) = -A_{ji}(U_j, U_i), D_{ij}(U_i, U_j) = -D_{ji}(U_j, U_i). \quad (2)$$

Summing over all nodes yields

$$\sum_{i \in V} \frac{dU_i}{dt} = 0. \quad (3)$$

Thus, the global conserved quantities are

$$M = \sum_i \rho_i, P = \sum_i m_i, E = \sum_i \left(e_i + \frac{|m_i|^2}{2\rho_i} \right). \quad (4)$$

Crucially, the conserved energy is the total energy, not merely the internal energy. Dissipation should therefore be understood as a transfer from kinetic to internal energy, rather than a loss of energy itself.

2.4. Entropy and the Second Law of Thermodynamics

Conservation laws alone do not determine the direction of time evolution. Since fluid motion is inherently irreversible, the second law of thermodynamics must be satisfied. Introduce the specific entropy $s_i = s(\rho_i, e_i)$, and define the total entropy as

$$S = \sum_i \rho_i s_i. \quad (5)$$

The second law requires

$$\frac{dS}{dt} \geq 0. \quad (6)$$

This inequality is treated not consequentially but as a design principle for admissible dynamics. Reversible fluxes must preserve entropy,

$$\sum_i \frac{\partial S}{\partial U_i} \cdot \sum_{j \in N(i)} A_{ij} = 0, \quad (7)$$

while dissipative fluxes must produce entropy,

$$\sum_i \frac{\partial S}{\partial U_i} \cdot \sum_{j \in N(i)} D_{ij} \geq 0. \quad (8)$$

2.5. Reversible-Irreversible Decomposition and the GENERIC Perspective

From nonequilibrium thermodynamics, time evolution can often be written as

$$\frac{dU}{dt} = L(U) \nabla E(U) + M(U) \nabla S(U), \quad (9)$$

where $L(U)$ is antisymmetric (reversible dynamics) and $M(U)$ is symmetric positive semidefinite (dissipation). The degeneracy conditions

$$M(U) \nabla E(U) = 0, L(U) \nabla S(U) = 0, \quad (10)$$

ensure compatibility between energy conservation and entropy production.

While not adopting GENERIC as a strict axiomatic framework, the present construction closely follows its design philosophy: reversible transport is entropy-neutral, while irreversible transport generates entropy without violating conservation laws.

2.6. Finite-Volume Realization

Partition the domain $\Omega \subset \mathbb{R}^3$ into cells $\Omega = \bigcup_i K_i$. The master equation can be expressed in finite-volume form as

$$\frac{d}{dt} (\int_{K_i} U_i) = - \sum_{j \in N(i)} F_{ij} + \sum_{j \in N(i)} G_{ij}, \quad (11)$$

where F_{ij} and G_{ij} denote conservative and dissipative fluxes across interfaces. A natural scaling for dissipative fluxes is

$$\mu_{ij} \sim \mu \frac{S_{ij}}{h_{ij}}, \quad (12)$$

which ensures consistency with diffusion terms in the continuum limit.

2.7. Entropy Production at the Interface Level

Entropy production can be expressed as

$$\frac{dS}{dt} = \frac{1}{2} \sum_{(i,j) \in E} \Sigma_{ij}, \quad (13)$$

with

$$\Sigma_{ij} \geq 0. \quad (14)$$

This implies

$$\frac{dS}{dt} \geq 0. \quad (15)$$

This discrete structure mirrors the dissipation inequality in continuum theory.

2.8. Continuum Limit and Conservation-Law-Type PDEs

In the limit $h \rightarrow 0$, the system converges formally to

$$\partial_t U + \nabla \cdot F(U) = \nabla \cdot G(U, \nabla U). \quad (16)$$

In fluid variables,

$$\partial_t \rho + \nabla \cdot (\rho u) = 0, \quad (17)$$

$$\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p = \nabla \cdot \tau, \quad (18)$$

$$\partial_t E + \nabla \cdot ((E + p)u) = \nabla \cdot (\tau u) + \dots. \quad (19)$$

Under the incompressible limit

$$\rho = \rho_0, \nabla \cdot u = 0, \quad (20)$$

and Newtonian constitute law

$$\tau = 2\nu D(u), \quad (21)$$

one recovers

$$\partial_t u + (u \cdot \nabla)u + \nabla p = \nu \Delta u, \nabla \cdot u = 0. \quad (22)$$

2.9. On Compactness: Established Results and Open Issues

The correspondence between discrete systems and continuum PDEs is structurally natural, but a full compactness theorem is not established here. In general, for a sequence of discrete solutions U^h , if one can establish a uniform energy bound,

$$\sup_{t \in [0, T]} \|U^h(t)\|_{L^2} \leq C, \quad (23)$$

a uniform dissipation bound,

$$\int_0^T \|\nabla_h U^h(t)\|_{L^2}^2 dt \leq C, \quad (24)$$

and uniform boundedness of the time derivative,

$$\partial_t U^h \text{ is bounded in a suitable negative norm,} \quad (25)$$

then one may expect, via an Aubin–Lions type compactness argument, to obtain strong convergence. However, at the level of general theory, this still involves issues that should be made fully rigorous in future work.

What is required for the present paper is not the completion of such a full discrete compactness theorem at this stage. Rather, the role of this chapter is to provide the structural foundation showing that the network master equation algebraically incorporates conservation laws, is consistent with the second law of thermodynamics, possesses a finite-volume/continuum correspondence, and leads naturally to the relaxation extension.

2.10. Natural Emergence of the Relaxation Extension

Even after the continuum limit has been reached, one further essential question remains. It is whether the viscous stress should, from the outset, be fixed by the instantaneous constitutive law $\tau = 2\nu D(u)$, or whether it should instead be treated dynamically as an independent nonequilibrium variable. Introduce a stress tensor r and define

$$w = r - 2\nu D(u). \quad (26)$$

Then

$$\partial_t u + P((u \cdot \nabla)u) = P\nabla \cdot r, \quad (27)$$

$$\varepsilon \partial_t r + r = 2\nu D(u) - \kappa(-\Delta)^\theta r. \quad (28)$$

This yields a triple dissipation structure:

- viscous dissipation,
- stress diffusion,
- relaxation of w .

2.11. Connection to Triadic Interaction Structures

In Fourier space, nonlinear interactions take the form $k + p + q = 0$. Thus,

$$\text{network interaction} \rightarrow \text{continuum equation} \rightarrow \text{Fourier triadic interaction}. \quad (29)$$

This establishes that High–High triadic interactions arise as a structural manifestation of the underlying interaction network.

2.12. Summary

This chapter established the structural origin of fluid equations from a network master equation. The key conclusions are:

- admissibility is governed by conservation laws and entropy production,
- the abstract structure admits a finite-volume realization,
- the continuum limit yields Navier–Stokes equations,
- the relaxation extension introduces a triple dissipation structure,
- the framework naturally connects to Fourier triadic interactions.

This provides the structural foundation linking the triadic analysis, High–High absorption theory, and relaxation-based dissipation mechanisms developed in the subsequent chapters.

3. The Relaxation System and the Emergence of Additional Dissipation

3.1. Purpose and Positioning of This Chapter

Chapter 2 established that Navier–Stokes-type equations arise, through finite-volume structure and continuum passage, from a network master equation viewed as a discrete interacting system satisfying conservation laws and the second law of thermodynamics. At that stage, however, the

viscous stress is still treated as a quantity given instantaneously by the Newtonian constitutive law. In other words, one assumes the static closure that, once the velocity gradient $D(u)$ is specified, the stress is determined immediately by $\tau = 2\nu D(u)$. The starting point of the present chapter is to reinterpret this constitutive law not as a mere substitution formula, but as the limiting form of a dynamical relaxation process toward equilibrium. This viewpoint makes it possible to introduce the stress as an independent nonequilibrium variable and thereby reveals an additional dissipative structure that is absent from the classical Navier–Stokes equations. In our earlier works [36,37], this relaxation system played an important role in the construction of weak solutions to the Navier–Stokes equations and in the theory of global strong solutions for small initial data. Here we go one step further and position this relaxation system as the dynamical background that renders natural the High–High Absorption Condition appearing later in Chapters 7 and 8.

The main points to be established in this chapter are threefold. First, we define a stress relaxation system in which the viscous stress of the Navier–Stokes equations are promoted to an independent variable. Second, we show that this system possesses, in addition to the usual viscous dissipation, a triple dissipation structure consisting of high-frequency diffusive dissipation of stress and relaxation dissipation associated with deviations from equilibrium. Third, we clarify that this structure may be interpreted as the dynamical background suppressing the high-wavenumber triadic transfer that later becomes central in Fourier space.

This chapter still does not contain the proof of the main theorem itself. Nevertheless, the defect stress introduced here, together with its dissipative structure, is of decisive importance for explaining, within the triadic regularity theory developed later, why High–High interactions may be dissipative absorbed.

3.2. A Dynamical Reinterpretation of the Newtonian Constitutive Law

The classical incompressible Navier–Stokes equations are given by

$$\partial_t u + (u \cdot \nabla)u + \nabla p = \nu \Delta u, \nabla \cdot u = 0. \quad (30)$$

Here $u(x, t)$ denotes the velocity field, $p(x, t)$ the pressure, and $\nu > 0$ the kinematic viscosity. These equations are usually understood as arising from the constitutive relation

$$\tau = 2\nu D(u), D(u) := \frac{1}{2}(\nabla u + (\nabla u)^T), \quad (31)$$

where the viscous stress tensor is assumed to respond instantaneously to the velocity gradient and to possess no independent dynamics.

From the viewpoint of nonequilibrium thermodynamics and relaxation-type modeling, however, such an instantaneous closure should be regarded as a limiting description rather than a fundamental one [38]. More generally, stress may be treated as a dynamical variable that approaches the equilibrium value $2\nu D(u)$ over a finite relaxation time. This idea is reminiscent of Maxwell-type viscoelastic relaxation, although our aim here is not to study viscoelasticity itself, but to extend the viscous stress of the Navier–Stokes equations into a nonequilibrium variable.

We therefore introduce the stress tensor $r = r(x, t)$ as an independent variable and consider, together with the velocity field u , the relaxation system

$$\partial_t u + P((u \cdot \nabla)u) = P\nabla \cdot r, \nabla \cdot u = 0, \quad (32)$$

$$\varepsilon \partial_t r + r = 2\nu D(u) - \kappa(-\Delta)^\theta r. \quad (33)$$

Here,

- P is the Leray projection,
- $\varepsilon > 0$ is the relaxation time,
- $\kappa > 0$ is the stress diffusion coefficient,
- $\theta > 0$ is the fractional diffusion exponent.

Equation (32) replaces the viscous term $\nu \Delta u$ in the momentum equation by the divergence of the independent stress r . Equation (33) states that stress relaxes toward its equilibrium value $2\nu D(u)$ while also undergoing additional dissipation of its high-frequency components through the term $-\kappa(-\Delta)^\theta r$.

In the limit $\varepsilon \rightarrow 0$, one obtains formally

$$r = 2\nu D(u) - \kappa(-\Delta)^\theta r. \quad (34)$$

If, furthermore, the effect of high-wavenumber stress diffusion is negligible, then

$$r \approx 2\nu D(u), \quad (35)$$

and equation (32) reduces to the classical Navier–Stokes equation (30). Thus, the system (32)–(33) is not an arbitrary extension of Navier–Stokes, but a natural extension obtained by reinterpreting the constitutive law as a relaxation process.

3.3. Introduction of the Defect Stress

To make the essential structure of this relaxation system transparent, we define the deviation from Newtonian equilibrium, namely the defect stress, by

$$w := r - 2\nu D(u). \quad (36)$$

Then $w = 0$ signifies that the stress is exactly at Newtonian equilibrium, whereas $w \neq 0$ indicates the presence of residual nonequilibrium stress. Using w , equation (33) becomes $\varepsilon \partial_t r + w + 2\nu D(u) = 2\nu D(u) - \kappa(-\Delta)^\theta r$, that is,

$$\varepsilon \partial_t r + w = -\kappa(-\Delta)^\theta r. \quad (37)$$

Hence the defect stress w is determined by the balance between the temporal variation of the stress and the high-frequency diffusion term.

Substituting (36) into the momentum equation (32) gives $\partial_t u + P((u \cdot \nabla)u) = P\nabla \cdot (2\nu D(u) + w)$. Using $\nabla \cdot u = 0$ together with the properties of the Leray projection, one has $P\nabla \cdot (2\nu D(u)) = \nu \Delta u$, and therefore

$$\partial_t u + P((u \cdot \nabla)u) = \nu \Delta u + P\nabla \cdot w. \quad (38)$$

This identity is crucial. The relaxation system can thus be regarded as the Navier–Stokes equations supplemented by the internally generated defect forcing

$$P\nabla \cdot w. \quad (39)$$

Accordingly, if w decays sufficiently rapidly, the system approaches Navier–Stokes. In this sense, w represents not only the deviation from equilibrium stress, but also the internal departure from the Navier–Stokes dynamics.

3.4. Derivation of the Basic Energy Identity

We now derive the energy structure carried by the relaxation system. Multiplying the velocity equation (32) by u and integrating over space, we obtain, on the periodic domain $\mathbb{T}^3$, the classical cancellation

$$\int_{\mathbb{T}^3} P((u \cdot \nabla)u) \cdot u \, dx = 0 \quad (40)$$

[40,43,46]. Hence

$$\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 = \int_{\mathbb{T}^3} (P\nabla \cdot r) \cdot u \, dx.$$

Using the self-adjointness of the Leray projection and the incompressibility condition $\nabla \cdot u = 0$, we obtain

$$\int_{\mathbb{T}^3} (P \nabla \cdot r) \cdot u \, dx = \int_{\mathbb{T}^3} (\nabla \cdot r) \cdot u \, dx = - \int_{\mathbb{T}^3} r : \nabla u \, dx.$$

Since r is regarded as a symmetric stress tensor, $r : \nabla u = r : D(u)$, and thus

$$\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 = - \int_{\mathbb{T}^3} r : D(u) \, dx. \quad (41)$$

Next, multiply the stress equation (33) by r and integrate. These yields

$$\varepsilon \int_{\mathbb{T}^3} \partial_t r : r \, dx + \int_{\mathbb{T}^3} |r|^2 \, dx = 2\nu \int_{\mathbb{T}^3} D(u) : r \, dx - \kappa \int_{\mathbb{T}^3} (-\Delta)^\theta r : r \, dx.$$

Therefore,

$$\frac{\varepsilon}{2} \frac{d}{dt} \|r\|_{L^2}^2 + \|r\|_{L^2}^2 + \kappa \|(-\Delta)^{\theta/2} r\|_{L^2}^2 = 2\nu \int_{\mathbb{T}^3} D(u) : r \, dx. \quad (42)$$

Combining (41) and (42), one sees that the structure becomes most transparent not by direct addition alone, but by completing the square using the defect stress $w = r - 2\nu D(u)$.

3.5. Completion of the Square Using the Defect Variable

By definition (36),

$$r = w + 2\nu D(u). \quad (43)$$

Therefore,

$$|w|^2 = |r - 2\nu D(u)|^2 = |r|^2 - 4\nu r : D(u) + 4\nu^2 |D(u)|^2. \quad (44)$$

Integrating into space yields

$$\|w\|_{L^2}^2 = \|r\|_{L^2}^2 - 4\nu \int_{\mathbb{T}^3} r : D(u) \, dx + 4\nu^2 \|D(u)\|_{L^2}^2. \quad (45)$$

Using

$$\|D(u)\|_{L^2}^2 = \frac{1}{2} \|\nabla u\|_{L^2}^2, \quad (46)$$

we obtain

$$4\nu^2 \|D(u)\|_{L^2}^2 = 2\nu^2 \|\nabla u\|_{L^2}^2. \quad (47)$$

On the other hand, from (41),

$$-4\nu \int_{\mathbb{T}^3} r : D(u) \, dx = 2\nu \frac{d}{dt} \|u\|_{L^2}^2. \quad (48)$$

Hence (45) becomes

$$\|w\|_{L^2}^2 = \|r\|_{L^2}^2 + 2\nu \frac{d}{dt} \|u\|_{L^2}^2 + 2\nu^2 \|\nabla u\|_{L^2}^2. \quad (49)$$

This identity allows one to derive from the combination of (41) and (42), an energy inequality that simultaneously contains the kinetic energy, the stress energy, and the dissipation associated with the defect variable.

3.6. Extended Energy and Triple Dissipation

We define the extended energy by

$$E_\varepsilon(t) = \frac{1}{2} \|u(t)\|_{L^2}^2 + \frac{\varepsilon}{2} \|r(t)\|_{L^2}^2. \quad (50)$$

From (41) and (42), we obtain

$$\frac{d}{dt} E_\varepsilon(t) = - \int_{\mathbb{T}^3} r : D(u) \, dx - \|r\|_{L^2}^2 - \kappa \|(-\Delta)^{\theta/2} r\|_{L^2}^2 + 2\nu \int_{\mathbb{T}^3} D(u) : r \, dx.$$

Hence

$$\frac{d}{dt} E_\varepsilon(t) = -\|r\|_{L^2}^2 - \kappa \|(-\Delta)^{\theta/2} r\|_{L^2}^2 + (2\nu - 1) \int D(u):r \, dx. \quad (51)$$

Reorganizing this expression using the defect variable, the essential dissipative structure takes the form

$$\frac{d}{dt} E_\varepsilon(t) + \nu \|\nabla u\|_{L^2}^2 + \frac{1}{\varepsilon} \|w\|_{L^2}^2 + \kappa \|(-\Delta)^{\theta/2} r\|_{L^2}^2 \leq 0. \quad (52)$$

This is the fundamental inequality for the relaxation system established in [37]. It contains the following three dissipative terms:

- the usual viscous dissipation associated with the velocity gradient,

$$\nu \|\nabla u\|_{L^2}^2, \quad (53)$$

- the relaxation dissipation associated with the defect stress,

$$\frac{1}{\varepsilon} \|w\|_{L^2}^2, \quad (54)$$

- the diffusive dissipation of the high-frequency components of the stress,

$$\kappa \|(-\Delta)^{\theta/2} r\|_{L^2}^2. \quad (55)$$

Thus, in addition to the ordinary viscous dissipation of the Navier–Stokes equations, the system possesses two further dissipative mechanisms acting directly on the nonequilibrium stress. We refer to this threefold structure as the triple dissipation structure.

Its significance is substantial. In the Navier–Stokes equations alone, the only mechanism suppressing high-frequency growth is the viscous term $\nu \Delta u$. In the relaxation system, by contrast, there are in addition

- a term that directly damps the high-frequency components of the stress itself,
- a term that damps the deviation from equilibrium as such.

When the suppression of High–High transfer is analyzed later, these additional dissipative mechanisms provide the dynamical background for why sustained amplification of high-wavenumber triadic interaction becomes less likely.

3.7. The Meaning of Defect Forcing

The structure obtained above admits an important interpretation when viewed again from the momentum equation. Recalling (38),

$$\partial_t u + P((u \cdot \nabla)u) = \nu \Delta u + P \nabla \cdot w. \quad (56)$$

This shows that the relaxation system is the Navier–Stokes system supplemented by the defect forcing $P \nabla \cdot w$. At the same time, however, (52) shows that w is strongly damped through the term $\frac{1}{\varepsilon} \|w\|_{L^2}^2$. The defect forcing is therefore not an external forcing, but one that is generated internally and simultaneously subjected to strong internal relaxation.

This point was already essential in our previous works [36,37]. In general, when one enlarges the Navier–Stokes system by introducing auxiliary variables, the theoretical value is limited if one merely adds arbitrary new degrees of freedom. Here, however, the auxiliary variable r

- is introduced as a dynamical reinterpretation of the constitutive law,
- gives rise to the defect w , which represents an internal forcing,
- and is itself dissipative suppressed.

In this sense, the system forms a theoretically closed structure. The same duality—an internal forcing controlled by internal dissipation—reappears later in the discussion of the absorption of High–High triadic transfer. Potentially dangerous nonlinear transfer may indeed arise, but it is not

to be regarded as a freely growing independent degree of freedom; rather, it should be viewed as something to be absorbed under suitable relaxation and dissipation mechanisms.

3.8. Connection to Weak and Strong Solutions

Because the relaxation system (32)–(33) possesses a stronger dissipative structure than the classical Navier–Stokes equations, it enjoys advantages in existence theory. In [37], by combining

- an extended energy inequality,
- compactness methods,
- higher-order Sobolev estimates,

we established the existence of Leray–Hopf-type weak solutions and, for small initial data, global strong solutions.

What matters here is that the purpose of the present chapter is not to reprove those existing results in full detail. Rather, its role is to make explicit that the relaxation system

- is a natural extension of the Navier–Stokes equations,
- possesses additional dissipation,
- suppresses high-wavenumber structure through the decay of the defect stress.

In the classical Navier–Stokes problem, the possibility of finite-time breakdown involves a direct competition between potentially dangerous high-frequency amplification and the viscous dissipation that attempts to suppress it. In the relaxation system, additional stress-level damping is superposed on this competition. This provides a dynamical intuition for why, once the high-frequency transfer is later observed shellwise through triadic family decomposition, High–High channels should be more readily absorbed by dissipation.

3.9. Implications of the Relaxation System in Fourier Space

We conclude this chapter by briefly indicating the implications of the relaxation system for high-wavenumber analysis in Fourier space. Taking the Fourier transform of the stress equation (33), one obtains

$$\varepsilon \partial_t \hat{r}(k, t) + \hat{r}(k, t) = 2\nu \widehat{D(u)}(k, t) - \kappa |k|^{2\theta} \hat{r}(k, t). \quad (57)$$

Therefore, for high wavenumbers $|k| \gg 1$, the term

$$\kappa |k|^{2\theta} \hat{r}(k, t), \quad (58)$$

acts strongly, and the high-frequency components of the stress decay rapidly. This constitutes a high-frequency damping mechanism distinct from the viscous damping $\nu |k|^2 \hat{u}(k)$ acting directly on the velocity field. In the language of triadic interaction, it creates room for same-scale transfer among high-wavenumber modes to be absorbed on the stress side.

At this stage, however, shellwise triadic transfer has not yet been defined rigorously. Accordingly, we do not claim the High–High Absorption Condition itself here. Equations (57)–(58) nevertheless show, at minimum, that the relaxation system possesses a stronger damping of high-wavenumber structure than the classical Navier–Stokes equations. This fact will later be reformulated in shellwise language in Chapter 10 and beyond and will then be transmitted to the Navier–Stokes limit through relative entropy.

3.10. Summary of This Chapter

In this chapter, we introduced a stress relaxation system by reinterpreting the constitutive law of the Navier–Stokes equations as a dynamical relaxation process. The main conclusions are as follows.

First, by introducing the viscous stress as an independent variable r , the momentum equation can be rewritten as an extended system containing a defect forcing.

Second, by defining the deviation from Newtonian equilibrium, $w = r - 2\nu D(u)$, as the defect stress, one obtains an internal variable that quantifies the extent to which the extended system departs from the Navier–Stokes equations.

Third, the extended energy inequality yields a triple dissipation structure consisting of

- the usual viscous dissipation,
- relaxation dissipation of the defect stress,
- high-frequency diffusive dissipation of stress.

Fourth, this additional dissipative structure promotes the decay of high-wavenumber Fourier modes and thereby provides a dynamical basis for the absorption mechanism of High–High triadic transfer developed later.

This chapter therefore serves as a central bridge linking

- the Fourier and triadic interaction analysis beginning in Chapter 4,
- the conditional regularity theorems based on High–High Absorption in Chapters 7 and 8,
- and the later naturalization of the absorption condition through relaxation damping and relative entropy in Chapter 10 and beyond.

In the next chapter, we finally pass to the Fourier representation of the Navier–Stokes equations and introduce the Leray projection, triadic interaction, helical decomposition, and dyadic shell decomposition. These structures make it possible to isolate the dangerous High–High channels within the nonlinear interaction mechanism.

4. The Structure of Nonlinear Interaction in the Navier–Stokes Equations in Fourier Space

4.1. Purpose and Positioning of This Chapter

Chapter 2 explained how continuum fluid equations emerge from interacting systems governed by conservation laws and the second law of thermodynamics, while Chapter 3 introduced, as a continuum continuation of that framework, the stress relaxation system and revealed the presence of an additional dissipative structure. To understand the regularity problem for the three-dimensional Navier–Stokes equations, however—and in particular the potentially dangerous transfer of energy toward high wavenumbers—it is indispensable to reorganize the nonlinear term in Fourier space.

Indeed, the nonlinear term $(u \cdot \nabla)u$ in the Navier–Stokes equations appears in physical space as a local advection term, but in Fourier space it is expressed as a superposition of interactions among triples of modes whose wave vectors sum to zero, namely triadic interactions [16,21,22,42]. Since the energy cascade in turbulence is intrinsically a multiscale energy-transfer phenomenon, the triadic representation is not merely a formal change of variables. It is the most natural framework in which to identify which frequency interactions are capable of driving amplification at high wavenumbers.

The objectives of this chapter are fivefold.

First, to formulate the Navier–Stokes equations in Fourier space and make explicit that the pressure term is eliminated by the Leray projection.

Second, to formalize the fact that the nonlinear term can be written as a sum of triadic interactions.

Third, to show that energy is conserved within each triad, namely, to establish triadic energy conservation.

Fourth, to introduce the helical decomposition, thereby preparing the geometric and sign-sensitive description of triadic interactions.

Fifth, to introduce the dyadic shell decomposition, thereby laying the foundation for the shell energies and shellwise transfers used later.

The results obtained in this chapter belong, in themselves, to the standard organization of Fourier analysis and turbulence theory [14]–[17,40,42,43,45,46]. In the present paper, however, they

are far more than preliminaries. To introduce rigorously, in later chapters, the classification of interactions into

- Low–Low,
- Low–High,
- High–High,

and then to define the absorption condition for the High–High channel, it is first necessary to close the logical structure of the triadic representation established here. In what follows, we take the incompressible Navier–Stokes equations on the periodic domain $\mathbb{T}^3$ as the starting point and proceed from the Fourier expansion to the formulation of triadic interaction.

4.2. Governing Equation and the Leray Projection

On the periodic domain $\Omega = \mathbb{T}^3$, the three-dimensional incompressible Navier–Stokes equations are

$$\partial_t u + (u \cdot \nabla)u + \nabla p = \nu \Delta u, \nabla \cdot u = 0. \quad (59)$$

Here,

- $u(x, t) \in \mathbb{R}^3$: velocity field,
- $p(x, t) \in \mathbb{R}$: pressure,
- $\nu > 0$: kinematic viscosity.

To preserve the incompressibility condition $\nabla \cdot u = 0$, we introduce the Leray projection

$$P: L^2(\mathbb{T}^3; \mathbb{R}^3) \rightarrow L^2_\sigma(\mathbb{T}^3), \quad (60)$$

where L^2_σ denotes the subspace of divergence-free vector fields. Applying the Leray projection to (59), the pressure gradient is eliminated through

$$P \nabla p = 0, \quad (61)$$

so that the equation becomes

$$\partial_t u + P((u \cdot \nabla)u) = \nu \Delta u. \quad (62)$$

This form is decisive for the later treatment of the pressure in Fourier space. The nonlinear term in the Navier–Stokes equations must therefore be understood not simply as $(u \cdot \nabla)u$, but as its divergence-free component after projection,

$$P((u \cdot \nabla)u). \quad (63)$$

4.3. Fourier Expansion and the Incompressibility Constraint

We expand the velocity field on $\mathbb{T}^3$ in Fourier series. With the convention

$$\hat{u}(k, t) = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} u(x, t) e^{-ik \cdot x} dx, k \in \mathbb{Z}^3, \quad (64)$$

the inverse transform is

$$u(x, t) = \sum_{k \in \mathbb{Z}^3} \hat{u}(k, t) e^{ik \cdot x}. \quad (65)$$

The incompressibility condition $\nabla \cdot u = 0$ becomes, in Fourier space,

$$k \cdot \hat{u}(k, t) = 0 (k \in \mathbb{Z}^3). \quad (66)$$

Moreover, since $u(x, t)$ is real-valued, its Fourier coefficients satisfy the conjugate symmetry

$$\hat{u}(-k, t) = \overline{\hat{u}(k, t)}. \quad (67)$$

The Leray projection is represented, for each nonzero wavenumber, by

$$P(k) = I - \frac{k \otimes k}{|k|^2}, (k \neq 0). \quad (68)$$

Hence

$$P(k)\hat{u}(k) = \hat{u}(k). \quad (69)$$

That is, each Fourier mode lies in the two-dimensional plane orthogonal to k .

4.4. Fourier Representation of the Nonlinear Term

We next derive the Fourier representation of the nonlinear term $(u \cdot \nabla)u$. Using the expansion (65), we write

$$(u \cdot \nabla)u = \left(\sum_p \hat{u}(p) e^{ip \cdot x} \right) \cdot \nabla \left(\sum_q \hat{u}(q) e^{iq \cdot x} \right).$$

Since

$$\nabla(\hat{u}(q) e^{iq \cdot x}) = iq \hat{u}(q) e^{iq \cdot x}, \quad (70)$$

it follows that

$$(u \cdot \nabla)u = \sum_{p,q} (\hat{u}(p) \cdot iq) \hat{u}(q) e^{i(p+q) \cdot x}. \quad (71)$$

Therefore, the Fourier coefficient at wavenumber k is

$$(\widehat{u \cdot \nabla} u)(k) = i \sum_{p+q=k} (q \cdot \hat{u}(p)) \hat{u}(q). \quad (72)$$

It is also common to write this, using $k = p + q$, in the equivalent form

$$(\widehat{u \cdot \nabla} u)(k) = i \sum_{p+q=k} (k \cdot \hat{u}(p)) \hat{u}(q), \quad (73)$$

because the difference between $q \cdot \hat{u}(p)$ and $k \cdot \hat{u}(p)$ vanishes by the incompressibility condition $p \cdot \hat{u}(p) = 0$. Indeed,

$$k \cdot \hat{u}(p) = (p + q) \cdot \hat{u}(p) = q \cdot \hat{u}(p). \quad (74)$$

Applying the Leray projection, the Fourier representation of the nonlinear term becomes

$$P(\widehat{u \cdot \nabla} u)(k) = i P(k) \sum_{p+q=k} (k \cdot \hat{u}(p)) \hat{u}(q). \quad (75)$$

Accordingly, the Navier–Stokes equations (62) take the form

$$\partial_t \hat{u}(k) = -\nu |k|^2 \hat{u}(k) - i P(k) \sum_{p+q=k} (k \cdot \hat{u}(p)) \hat{u}(q). \quad (76)$$

The significance of this formula is immediate. The evolution of the mode at wavenumber k is never determined in isolation; it is always governed by interaction with pairs of modes p, q satisfying

$$p + q = k. \quad (77)$$

The triple (k, p, q) thus forms a triad.

4.5. Definition of Triadic Interaction

Based on (76), we define the three-wave interaction explicitly. Whenever a triple of wave vectors (k, p, q) satisfies

$$k = p + q, \quad (78)$$

we call (k, p, q) a triad. The corresponding triadic contribution is defined by

$$N(k | p, q) := -i P(k) ((k \cdot \hat{u}(p)) \hat{u}(q)). \quad (79)$$

Then the Fourier equation may be written as

$$\partial_t \hat{u}(k) = -\nu |k|^2 \hat{u}(k) + \sum_{p+q=k} N(k|p, q). \quad (80)$$

This representation shows that the nonlinear term in the Navier–Stokes equations is naturally interpreted as a sum of triadic interactions. In turbulence theory, these triadic interactions constitute the fundamental units of the energy cascade, and the transfer of energy from one scale to another depends on the geometry, amplitudes, and phases of the triads [14]–[17,21,22,42].

What must be emphasized is that a triad is not merely a triple of wave vectors. It is the minimal closed unit of energy transfer. It is precisely this triadic closedness that later makes it possible, after introducing shell decomposition, to isolate the dangerous High–High transfer channels.

4.6. Definition of Triadic Energy Transfer

We define the energy of each Fourier mode by

$$E(k) := \frac{1}{2} |\hat{u}(k)|^2. \quad (81)$$

Multiplying (80) by $\hat{u}(\bar{k})$ and taking the real part, we obtain

$$\frac{d}{dt} E(k) = -\nu |k|^2 |\hat{u}(k)|^2 + \operatorname{Re} \left(\sum_{p+q=k} N(k|p, q) \cdot \hat{u}(\bar{k}) \right). \quad (82)$$

We then define the triadic energy transfer

$$T(k|p, q) := \operatorname{Re} (N(k|p, q) \cdot \hat{u}(\bar{k})). \quad (83)$$

Hence

$$\frac{d}{dt} E(k) = -\nu |k|^2 |\hat{u}(k)|^2 + \sum_{p+q=k} T(k|p, q). \quad (84)$$

The quantity $T(k|p, q)$ represents the rate at which the triad (k, p, q) supplies energy to the mode k . Its sign has a transparent meaning:

- $T(k|p, q) > 0$: the triad feeds energy into mode k ,
- $T(k|p, q) < 0$: the triad extracts energy from mode k .

4.7. Energy Conservation Within Each Triad

A fundamental property of the nonlinear structure of the Navier–Stokes equations is that energy is conserved within each triad. Specifically,

$$T(k|p, q) + T(p|q, k) + T(q|k, p) = 0. \quad (85)$$

This identity is the triadic version of the fact that the nonlinear term neither creates nor destroys total energy. Physically, energy is merely redistributed among the three modes within a triad; there is no net generation or annihilation.

To verify this, one may localize the total energy conservation of the inviscid Euler equation to each individual Fourier triad. Restricting attention to the nonlinear term alone, one has

$$\sum_k \sum_{p+q=k} T(k|p, q) = 0, \quad (86)$$

and regrouping the contributions triad by triad yields (85) [14,42,43].

This triadic conservation law is foundational for the entire paper. Dangerous high-frequency growth does not mean that energy is created from nothing; rather, it means that, within certain triadic configurations, energy is redistributed preferentially from lower to higher wavenumbers. To ask

which triads are dangerous is therefore equivalent to asking which triads can most efficiently induce a high-frequency bias in the redistribution of energy.

4.8. Helical Decomposition

Each Fourier mode lies in the two-dimensional plane orthogonal to its wave vector k . One may therefore introduce, on this plane, an eigenbasis associated with helicity [42,46]. For each $k \neq 0$, let the complex vectors $h^\pm(k)$ be defined as eigenvectors satisfying

$$ik \times h^\pm(k) = \pm |k| h^\pm(k). \quad (87)$$

We further impose the normalization

$$\|h^\pm(k)\| = 1. \quad (88)$$

Then any divergence-free Fourier mode can be expanded as

$$\hat{u}(k) = u^+(k)h^+(k) + u^-(k)h^-(k), \quad (89)$$

where $u^\pm(k) \in \mathbb{C}$ are the helical amplitudes.

The advantage of the helical decomposition is that it allows a more refined description of the sign and geometric structure of triadic interactions. For a triad (k, p, q) , the interaction coefficient is known to depend in strength and sign tendency on the helicity-sign combination

$$(s_k, s_p, s_q) \in \{+, -\}^3. \quad (90)$$

In the present paper, the helical decomposition is not used directly in the proof of the main theorem. It is introduced here to fix the underlying basis structure for later discussions of coherence and geometric cancellation within the High-High families.

Moreover, the Leray projection satisfies

$$P(k)h^\pm(k) = h^\pm(k), \quad (91)$$

so, the helical decomposition is fully compatible with the divergence-free constraint.

4.9. Dyadic Shell Decomposition

To analyze energy transfer toward high wavenumbers, we partition Fourier space into dyadic shells. For each integer $j \in \mathbb{Z}$, define

$$\Lambda_j := \{k \in \mathbb{Z}^3 : 2^j \leq |k| < 2^{j+1}\}, \quad (92)$$

as the j -th shell. The corresponding shell component is

$$u_j(x, t) := \sum_{k \in \Lambda_j} \hat{u}(k, t) e^{ik \cdot x}, \quad (93)$$

so that

$$u(x, t) = \sum_j u_j(x, t). \quad (94)$$

We define the shell energy by

$$E_j(t) := \frac{1}{2} \|u_j(t)\|_{L^2}^2. \quad (95)$$

By Parseval's identity,

$$E_j(t) = \frac{1}{2} \sum_{k \in \Lambda_j} |\hat{u}(k, t)|^2. \quad (96)$$

The corresponding shellwise viscous dissipation is

$$D_j(t) := \nu \sum_{k \in \Lambda_j} |k|^2 |\hat{u}(k, t)|^2. \quad (97)$$

The dyadic shell decomposition allows the mode-by-mode triadic transfer to be gathered into scale bands of comparable wavenumber. This is the basis upon which the notions of triadic family, shellwise transfer, and High–High absorption is built later. What matters here is that the basic strategy of this paper—namely, to observe the dangerous mechanism not strictly mode by mode, but through shellwise coarse-grained quantities—already begins at this stage.

4.10. Basic Form of the Shell Energy Equation

Summing (84) over the shell Λ_j , we obtain

$$\frac{d}{dt} E_j(t) = -D_j(t) + \sum_{k \in \Lambda_j} \sum_{p+q=k} T(k | p, q). \quad (98)$$

If we define the shellwise nonlinear transfer by

$$T_j(t) := \sum_{k \in \Lambda_j} \sum_{p+q=k} T(k | p, q), \quad (99)$$

then

$$\frac{d}{dt} E_j(t) = -D_j(t) + T_j(t). \quad (100)$$

This equation shows that the change in energy of each shell is governed by the competition between

- the viscous dissipation D_j ,
- the nonlinear transfer T_j .

If blow-up were to occur in a high-wavenumber shell, it would mean that T_j persistently exceeds D_j . Accordingly, when the High–High Absorption Condition is defined in later chapters, the central issue will be precisely the comparison

$$T_j \text{ vs. } D_j. \quad (101)$$

4.11. Summary of This Chapter

In this chapter, we organized the nonlinear structure of the three-dimensional incompressible Navier–Stokes equations in Fourier space and established the basis for the triadic regularity theory developed later. The principal conclusions are as follows.

First, by means of the Leray projection, the Navier–Stokes equations can be written as a divergence-free equation with the pressure removed.

Second, the nonlinear term can be formulated in Fourier space as a sum of triadic interactions.

Third, the triadic energy transfer was defined, and it was confirmed that energy is conserved within each triad. Dangerous high-frequency amplification must therefore be understood not as the creation of energy, but as a biased redistribution within a triad toward the high-wavenumber side.

Fourth, the helical decomposition was introduced, thereby fixing the basis needed for later discussions of triadic geometry and cancellation.

Fifth, the dyadic shell decomposition was introduced, and the basic shell energy equation

$$\frac{d}{dt} E_j = -D_j + T_j,$$

was obtained. Consequently, the regularity problem for the Navier–Stokes equations is translated, in the subsequent chapters, into the problem of

- classifying the shellwise transfer T_j ,

- according to the geometry of the interactions,
- and determining which channel can sustain high-wavenumber blow-up.

In the next chapter, this shellwise transfer will be classified, based on triadic geometry, into

- Low–Low,
- Low–High,
- High–High,

and the potentially dangerous interaction classes will be analyzed. Only there does the rigorous preparation become complete for the central claim of this paper: dangerous nonlinear amplification is localized in the High–High channel.

5. Scale Classification of Triadic Interaction and Localization of the Dangerous Mechanism

5.1. Purpose and Positioning of This Chapter

In Chapter 4, we represented the nonlinear term of the three-dimensional incompressible Navier–Stokes equations in Fourier space and showed that it can be written as a sum of triadic interactions. We derived the shellwise energy equation

$$\frac{d}{dt}E_j(t) = -D_j(t) + T_j(t), \quad (102)$$

thereby confirming that the energy variation in each dyadic shell is determined by the competition between the viscous dissipation D_j and the nonlinear energy transfer T_j .

However, the nonlinear transfer T_j appearing in (102) is still an aggregate quantity that combines all triadic interactions, and it therefore contains several interaction channels of fundamentally different characters. To discuss the danger of high-frequency blow-up, one must further decompose T_j according to scale geometry and determine which interaction class can genuinely be dangerous.

The aim of this chapter is to classify triadic interactions based on the relative scales of the participating wave numbers, thereby preparing the framework in which the dangerous mechanism is localized before we proceed to the main theorem. More specifically, the purposes of this chapter are:

- to classify triadic interactions into Low–Low, Low–High, and High–High types,
- to organize the dynamical meaning of each interaction class,
- to identify where the candidate mechanism for high-frequency self-amplification may arise,
- and, consequently, to show that the essential channel that must later be controlled is the High–High family.

What is important here is that we do not seek to make an excessive claim such as “Low–Low and Low–High make no contribution whatsoever.” Rather, the role of this chapter is to separate the interaction classes accurately and to prepare the later logical structure in which

- Low–Low and Low–High is treated as controllable perturbative terms through weighted paraproduct estimates and Sobolev closure,
- whereas the genuinely unresolved dangerous channel remains on the High–High side.

Indeed, in the final conditional regularity theorem of this paper, the contributions of Low–Low and Low–High are handled through estimates in the main text and appendices, and the only external assumption that remains is the High–High Absorption Condition. In this sense, the present chapter serves as an essential bridge establishing the logical prerequisites for the main theorem.

5.2. Description of Triads by Dyadic Shell Indices

Following the dyadic shell decomposition introduced in Chapter 4,

$$\Lambda_j = \{k \in \mathbb{Z}^3: 2^j \leq |k| < 2^{j+1}\}, \quad (103)$$

we write, for any triad (k, p, q) ,

$$k \in \Lambda_j, p \in \Lambda_l, q \in \Lambda_m. \quad (104)$$

Then the triadic transfer may be described not only in terms of the vector triple (k, p, q) , but also, after coarse-graining, in terms of the shell-index triple

$$(j, l, m). \quad (105)$$

This constitutes the starting point for reorganizing the shellwise transfer later as a family-level observable. Because of the triad condition

$$k = p + q, \quad (106)$$

the magnitudes of the three wave numbers are not independent. In particular, the triangle inequality gives

$$|k| \leq |p| + |q|, \quad (107)$$

and hence the shell indices are subject to intrinsic geometric constraints. For example, it is impossible for $|p|$ and $|q|$ both to be very small while $|k|$ alone is extremely large. The classification of interaction classes is therefore not arbitrary labeling, but a natural classification dictated by triadic geometry.

To make this more transparent, we fix from now on a high-frequency shell j and consider those triads contributing to it. That is, we decompose

$$T_j = \sum_{k \in \Lambda_j} \sum_{p+q=k} T(k | p, q), \quad (108)$$

according to the relative positions of l and m .

5.3. Basic Classification of Interaction Classes

For a high-frequency shell Λ_j , we classify the contributing triadic interactions according to the relative magnitudes of l , m , and j . The basic classes are the following three.

(i) Low–Low interaction

When both input wave numbers are sufficiently lower than j , namely,

$$l \ll j, m \ll j, \quad (109)$$

we call the corresponding interaction Low–Low.

However, once the triad condition $k = p + q$ is considered, geometry strongly restricts the possibility that two low-frequency modes suddenly generate a mode in the high-frequency shell Λ_j . In practice, the contribution of this regime is much closer to low-frequency transport or rearrangement than to essential self-amplification in the high-frequency shell.

(ii) Low–High interaction

When one wave number is low and the other is high, namely,

$$l \ll j, m \sim j, \text{ or } m \ll j, l \sim j, \quad (110)$$

we call the corresponding interaction Low–High.

This class corresponds to the situation in which a high-frequency mode is transported or modulated by a smooth low-frequency field. In the language of Bony's paraproduct decomposition, the low-frequency factor acts as a smooth coefficient [45]. Accordingly, this channel should later be controlled through paraproduct estimates in weighted Sobolev energies and is fundamentally distinct from strong resonant self-amplification at the same scale.

(iii) High–High interaction

When both input wave numbers are comparable to j , namely,

$$l \sim j, m \sim j, \quad (111)$$

we call the corresponding interaction High–High.

In this case, all wave numbers forming the triad are concentrated on essentially the same dyadic scale, and same-scale interactions give rise to local energy redistribution. As will become clear later, this class is capable of actively redistributing energy within the high-frequency shell itself and therefore emerges as the only unresolved channel that can sustain dangerous self-amplification on the high-frequency side.

5.4. Decomposition of the Shellwise Transfer

According to the above classification, we decompose the shellwise nonlinear transfer T_j as

$$T_j = T_j^{LL} + T_j^{LH} + T_j^{HH}. \quad (112)$$

Here,

- T_j^{LL} : contribution from the Low–Low class,
- T_j^{LH} : contribution from the Low–High class,
- T_j^{HH} : contribution from the High–High class.

More concretely, fix a proximity width $c_* \geq 1$. Then one may define:

- as LL, those interactions for which both $|l - j| > c_*$ and $|m - j| > c_*$, with both indices lying on the low-frequency side,
- as LH, those for which only one index lies near j and the other is sufficiently lower,
- as HH, those satisfying $|l - j| \leq c_*, |m - j| \leq c_*$.

A fully rigorous definition of this notation will be given again later in the triadic family decomposition. At the present stage, however, we adopt (112) as the conceptual classification.

The shell energy equation then becomes

$$\frac{d}{dt} E_j = -D_j + T_j^{LL} + T_j^{LH} + T_j^{HH}. \quad (113)$$

The problem is now clear: one must determine which channel can persistently exceed D_j .

5.5. Character of the Low–Low Interaction

The Low–Low interaction represents the contribution of two low-frequency components to a high-frequency shell. Yet triadic geometry makes this class unlikely to constitute an essential self-amplification channel for high frequencies.

The reason is geometric. If $k \in \Lambda_j$ is a high-frequency mode while p and q are both sufficiently low-frequency, then

$$|k| = |p + q| \leq |p| + |q|, \quad (114)$$

so $|k|$ cannot become vastly larger than both $|p|$ and $|q|$. Hence, in the genuine regime $l, m \ll j$, the Low–Low interaction cannot serve as a channel that directly injects energy into the high-frequency shell.

Analytically as well, because the Low–Low interaction is composed entirely of low-frequency components, it appears as a lower-order term in weighted energy estimates for high-frequency shells. In particular, after multiplication by Sobolev weights 2^{2sj} and summation over j , the Low–Low contribution does not create direct same-scale resonance in the high-frequency sector, but is instead controlled later through Hölder, Bernstein, and Young inequalities together with Littlewood–Paley theory [40,45,46].

Thus, from the standpoint of the present paper, the Low–Low interaction is indeed one possible source term in the high-frequency balance, but it is to be treated as a channel absorbable by weighted estimates, not as the essential unresolved mechanism.

5.6. Character of the Low–High Interaction

Among the three classes, the Low–High interaction requires the greatest care. Since one of the participating modes is high-frequency, it may at first sight appear capable of contributing directly to high-frequency amplification. Yet a closer inspection shows that its principal role is not same-scale self-amplification but rather transport and modulation.

To be specific, consider the case $l \ll j$ and $m \sim j$. The low-frequency component u_l then behaves as a spatially smooth field and acts as a coefficient transporting the high-frequency component u_m . In the language of Bony's paraproduct decomposition,

$$u \cdot \nabla u = T_u(\nabla u) + T_{\nabla u}u + R(u, \nabla u), \quad (115)$$

and the Low–High interaction corresponds primarily to the paraproduct term

$$T_{u_{\text{low}}}(\nabla u_{\text{high}}). \quad (116)$$

The low-frequency factor therefore acts as a smooth coefficient, and the interaction is fundamentally different from a resonant same-scale interaction.

As will be shown later, the Low–High interaction satisfies a weighted paraproduct estimate of the form

$$|T_j^{LH}| \leq C a_j(t) E_j + \text{lower-order remainder}, \quad (117)$$

where $a_j(t)$ is controlled by an L^∞ -type norm of the low-frequency component, or by a suitable Sobolev norm.

In this sense, the Low–High interaction does influence the high-frequency shell, but not as an independent blow-up mechanism. Rather, it is a perturbative channel to be handled within Sobolev closure. This is precisely why it does not remain as an external condition in the final theorem.

5.7. The Singular Role of the High–High Interaction

The High–High interaction corresponds to the regime

$$l \sim j, m \sim j, \quad (118)$$

that is, the wave numbers forming the triad all lie on essentially the same dyadic scale. The singular role of this class consists in the following three points.

(i) It is a same-scale interaction

Unlike the Low–High case, where one factor acts as a smooth coefficient, all factors here are of the same high order. The interaction is therefore genuinely nonlinear, and the essential strength of the nonlinearity appears in its most direct form.

(ii) It redistributes energy within the high-frequency shell itself

Energy is conserved inside each triad:

$$T(k | p, q) + T(p | q, k) + T(q | k, p) = 0. \quad (119)$$

In a High–High triad, however, this redistribution takes place entirely near the high-frequency shell. It is therefore not a matter of low-frequency sweeping or transport, but of local concentration and redistribution within the high-frequency band itself.

(iii) It requires same-scale coherence

As will be explained in detail later, a large High–High transfer cannot be produced merely by large amplitudes. It also requires a certain coherence involving triadic phases and helical geometry. In other words, the High–High interaction is not “always dangerous,” but rather “the only

unresolved channel that can become dangerous.” For this reason, the High–High interaction must ultimately be quantified only after being linked to the notions introduced later—triadic family, coherence, residence time, and shell defect. At the present stage, however, it is already essential to make explicit the following structural point: the candidate for genuinely unresolved nonlinear self-amplification on the high-frequency side is concentrated in the High–High channel.

5.8. Meaning of the “Localization of the Dangerous Mechanism”

It is important to state carefully what the conclusion of this chapter does and does not mean. The argument developed here does not imply that “Low–Low and Low–High make no contribution at all.”

Indeed, Low–High interactions do appear in the energy balance of high-frequency shells, and Low–Low interactions may also exert indirect influence through low-frequency rearrangement. What this chapter establishes is a more precise and more limited statement, but one that is essential to the main logical line of the paper:

- Low–Low and Low–High are channels to be handled by later analytical estimates,
- the unresolved channel responsible for same-scale high-frequency self-amplification is High–High,
- and therefore, the essential quantity to be compared directly with shellwise dissipation is T_j^{HH} .

It is in this sense that we speak of the localization of the dangerous mechanism. The claim is not that blow-up has already been completely classified, but that the channel which must remain at the end as an external condition in the main theorem is localized to the High–High class.

This is crucial for the logic of the entire paper. If Low–Low or Low–High was also to remain as independent unresolved conditions, then the final continuation criterion could not be stated in terms of “High–High absorption only.” To avoid precisely this situation, we treat Low–Low and Low–High through later weighted estimates, so that the final theorem retains only the High–High Absorption Condition as an external assumption. The logical preparation for that structure is the purpose of the present chapter.

5.9. Preparation for the Transition to Triadic Families

In Chapters 4 and 5, interactions have been viewed primarily at the level of individual triads. For the actual quantitative closure of shellwise estimates and the measurement of coherence, however, it is not sufficient to follow triads one by one. Even for a fixed shell j , the number of triads participating in it is enormous, and among them many share closely related geometric configurations.

For this reason, the next chapter reorganizes individual triads into coarser units called triadic families. More specifically, we will introduce:

- sets of triads belonging to the same dyadic-scale neighborhood,
- family transfer,
- family coherence,
- coherent time sets,
- family residence time,
- and shell defect observables,

so that dangerous High–High interactions can be described in terms of observable shell-level quantities.

The classification obtained in this chapter,

$$T_j = T_j^{LL} + T_j^{LH} + T_j^{HH}, \quad (120)$$

will serve directly as the point of departure for that family decomposition.

5.10. Summary of This Chapter

In this chapter, we classified the shellwise transfer of the Navier–Stokes nonlinearity into Low–Low, Low–High, and High–High components on the basis of triadic geometry. The principal conclusions are the following.

First, the shellwise nonlinear transfer T_j can be decomposed according to interaction geometry as $T_j = T_j^{LL} + T_j^{LH} + T_j^{HH}$.

Second, the Low–Low interaction is not an essential channel for high-frequency self-amplification, but is treated geometrically and analytically as a lower-order contribution.

Third, the Low–High interaction does affect the high-frequency shell, but its essential character is transport and modulation; it is therefore a perturbative channel to be controlled later by weighted paraproduct estimates.

Fourth, the unresolved dangerous channel, as a genuinely nonlinear same-scale interaction, is concentrated in the High–High interaction.

Fifth, in this sense, the continuation problem studied in this paper should ultimately be reduced to the control of the High–High channel, and for this purpose the next chapter introduces the coarse-grained notion of a triadic family.

Accordingly, this chapter is the one that localizes the dangerous mechanism and connects, within the overall logic of the paper,

- the triadic representation of Chapter 4,
- the triadic family decomposition of Chapter 6,
- the High–High Absorption Condition of Chapter 7,
- and the conditional regularity theorem of Chapter 8.

In the next chapter, the High–High interaction will be reorganized into triadic families, and by defining family transfer, family coherence, residence time, and shell defect, we will introduce the observables needed for a quantitative formulation of the High–High Absorption Condition.

6. Triadic Family Decomposition and Shell-Level Observables

6.1. Purpose and Positioning of This Chapter

Chapter 4 showed that the nonlinear term in the Navier–Stokes equations can be represented in Fourier space as a sum of triadic interactions, and Chapter 5 classified those triadic interactions into Low–Low, Low–High, and High–High according to the relative scales of the participating wave numbers. As a result, we arrived at the structural conclusion that the only unresolved dangerous channel remaining on the high-frequency side is the High–High interaction.

However, tracking individual triads one by one is insufficient for the quantitative estimates required later. There are two reasons for this. First, even for a fixed dyadic shell, the number of triads that may contribute is extremely large, so it is impractical to treat the signed contribution of each individual triad directly. Second, what is needed for the main theorem is not the danger posed by any single triad, but the effective danger of the entire collection of High–High interactions associated with a given shell. What is therefore essential is not a triad-by-triad description, but a coarse-grained description at the level of families.

This is precisely the aim of the present chapter. We reorganize the High–High interactions localized in Chapter 5 into collections of individual triads and translate them into shell-level quantities that can later be directly observed and estimated. To this end, we introduce, in sequence,

- neighboring shell sets,
- triadic families,
- strict High–High families,
- family transfer,
- family coherence,
- coherent time sets,

- family residence time,
- and shell defect observables.

The role of this chapter is not to prove the main theorem itself. Rather, it converts the qualitative claim established in Chapter 5—the localization of the dangerous mechanism—into quantitative shellwise observables that can be used in the subsequent chapters. In other words, this chapter serves as the bridge from

- detailed description of triadic interactions

to

- the observables required for shellwise regularity theory.

Of particular importance is the fact that the family coherence and family residence time introduced here will later provide sufficient conditions for the High–High Absorption Condition in Chapter 7. This chapter is therefore not a merely technical reorganization but occupies a central position in the logic of the paper as a whole.

6.2. Basic Setting and Shell-Level Notation

Throughout this chapter, the domain is the three-dimensional torus

$$\Omega = \mathbb{T}^3. \quad (121)$$

The velocity field $u(x, t)$ is decomposed into dyadic shells, as introduced in Chapter 4:

$$u(x, t) = \sum_{j \in \mathbb{Z}} u_j(x, t). \quad (122)$$

Here,

$$u_j(x, t) = \sum_{k \in \Lambda_j} \hat{u}(k, t) e^{ik \cdot x}, \Lambda_j := \{k \in \mathbb{Z}^3 : 2^j \leq |k| < 2^{j+1}\}. \quad (123)$$

The corresponding shell energy is defined by

$$E_j(t) := \frac{1}{2} \|u_j(t)\|_{L^2}^2, \quad (124)$$

and shellwise viscous dissipation by

$$D_j(t) := \nu \sum_{k \in \Lambda_j} |k|^2 |\hat{u}(k, t)|^2. \quad (125)$$

Using the triadic transfer introduced in Chapter 4,

$$T(k | p, q), \quad (126)$$

we write the shellwise nonlinear transfer as

$$T_j(t) := \sum_{k \in \Lambda_j} \sum_{p+q=k} T(k | p, q). \quad (127)$$

By the results of Chapter 5, T_j decomposes as

$$T_j = T_j^{LL} + T_j^{LH} + T_j^{HH}. \quad (128)$$

Here, T_j^{HH} is the unresolved essential dangerous channel. Yet T_j^{HH} is still only an aggregate over many triads, and its internal structure must therefore be further organized. For this reason, the present chapter reorganizes T_j^{HH} as a sum over triadic families.

6.3. Neighboring Shell Sets

When working with dyadic shell decompositions, it is more natural to group together finitely many neighboring shells rather than relying on a single shell, to incorporate triads lying near shell boundaries in a stable manner. Accordingly, for a fixed proximity width $c_* \in \mathbb{N}$, we define the neighboring shell set of an integer j by

$$N_*(j) := \{l \in \mathbb{Z} : |l - j| \leq c_*\}. \quad (129)$$

This set provides a coarse-grained representation designed to capture triadic interactions that arise near the boundaries of dyadic shells. In particular, the High–High family introduced later will collect triads satisfying

$$l, m \in N_*(j). \quad (130)$$

Here c_* is a fixed finite constant, used consistently throughout the paper. Its precise numerical value is not essential. What matters is that the neighboring-shell construction has finite overlaps. Indeed, the use of $N_*(j)$ guarantees that, when family-level estimates are later converted back into shell-level estimates, a given family can affect only finitely many shells.

6.4. Definition of Triadic Families

Instead of tracing individual triads (k, p, q) directly, we introduce families consisting of triads belonging to the same scale band. We first define the local family.

For each integer j , the triadic family centered on the j -th shell is defined by

$$F_j := \{(k, p, q) \in (\mathbb{Z}^3)^3 : k + p + q = 0, k \in \Lambda_j, p \in \Lambda_l, q \in \Lambda_m, l, m \in N_*(j)\}. \quad (131)$$

The set F_j thus consists of all triads in which the wave number k lies in shell j , while the remaining two wave numbers belong to finitely many shells neighboring j . It may therefore be interpreted as the family of local triadic interactions localized around shell j .

Since the object of essential importance in this paper is the High–High interaction, we next isolate the genuinely High–High–High contributions by defining the strict family

$$F_j^{HH} := \{(k, p, q) \in F_j : p \in \Lambda_l, q \in \Lambda_m, l, m \in N_*(j)\}. \quad (132)$$

Since $k \in \Lambda_j$ is already built into F_j , this condition effectively imposes

$$|k| \sim |p| \sim |q| \sim 2^j. \quad (133)$$

Thus, F_j^{HH} is the collection of triads concentrated on the same dyadic scale, namely the object to be treated at later stages as the dangerous family.

6.5. Definition of Family Transfer

Chapter 4 defined the transfer

$$T(k | p, q), \quad (134)$$

for each individual triad. We now lift this notion to family level. We define the effective transfer of all triads belonging to the j -th family by

$$T_j^{\text{fam}}(t) := \sum_{(k,p,q) \in F_j} T(k | p, q). \quad (135)$$

Similarly, for the strict High–High family, we define

$$T_j^{HH}(t) := \sum_{(k,p,q) \in F_j^{HH}} T(k | p, q). \quad (136)$$

The quantity T_j^{HH} is the central object in the High–High Absorption Condition introduced later in Chapter 7. In the present paper, the dangerous nonlinear transfer in shell j is ultimately represented by

$$T_j^{HH}. \quad (137)$$

The crucial point is that T_j^{HH} is not the contribution of a single triad, but an aggregate over many triads belonging to the same scale band. Thanks to this coarse-graining, the later absorption condition can be formulated not in terms of the sign-indefinite behavior of individual triads, but as a shell-level observable measuring effective danger.

6.6. Family–Shell Localization

To treat family-level transfer as a shellwise quantity, one must verify how many shells may be influenced by a single family. The key point here is family–shell localization.

Lemma 6.1 (Family–shell localization).

For every j , the contribution of triads in F_j can affect only finitely many neighboring shells. More precisely, there exists an absolute constant C_{ov} such that the contribution of F_j can appear in at most C_{ov} shell indices.

Proof. If $(k, p, q) \in F_j$, then

$$k \in \Lambda_j, p \in \Lambda_l, q \in \Lambda_m, l, m \in N_*(j). \quad (138)$$

Hence l and m are confined to a finite neighborhood of j , and the magnitudes of the participating wave numbers remain within finite multiples and finite fractions of 2^j . It follows that the shell indices affected by contributions from the family F_j are localized in a finite neighborhood of j , and their number is bound by a finite constant depending only on c_* .

The meaning of this lemma is straightforward. The coarse graining performed at the family level is compatible with shellwise energy estimates, and a single family can never spread across infinitely many shells. This fact allows family-transfer estimates to be reinserted later into the shell energy equation.

6.7. Individual Coherence and Family Coherence

As observed in Chapter 5, a High–High interaction can produce strong transfer only if large amplitudes are accompanied by an alignment of triadic phases and geometric structure, namely by coherence. We now lift this notion from individual triads to the family level.

For an individual triad (k, p, q) , define its triadic phase by

$$\Phi(k, p, q; t) := \arg \hat{u}(k, t) + \arg \hat{u}(p, t) + \arg \hat{u}(q, t), \quad (139)$$

where $\arg \hat{u}(k, t)$ denotes the phase of the corresponding Fourier mode. We then define the individual triadic coherence by

$$c(k, p, q; t) := |\cos \Phi(k, p, q; t)|. \quad (140)$$

Clearly,

$$0 \leq c(k, p, q; t) \leq 1. \quad (141)$$

The quantity $c(k, p, q; t)$ measures the degree of phase alignment within the triad. Strong phase alignment enhances the transfer amplitude, whereas randomization of phases causes the signed contributions to cancel on average.

We now lift this notion to family level. For the j -th family, define its maximal coherence by

$$C_j(t) := \sup_{(k, p, q) \in F_j^{HH}} c(k, p, q; t). \quad (142)$$

We also define the average family coherence by the weighted average

$$\bar{C}_j(t) := \frac{\sum_{(k,p,q) \in F_j^{HH} w_{kpq}(t) c(k,p,q;t)}{\sum_{(k,p,q) \in F_j^{HH} w_{kpq}(t)}. \quad (143)$$

Here the weights $w_{kpq}(t) \geq 0$ are appropriate nonnegative quantities reflecting the transfer amplitude of the triad. A typical choice is

$$w_{kpq}(t) := |\hat{u}(k, t)| |\hat{u}(p, t)| |\hat{u}(q, t)|, \quad (144)$$

or any equivalent quantity.

The quantity $C_j(t)$ represents the worst-case coherence within the family, whereas $\bar{C}_j(t)$ measures how coherent the family is on average. In later quantitative estimates of the High–High transfer, these coherence indicators will appear as prefactors governing the effective size of the transfer.

6.8. Basic Estimate for Family Transfer

Once family coherence has been introduced, one obtains a basic inequality bounding the High–High family transfer in terms of shell energy. We state here its qualitative form.

Proposition 6.2 (Basic estimate for family transfer). There exists an absolute constant $C_B > 0$ such that, for every shell j ,

$$|T_j^{HH}(t)| \leq C_B C_j(t) V_j(t) + Q_j(t), \quad (145)$$

where

- $C_j(t)$ is the family coherence,
- $V_j(t)$ is the natural viscous scale in shell j ,
- $Q_j(t)$ is a remainder term.

The viscous scale V_j will later be specified as

$$V_j(t) := 2\nu 2^{2j} E_j(t). \quad (146)$$

This is the natural scale of viscous dissipation in shell j . The term $Q_j(t)$ is the remainder arising from the regrouping of families; its detailed structure will be given later and in the appendices. What matters here is that Q_j is designed to be summable under later Sobolev-weighted summation. The significance of (145) is essentially that the magnitude of the High–High transfer grows proportionally to the family coherence. In other words, the High–High interaction is dangerous not merely because it is same-scale, but because same-scale interactions can produce large transfer only when they become coherently organized.

This proposition is the starting point for the later pointwise formulation of the High–High Absorption Condition in Chapter 7.

6.9. Coherent Time Set and Family Residence Time

The danger posed by a High–High interaction is not determined solely by the fact that the coherence is large at a given instant. If a high coherence persists only for a very short time, its effect may remain small after time integration. For this reason, we must measure not only the magnitude of coherence, but also its duration.

Fix a threshold $\eta \in (0,1)$. We say that the j -th family is coherent at time t if

$$C_j(t) \geq \eta. \quad (147)$$

The corresponding coherent time set is defined by

$$I_{j,\eta} := \{ t \in [0, T] : C_j(t) \geq \eta \}. \quad (148)$$

We then define its Lebesgue measure

$$R_j(\eta; T) := |I_{j,\eta}|, \quad (149)$$

and call it the residence time of the j -th family.

The quantity $R_j(\eta; T)$ represents the total time during which the High–High family in shell j maintains coherence above the threshold η on the interval $[0, T]$. Later chapters will show that, as the wave number increases, this residence time becomes shorter, and that this leads naturally to sufficient conditions under which the integrated High–High Absorption Condition is satisfied. In this sense, $C_j(t)$ is an instantaneous indicator of danger, while $R_j(\eta; T)$ is a cumulative indicator of danger.

6.10. Shell Defect Observable

To formulate the High–High Absorption Condition later, one needs a quantity that measures whether the family transfer exceeds the shellwise viscous dissipation. We therefore introduce the shell defect observable. For a fixed absorption coefficient $\alpha \in (0, 1)$, define the defect in shell j by

$$\delta_j(t) := T_j^{HH}(t) - \alpha V_j(t), \quad (150)$$

where $V_j(t)$ is the viscous scale defined in (146).

The meaning of this quantity is immediate:

- if $\delta_j(t) \leq 0$, then at that time the High–High transfer is absorbed by the viscous scale.
- if $\delta_j(t) > 0$, then at that time the High–High transfer exceeds the viscous dissipation in that shell.

Thus, δ_j is the observable measuring the dangerous excess transfer in shell j .

In Chapter 7, this defect variable will be used to formulate in a unified way

- pointwise absorption,
- integral absorption,
- coherence-smallness conditions,
- and residence-time budget conditions.

6.11. Significance of Family Regrouping

It is worth restating the mathematical significance of the triadic family decomposition introduced in this chapter. Individual triads are sign-indefinite and controlling them one by one is extremely difficult. Once triads belonging to the same dyadic scale are regrouped into families, however, their effective danger can be described in terms of the finite collection of shell-level observables

- family transfer T_j^{HH} ,
- family coherence $C_j, \bar{C}_j$,
- coherent time set $I_{j,\eta}$,
- residence time $R_j(\eta; T)$,
- shell defect δ_j .

In other words, the triadic family decomposition provides a coarse graining from

$$\text{the detailed structure of individual triads} \rightarrow \text{effective shellwise indicators of danger.} \quad (151)$$

This coarse graining is one of the central ideas of the paper. If one were to treat individual triads directly, the dangerous mechanism would remain buried under a multitude of sign-indefinite contributions. By regrouping into families, however, the dangerous channel becomes visible at the level of shellwise observables. This is the decisive prerequisite that allows the High–High Absorption Condition to be formulated and ultimately connected to the conditional regularity theorem.

6.12. Summary of This Chapter

In this chapter, the High–High interactions localized in Chapter 5 were reorganized from individual triads into family-level quantities. The principal points are as follows.

First, the neighboring shell set $N_*(j)$ was introduced so that local triadic interactions could be described in terms of finitely many nearby shells.

Second, the triadic family F_j and the strict High–High family F_j^{HH} were defined, making explicit the collection of same-scale interactions localized around shell j .

Third, the family transfer T_j^{fam} and the High–High family transfer T_j^{HH} were defined, thereby identifying the essential quantity to be compared in the later absorption condition.

Fourth, the family coherence $C_j, \bar{C}_j$, the coherent time set $I_{j,\eta}$, and the residence time $R_j(\eta; T)$ were introduced, thereby providing a framework for measuring the time persistence of dangerous transfer.

Fifth, the shell defect observable δ_j was defined, introducing a direct indicator of whether the High–High transfer exceeds the shellwise viscous dissipation.

Accordingly, the qualitative statement obtained up to Chapter 5—the localization of the dangerous mechanism—has now been translated into quantitative shell-level observables suitable for the formulation of the High–High Absorption Condition in Chapter 7.

In the next chapter, these observables will be used to formulate conditions controlling the High–High transfer in pointwise, integrated, coherence-smallness, and residence-time budget forms. In other words, the description of the dangerous mechanism will at last be converted into the assumptions of the regularity theorem itself.

7. Formulation of the High–High Absorption Condition

7.1. Purpose and Positioning of This Chapter

Chapter 5 established that the dangerous mechanism in the Navier–Stokes nonlinearity is localized in the High–High interaction, while Chapter 6 coarse-grained that interaction at the level of triadic families and introduced the shellwise observables

- the High–High family transfer T_j^{HH} ,
- the family coherences C_j and $\bar{C}_j$,
- the coherent time set $I_{j,\eta}$,
- the family residence time $R_j(\eta; T)$,
- and the shell defect δ_j .

In this way, the qualitative question of *which nonlinear channel is dangerous* was translated into the quantitative question of *under what conditions dissipation dominates that channel*.

The purpose of this chapter is to complete that translation explicitly. Using the observables defined in Chapter 6, we formulate the High–High Absorption Condition in several equivalent or sufficient forms, thereby making explicit the assumption needed in Chapter 8 to close the Sobolev energy estimates. Logically, the chapter proceeds in four steps.

First, we derive a quantitative High–High estimate bounding the High–High family transfer by the shellwise viscous scale and a remainder.

Second, taking this as the starting point, we define the pointwise absorption condition, which becomes the principal assumption of the regularity theorem.

Third, we introduce an integral absorption condition, expressing the fact that even if coherence is momentarily large, it is still sufficient that its duration be short.

Fourth, we derive sufficient conditions based on the smallness of coherence and on a residence-time budget, thereby showing that the observables introduced in Chapter 6 indeed provide the natural origin of the absorption hypothesis. The central point of the present chapter is that the absorption conditions defined here appear later as the only external assumptions in the main theorem. The Low–Low and Low–High contributions are treated internally in Chapter 8 and in the appendices

by weighted paraproduct estimates. The role of this chapter is therefore to convert the *description of the dangerous mechanism* into the *shellwise conditions needed to prevent blow-up of strong solutions*.

7.2. Viscous Scale and the Quantitative High–High Estimate

Consider the High–High family transfer introduced in Chapter 6,

$$T_j^{HH}(t) = \sum_{(k,p,q) \in F_j^{HH}} T(k \mid p, q). \quad (152)$$

We define the corresponding shellwise viscous scale by

$$V_j(t) := 2\nu 2^{2j} E_j(t), \quad (153)$$

where $E_j(t)$ is the shell energy

$$E_j(t) = \frac{1}{2} \|u_j(t)\|_{L^2}^2. \quad (154)$$

The coefficient $2\nu 2^{2j}$ in (153) represents the natural scale of viscous dissipation on the dyadic shell Λ_j . Indeed, for the shellwise viscous dissipation

$$D_j(t) = \nu \sum_{k \in \Lambda_j} |k|^2 |\hat{u}(k, t)|^2, \quad (155)$$

the relation $|k| \sim 2^j$ implies that

$$c_0 V_j(t) \leq D_j(t) \leq C_0 V_j(t), \quad (156)$$

for absolute constants $c_0, C_0 > 0$. Thus V_j may be regarded as a shellwise viscous scale equivalent to D_j .

We now quantify the basic estimate based on family coherence stated in Chapter 6. Namely, assume that there exist an absolute constant $C_B > 0$ and a nonnegative remainder $Q_j(t)$ such that

$$|T_j^{HH}(t)| \leq C_B C_j(t) V_j(t) + Q_j(t). \quad (157)$$

Here,

- $C_j(t)$ is the family coherence,
- $V_j(t)$ is the shellwise viscous scale,
- $Q_j(t)$ is the remainder associated with regrouping.

Equation (157) is the point of departure of this chapter. Its meaning is straightforward: the magnitude of the High–High transfer is amplified when coherence is large, but its natural scale is nevertheless measured by the viscous scale V_j . Thus, the High–High interaction becomes dangerous only when $C_j(t)$ is sufficiently large and the remainder accumulates in a non-negligible manner.

What matters in the later argument is that Q_j be summable under Sobolev-weighted summation. Accordingly, we require the remainder to satisfy the weighted summability condition

$$\sum_{j \geq j_0} 2^{2sj} Q_j(t) \leq C_Q (1 + \|u(t)\|_{H^s}^2), \quad (158)$$

for some $s > 5/2$, where j_0 denotes the starting shell on the high-frequency side and $C_Q > 0$ is a constant.

This condition allows shellwise remainders to be present but prevents them from becoming so large that they obstruct closure of the Sobolev energy estimate. In Chapter 8, it is precisely this weighted summability that allows the remainder to be absorbed on the right-hand side of the final differential inequality.

7.3. Pointwise High–High Absorption Condition

The most basic assumption used in the main theorem of Chapter 8 is the pointwise statement that, for every time and every shell, the High–High transfer is dominated by the viscous scale. Fix an absorption coefficient $\eta \in (0,1)$, and define the following.

Definition 7.1 (Pointwise High–High Absorption Condition). We say that the shellwise pointwise High–High Absorption Condition holds if there exist $\eta \in (0,1)$ and a nonnegative sequence $\rho_j(t) \geq 0$ such that, for every $j \geq j_0$ and every $t \in [0, T]$,

$$T_j^{HH}(t) \leq \eta V_j(t) + \rho_j(t). \quad (159)$$

Here the remainder ρ_j is assumed to satisfy, for some $s > 5/2$,

$$\sum_{j \geq j_0} 2^{2sj} \rho_j(t) \leq C_R(1 + \|u(t)\|_{H^s}^2). \quad (160)$$

Equation (159) is the central assumption of the paper. It means that, in shell j , the dangerous energy transfer generated by the High–High interaction is controlled by the viscous scale V_j of the same shell. The requirement $\eta < 1$ is essential, because in the later weighted Sobolev closure it guarantees that a strictly positive fraction of the dissipative term remains available. In this sense, η represents the proportion to which the High–High transfer is absorbed by dissipation.

It is equally essential that a remainder ρ_j is allowed. In practice, owing to triadic regrouping, finite overlaps, and secondary effects beyond coherence, it is not natural to expect a strict absorption condition of the form $T_j^{HH} \leq \eta V_j$. However, if the excess is summable under Sobolev-weighted summation, then it is still harmless within the framework of the main theorem. For this reason, the present paper adopts as its basic form not strict absorption, but an absorption condition with remainder.

7.4. The Shell-Defect Form and Its Meaning

Using the shell defect observable introduced in Chapter 6,

$$\delta_j(t) := T_j^{HH}(t) - \eta V_j(t), \quad (161)$$

the pointwise absorption condition (159) may be rewritten as

$$\delta_j(t) \leq \rho_j(t). \quad (162)$$

The advantage of this form is that the dangerous excess transfer in shell j becomes explicitly visible. If $\delta_j(t) \leq 0$, then the High–High transfer is completely absorbed by dissipation at that time. If $\delta_j(t) > 0$, but is uniformly controlled by $\rho_j(t)$, and ρ_j is weighted summable, then the total Sobolev energy budget remains controllable. In this sense, the shell defect δ_j is the observable measuring the *violation amount* of pointwise absorption, while the remainder ρ_j measures how harmless that violation is.

The pointwise absorption condition of this chapter may therefore be understood, in essence, as the statement that the excess:

High–High transfer – viscous scale

is bounded by a weighted summable remainder.

7.5. Integral High–High Absorption Condition

The pointwise absorption condition is the most direct form, but not necessarily the most natural. In practice, the High–High transfer and the family coherence may become large instantaneously, yet their cumulative effect may remain small provided the duration is sufficiently short. For this reason, we introduce a more flexible, integrated form.

Definition 7.2 (Integral High–High Absorption Condition). We say that the integral High–High Absorption Condition holds if there exist $\eta \in (0,1)$ and nonnegative functions $\rho_j(T) \geq 0$ such that, for every $T > 0$ and every $j \geq j_0$,

$$\int_0^T T_j^{HH}(t) dt \leq \eta \int_0^T V_j(t) dt + \rho_j(T). \quad (163)$$

The remainder $\rho_j(T)$ is assumed to satisfy, for some $s > 5/2$,

$$\sum_{j \geq j_0} 2^{2sj} \rho_j(T) < \infty. \quad (164)$$

Equation (163) means that the cumulative effect of the High–High transfer in shell j is controlled by the cumulative viscous scale. In contrast to the pointwise version, it allows momentary violations of $T_j^{HH}(t) \leq \eta V_j(t)$, provided that such excesses are short-lived in time or cancel in the signed time integral.

This integral form is naturally connected to the sufficient conditions based on residence time developed below. Even if there are intervals on which coherence is large, the time-integrated High–High transfer may still remain dissipatively controlled if such intervals become shorter at higher wave numbers.

7.6. Sufficient Condition from Coherence Smallness

Definition 7.1 gives a direct condition on the transfer itself. However, the quantitative High–High estimate (157) from Chapter 6 allows one to derive pointwise absorption from the smallness of the family coherence.

Recalling

$$|T_j^{HH}(t)| \leq C_B C_j(t) V_j(t) + Q_j(t), \quad (165)$$

we see that if family coherence satisfies

$$C_j(t) \leq \frac{\eta}{C_B}, \quad (166)$$

then it follows immediately that

$$T_j^{HH}(t) \leq \eta V_j(t) + Q_j(t). \quad (167)$$

Hence, if one identifies Q_j with the remainder ρ_j , the pointwise absorption condition (159) follows. We summarize this as follows.

Proposition 7.3 (Pointwise absorption from small coherence). Let $\eta \in (0,1)$ be given. If, for every $j \geq j_0$ and every $t \in [0, T]$,

$$C_j(t) \leq \frac{\eta}{C_B}, \quad (168)$$

and if Q_j satisfies

$$\sum_{j \geq j_0} 2^{2sj} Q_j(t) \leq C_Q(1 + \|u(t)\|_{H^s}^2), \quad (169)$$

then the pointwise High–High Absorption Condition holds.

The essential meaning of this proposition is that the High–High interaction is dangerous not merely because it is same-scale, but only when it is both same-scale and coherent. If the coherence remains uniformly small, then the High–High transfer is necessarily smaller than the viscous scale,

and the dangerous channel is absorbed immediately. This condition is, however, only sufficient, not necessary. Coherence may become large temporarily, and yet the integral version of absorption may still hold if the residence-time budget remains sufficiently small.

7.7. An Integral Sufficient Condition via Residence-Time Budget

We next derive the integral High–High Absorption Condition (163) from the coherent time set and the residence time. In Chapter 6, for a threshold $\eta_0 \in (0,1)$, we defined the coherent time set by

$$I_{j,\eta_0} := \{t \in [0, T] : C_j(t) \geq \eta_0\}, \quad (170)$$

and its Lebesgue measure by

$$R_j(\eta_0; T) := |I_{j,\eta_0}|. \quad (171)$$

Decomposed to the time interval $[0, T]$ into its coherent and incoherent parts, we write

$$\int_0^T T_j^{HH}(t) dt = \int_{I_{j,\eta_0}} T_j^{HH}(t) dt + \int_{[0,T] \setminus I_{j,\eta_0}} T_j^{HH}(t) dt. \quad (172)$$

On the incoherent part, $C_j(t) < \eta_0$, so by (165),

$$\int_{[0,T] \setminus I_{j,\eta_0}} T_j^{HH}(t) dt \leq C_B \eta_0 \int_0^T V_j(t) dt + \int_0^T Q_j(t) dt. \quad (173)$$

On the coherent part, C_j may be large, so we use the rougher bound

$$T_j^{HH}(t) \leq C_H V_j(t), \quad (174)$$

for some constant $C_H > 0$. Then

$$\int_{I_{j,\eta_0}} T_j^{HH}(t) dt \leq C_H \int_{I_{j,\eta_0}} V_j(t) dt. \quad (175)$$

If, moreover, a coherent-set budget of the form

$$\int_{I_{j,\eta_0}} V_j(t) dt \leq \beta_j(T), \quad (176)$$

holds, then

$$\int_{I_{j,\eta_0}} T_j^{HH}(t) dt \leq C_H \beta_j(T). \quad (177)$$

Substituting (173) and (177) into (172), we obtain

$$\int_0^T T_j^{HH}(t) dt \leq C_B \eta_0 \int_0^T V_j(t) dt + \int_0^T Q_j(t) dt + C_H \beta_j(T). \quad (178)$$

Therefore, if η_0 satisfies

$$C_B \eta_0 \leq \eta < 1, \quad (179)$$

and if

$$\rho_j(T) := \int_0^T Q_j(t) dt + C_H \beta_j(T), \quad (180)$$

is weighted summable, then the integral High–High Absorption Condition (163) follows. We record this as follows.

Proposition 7.4 (Integral absorption from residence-time budget). Fix $\eta_0 \in (0,1)$. If the incoherent part satisfies (173), the coherent part satisfies (176), and moreover

$$\sum_{j \geq j_0} 2^{2sj} \left(\int_0^T Q_j(t) dt + C_H \beta_j(T) \right) < \infty, \quad (181)$$

then the integral High–High Absorption Condition holds.

The substance of this proposition is that what is dangerous is not the mere occurrence of high coherence, but the persistence of a highly coherent state. If the coherent residence time becomes shorter at higher wavenumbers, then even large instantaneous High–High transfer remains cumulatively dominated by dissipation.

7.8. Relation Between the Pointwise and Integral Forms

We now clarify the relation between the pointwise form (159) and the integral form (163). If the pointwise absorption condition holds, then integrating in time immediately gives

$$\int_0^T T_j^{HH}(t) dt \leq \eta \int_0^T V_j(t) dt + \int_0^T \rho_j(t) dt. \quad (182)$$

Thus, if one sets

$$\rho_j(T) := \int_0^T \rho_j(t) dt, \quad (183)$$

the integral absorption condition follows.

The converse, however, is false in general. The integral form requires only time-averaged domination and therefore does not imply pointwise control. The logical hierarchy in this paper is therefore

$$\text{coherence smallness} \Rightarrow \text{pointwise absorption} \Rightarrow \text{integral absorption}, \quad (184)$$

and, separately,

$$\text{residence-time budget} \Rightarrow \text{integral absorption}. \quad (185)$$

This hierarchy is important for the overall claim of the paper. In Chapter 8, it will be enough to assume either the pointwise or the integral form. Behind these assumptions, however, one may view the absorption condition as arising from dynamical mechanisms such as

- uniformly small coherence,
- large coherence with sufficiently short residence time,
- summable remainders induced by relaxation damping.

In this sense, the present chapter does not merely posit the assumptions of the main theorem but also exhibits the routes by which such assumptions may be generated.

7.9. Reformulation in Terms of the Shell Defect

The content of this chapter may be summarized cleanly in terms of the shell defect δ_j . The pointwise form becomes

$$\delta_j(t) = T_j^{HH}(t) - \eta V_j(t) \leq \rho_j(t), \quad (186)$$

while the integral form becomes

$$\int_0^T \delta_j(t) dt \leq \rho_j(T). \quad (187)$$

Thus, the shell defect is the universal observable measuring the degree of violation of High–High absorption. From this viewpoint, all of the conditions introduced in this chapter are, in essence, reductions of the single question:

How can one control δ_j , or its time integral $\int_0^T \delta_j$?

In Chapter 8, the shell energy equation will be written in defect form, and the Low–Low and Low–High contributions will be treated by weighted paraproduct estimates. This will lead to the final differential inequality for the weighted Sobolev energy. The shell-defect formulation given here therefore also serves as preparation for the most efficient presentation of the calculations in the next chapter.

7.10. Summary of This Chapter

In this chapter, using the family-level observables introduced in Chapter 6, we formulated conditions controlling the High–High transfer in a form suitable for the regularity theorem. The principal points are as follows.

First, we defined the shellwise viscous scale $V_j = 2\nu 2^{2j} E_j$ and took as the starting point the quantitative estimate $|T_j^{HH}| \leq C_B C_j V_j + Q_j$ for the High–High family transfer.

Second, we defined the pointwise High–High Absorption Condition $T_j^{HH}(t) \leq \eta V_j(t) + \rho_j(t)$, and identified it as the principal shellwise assumption of the paper.

Third, we introduced the integral absorption condition $\int_0^T T_j^{HH}(t) dt \leq \eta \int_0^T V_j(t) dt + \rho_j(T)$, thereby showing that it is sufficient for absorption to hold in a time-averaged sense, even if instantaneous excess is allowed.

Fourth, we showed that small coherence is a sufficient condition for pointwise absorption, while a residence-time budget is a sufficient condition for integral absorption.

Fifth, we showed that, through the shell defect observable $\delta_j = T_j^{HH} - \eta V_j$, all the absorption conditions can be unified as instances of controlling the excess transfer.

Accordingly, the dangerous mechanism described in Chapters 5 and 6 has now been converted into the quantitative assumptions needed for the regularity theorem proved in Chapter 8.

In the next chapter, assuming either the pointwise or the integral High–High Absorption Condition, we lift the dyadic shell energy equation to the weighted Sobolev energy. There, the Low–Low and Low–High interactions will be treated through weighted paraproduct estimates, and a Grönwall-type inequality will ultimately be derived, yielding the conditional regularity theorem for the three-dimensional Navier–Stokes equations.

8. A Conditional Regularity Theorem Under High–High Absorption

8.1. Purpose and Positioning of This Chapter

Chapter 4 represented the nonlinear term in the Navier–Stokes equations as a triadic interaction in Fourier space, while Chapter 5 classified those interactions into three classes: Low–Low, Low–High, and High–High. Chapter 6 then coarse-grained the High–High interaction into triadic families and introduced shell-level observables such as family transfer, coherence, residence time, and shell defect. In Chapter 7, these observables were used to formulate, in both pointwise and integral forms, the quantitative assumption under which dissipation dominates the dangerous High–High transfer, namely the High–High Absorption Condition.

The purpose of the present chapter is to show that, under this absorption condition, a strong solution of the three-dimensional incompressible Navier–Stokes equations cannot blow up in finite time. More precisely, we lift the dyadic shell energy equation to a weighted Sobolev energy, treat the Low–Low and Low–High interactions by internal estimates, and ultimately derive finite-time boundedness of the Sobolev norm, leaving the High–High channel as the only external assumption.

The logical structure of the argument is transparent.

First, we take as the starting point the shellwise energy equation

$$\frac{d}{dt} E_j = -D_j + T_j^{LL} + T_j^{LH} + T_j^{HH}. \quad (188)$$

Second, we apply weighted paraproduct estimates to the Low–Low and Low–High terms and show that they may be treated as lower-order controllable contributions in the Sobolev energy balance.

Third, we apply the absorption condition from Chapter 7 to the High–High term and absorb it into a portion of the viscous scale.

Fourth, using the weighted summability of the remainder, we sum the shellwise inequalities with Sobolev weights and derive a closed differential inequality for the Sobolev energy X_s and dissipation Y_s .

Fifth, by Grönwall's inequality [44], we obtain finite-time boundedness of $X_s(t)$, and then exclude finite-time blow-up by the continuation principle for local strong solutions [40,43,46].

The crucial point is that the Low–Low and Low–High interactions do not appear as assumptions in the main theorem. They are handled within this chapter and in the appendices by weighted estimates. Thus, the only external condition that remains is the absorption condition on the High–High channel. This is the precise mathematical meaning of the central conclusion of the paper: the continuation problem for strong solutions reduces to the High–High absorption condition.

8.2. Restatement of the Shellwise Energy Equation

We first recall the shellwise energy equation obtained in Chapter 4. For each dyadic shell Λ_j , let

$$E_j(t) := \frac{1}{2} \|u_j(t)\|_{L^2}^2. \quad (189)$$

Then

$$\frac{d}{dt} E_j(t) = -D_j(t) + T_j(t), \quad (190)$$

where

$$D_j(t) := \nu \sum_{k \in \Lambda_j} |k|^2 |\hat{u}(k, t)|^2, \quad (191)$$

is the shellwise viscous dissipation, and T_j is the nonlinear transfer into shell j .

According to the classification of Chapter 5,

$$T_j = T_j^{LL} + T_j^{LH} + T_j^{HH}, \quad (192)$$

and therefore

$$\frac{d}{dt} E_j(t) = -D_j(t) + T_j^{LL}(t) + T_j^{LH}(t) + T_j^{HH}(t). \quad (193)$$

Moreover, the viscous scale introduced in Chapter 7,

$$V_j(t) := 2\nu 2^{2j} E_j(t), \quad (194)$$

is equivalent to D_j : there exist absolute constants $c_0, C_0 > 0$ such that

$$c_0 V_j(t) \leq D_j(t) \leq C_0 V_j(t). \quad (195)$$

Thus, in analyzing (193), the essential point is to determine how dangerous each of the terms T_j^{LL} , T_j^{LH} , and T_j^{HH} is when compared with the viscous scale V_j .

8.3. Definition of the Weighted Sobolev Energy and Dissipation

Throughout this chapter we fix an exponent

$$s > \frac{5}{2}. \quad (196)$$

This condition is natural in three dimensions, since H^s is then an algebra and the embedding $H^s \hookrightarrow W^{1,\infty}$ becomes available [40,45,46].

We define the weighted Sobolev energy associated with the dyadic shell decomposition by

$$X_s(t) := \sum_{j \geq j_0} 2^{2sj} E_j(t). \quad (197)$$

By Littlewood–Paley theory [45],

$$X_s(t) \simeq \|u(t)\|_{H^s}^2, \quad (198)$$

where $\simeq$ denotes equivalence up to constants depending only on s .

The corresponding weighted dissipation is defined by

$$Y_s(t) := \sum_{j \geq j_0} 2^{2sj} V_j(t). \quad (199)$$

Using (194), this becomes

$$Y_s(t) = 2\nu \sum_{j \geq j_0} 2^{2(s+1)j} E_j(t), \quad (200)$$

and again, by Littlewood–Paley equivalence,

$$Y_s(t) \simeq \nu \|u(t)\|_{H^{s+1}}^2. \quad (201)$$

The aim of this chapter is to derive a closed differential inequality of the form

$$\frac{d}{dt} X_s(t) + c_1 Y_s(t) \leq C_1 (1 + X_s(t)), \quad (202)$$

for some constants $c_1, C_1 > 0$. Once this is established, finite-time blow-up is excluded by Grönwall's inequality together with the continuation principle for local strong solutions.

8.4. Weighted Paraproduct Estimates for the Low–Low and Low–High Terms

One of the central points of this chapter is that the Low–Low and Low–High terms are not external assumptions of the main theorem but are handled internally by estimates. To make this explicit, we first state the needed weighted bound.

By the classification of Chapter 5, T_j^{LL} and T_j^{LH} belong not to genuinely same-scale resonant interactions, but to paraproduct-type interactions in which a low-frequency factor acts as a smooth coefficient. Accordingly, using Bony's decomposition [45],

$$u \cdot \nabla u = T_u(\nabla u) + T_{\nabla u} u + R(u, \nabla u), \quad (203)$$

the Low–Low and Low–High terms can be estimated as lower-order perturbations on the high-frequency shells.

We state the resulting bound as follows.

Proposition 8.1 (Weighted paraproduct estimate). Let $s > 5/2$. Then, for every $j \geq j_0$,

$$|T_j^{LL}(t) + T_j^{LH}(t)| \leq \frac{c_0}{4} V_j(t) + a_j(t) E_j(t) + b_j(t), \quad (204)$$

where $a_j(t) \geq 0$ and $b_j(t) \geq 0$ satisfy

$$\sum_{j \geq j_0} 2^{2sj} a_j(t) E_j(t) \leq C_A X_s(t), \quad (205)$$

and

$$\sum_{j \geq j_0} 2^{2sj} b_j(t) \leq C_B(1 + X_s(t)). \quad (206)$$

The proof is deferred to the appendix, but its essential content is that in the Low–High interaction the low-frequency factor behaves as an L^∞ -like smooth coefficient and therefore does not possess the genuinely same-scale resonant self-amplification characteristic of the High–High channel. Consequently, these terms may be absorbed partly into the dissipative term V_j , while the remainder is treated as a lower-order contribution proportional to X_s . It should be stressed that, by means of (204)–(206), the Low–Low and Low–High channels are handled entirely within the present chapter. They therefore do not appear later as additional assumptions in the main theorem.

8.5. Application of the High–High Absorption Condition

We now apply the pointwise High–High Absorption Condition introduced in Chapter 7. Namely, assume that there exist $\eta \in (0,1)$ and a nonnegative sequence $\rho_j(t)$ such that

$$T_j^{HH}(t) \leq \eta V_j(t) + \rho_j(t), \quad (207)$$

for all $j \geq j_0$ and all $t \in [0, T]$. Assume further that the remainder satisfies the weighted summability condition

$$\sum_{j \geq j_0} 2^{2sj} \rho_j(t) \leq C_R(1 + X_s(t)). \quad (208)$$

Substituting (204) and (207) into the shellwise energy equation (193), we obtain

$$\frac{d}{dt} E_j \leq -D_j + \frac{c_0}{4} V_j + a_j E_j + b_j + \eta V_j + \rho_j. \quad (209)$$

Using the lower bound $D_j \geq c_0 V_j$ from (195), it follows that

$$\frac{d}{dt} E_j \leq -(c_0 - \frac{c_0}{4} - \eta) V_j + a_j E_j + b_j + \rho_j. \quad (210)$$

If η is chosen so that

$$0 < \eta < \frac{3}{4} c_0, \quad (211)$$

then there exists a positive constant $c_1 > 0$ such that

$$\frac{d}{dt} E_j + c_1 V_j \leq a_j E_j + b_j + \rho_j. \quad (212)$$

This is the fundamental shellwise inequality. The High–High transfer has been absorbed by assumption, while the Low–Low and Low–High terms have been handled by internal estimates. What remains is therefore only a controllable lower-order term together with a summable remainder.

8.6. Sobolev Closure by Weighted Summation

We now multiply (212) by the weight 2^{2sj} and sum over all $j \geq j_0$. This yields

$$\frac{d}{dt} \sum_{j \geq j_0} 2^{2sj} E_j + c_1 \sum_{j \geq j_0} 2^{2sj} V_j \leq \sum_{j \geq j_0} 2^{2sj} a_j E_j + \sum_{j \geq j_0} 2^{2sj} b_j + \sum_{j \geq j_0} 2^{2sj} \rho_j. \quad (213)$$

Using the definitions (197) and (199), together with the bounds (205), (206), and (208), we obtain

$$\frac{d}{dt} X_s(t) + c_1 Y_s(t) \leq C_A X_s(t) + C_B(1 + X_s(t)) + C_R(1 + X_s(t)). \quad (214)$$

Collecting the constants into a single constant $C_2 > 0$, we arrive at

$$\frac{d}{dt}X_s(t) + c_1Y_s(t) \leq C_2(1 + X_s(t)). \quad (215)$$

This is the central Sobolev closure inequality of the chapter. Its logical structure is crucial:

- Y_s is a higher-order dissipative term and remains on the left-hand side with positive sign.
- the right-hand side contains only X_s and a constant.
- the danger of the Low–Low and Low–High terms has already been eliminated internally.
- the High–High contribution has been neutralized only through the absorption condition.

Accordingly, (215) is the precise analytic realization of the central claim of the paper: the continuation problem for strong solutions reduces to the condition of High–High absorption.

8.7. A Grönwall-Type Inequality

Dropping the positive term c_1Y_s from (215), we obtain

$$\frac{d}{dt}X_s(t) \leq C_2(1 + X_s(t)). \quad (216)$$

This is a standard Grönwall-type inequality [44]. Integrating from the initial time 0 to t , we find

$$X_s(t) \leq X_s(0) + C_2 \int_0^t (1 + X_s(\tau)) d\tau. \quad (217)$$

Grönwall's inequality therefore yields

$$X_s(t) \leq (X_s(0) + C_2t)e^{C_2t}. \quad (218)$$

For every finite time $T > 0$,

$$\sup_{0 \leq t \leq T} X_s(t) < \infty. \quad (219)$$

By (198), since $X_s(t) \simeq \|u(t)\|_{H^s}^2$, it follows that

$$\sup_{0 \leq t \leq T} \|u(t)\|_{H^s} < \infty. \quad (220)$$

Because $s > 5/2$, boundedness of the H^s -norm is sufficient to invoke the continuation principle for strong solutions [40,43,46]. Hence the local strong solution cannot blow up in finite time and may be continued to any finite interval.

8.8. Conditional Regularity Theorem

We now summarize the preceding argument as a theorem.

Theorem 8.2 (Conditional regularity under High–High absorption). Let $\Omega = \mathbb{T}^3$, and let $s > 5/2$. Let u be a local strong solution of the three-dimensional incompressible Navier–Stokes equations

$$\partial_t u + (u \cdot \nabla)u + \nabla p = \nu \Delta u, \nabla \cdot u = 0. \quad (221)$$

Assume that, with respect to the corresponding dyadic shell decomposition, there exist $\eta \in (0,1)$ and a nonnegative sequence $\rho_j(t)$ such that, for every $j \geq j_0$ and every $t \in [0, T]$,

$$T_j^{HH}(t) \leq \eta V_j(t) + \rho_j(t), \quad (222)$$

and that the remainder satisfies

$$\sum_{j \geq j_0} 2^{2sj} \rho_j(t) \leq C_R(1 + \|u(t)\|_{H^s}^2). \quad (223)$$

(223)

Then, for every finite time $T > 0$,

$$\sup_{0 \leq t \leq T} \|u(t)\|_{H^s} < \infty. \quad (224)$$

Finite-time blow-up does not occur.

Proof. Starting from the shellwise energy equation (193) from Chapter 4, one treats the Low–Low and Low–High terms by the weighted paraproduct estimate of Section 8.4. The High–High term is then absorbed by the assumption (222), and the weighted summability (223) is used to control the shellwise remainder under Sobolev-weighted summation. This yields a closed inequality for the weighted Sobolev energy X_s : $\frac{d}{dt}X_s(t) + c_1Y_s(t) \leq C_2(1 + X_s(t))$. Grönwall’s inequality implies that $X_s(t)$ remains bounded on every finite time interval. Since $X_s \simeq \|u\|_{H^s}^2$, this gives (224). Because $s > 5/2$, the continuation principle for local strong solutions applies, and finite-time blow-up is excluded.

This theorem is the principal result of the paper. What should be emphasized is that the only external assumption that remains is the High–High Absorption Condition. The Low–Low and Low–High interactions do not appear among the assumptions of the theorem; they are handled entirely within the weighted estimates of this chapter. Thus Theorem 8.2 is the precise mathematical formulation of the claim that the continuation problem for strong solutions reduces to the absorption condition for the High–High transfer.

8.9. Conditional Regularity Under the Integral Form

A nearly identical argument applies when one uses the integral High–High Absorption Condition introduced in Chapter 7. Namely, assume that

$$\int_0^T T_j^{HH}(t) dt \leq \eta \int_0^T V_j(t) dt + \rho_j(T), \quad (225)$$

And

$$\sum_{j \geq j_0} 2^{2sj} \rho_j(T) < \infty. \quad (226)$$

Then, by integrating the shellwise inequality in time and taking the weighted sum, one obtains

$$X_s(T) + c_1 \int_0^T Y_s(t) dt \leq X_s(0) + C \int_0^T (1 + X_s(t)) dt. \quad (227)$$

Grönwall’s inequality again yields boundedness of $X_s(T)$. Hence the integral version of the absorption condition is likewise sufficient to deduce conditional regularity.

This shows that the theory developed in the present paper does not rely excessively on strict absorption at every instant. Temporary increases in coherence are allowed, provided that their cumulative effect remains sufficiently small through the residence-time budget. Regularity is then preserved.

8.10. Summary of This Chapter

In this chapter, assuming the High–High Absorption Condition formulated in Chapter 7, we proved that strong solutions of the three-dimensional incompressible Navier–Stokes equations cannot develop finite-time singularities.

The principal points are as follows.

First, we took as the starting point the shellwise energy equation

$$\frac{d}{dt}E_j = -D_j + T_j^{LL} + T_j^{LH} + T_j^{HH}.$$

Second, the Low–Low and Low–High interactions were estimated by weighted paraproduct bounds of the form

$$|T_j^{LL} + T_j^{LH}| \leq \frac{c_0}{4} V_j + a_j E_j + b_j,$$

and were therefore treated internally rather than as external assumptions.

Third, for the High–High interaction, we applied the pointwise absorption condition

$$T_j^{HH} \leq \eta V_j + \rho_j,$$

which yielded the shellwise inequality

$$\frac{d}{dt} E_j + c_1 V_j \leq a_j E_j + b_j + \rho_j.$$

Fourth, multiplying by the Sobolev weights 2^{2sj} and summing over the shells, we derived the closed inequality

$$\frac{d}{dt} X_s + c_1 Y_s \leq C_2(1 + X_s)$$

for the weighted Sobolev energy.

Fifth, by Grönwall's inequality and the continuation principle for local strong solutions, we showed that $\|u(t)\|_{H^s}$ remains bounded on every finite time interval and therefore excluded finite-time blow-up.

Accordingly, the central claim of the paper has now been established as a rigorous theorem:

the continuation problem for strong solutions of the three-dimensional Navier–Stokes equations ultimately reduces to the absorption condition for the High–High transfer.

The phrase “reduces to” does not mean that the Low–Low and Low–High channels are ignored. Rather, it means that they are fully treated within the weighted estimates of this chapter, so that the only condition remaining externally in the main theorem is the High–High Absorption Condition.

In the next chapter, we discuss why this absorption condition should be viewed not as a merely formal hypothesis, but as something natural from the standpoint of triadic geometry, helical cancellation, and turbulence phenomenology. Thereafter, beginning in Chapter 10, we return to the network master equation and the relaxation system, and use the additional dissipative structure together with relative entropy to provide a dynamical origin for the absorption condition.

9. Logical Significance of the Conditional Regularity Theorem and the Position of the Present Theory

9.1. Purpose and Positioning of This Chapter

Chapter 8 established that, for the three-dimensional incompressible Navier–Stokes equations, the Sobolev norm cannot diverge in finite time under the High–High Absorption Condition, thereby yielding a conditional regularity theorem. At this stage, the principal mathematical line of the present work has been completed, namely,

- the Fourier representation of triadic interaction,
- the classification of interaction classes,
- the coarse-graining into High–High families,
- the shellwise absorption condition,
- the weighted Sobolev closure,
- and the Grönwall-type argument.

Even once the main theorem has been obtained, however, several points still require logical clarification. Most importantly, it is essential to distinguish with precision what the theorem establishes within the Navier–Stokes regularity problem, and how it should be situated conceptually. Three points must be made explicit:

- the present theory does not assert an unconditional global regularity theorem for the three-dimensional Navier–Stokes equations,

- nor does it merely restate a conventional condition in terms of a global norm,
- rather, its essential content is the reduction of the strong-solution continuation problem to a shellwise absorption condition for the High–High transfer.

The purpose of this chapter is to provide that clarification. More specifically, we proceed as follows:

- we first restate the precise mathematical content of the theorem proved in Chapter 8,
- we then distinguish once again the roles played by the Low–Low / Low–High channels and by the High–High channel,
- we show that the contribution of the present theory lies in a structural reduction of the continuation problem,
- we explain why this reduction nevertheless has substantial theoretical significance,
- and we indicate how the later chapters will render the High–High absorption condition dynamically natural.

This chapter is therefore not a supplement to the proof, but a chapter devoted to the logical positioning of the theory developed in this paper.

9.2. Restatement of the Main Theorem of Chapter 8

We briefly recall the results obtained in Chapter 8. Consider a local strong solution u of the three-dimensional incompressible Navier–Stokes equations

$$\partial_t u + (u \cdot \nabla)u + \nabla p = \nu \Delta u, \nabla \cdot u = 0. \quad (228)$$

With respect to the corresponding dyadic shell decomposition, we assumed that the shellwise High–High transfer $T_j^{HH}(t)$ is absorbed by the viscous scale $V_j(t)$, namely,

$$T_j^{HH}(t) \leq \eta V_j(t) + \rho_j(t), 0 < \eta < 1, \quad (229)$$

and that the remainder ρ_j satisfies the weighted summability condition

$$\sum_{j \geq j_0} 2^{2sj} \rho_j(t) \leq C_R(1 + \|u(t)\|_{H^s}^2). \quad (230)$$

Under these assumptions, Chapter 8 derived for the weighted Sobolev energy

$$X_s(t) := \sum_{j \geq j_0} 2^{2sj} E_j(t), \quad (231)$$

the closed differential inequality

$$\frac{d}{dt} X_s(t) + c_1 Y_s(t) \leq C_2(1 + X_s(t)), \quad (232)$$

from which Grönwall's inequality yields

$$\sup_{0 \leq t \leq T} X_s(t) < \infty. \quad (233)$$

Since $X_s(t) \simeq \|u(t)\|_{H^s}^2$, it follows that

$$\sup_{0 \leq t \leq T} \|u(t)\|_{H^s} < \infty, \quad (234)$$

and finite-time blow-up is therefore excluded by the continuation principle for local strong solutions. The structure of this theorem may be summarized in one line:

$$\text{High–High absorption} \Rightarrow \text{weighted Sobolev closure} \Rightarrow \text{no finite-time blow-up.} \quad (235)$$

This logical chain is precise and entirely closed within the framework of the present paper. There is therefore no ambiguity in the mathematical content of the theorem itself. The issue addressed in this

chapter is rather the place of that theorem within the broader regularity problem for the Navier–Stokes equations.

9.3. Distinct Logical Roles of the Low–Low / Low–High and High–High Channels

One of the most important points for understanding the present theory is the precise distinction between the roles of the Low–Low / Low–High channels and that of the High–High channel.

As stated in Chapter 5, the shellwise transfer decomposes as

$$T_j = T_j^{LL} + T_j^{LH} + T_j^{HH}. \quad (236)$$

The Low–Low interaction is not, either geometrically or analytically, a genuinely dangerous same-scale amplification channel for the high-frequency shells. It appears instead as a lower-order contribution. The Low–High interaction does indeed influence the high-frequency components, but its principal role is transport and modulation; under Bony’s paraproduct decomposition, it is treated as a smooth-coefficient-type term [45].

By contrast, the High–High interaction is a genuinely nonlinear interaction among three modes belonging to the same dyadic scale and therefore carries the possibility of same-scale high-frequency self-amplification. For this reason, the present theory localizes the unresolved dangerous channel to

$$T_j^{HH}. \quad (237)$$

What must be emphasized, however, is the precise meaning of this localization. The present theory does not claim that the Low–Low and Low–High channels exert no influence whatsoever. They do appear in the shell energy equation, and they are treated in Chapter 8 by weighted estimates. The correct statement is therefore the following:

- the Low–Low and Low–High channels do not remain as external assumptions in the theorem,
- whereas the High–High channel is the only unresolved structural condition that survives at the theorem level.

In logical form, the structure of the argument may be written as

$$[\text{LL/LH are internally controllable}] + [\text{HH is absorbed}] \Rightarrow \text{regularity}. \quad (238)$$

The first component on the left-hand side is handled internally in the paper through the weighted paraproduct estimates of Chapter 8 and the appendices. It therefore does not remain as an external hypothesis. Consequently, the only structural condition remaining at the theorem level is

$$\text{HH is absorbed}. \quad (239)$$

This is the exact meaning of the statement that, in the present theory, the continuation problem for strong solutions reduces to the High–High absorption condition.

9.4. Established Result and Scope of the Reduction

Considering the preceding discussion, it is appropriate to state clearly the precise scope of the present result.

The theory developed here establishes that the continuation problem for strong solutions to the three-dimensional Navier–Stokes equations can be formulated not in terms of a global norm condition or a pointwise vorticity criterion, but in terms of a shellwise absorption condition for the High–High triadic transfer in Fourier space. More concretely, the paper

- localizes the dangerous nonlinear mechanism through triadic geometry,
- coarse-grains it into family-level observables,
- formulates it as a shellwise absorption condition,
- and, under that condition, excludes blow-up of the Sobolev norm.

This provides a new structural interpretation of conditional regularity theory. In particular, the criterion obtained here is not a global statement of the form “which norm must remain bounded,” but one that directly describes

which nonlinear transfer, in which shell, competes with which dissipation. (240)

At the same time, the theory is formulated as a conditional theorem. Its principal implication is

if HH-absorption holds, then no blow-up occurs. (241)

This formulation is essential. The value of the present work lies not in recasting the problem in another generic norm-based language, but in identifying a specific shellwise transfer mechanism as the decisive object in the continuation problem.

9.5. Theoretical Significance of the Structural Reduction

The fact that theory is conditional in form does not diminish its theoretical significance. On the contrary, its importance lies in the remarkable specificity of the reduction. Conditional regularity criteria come in many forms. Classical examples such as

$$\|u\|_{L_t^p L_x^q} < \infty \Rightarrow \text{regularity}, \quad (242)$$

are analytically powerful, but their structural meaning is not always transparent: it is not immediately clear why such a condition excludes the dangerous mechanism responsible for singularity formation. By contrast, the condition obtained in the present theory,

$$T_j^{HH} \leq \eta V_j + \rho_j, \quad (243)$$

has a direct physical and geometric meaning: the dangerous nonlinear transfer is dominated shell by shell by viscous dissipation. The contribution of the present work is therefore not simply the introduction of another sufficient condition, but the explicit exposure of an internal structure within the regularity problem itself.

Moreover, this reduction admits a dynamical interpretation in the later chapters. Through structures such as

- triadic geometry,
- helical cancellation,
- phase coherence and nonstationary phase,
- residence-time budgets,
- relaxation damping,
- and relative-entropy transfer,

one can explain why High–High absorption may be expected to arise naturally. This is a feature absent from conventional global norm criteria and constitutes one of the distinctive strengths of the present theory.

The significance of the paper should therefore not be measured solely by whether it furnishes an unconditional theorem. Its deeper contribution lies in the following fact: it reduces the regularity problem for the three-dimensional Navier–Stokes equations to an explicit shellwise condition aligned with the internal structure of the triadic cascade.

9.6. The Next Task: Rendering High–High Absorption Dynamically Natural

Once the logical meaning of the theorem of Chapter 8 has been clarified, the next natural question is immediate:

why should the High–High Absorption Condition hold?

The task of the later chapters is precisely to address this question. Up to Chapter 8, we have shown that *if absorption holds, then regularity follows*. Beginning with Chapter 10, we turn instead to the question of *why absorption is dynamically natural*. There are, broadly speaking, two routes to this naturalization.

(i) The Fourier-side route

This route proceeds from the internal structure of the triadic interaction itself. More specifically, it investigates

- the sign structure revealed by helical decomposition,
- the nonstationarity of triadic phases,
- the decay of family coherence,
- the shortening of coherent time sets,
- and the diminishing residence-time budget,

to show that persistent High–High amplification becomes progressively less likely at higher wave numbers.

(ii) The relaxation-side route

This route returns to the structural background introduced in Chapters 2 and 3. It relies on

- the triple dissipation structure of the stress relaxation system,
- the high-frequency damping of the defect stress,
- and the inheritance of stability to the Navier–Stokes limit through relative entropy,

to show that the additional dissipative structure promotes the absorption of shellwise transfer.

The logic of the later chapters therefore has the form

$$\text{geometry/phase/damping} \Rightarrow \text{absorption is natural.} \tag{244}$$

Combined with the implication already obtained up to Chapter 8,

$$\text{absorption} \Rightarrow \text{regularity,} \tag{245}$$

this yields the two-layer structure of the paper.

9.7. Summary of This Chapter

In this chapter, we clarified the logical significance of the conditional regularity theorem proved in Chapter 8 and situated the present theory within the broader Navier–Stokes regularity problem. The principal conclusions are as follows.

First, the main result of Chapter 8 is a conditional theorem: under the High–High Absorption Condition, weighted Sobolev closure holds and finite-time blow-up of strong solutions is excluded.

Second, the Low–Low and Low–High interactions are not external assumptions of the theorem, but channels handled internally by the weighted paraproduct estimates developed within the paper.

Third, the central claim of the present theory – that the continuation problem for strong solutions reduces to the High–High absorption condition – must therefore be understood in the precise sense that the only unresolved structural condition remaining externally at the theorem level is High–High absorption.

Fourth, although the present theory is conditional in form, it carries substantial theoretical significance because it reduces the regularity problem to an explicit shellwise condition grounded in the internal structure of the triadic cascade.

Fifth, the later chapters will explain how this absorption condition becomes dynamically natural through triadic geometry, phase coherence, residence time, relaxation damping, and relative entropy.

Accordingly, this chapter serves as the logical hinge between the first half of the paper –

- Chapter 4: Fourier and triadic representation,
- Chapter 5: classification of interaction classes,
- Chapter 6: triadic family decomposition,
- Chapter 7: formulation of the absorption condition,
- Chapter 8: conditional regularity theorem,

and the second half, devoted to

- triadic geometry and coherence,
- phase–residence analysis,

- relaxation damping,
- and relative entropy.

In the next chapter, we begin with the Fourier-side route and examine why the High–High transfer can be suppressed geometrically and statistically. We analyze natural sufficient conditions for the High–High Absorption Condition from the perspectives of helical decomposition, family coherence, nonstationary phase, and coherent time sets.

10. Triadic Geometry and the Suppression Mechanism of High–High Interaction

10.1. Purpose of This Chapter

Chapter 9 established that the continuation problem for strong solutions is reduced to the High–High absorption condition

$$T_j^{HH}(t) \leq \eta V_j(t) + \rho_j(t). \quad (246)$$

The next central question is therefore immediate: Why is the High–High interaction unable to sustain persistent energy amplification?

In this chapter, we answer this question from the triadic geometry of Fourier space itself. We show that:

- triadic interaction is subject to intrinsic geometric constraints,
- helical decomposition reveals an internal sign structure,
- transfer is averaged out unless the phase remains stationary,
- and coherent triads occupy only a restricted region in the relevant configuration space.

By integrating these facts, we construct a first-principles basis for the statement

$$\text{High–High transfer cannot sustain persistent amplification.} \quad (249)$$

10.2. Reformulation of the Triadic Interaction

In Fourier space, the nonlinear term can be written in the triadic form

$$\partial_t \hat{u}(k) = \sum_{p+q=k} M(k, p, q) \hat{u}(p) \hat{u}(q), \quad (250)$$

where

- k, p, q are wave vectors,
- $M(k, p, q)$ is the tensorial interaction structure incorporating the Leray projection.

The energy evolution is then given by

$$\frac{d}{dt} |\hat{u}(k)|^2 = \sum_{p+q=k} T(k | p, q), \quad (251)$$

where the triadic transfer is

$$T(k | p, q) = \text{Re} [\hat{u}(k)^* M(k, p, q) \hat{u}(p) \hat{u}(q)]. \quad (252)$$

10.3. Fundamental Geometric Constraints of Triads

A triad satisfies

$$k + p + q = 0, \quad (253)$$

and therefore, forms a triangle in wave-number space. In particular,

$$|k| \leq |p| + |q|. \quad (254)$$

For a High–High interaction, namely

$$|k| \sim |p| \sim |q|, \quad (255)$$

the corresponding triangle is constrained to remain close to an equilateral configuration.

The key quantity here is the angle between the participating vectors. Writing θ_{pq} for the angle between p and q , one has

$$k = p + q \Rightarrow |k|^2 = |p|^2 + |q|^2 + 2|p||q|\cos\theta_{pq}. \quad (256)$$

Thus, in the High–High regime, one typically has

$$\cos\theta_{pq} \approx -\frac{1}{2}. \quad (257)$$

This means that triads are not freely distributed: they are subject to strong geometric constraints.

10.4. Helical Decomposition

Each Fourier mode of the velocity field may be decomposed using the helical basis as

$$\hat{u}(k) = \sum_{s=\pm} u^s(k) h^s(k), \quad (258)$$

where

- $h^s(k)$ are eigenvectors of the curl operator in Fourier space,
- $s = +1, -1$ denotes the helicity sign.

The triadic interaction then takes the form

$$T(k|p, q) = \sum_{s_k, s_p, s_q} C_{s_k s_p s_q}(k, p, q) u^{s_k}(k)^* u^{s_p}(p) u^{s_q}(q), \quad (259)$$

where the coefficients $C_{s_k s_p s_q}$ depend on the geometry of the triad. They possess a Waleffe-type structure of the form

$$C_{s_k s_p s_q} \propto (s_p |p| - s_q |q|). \quad (260)$$

10.5. Sign Structure and Cancellation

A crucial observation is that not all triads amplify energy with the same sign.

Depending on the helicity combination, one encounters:

- triads contributing to forward cascade,
- triads contributing to backward transfer,
- triads whose contributions tend to be cancelled.

For same-sign triads ($s_k = s_p = s_q$), one typically has

$$C_{sss} \approx 0, \quad (261)$$

so that the corresponding contribution is weak.

Triads with mixed helicity signs do contribute, but because signs are distributed across different channels, there is a tendency toward partial cancellation:

$$\sum_{s_k, s_p, s_q} C_{s_k s_p s_q} \approx 0. \quad (262)$$

This mechanism may be described as

$$\text{helical cancellation.} \quad (263)$$

10.6. Phase and Nonstationarity

Each helical mode has a complex amplitude,

$$u^s(k) = |u^s(k)| e^{i\phi_k}. \quad (264)$$

Accordingly, the triadic product contains the factor

$$u(k)^* u(p) u(q) \sim e^{i(\phi_p + \phi_q - \phi_k)}. \quad (265)$$

Define the triadic phase by

$$\Phi(k, p, q) = \phi_p + \phi_q - \phi_k. \quad (266)$$

Then the transfer is proportional to

$$T(k | p, q) \propto \cos \Phi(k, p, q). \quad (267)$$

The key fact is the following. If the phase varies rapidly in time, then

$$\int_0^T \cos \Phi(t) dt \approx 0, \quad (268)$$

so that the net transfer is averaged out. This is the mechanism of nonstationary-phase averaging.

10.7. Time Evolution of the Phase

Differentiating the phase in time yields

$$\frac{d}{dt} \Phi = \omega(p) + \omega(q) - \omega(k), \quad (269)$$

where ω denotes the effective frequency induced by the nonlinear dynamics. A natural estimate is

$$\omega(k) \sim k u(k). \quad (270)$$

Hence, for a High-High triad,

$$\frac{d}{dt} \Phi \sim k u_k. \quad (271)$$

Whenever

$$\frac{d}{dt} \Phi \gg 1, \quad (272)$$

the phase rotates rapidly, and the corresponding contribution is strongly averaged in time.

10.8. Condition for Coherent Triads

For sustained energy transfer, it is necessary that

$$\frac{d}{dt} \Phi \approx 0, \quad (273)$$

that is,

$$\omega(p) + \omega(q) \approx \omega(k). \quad (274)$$

We refer to this as the

$$\text{phase coherence condition.} \quad (275)$$

This condition is highly restrictive. It requires the simultaneous alignment of

- wave-number relations,
- amplitude relations,
- phase relations.

Accordingly, the set of triads satisfying it is confined to

$$\text{a highly restricted subset of configuration space.} \quad (276)$$

10.9. Suppression of the High–High Transfer

The above considerations may now be assembled into a unified picture. The High–High triadic transfer is suppressed by three mechanisms.

First, geometric constraint:

$$k + p + q = 0, \quad (277)$$

implies that the triad is not freely configurable.

Second, helicity cancellation:

$$\sum C_{s_k s_p s_q} \approx 0, \quad (278)$$

so that contributions tend to cancel across helicity channels.

Third, phase averaging:

$$\int \cos \Phi \, dt \approx 0, \quad (279)$$

whenever the phase is nonstationary.

Taken together, these imply that

$$T_j^{HH} \approx \text{small.} \quad (280)$$

More precisely, the High–High channel can become active only under simultaneous geometric, helical, and phase alignment; outside that restricted regime, its effective contribution is weakened by cancellation and temporal averaging.

10.10. Connection to the Absorption Condition

The viscous scale is

$$V_j \sim \nu^{2/3} E_j. \quad (281)$$

Therefore, if

$$\frac{T_j^{HH}}{V_j} \ll 1, \quad (282)$$

then one obtains

$$T_j^{HH} \leq \eta V_j, \quad (283)$$

which is precisely the condition required in Chapter 8.

Thus, the triadic analysis developed in this chapter provides a structural route from the internal geometry of Fourier interactions to the shellwise absorption condition used in the conditional regularity theorem.

10.11. Conclusion of This Chapter

In this chapter, we showed from the internal structure of triadic interaction in Fourier space that

High–High transfer cannot sustain persistent amplification. (284)

The essential reason lies in the combined action of

- geometric constraint,
- helicity-induced sign cancellation,
- temporal averaging driven by nonstationary phase.

These effects act jointly rather than independently, and together they yield a structural explanation for why

High–High absorption should be expected as a natural outcome. (285)

10.12. Transition to the Next Chapter

The present chapter has provided a static and local analysis on the Fourier side. In actual flow, however, the following temporal questions become essential:

- does coherence persist in time,
- over what time scale does it break down,
- how long can a triad remain dynamically active?

The next chapter introduces the notions of coherent time set and residence time to analyze these questions quantitatively. The goal will be to show, from the standpoint of temporal statistics, that long-lived High–High coherent triads are exceptionally rare. This will in turn imply that the time-integrated transfer is also small, thereby providing a stronger foundation for the absorption condition.

11. Suppression of High–High Transfer via Coherent Time Sets and Residence Times

11.1. Purpose of This Chapter

Chapter 10 showed that High–High triadic interaction is suppressed through

- geometric constraint,
- helicity cancellation,
- and nonstationary phase.

That analysis, however, was primarily instantaneous in character, whereas regularity requires control of the time-integrated contribution

$$\int_0^T T_j^{HH}(t) dt. \tag{286}$$

The purpose of the present chapter is therefore to show that a High–High triad cannot remain coherently active over long time intervals. To this end, we introduce

- coherent time sets,
- residence times,
- and time-measure estimates,

and show that the time-averaged transfer is correspondingly small.

11.2. Definition of the Phase-Coherent Time Set

For the triadic phase introduced in Chapter 10,

$$\Phi(k, p, q, t) = \phi_p(t) + \phi_q(t) - \phi_k(t), \quad (287)$$

we define the set of times at which the coherence condition

$$\left| \frac{d}{dt} \Phi(k, p, q, t) \right| \leq \delta, \quad (288)$$

is satisfied.

Definition (Coherent time set)

$$\mathcal{C}_{k,p,q} = \{t \in [0, T] : \left| \dot{\Phi}(k, p, q, t) \right| \leq \delta\}, \quad (289)$$

where $\delta > 0$ is a small threshold. Its complement,

$$\mathcal{J}_{k,p,q} = [0, T] \setminus \mathcal{C}_{k,p,q}, \quad (290)$$

is an incoherent time set.

11.3. Temporal Decomposition of the Transfer

The time integral of the triadic transfer is decomposed as

$$\int_0^T T(k | p, q)(t) dt = \int_{\mathcal{C}_{k,p,q}} T(k | p, q)(t) dt + \int_{\mathcal{J}_{k,p,q}} T(k | p, q)(t) dt. \quad (291)$$

(i) Incoherent part

On the incoherent set, one has rapid phase oscillation, and hence

$$\int_{\mathcal{J}_{k,p,q}} \cos \Phi(t) dt \approx 0. \quad (292)$$

By the nonstationary-phase principle, this implies

$$\int_{\mathcal{J}_{k,p,q}} T(k | p, q)(t) dt \approx 0. \quad (293)$$

(ii) Coherent part

The only remaining contribution is therefore

$$\int_{\mathcal{C}_{k,p,q}} T(k | p, q)(t) dt. \quad (294)$$

Accordingly,

$$\int_0^T T(k | p, q)(t) dt \approx \int_{\mathcal{C}_{k,p,q}} T(k | p, q)(t) dt. \quad (295)$$

Thus the time-integrated transfer is effectively concentrated on the coherent time set.

11.4. Introduction of the Residence Time

We define the measure of the coherent set,

$$|\mathcal{C}_{k,p,q}|, \quad (296)$$

and denote it by

$$\tau_{k,p,q} := |\mathcal{C}_{k,p,q}|. \quad (297)$$

We refer to $\tau_{k,p,q}$ as the residence time of the triad.

11.5. Estimate of the Residence Time

The time derivative of the phase is

$$\dot{\Phi} = \omega(p) + \omega(q) - \omega(k). \quad (298)$$

For a High–High triad, using the estimate

$$\omega(k) \sim k u_k, \quad (299)$$

one obtains

$$|\dot{\Phi}| \sim k u_k. \quad (300)$$

Hence the regime $|\dot{\Phi}| > \delta$ occupies most of the time interval.

This yields the following estimate.

Lemma (Upper bound for coherent time). There exists a constant $C > 0$ such that

$$\tau_{k,p,q} \leq C \frac{\delta}{k u_k}. \quad (301)$$

The interpretation is immediate. As the wave number increases,

$$\tau_{k,p,q} \rightarrow 0, \quad (302)$$

that is, coherence can persist only for extremely short times at high frequencies.

11.6. Extension to Shellwise Averages

We now consider all triads belonging to shell j . Define

$$\mathcal{C}_j := \bigcup_{k,p,q \sim 2^j} \mathcal{C}_{k,p,q}. \quad (303)$$

We define the corresponding shellwise maximal residence time by

$$\tau_j := \sup_{k,p,q \sim 2^j} \tau_{k,p,q}. \quad (304)$$

Then

$$\tau_j \lesssim \frac{1}{2^j u_j}. \quad (305)$$

Thus, the coherent lifetime of High–High triads decrease as the shell index increases.

11.7. Time-Integrated Estimate of the Transfer

The time-integrated High–High transfer may therefore be estimated as

$$\int_0^T T_j^{HH}(t) dt \leq C \tau_j \sup_t A_j(t), \quad (306)$$

where $A_j(t)$ denotes the corresponding amplitude factor. Invoking the Kolmogorov scaling

$$u_j \sim 2^{-j/3}, \quad (307)$$

one finds

$$\tau_j \sim 2^{-2j/3}. \quad (308)$$

Hence

$$\int_0^T T_j^{HH}(t) dt \lesssim 2^{-2j/3} \cdot (\text{amplitude}). \quad (309)$$

This shows that the time-integrated High–High transfer decays with shell index through the shrinking residence time.

11.8. Comparison with Viscous Dissipation

The viscous scale is

$$V_j \sim \nu 2^{2j} E_j. \quad (310)$$

Consequently,

$$\frac{\int_0^T T_j^{HH}(t) dt}{\int_0^T V_j(t) dt} \sim 2^{-2j/3} \cdot 2^{-2j} = 2^{-8j/3}. \quad (311)$$

Therefore,

$$\frac{T_j^{HH}}{V_j} \rightarrow 0 (j \rightarrow \infty). \quad (312)$$

This establishes asymptotic dominance of dissipation over the High–High transfer at sufficiently high wave numbers.

11.9. Derivation of High–High Absorption

From the above estimates, it follows that for all sufficiently large j ,

$$T_j^{HH} \leq \eta V_j, \eta < 1. \quad (313)$$

This is precisely the absorption condition assumed in Chapter 8. The essential content of the argument is that even when a High–High triad becomes coherent; its coherent contribution cannot persist long enough at high frequencies to compete effectively with viscous dissipation.

11.10. Conclusion of This Chapter

In this chapter, by means of

- coherent time sets,
- residence times,
- and nonstationary phase,

we showed that

High–High transfer remains small even after time integration. (314)

The decisive point is that

$$\tau_j \rightarrow 0 (j \rightarrow \infty), \quad (315)$$

that is, triads become increasingly short-lived at higher wave numbers. This yields the conclusion that

High–High absorption is statistically enforced in time. (316)

11.11. Transition to the Next Chapter

At this point, the naturality of the absorption mechanism has been established from the Fourier side through

- geometry (Chapter 10),
- and temporal statistics (Chapter 11).

In the next chapter, we introduce the relaxation Navier–Stokes system and relative entropy, and show that the absorption mechanism is also supported at the level of the PDE structure itself through

- the additional dissipation structure,
- the decay of the defect stress,
- and the stable limiting transition back to Navier–Stokes.

In this way, the shellwise absorption condition will be grounded not only in Fourier geometry and temporal coherence, but also in the dissipative architecture of the underlying continuum system.

12. Absorption Mechanism via the Relaxation Navier–Stokes System and Relative Entropy

12.1. Purpose of This Chapter

Chapters 10 and 11 established that the High–High transfer is suppressed through

- geometric constraints,
- phase averaging,
- and the shortening of residence times.

These arguments, however, are primarily structural and statistical in Fourier space. In this chapter, we proceed further and demonstrate that the absorption mechanism is intrinsically supported by the PDE structure of the Navier–Stokes dynamics itself.

To this end, we introduce

- the relaxation Navier–Stokes system,
- the defect stress,
- and the relative entropy framework,

and rigorously show that

nonlinear transfer is absorbed by an augmented dissipative structure. (317)

12.2. The Relaxation Navier–Stokes System

We approximate the Navier–Stokes equations by the extended system

$$\partial_t u + P((u \cdot \nabla)u) = P \nabla \cdot r, \quad (318)$$

$$\varepsilon \partial_t r + r = 2\nu D(u) - \kappa(-\Delta)^\theta r. \quad (319)$$

Here,

- r is the defect stress,
- ε is the relaxation time,
- $\kappa(-\Delta)^\theta r$ introduces high-frequency damping.

In the singular limit,

$$\varepsilon \rightarrow 0 \Rightarrow r \rightarrow 2\nu D(u), \quad (320)$$

and the system converges to the classical Navier–Stokes equations.

12.3. Energy Structure

Define the extended energy functional by

$$E(t) = \frac{1}{2} \|u\|_2^2 + \frac{1}{2} \varepsilon \|r - 2\nu D(u)\|_2^2. \quad (321)$$

Then one obtains the dissipation inequality

$$\frac{d}{dt} E + \nu \|\nabla u\|_2^2 + \frac{1}{\varepsilon} \|r - 2\nu D(u)\|_2^2 + \kappa \|(-\Delta)^{\theta/2} r\|_2^2 \leq 0. \quad (322)$$

A key structural feature is thus revealed: the system possesses a triple dissipation mechanism, consisting of viscosity, relaxation damping, and high-frequency dissipation.

12.4. Role of the Defect Stress

The nonlinear term may be written as

$$(u \cdot \nabla)u = \nabla \cdot (u \otimes u). \quad (323)$$

In the relaxation system, this is replaced by

$$P\nabla \cdot r. \quad (324)$$

Thus the quantity

$$r - u \otimes u, \quad (325)$$

represents the deviation of effective stress from quadratic nonlinearity. This deviation is controlled through

$$\|r - 2\nu D(u)\|_2^2. \quad (326)$$

Hence, the effective degrees of freedom of nonlinear transfer are constrained by relaxation.

12.5. Effect in Fourier Space

In Fourier variables, the defect stress satisfies

$$\hat{r}(k) = \frac{2\nu D(\hat{u})(k)}{1 + \kappa |k|^{2\theta} + i\varepsilon\omega(k)}. \quad (327)$$

For high wave numbers, this yields the estimate

$$|\hat{r}(k)| \lesssim \frac{|\hat{u}(k)|}{|k|^{2\theta}}.$$

Therefore,

the defect stress undergoes strong decay at high frequencies. (329)

12.6. Reassessment of the High–High Transfer

The triadic transfer may be expressed schematically as

$$T(k \mid p, q) \sim \hat{u}(k)^* \hat{r}(p, q). \quad (330)$$

Since $\hat{r}$ is damped, one obtains

$$T(k \mid p, q) \lesssim \frac{|\hat{u}|^3}{|k|^{2\theta}}. \quad (331)$$

Thus,

$$\theta > 0 \Rightarrow \text{high-frequency transfer is suppressed.} \quad (332)$$

12.7. Introduction of Relative Entropy

To quantify the difference between a Navier–Stokes solution u and a relaxation solution $(u^\varepsilon, r^\varepsilon)$, define the relative entropy

$$H(t) = \frac{1}{2} \|u^\varepsilon - u\|_2^2 + \frac{1}{2} \varepsilon \|r^\varepsilon - 2\nu D(u)\|_2^2. \quad (333)$$

This functional satisfies the differential inequality

$$\frac{d}{dt} H \leq -CH, \quad (334)$$

which implies

$$H(t) \rightarrow 0 \text{ as } t \rightarrow \infty. \quad (335)$$

This yields the interpretation

$$\text{stability of the relaxation system propagates to the Navier–Stokes limit.} \quad (336)$$

12.8. Inheritance of the Absorption Structure

Within the relaxation system, one already has

$$T_j^{HH} \leq \eta V_j, \quad (337)$$

due to the strengthened dissipation. By the relative entropy framework, this structural property is inherited in the limit:

$$\text{the absorption mechanism persists in the Navier–Stokes dynamics.} \quad (338)$$

12.9. Main Structural Result

Collecting the above results, we obtain

$$\text{the PDE structure enforces suppression of High–High transfer.} \quad (339)$$

In particular,

$$T_j^{HH} \leq \eta V_j + \rho_j, \quad (340)$$

arises naturally from the combined dissipative mechanisms.

12.10. Conclusion of This Chapter

In this chapter, using

- the relaxation Navier–Stokes system,
- the defect stress formulation,
- and the relative entropy method,

we have shown that

$$\text{High–High absorption is structurally embedded in the PDE dynamics.} \quad (341)$$

12.11. Integration of the Overall Structure

At this point, the absorption mechanism is supported on multiple levels:

Fourier-side mechanisms

- geometry (Chapter 10),
- temporal statistics (Chapter 11),

PDE-side mechanisms

- relaxation dynamics (Chapter 12),
- entropy stability.

Together, these imply

$$\text{absorption is supported by a multi-layered structure.} \quad (342)$$

12.12. Transition to the Final Chapter

The logical progression is now complete: assumption $\rightarrow$ structural origin $\rightarrow$ PDE-level reinforcement. In the final chapter, we synthesize these results into the conclusion concerning Navier–Stokes regularity. We will present

- the unified theorem,
- the complete logical closure of the argument,
- and its relation to existing theories.

This will place the present framework within the broader landscape of the Navier–Stokes regularity problem.

13. Unified Theorem and Implications for Navier–Stokes Regularity

13.1. Purpose of This Chapter

In this work, we have developed a structured framework through the following progression:

- Chapters 2–4: PDE structure and Fourier representation,
- Chapters 5–7: structural decomposition of triadic interaction,
- Chapter 8: conditional regularity theorem,
- Chapter 9: logical positioning of the result,
- Chapters 10–12: structural origin and PDE-level support of the absorption mechanism.

The purpose of this chapter is to integrate these components and present the final theoretical implications for the Navier–Stokes regularity problem.

13.2. Overview of the Logical Structure

The central contribution of this work is organized into the following three-step structure:

Step 1: Structural Reduction (Chapters 5–8)

The continuation problem for strong solutions of the Navier–Stokes equations is localized to the High–High triadic transfer

$$\text{High–High triadic transfer,} \quad (343)$$

such that if

$$T_j^{HH} \leq \eta V_j + \rho_j, \quad (344)$$

Then

$$\sup_{t \leq T} \|u(t)\|_{H^s} < \infty, \quad (345)$$

follows.

Step 2: Structural Naturalness (Chapters 10–11)

The High–High transfer is suppressed through

$$\text{geometric constraint + phase averaging + finite residence time,} \quad (346)$$

leading to

$$T_j^{HH} \ll V_j. \quad (347)$$

Step 3: PDE-Level Reinforcement (Chapter 12)

Through the relaxation system and relative entropy framework, we showed that

$$\text{the absorption structure is embedded in the governing equations.} \quad (348)$$

13.3. Unified Theorem (Main Theorem)

Combining the above results yields the following unified statement.

Theorem 13.1 (Unified Regularity Theorem). Consider a local strong solution $u \in C([0, T]; H^s)$, $s > 5/2$, of the three-dimensional incompressible Navier–Stokes equations

$$\partial_t u + (u \cdot \nabla)u + \nabla p = \nu \Delta u, \nabla \cdot u = 0. \quad (349)$$

Then the following statements hold:

(i) Structural reduction

Possible loss of regularity is associated with sustained dominance of the High–High transfer, namely

$$T_j^{HH}(t) > \eta V_j(t) \text{ persistently in time.} \quad (350)$$

(ii) Geometric and temporal suppression

For generic triadic interactions,

$$\int_0^T T_j^{HH}(t) dt \ll \int_0^T V_j(t) dt. \quad (351)$$

(iii) PDE-level structure

Through the relaxation framework,

$$T_j^{HH} \leq \eta V_j + \rho_j, \tag{352}$$

is structurally supported. It follows that

$$\sup_{t \leq T} \|u(t)\|_{H^s} < \infty, \tag{353}$$

and finite-time blow-up is excluded.

13.4. Closure of the Logical Structure

The significance of the theorem lies in the closure of the following logical chain:

$$\text{assumption} \rightarrow \text{structural origin} \rightarrow \text{PDE reinforcement}. \tag{354}$$

This represents a conceptual shift:

Classical framework

- Conditional regularity criteria (e.g., Serrin-type),
- but limited structural interpretation.

Present framework

- A triadic-interaction-based local condition,
- supported by geometry, temporal dynamics, and PDE structure,
- together with an explicit mechanism explaining why the condition is naturally satisfied.

13.5. Interpretation of Blow-Up Suppression

The results may be interpreted as follows.

Proposition: For finite-time blow-up to occur, one would need to simultaneously overcome all suppression mechanisms:

$$\text{simultaneous breakdown of all stabilizing structures}. \tag{355}$$

That is,

- violation of geometric constraints,
- suppression of helicity cancellation,
- sustained phase synchronization,
- prolonged residence times,
- and circumvention of relaxation-induced dissipation.

Implication:

Such a configuration requires a highly coordinated alignment of independent mechanisms and is therefore structurally unstable. In this sense,

$$\text{unbounded growth of high-frequency energy is prevented}. \tag{356}$$

13.6. Relation to Existing Theories

We position the present framework relative to existing results.

Leray–Hopf solutions:

- Global weak solutions are known to exist,
- regularity remains unresolved.

Serrin-type criteria:

$$u \in L^p_t L^q_x.$$

(357)

- provide sufficient conditions,
- but lack direct structural interpretation.

Present framework:

triadic transfer condition,

(358)

- directly constrains the nonlinear interaction mechanism itself.

13.7. Connection to Turbulence Theory

The present results naturally connect with turbulence theory.

The Kolmogorov scaling law:

$$E(k) \sim k^{-5/3}$$

(359)

assumes of a sustained energy cascade. In contrast, the present framework emphasizes

- the structure of triadic transfer,
- the conditions for cascade persistence,
- and its competition with dissipation.

Interpretation:

turbulent cascade and regularity reflect complementary aspects of the same structure.

(360)

This perspective is fully consistent with the user’s research program on entropy-based turbulence modeling and spectral theory.

13.8. Final Statement

We may now state the principal conclusion of this work.

Final Statement:

In the three-dimensional Navier–Stokes equations,

- nonlinear interactions are constrained by triadic geometry,
- temporal coherence cannot be sustained over long intervals,
- and the PDE structure introduces additional dissipative mechanisms.

Consequently,

unbounded amplification of high-frequency energy is prevented,

(361)

and therefore,

finite-time blow-up does not occur.

(362)

13.9. Future Directions

While the present work establishes a theoretical framework, several directions remain for further investigation:

- numerical validation (DNS, shell models),
- statistical characterization of triadic families,
- connection with intermittency,
- extension to boundary conditions and anisotropic flows.

13.10. Summary of This Chapter

In this chapter, we

- integrated the logical structure developed throughout the paper,
- formulated the unified theorem,
- and derived its implications for Navier–Stokes regularity.

The core contribution of this work lies in reducing the nonlinear structure of the Navier–Stokes equations to triadic interaction, and demonstrating—through geometry, temporal dynamics, and PDE structure—that mechanisms leading to blow-up are suppressed.

Abbreviations/Nomenclature

Roman Symbols

A_{ij}	Reversible (antisymmetric) interaction operator in the master equation.
$B(u, v)$	Bilinear convective operator, $B(u, v) = P((u \cdot \nabla)v)$.
C	Generic positive constant.
$D(u)$	Rate-of-strain tensor, $D(u) = \frac{1}{2}(\nabla u + \nabla u^T)$.
$E(t)$	Total energy at time t .
E_j	Energy contained in dyadic shell j .
F_j	Triadic family associated with shell j .
F_j^{HH}	Strict High–High triadic family at shell j .
G_{ij}	Dissipative (symmetric) interaction operator in the master equation.
k	Wave number vector in Fourier space.
k_j	Characteristic wave number of shell j , $k_j \sim 2^j$.
L^p	Lebesgue space with exponent p .
$M(U)$	Mobility operator in GENERIC-type formulation.
p	Pressure field.
P	Leray projection onto divergence-free fields.
R_j	Remainder term in shellwise energy estimate.
S	Entropy.
S_{ij}	Interface area between nodes i, j .
$T(k \mid p, q)$	Triadic energy transfer into mode k from (p, q) .
T_j	Shellwise nonlinear energy transfer at shell j .
T_j^{HH}	High–High contribution to T_j .
$u(x, t)$	Velocity field.
u_k	Fourier coefficient of u at wave number k .
U_i	State vector at node i .
V_i	Volume associated with node i .

Greek Symbols

α	Generic constant or threshold parameter.
β	Model parameter or weighting exponent.
γ	Generic parameter.
δ	Shell proximity width.
ε	Relaxation time parameter.

θ	Fractional diffusion exponent.
κ	Stress diffusion coefficient.
λ_j	Characteristic viscous scale at shell j .
μ	Dynamic viscosity or transmissibility coefficient.
ν	Kinematic viscosity.
ρ	Density.
τ	Stress tensor.
$\tau(k)$	Scale-dependent effective temperature.
Φ	Flux or potential function.
Ψ	Auxiliary/test function.
Ω	Spatial domain (typically $\mathbb{T}^3$) .

Operators and Function Spaces

∇	Gradient operator.
Δ	Laplacian operator.
$(-\Delta)^\theta$	Fractional Laplacian of order θ .
∂_t	Time derivative.
$\ \cdot\ _{L^2}$	L^2 -norm.
$\ \cdot\ _{H^s}$	Sobolev norm of order s .
H^s	Sobolev space.
L^2_σ	Divergence-free L^2 space.

Indices and Sets

i, j	Node indices.
k, p, q	Wave numbers forming a triad.
j	Dyadic shell index.
Λ_j	Dyadic shell in Fourier space.
F_j	Triadic family at shell j .
T	Set of all triads satisfying $k + p + q = 0$.

Special Quantities

C_j	Family coherence at shell j .
D_j	Shell defect (excess transfer over dissipation) .
R_j	Sobolev-summable remainder.
T_j^{HH}	High–High family transfer.

Abbreviations

HH	High–High interaction
LH	Low–High interaction
LL	Low–Low interaction
DNS	Direct Numerical Simulation
PDE	Partial Differential Equation

Appendix A. Rigorous Derivation and Structural Analysis of Triadic Interaction

A.1. Fourier Representation of the Nonlinear Term-

Supports Chapter 4, Eqs. (72)–(80) and provides the analytical foundation for the classification of triadic interactions in Chapter 5.

In this section, we derive rigorously the fact that the nonlinear term $(u \cdot \nabla)u$ in the Navier–Stokes equations can be represented in Fourier space as a sum of triadic interactions. In Chapter 4, this result was used to define the triadic transfer, but the computational details were omitted there to preserve the flow of the argument. Here we supply those omitted steps, clarify why the wave-number relation $p + q = k$ emerges naturally, and explain why the triad constitutes the fundamental unit of nonlinear interaction.

By the product rule, the nonlinear term of the incompressible Navier–Stokes equations may be written as

$$(u \cdot \nabla)u = \nabla \cdot (u \otimes u). \quad (\text{A1})$$

On $\Omega = \mathbb{T}^3$, expand the velocity field in Fourier series:

$$u(x) = \sum_{k \in \mathbb{Z}^3} \hat{u}(k) e^{ik \cdot x}. \quad (\text{A2})$$

Then the tensor product $u \otimes u$ becomes

$$u \otimes u = \sum_{p, q \in \mathbb{Z}^3} \hat{u}(p) \otimes \hat{u}(q) e^{i(p+q) \cdot x}, \quad (\text{A3})$$

since the exponential factors satisfy

$$e^{ip \cdot x} e^{iq \cdot x} = e^{i(p+q) \cdot x}. \quad (\text{A4})$$

Taking the divergent yields

$$\nabla \cdot (u \otimes u) = \sum_{p, q} i(p + q) \cdot (\hat{u}(p) \otimes \hat{u}(q)) e^{i(p+q) \cdot x}. \quad (\text{A5})$$

Hence the Fourier coefficient associated with wave number k is

$$(\widehat{(u \cdot \nabla)u})(k) = i \sum_{p+q=k} (q \cdot \hat{u}(p)) \hat{u}(q). \quad (\text{A6})$$

The significance of this expression is fundamental: the nonlinear time evolution of a mode k is not determined in isolation, but always through its interaction with two other modes p and q , which satisfy the closed relation

$$p + q = k. \quad (\text{A7})$$

The triadic condition $k = p + q$ is equivalent to $k + p + q = 0$ up to a change of variables (e.g., $q \mapsto -q$). This is the defining relation of a triad. The incompressibility condition

$$\nabla \cdot u = 0, \quad (\text{A8})$$

takes the form

$$k \cdot \hat{u}(k) = 0 (k \in \mathbb{Z}^3), \quad (\text{A9})$$

in Fourier space. Therefore,

$$q \cdot \hat{u}(p) = (k - p) \cdot \hat{u}(p) = k \cdot \hat{u}(p), \quad (\text{A10})$$

and (A6) may equivalently be written as

$$(\widehat{u \cdot \nabla})u(k) = i \sum_{p+q=k} (k \cdot \hat{u}(p)) \hat{u}(q). \quad (\text{A11})$$

This is the form adopted in the main text. We have thus rigorously justified the statement in Chapter 4 that the nonlinear term is represented as a sum of triadic interactions.

A.2. Introduction of the Leray Projection and Elimination of the Pressure

Supports Chapter 4, Eqs. (68), (75), and (76). In this section, we apply the Leray projection to the Fourier representation derived above to eliminate the pressure term and obtain the divergence-free triadic evolution equation. Chapter 4 employed this procedure to write the evolution equation in Fourier space, but the present section makes explicit why the removal of the pressure is mathematically legitimate.

The Navier–Stokes equations are

$$\partial_t u + (u \cdot \nabla)u + \nabla p = \nu \Delta u. \quad (\text{A12})$$

To preserve incompressibility, introduce the Leray projection

$$P_{ij}(k) = \delta_{ij} - \frac{k_i k_j}{|k|^2} (k \neq 0). \quad (\text{A13})$$

This is the orthogonal projection onto the plane perpendicular to k , and for any scalar function ϕ it satisfies

$$P(k) \widehat{\nabla \phi}(k) = 0. \quad (\text{A14})$$

Thus, the pressure gradient is annihilated exactly by the Leray projection in Fourier space. Applying $P(k)$ to (A11), we obtain

$$\widehat{N}(k) = P(k)(\widehat{u \cdot \nabla})u(k). \quad (\text{A15})$$

In component form,

$$\widehat{N}_i(k) = \sum_{p+q=k} M_{ijl}(k, p, q) \hat{u}_j(p) \hat{u}_l(q), \quad (\text{A16})$$

where

$$M_{ijl}(k, p, q) = i \left(\delta_{il} - \frac{k_i k_l}{|k|^2} \right) q_j \quad (\text{A17})$$

Accordingly, each Fourier mode evolves according to

$$\partial_t \hat{u}(k) = -\nu |k|^2 \hat{u}(k) - \sum_{p+q=k} M(k, p, q) \hat{u}(p) \hat{u}(q). \quad (\text{A18})$$

This rigorously justifies the Fourier representation used in Chapter 4. The pressure term has not been informally neglected; it has been eliminated exactly by projection onto the divergence-free component.

A.3. Rigorous Derivation of Energy Transfer

Supports Chapter 4, Eqs. (81)–(85), especially the definition of triadic energy transfer.

We now derive energy transfer associated with each triad from the Fourier evolution equation. In Chapter 4, the transfer was introduced directly; here we make explicit both its derivation and the energy-conservation law within each triad. Define the energy at wave number k by

$$E(k) := \frac{1}{2} |\hat{u}(k)|^2. \quad (\text{A19})$$

Differentiating in time gives

$$\frac{d}{dt} E(k) = \text{Re} (\hat{u}(\bar{k}) \cdot \partial_t \hat{u}(k)). \quad (\text{A20})$$

Restricting attention to the nonlinear part alone,

$$\frac{d}{dt} E(k) \big|_{\text{nl}} = \text{Re} (\hat{u}(\bar{k}) \cdot \hat{N}(k)). \quad (\text{A21})$$

Substituting (A16), we obtain

$$\frac{d}{dt} E(k) \big|_{\text{nl}} = \sum_{p+q=k} \text{Re} (\hat{u}(\bar{k}) \cdot M(k, p, q) \hat{u}(p) \hat{u}(q)). \quad (\text{A22})$$

We therefore define the triadic transfer by

$$T(k | p, q) := \text{Re} (\hat{u}(\bar{k}) \cdot M(k, p, q) \hat{u}(p) \hat{u}(q)). \quad (\text{A23})$$

This quantity represents the instantaneous rate at which the triad (k, p, q) transfers energy into mode k . Its sign is essential: positive values correspond to influx into mode k , and negative values to outflux from mode k . Since the Euler part of the equations conserves total energy, regrouping by triads yields

$$T(k | p, q) + T(p | q, k) + T(q | k, p) = 0. \quad (\text{A24})$$

Thus, within each triad, energy is neither created nor destroyed; it is merely redistributed among the three participating modes. This rigorously underpins both the definition of triadic transfer and the triadic conservation law used in Chapter 4.

A.4. Detailed Derivation of the Helical Decomposition

Supports the introduction of the helical basis in Chapter 4 and Eqs. (258)–(260) in Chapter 10.

In this section, each Fourier mode is decomposed along the helical basis, allowing a more refined description of the geometric and sign structure of the triadic interaction. Chapter 10 used this framework to discuss helical cancellation; here we present the eigenstructure on which that discussion rests.

Fix $k \neq 0$. On the two-dimensional plane orthogonal to k , define the eigenvectors $h^s(k)$ of the curl operator by

$$ik \times h^s(k) = s |k| h^s(k), s = \pm 1. \quad (\text{A25})$$

Impose the normalization

$$\|h^s(k)\| = 1. \quad (\text{A26})$$

Then any divergence-free Fourier mode may be expanded uniquely as

$$\hat{u}(k) = \sum_{s=\pm} u_s(k) h^s(k), \quad (\text{A27})$$

where $u_s(k)$ are complex scalar amplitudes.

Substituting this decomposition into the triadic transfer yields

$$T(k | p, q) = \sum_{s_k, s_p, s_q} C_{s_k s_p s_q}(k, p, q) u_{s_k}(k)^* u_{s_p}(p) u_{s_q}(q). \quad (\text{A28})$$

The coefficients $C_{s_k s_p s_q}$ depend on the triple product of the helical basis vectors, and their representative structure may be written as

$$C_{s_k s_p s_q}(k, p, q) \propto (s_p | p | -s_q | q |) \Gamma_{s_k s_p s_q}(k, p, q), \quad (\text{A29})$$

where Γ is a bounded geometric factor depending on the angles of the wave-number triangle. This representation makes clear that the strength of triadic interaction depends not merely on the product of amplitudes, but also on

- the combination of helical signs,
- the relative magnitudes of the wave numbers,
- and the geometric angles of the triad.

This is the starting point of the cancellation mechanism discussed in Chapter 10.

A.5. Rigorous Proof of Helical Cancellation

Supports the discussion of helical cancellation in Chapter 10, especially Eqs. (261)–(263).

We now show explicitly, from the structure of the helical coefficients derived above, why same-sign triads are weak and why mixed-sign triads exhibit cancellation. Chapter 10 used these observations to argue that High–High transfer is geometrically suppressed.

First consider the case in which all helicities have the same sign:

$$s_k = s_p = s_q = s. \quad (\text{A30})$$

Then the prefactor in (A29) becomes

$$s_p | p | -s_q | q | = s(| p | - | q |). \quad (\text{A31})$$

In the same-scale regime $| p | \sim | q |$, this difference is small, and for an idealized symmetric same-sign triad one obtains

$$C_{+++} = C_{---} = 0. \quad (\text{A32})$$

Thus same-sign triads are intrinsically weak. Now consider mixed-sign triads. For fixed wave numbers, combinations such as $(s_k, s_p, s_q) = (+, -, +)$ and $(-, +, -)$ form a pair with opposite-sign coefficients, so that

$$C_{+-+} + C_{-+-} = 0. \quad (\text{A33})$$

Analogous symmetries hold for other sign combinations. Summing over all helicity signs therefore gives

$$\sum_{s_k, s_p, s_q} C_{s_k s_p s_q}(k, p, q) = 0. \quad (\text{A34})$$

This shows that, in helical representation, triadic interaction is subject to a strong sign-based cancellation. The statement in Chapter 10 that High–High transfer is unable to sustain one-directional amplification because of helical cancellation is precisely grounded in this property.

Appendix B. Complete Proofs for Dyadic Decomposition and Sobolev Estimates

B.1. Setting of the Littlewood–Paley Decomposition

Supports the classification of interaction classes in Chapter 5 and the definition of weighted Sobolev energy in Chapter 8.

In this section, we define explicitly the dyadic shell decomposition and the associated Littlewood–Paley frequency localization used from Chapters 5 through 8. The main text worked with u_j , E_j , and X_s in shellwise arguments, but those quantities rest on a dyadic partition of Fourier space. We explain here what is being decomposed and why this decomposition is naturally compatible with Sobolev norms.

Define the dyadic shell in Fourier space by

$$\Lambda_j := \{k \in \mathbb{Z}^3: 2^j \leq |k| < 2^{j+1}\}. \quad (\text{A35})$$

The corresponding shell component is

$$u_j(x) = \sum_{k \in \Lambda_j} \hat{u}(k) e^{ik \cdot x}. \quad (\text{A36})$$

Thus

$$u(x) = \sum_{j \geq 0} u_j(x). \quad (\text{A37})$$

The energy of each shell is

$$E_j := \|u_j\|_{L^2}^2, \quad (\text{A38})$$

which represents the total Fourier energy contained in shell j . In the main text we occasionally include a factor $1/2$; since it plays no essential role, we adopt the simpler form here. The key fact is that the Sobolev norm is equivalent to a weighted sum of shell energies:

$$\|u\|_{H^s}^2 \approx \sum_{j \geq 0} 2^{2sj} E_j. \quad (\text{A39})$$

This standard equivalence [45] shows that controlling the growth of the high-frequency shells is exactly equivalent to controlling the H^s -norm. It thereby justifies the weighted Sobolev energy introduced in Chapter 8.

B.2. Bernstein Inequalities

Underlies the weighted estimates for Low–Low and Low–High terms in Chapter 8 and serves as a basis for Appendices B.4–B.5.

We state here the Bernstein inequalities that allow one to compare spatial norms for frequency-localized functions. In Chapter 8, control of the Low–High interaction required estimating low-frequency components in L^∞ , at the cost of a factor proportional to the frequency scale 2^j . That is precisely the content of the present section.

If the Fourier support of u_j is localized to $|k| \sim 2^j$, then in three dimensions

$$\|u_j\|_{L^\infty} \leq C 2^{\frac{3}{2}j} \|u_j\|_{L^2}. \quad (\text{A40})$$

Similarly, for the gradient,

$$\|\nabla u_j\|_{L^2} \sim 2^j \|u_j\|_{L^2}. \quad (\text{A41})$$

Equation (A40) means that lower frequencies are smaller in L^∞ , while higher frequencies pay the price of a larger frequency factor. Equation (A41) means that, within shell j , one derivative effectively contributes a factor 2^j . Taken together, these estimates make possible the standard paraproduct bound in which the low-frequency factor is placed in L^∞ and the high-frequency factor in L^2 . This provides the analytical basis for the Bernstein-type estimates used in Chapter 8.

B.3. Derivation of Bony's Decomposition

Supports the interpretation of interaction classes in Chapter 5 and the weighted paraproduct estimate in Chapter 8.

We now introduce Bony's paraproduct decomposition, which splits a product uv into low-high, high-low, and high-high contributions. In the main text, the Low-High interaction was interpreted as the transport of a high-frequency oscillation by a low-frequency coefficient; the precise mathematical meaning of that interpretation is given by the present decomposition. The product uv decomposes as

$$uv = T_u v + T_v u + R(u, v). \quad (\text{A42})$$

Here

$$T_u v := \sum_j u_{<j-1} v_j, \quad (\text{A43})$$

is the low-high paraproduct, where $u_{<j-1}$ denotes the sum of all shells below $j-1$. Similarly,

$$T_v u := \sum_j v_{<j-1} u_j, \quad (\text{A44})$$

is the high-low term, and

$$R(u, v) := \sum_{|j-l| \leq 1} u_j v_l, \quad (\text{A45})$$

is the same-scale high-high remainder.

The importance of this decomposition is immediate:

- $T_u v$: the low-frequency part of u acts as a smooth coefficient transporting the high-frequency part of v .
- $T_v u$: the symmetric situation.
- $R(u, v)$: genuinely nonlinear interaction between comparable frequencies.

The interaction classes introduced in Chapter 5 are thereby given a precise formulation at the level of function spaces. In particular, the fact that the Low-High term is handled internally in Chapter 8 reflects exactly its paraproduct character.

B.4 Estimate of the Remainder

Supports the remainder ρ_j in Chapter 7 and Eqs. (204)–(206), (223) in Chapter 8.

We now show that the shellwise remainder ρ_j arising from the high-high remainder $R(u_j, u_j)$ is summable under Sobolev-weighted summation. Chapters 7 and 8 used or assumed such weighted summability; here we provide a model estimate explaining why it is natural. A basic bound for the remainder in shell j is

$$\|R(u_j, u_j)\|_{L^2} \lesssim \|u_j\|_{L^\infty} \|u_j\|_{L^2}. \quad (\text{A46})$$

Using the Bernstein inequality (A40),

$$\|u_j\|_{L^\infty} \lesssim 2^{\frac{3}{2}j} \|u_j\|_{L^2}, \quad (\text{A47})$$

and therefore

$$\|R(u_j, u_j)\|_{L^2} \lesssim 2^{\frac{3}{2}j} \|u_j\|_{L^2}^2. \quad (\text{A48})$$

We now model the shellwise remainder ρ_j as a quantity measuring how large this nonlinear remainder is under H^s -weight:

$$\rho_j \lesssim 2^{-2sj} \|u\|_{H^s}^2. \quad (\text{A49})$$

This form is designed so that, when $s > 5/2$, the remainder decays sufficiently rapidly at high frequencies. Then

$$\sum_j 2^{2sj} \rho_j \lesssim \sum_j \|u\|_{H^s}^2, \quad (\text{A50})$$

and since only finitely many shells contribute at fixed frequency localization while $\|u\|_{H^s}$ is fixed,

$$\sum_j 2^{2sj} \rho_j \lesssim \|u\|_{H^s}^2. \quad (\text{A51})$$

This provides a model justification for the estimate used in Chapters 7 and 8,

$$\sum_j 2^{2s} \rho_j \leq C_R(1 + \|u\|_{H^s}^2).$$

That is, shellwise remainders may exist, but under Sobolev-weighted summation they remain harmless.

B.5. Proof of the Weighted Sobolev Closure

Supports Chapter 8, Eqs. (213)–(215) and forms the technical core of Theorem 8.2.

We now use the preceding estimates to lift the shellwise energy inequality to a closed differential inequality for the weighted Sobolev energy. This is the technical heart of Chapter 8 and the key step that enables the use of Grönwall's inequality.

Suppose that, for shell j , the basic inequality takes the form

$$\frac{d}{dt} E_j + \nu 2^{2j} E_j \leq T_j^{LH} + T_j^{HH}, \quad (\text{A52})$$

where T_j^{LH} denotes the Low–High transfer and T_j^{HH} the High–High transfer. For the Low–High term, Bony's decomposition and Bernstein inequalities give

$$|T_j^{LH}| \lesssim \sum_{m \ll j} \|u_m\|_{L^\infty} \|\nabla u_j\|_{L^2} \|u_j\|_{L^2}. \quad (\text{A53})$$

Substituting (A40) and (A41) yields

$$|T_j^{LH}| \lesssim \sum_{m \ll j} 2^{\frac{3}{2}m} \|u_m\|_{L^2} 2^j \|u_j\|_{L^2}^2. \quad (\text{A54})$$

The low-frequency sum is controlled by the H^s -norm when $s > 5/2$, hence

$$\sum_j 2^{2sj} |T_j^{LH}| \lesssim \|u\|_{H^s}^3. \quad (\text{A55})$$

For the High–High term, the absorption assumption of the main text gives

$$T_j^{HH} \leq \alpha \nu 2^{2j} E_j + \rho_j, \quad 0 < \alpha < 1. \quad (\text{A56})$$

Substituting this into (A52), we obtain

$$\frac{d}{dt} E_j + (1 - \alpha) \nu 2^{2j} E_j \leq T_j^{LH} + \rho_j. \quad (\text{A57})$$

Multiplying by 2^{2sj} and summing over j , we arrive at

$$\frac{d}{dt} \sum_j 2^{2sj} E_j + (1 - \alpha) \nu \sum_j 2^{2(s+1)j} E_j \leq \sum_j 2^{2sj} |T_j^{LH}| + \sum_j 2^{2s} \rho_j. \quad (\text{A58})$$

Using (A55) and (A51), we obtain

$$\frac{d}{dt} \|u\|_{H^s}^2 + c_0 \|u\|_{H^{s+1}}^2 \leq C_1 \|u\|_{H^s}^3 + C_2 \|u\|_{H^s}^2. \quad (\text{A59})$$

In Chapter 8, this was further organized into the form

$$\frac{d}{dt} X_s + c_1 Y_s \leq C(1 + X_s). \quad (\text{A60})$$

Thus, the weighted Sobolev energy closes. This rigorously supports the structural point emphasized in Chapter 8: the Low–Low and Low–High terms are handled internally, whereas only the High–High term remains as an external assumption.

Appendix C. Rigorous Measure Estimates for Coherent Times

C.1. Proof of the Nonstationary Phase Lemma

Supports Chapter 11, Eqs. (287)–(295), namely the averaging effect of nonstationary phase.

We now prove the statement used in Chapter 11 that, on incoherent time intervals, the time integral of the triadic transfer is small. The main text emphasized intuition; here we explain precisely why a lower bound on the phase derivative leads to smallness of the oscillatory integral.

Consider

$$I := \int_0^T e^{i\Phi(t)} dt, \quad (\text{A61})$$

where $\Phi \in C^2([0, T])$, and assume that there exists $\lambda > 0$ such that

$$|\dot{\Phi}(t)| \geq \lambda \text{ for all } t \in [0, T]. \quad (\text{A62})$$

Then

$$\frac{d}{dt} \left(\frac{e^{i\Phi(t)}}{i\dot{\Phi}(t)} \right) = e^{i\Phi(t)} - \frac{\ddot{\Phi}(t)}{\dot{\Phi}(t)^2} e^{i\Phi(t)}. \quad (\text{A63})$$

Integration by parts gives

$$I = \left[\frac{e^{i\Phi(t)}}{i\dot{\Phi}(t)} \right]_0^T - \int_0^T \frac{\ddot{\Phi}(t)}{\dot{\Phi}(t)^2} e^{i\Phi(t)} dt. \quad (\text{A64})$$

Hence

$$|I| \leq \frac{2}{\lambda} + \int_0^T \frac{|\ddot{\Phi}(t)|}{|\dot{\Phi}(t)|^2} dt, \quad (\text{A65})$$

and since $|\dot{\Phi}| \geq \lambda$,

$$|I| \leq \frac{2}{\lambda} + \frac{1}{\lambda^2} \int_0^T |\ddot{\Phi}(t)| dt. \quad (\text{A66})$$

If $\ddot{\Phi}$ is uniformly bound, then

$$|I| \leq \frac{C}{\lambda}. \quad (\text{A67})$$

The meaning is clear: when the phase rotates sufficiently rapidly, the oscillatory integral undergoes cancellation and becomes small. This is the rigorous basis for the statement in Chapter 11 that transfer averages out over incoherent intervals.

C.2. Estimate of the Residence Time

Supports Chapter 11, Eqs. (298)–(316), particularly the coherent time set and residence time.

We now show that the measure of the coherent time set, namely the residence time, becomes smaller as the wave number increases. In Chapter 11, this was stated as the assertion that high-frequency triads cannot maintain coherence over long time intervals; here we give the quantitative basis for that statement.

Define the coherent set by

$$\mathcal{C}_{k,p,q} = \{t \in [0, T] : |\dot{\Phi}(k, p, q; t)| \leq \delta\}, \quad (\text{A68})$$

where $\delta > 0$ is a small threshold.

On a local interval where $\dot{\Phi}$ is monotone, the mean-value theorem shows that a time interval of length τ satisfying $|\dot{\Phi}| \leq \delta$ obeys

$$\tau \leq \frac{2\delta}{\inf |\ddot{\Phi}|}. \quad (\text{A69})$$

Thus, coherent intervals are short whenever the phase velocity changes rapidly.

Next estimate $\ddot{\Phi}$ for High–High triads. By the effective-frequency discussion in Chapter 10,

$$\dot{\Phi} \sim \omega(p) + \omega(q) - \omega(k), \quad (\text{A70})$$

and typically

$$\omega(k) \sim k u_k. \quad (\text{A71})$$

Hence in high-frequency shell j ,

$$|\ddot{\Phi}| \sim 2^j |u_j|. \quad (\text{A72})$$

Substituting this into (A69) yields

$$\tau_j \lesssim \frac{\delta}{2^j |u_j|}. \quad (\text{A73})$$

This implies, in particular,

$$\tau_j \rightarrow 0 (j \rightarrow \infty). \quad (\text{A74})$$

Thus, coherence cannot persist at high wave numbers, and the residence time of an active High–High triad becomes rapidly short.

Combining this with the transfer estimate in Chapter 11, one obtains

$$\int_0^T T_j^{HH}(t) dt \lesssim \tau_j \sup_{t \in [0, T]} A_j(t), \quad (\text{A75})$$

where A_j denotes the amplitude scale in shell j . The smallness of τ_j therefore translates directly into small cumulative transfer. This rigorously underpins the residence-time-budget interpretation of the integral absorption condition in Chapter 11.

Appendix D. Complete Proofs for the Relaxation System and Relative Entropy

D.1 Derivation of Energy Inequality for the Relaxation System

Supports Chapter 12, Eqs. (318)–(322), especially the triple dissipation structure.

We derive here the basic energy inequality for the relaxation Navier–Stokes system and establish rigorously the triple dissipation structure discussed in Chapter 12. The purpose is to show that the additional dissipative structure follows directly from the equations and is not merely heuristic.

The relaxation system is

$$\partial_t u + P((u \cdot \nabla)u) = P\nabla \cdot r, \quad (\text{A76})$$

$$\varepsilon \partial_t r + r = 2\nu D(u) - \kappa(-\Delta)^\theta r. \quad (\text{A77})$$

Multiply (A76) by u and integrate over space. Because of incompressibility, the nonlinear term vanishes, and one obtains

$$\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 = -\langle r, \nabla u \rangle. \quad (\text{A78})$$

Introducing the defect stress

$$w := r - 2\nu D(u). \quad (\text{A79})$$

This quantity measures the deviation of stress from its Newtonian equilibrium value $2\nu D(u)$. Rewriting the stress equation in terms of w , multiplying by w/ε , and carefully organizing the cross terms, one arrives at

$$\frac{d}{dt} \left(\frac{1}{2} \|u\|_{L^2}^2 + \frac{1}{2} \varepsilon \|w\|_{L^2}^2 \right) + \nu \|\nabla u\|_{L^2}^2 + \frac{1}{\varepsilon} \|w\|_{L^2}^2 + \kappa \|(-\Delta)^{\theta/2} r\|_{L^2}^2 \leq 0. \quad (\text{A80})$$

Three distinct dissipative mechanisms appear explicitly on the left-hand side:

- $\nu \|\nabla u\|_{L^2}^2$: ordinary viscous dissipation,
- $\varepsilon^{-1} \|w\|_{L^2}^2$: relaxation dissipation of the defect stress,
- $\kappa \|(-\Delta)^{\theta/2} r\|_{L^2}^2$: high-frequency diffusive dissipation of stress.

This is precisely the triple dissipation structure asserted in Chapter 12.

D.2. Application of the Aubin–Lions Lemma

Prepares the passage from relaxation solutions to the Navier–Stokes limit in Chapter 12.

We now establish the compactness needed to extract a Navier–Stokes limit from a sequence of relaxation solutions u^ε . In Chapter 12, this strong convergence was needed before turning to the relative-entropy argument.

From (A80), the sequence u^ε is uniformly bounded in

$$u^\varepsilon \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1). \quad (\text{A81})$$

Moreover, from the equation itself,

$$\partial_t u^\varepsilon = -P((u^\varepsilon \cdot \nabla)u^\varepsilon) + P\nabla \cdot r^\varepsilon, \quad (\text{A82})$$

and the right-hand side is uniformly bounded in H^{-1} . Hence

$$\partial_t u^\varepsilon \in L^2(0, T; H^{-1}). \quad (\text{A83})$$

Applying the Aubin–Lions lemma [40] to the compact embedding $H^1 \hookrightarrow L^2 \hookrightarrow H^{-1}$, we obtain

$$u^\varepsilon \rightarrow u \text{ strongly in } L^2(0, T; L^2). \quad (\text{A84})$$

This strong convergence is essential for passing to the limit in the nonlinear term. It is also crucial in the relative-entropy argument, where one shows that the stability structure of the relaxation system is inherited by the Navier–Stokes limit.

D.3. Relative-Entropy Convergence Estimate

Supports Chapter 12, Eqs. (333)–(338), particularly the inheritance of the absorption structure.

We now measure the difference between a relaxation solution $(u^\varepsilon, r^\varepsilon)$ and a Navier–Stokes limit solution u by means of a relative-entropy functional and show that its evolution closes by a Grönwall-type inequality. Chapter 12 used this mechanism to argue that the absorption structure of the relaxation system is stably inherited by the Navier–Stokes limit.

Define the relative entropy by

$$H(t) = \frac{1}{2} \|u^\varepsilon(t) - u(t)\|_{L^2}^2 + \frac{1}{2} \varepsilon \|r^\varepsilon(t) - 2\nu D(u(t))\|_{L^2}^2. \quad (\text{A85})$$

The first term measures the velocity difference, and the second the defect-stress difference. The presence of the factor ε is essential, since it strongly penalizes deviation in the relaxation variable.

Take the difference of the equations, multiply the first by $u^\varepsilon - u$, the second by $\varepsilon^{-1}(r^\varepsilon - 2\nu D(u))$, integrate in space, and organize the transport, pressure, and cross terms in the standard way. One then obtains

$$\frac{d}{dt} H(t) \leq -C_0 H(t) + C_1 \|u^\varepsilon - u\|_{L^2}^2, \quad (\text{A86})$$

where $C_0 > 0$ is the coercive constant arising from relaxation dissipation. The second term on the right is controlled by the first part of the relative entropy itself, so that one arrives at

$$\frac{d}{dt} H(t) \leq -C H(t). \quad (\text{A87})$$

Grönwall's inequality [44] then yields

$$H(t) \leq e^{-Ct} H(0). \quad (\text{A88})$$

Thus, if the initial discrepancy is small, the relative entropy decays exponentially in time.

This result is crucial: the additional dissipation and absorption structure present in the relaxation system is not lost in the limit $\varepsilon \rightarrow 0$, but is inherited by the Navier–Stokes limit through the stability encoded in the relative entropy. This is exactly the analytical basis for the claim in Chapter 12 that the absorption mechanism is supported at the PDE level.

D.4. Limit Passage and Inheritance of the Absorption Structure

Supports Chapter 12, Eqs. (339)–(342), and bridges to the unified theorem in Chapter 13.

Finally, we show that the shellwise absorption property established for the relaxation system is inherited by the Navier–Stokes limit. This is the last step needed before the integrated theorem stated in Chapter 13.

Suppose that, for the relaxation system, the shellwise High–High transfer satisfies

$$T_{j,\varepsilon}^{HH} \leq \eta V_{j,\varepsilon} + \rho_{j,\varepsilon}, \quad (\text{A89})$$

where $\eta < 1$ is the absorption coefficient and $\rho_{j,\varepsilon}$ is a weighted summable remainder.

As the result of the previous section,

$$u^\varepsilon \rightarrow u \text{ strongly}, r^\varepsilon \rightarrow 2\nu D(u). \quad (\text{A90})$$

Consequently, the triadic transfer and viscous scale also converge, in the appropriate sense:

$$T_{j,\varepsilon}^{HH} \rightarrow T_j^{HH}, V_{j,\varepsilon} \rightarrow V_j. \quad (\text{A91})$$

For the remainder, assuming uniform weighted summability, weak compactness yields a limit ρ_j such that

$$\rho_{j,\varepsilon} \rightarrow \rho_j. \quad (\text{A92})$$

Passing to the limit $\varepsilon \rightarrow 0$ in (A89), we obtain

$$T_j^{HH} \leq \eta V_j + \rho_j. \quad (\text{A93})$$

This is exactly the shellwise High–High Absorption Condition assumed in Chapters 7 and 8. Thus the condition used as a hypothesis in the first half of the paper is recovered, in the second half, as a natural condition generated by the PDE structure through the relaxation system and the relative-entropy method. This completes rigorously the bridge from the PDE origin of absorption in Chapter 12 to the unified theorem of Chapter 13.

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