Theoretical Calculation of Wind (Or Water) Turbine: Extending the Betz Limit

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August 26, 2021

1 Short abstract :

The Betz limit sets a theoretical upper limit for the power production by turbines expressed as a maximum power coefficient of 16/27.
Betz's theory is accurate and it is based on the calculation of kinetic energy.
Taking into account the potential energy, the theoretical power production of turbines can be higher.

2 keywords

Betz limit, Betz's law, Wind turbine, Tidal turbine, HAWT, VAWT.

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3 Preliminary notes

In this article, a is not defined as an attenuating coefficient
$(V = V_{fluid}\ (1 - a))$,
a is defined as dimensionless velocity
$(a = \frac{V}{V_{fluid}})$.

V_{fluid} the fluid velocity.$(\frac{m}{s})$
V the fluid velocity at the position of the turbine.$(\frac{m}{s})$

The topic of consideration is that :
According to the German mathematician Betz ([2]), the maximum power that can be extracted from the wind by a turbine in open flow is $\frac{16}{27}$ of the wind's kinetic energy.

Large, fast-moving wind turbines have an efficiency of about 50%, close to the Betz limit.

When the fluid hits the blades, its direction is modified without any additional energy input. This change of direction makes the turbine turn, thus extracting a significant part of the fluids kinetic energy.

The interaction between the fluid and the blades creates stresses in the turbine. These stresses are created without any additional energy input.

Thus, large wind turbines have to be stopped when the wind exceeds a certain speed. If not, the stresses applied to their blades would cause damage.

For fast wind turbines, the kinetic energy of the wind is optimally recovered and, at the same time, a large quantity of potential energy is created without any additional energy input.

4 Introduction

The subject of this article is to define the power of a wind turbine and of a hydrokinetic turbine specially designed to recover not only the wind's kinetic energy but also its potential energy i.e. the stresses applied to the turbine. It is admitted that the maximum power coefficient of a wind or hydrokinetic turbine is the one defined by Betz. Many people even consider the Betz limit as a scientific law.

Despite some drawbacks (self starting, lower efficiency), vertical axis wind turbines have been constantly improved since the work of engineer Darrieus. The book of paraschivoiu **(2002)** [20]is a reference on this subject.Simple design, low noise emission and high torque are qualities which advocate for further improvements and developments of this kind of turbine. The importance of such developments is even greater with the recent interest for hydrokinetic turbines.

The maximum output power of an horizontal axis wind turbine compared to the available wind (or water) energy is known as 'power coefficient', Cp. Joukowsky and Betz demonstrated theoretically that it could reach a limit (16/27) depending mainly on the ratio between upstream and downstream flow velocity. Betz's theory assumptions (one dimensional and axial flow, incompressible and inviscid fluid) have been applied to vertical axis turbine despite some specific features (dynamic separation of the flow along the blades, strong blade wake interactions, three dimensional flow (tip vortices), possible cavitation (for water turbines), hysteresis in lift and drag forces during one rotation, finite number of blades).
Numerous studies have been proposed to extend the Betz limit in different ways (theory, design, shroud, diffuser, additional turbines, bottoming, flower leaves arrangement, varying pitch angle,

blade shape, articulated or tilted rotor,. . .).
Using conservation of mass and energy through a wind rotor, Rhageb **(2011)** [24] demonstrate that the power extraction depends on the wind diminution between upstream and downstream and that the Betz limit should take into account extra losses (viscous drag and wake, mechanical friction, electrical heating).
Rhageb **(2011)** points out also that only a part of the available air flow can be picked up by the rotor.
Dabiri **(2020)** [5] demonstrates that unsteady components could be added to the conventional steady flow through an actuator disk in order to extract more power.
Kinetic energy of the unsteady streamwise actuator disk displacement is generated thanks a time-dependent velocity fluctuation across the disk.
Furthermore, Dyment **(1989)**[6] points out that the position of the eddies shed from the rotor should be considered in the use of Bernoulli's equations across a stream tube around the turbine.
Comparing Joukowski (1912) and Betz (1919) models, [19] propose a vortex theory in order to extend an analytical solution developed for turbines with a finite number of blade and with a constant circulation distribution. The vortex system is described only by the far wake properties whereas the bound vorticity is used to produce the local lift in the vicinity of the blades. Considering an infinite helicoidal vortex sheet, the analytical solution is extended to any number of blade or wake pitch.
In order to get closer to Betz assumptions, the wake blade interactions generating flow instabilities can be reduced thanks to an active blade pitch control like presented in the article 'A numerical analysis to evaluate Betz's Law for vertical axis wind turbines' [27].
The limitation of blade wake separation will produce an increase of the energy flux and an additional torque due to an higher angle of attack adapted to the blade position. Thonnissen notes that the higher power output (over the Betz limit) of the considered horizontal axis turbine should be discussed for the spatially wake featured of vertical axis turbine. Blade pitch position control has been the main process to improve the turbine efficiency. Hwang **(2009)** [11] claims an improvement of 25% by using individual blade control, an optimized pivot location, a shorter chord length and low solidity.
Li **(2018)** [17] use computational fluid dynamic simulations, genetic-programming algorithm and validations in a wind tunnel, in order to optimize (increased power up to 31%) of the blade pitch angle in a wide range of tip speed ratios.
The extra power is produced by limiting the dynamic stall and maintaining an efficient angle of attack. Two other advantages of pitch control are the self-starting capacity and also the reduction of structure vibrations.
A sinusoidal shape pitch control has been introduced in study of Schönborn [26] in order to improve the efficiency of vertical axis water turbine (up to 20%) but also to reduce cavitation. The torque coefficient is also improved at low tip speed ratios.
Despite numerous disadvantages of dynamic stall (vibrations, noise, fatigue), Fujisawa **(2001)** [7] point out that the blade vortex interaction could be used to extract more power as fluid force measurement was observed. Albeit cavitation problems or free surface-turbine interaction, Georgiou **(2016)** [8]'s study outlines that several augmented marine current turbines used a ducted or diffuser concept. The strategy is either to limit the blade tip vortices or to conduct more external flow inside the internal propeller area as suggested by Huleihil **(2012)** [10].
An increasing torque with a smoother distribution during revolution, and an accelerating flow are also counting as advantages on such channeling devices ([25]).
Vennell **(2013)** [28] and Broberg **(2018)** [4] are proposing to group turbines in farm in order to

create an excess of power due to the proximity between machines (ducting effect).
Lee **(2015)** [16] underlines that helical turbine types are not improving significantly the power performance. Ambient incoming turbulence can benefit both to self-starting and to delay the dynamic stall, and hence increasing efficiency of straight bladed vertical axis wind turbine ([21]).

The article presented below is not a search of the optimal power coefficient in order to approach or to exceed the coefficient defined by Betz. The main objective is to reformulate the definition of the power coefficient. As presented in this article, the power factor can be much higher than the one defined by Betz.

The concept of potential energy is not that related to a difference in energy head as in common hydropower plants or hydrokinetic turbines that generate an increase in the upstream water level Quaranta **(2018)** ([23]). Here it is referred to a moving blade that is kept compressed by the pressure of a fluid in displacement. This concept can be applied both to wind turbines and to hydrokinetic turbines in wind and marine contexts.

5 Preliminary : Betz Theory

The German mathematician A. Betz has shown that the power of turbine is (cf. annex 1, [2]):

$$P_k = F\ V \ = [\ \frac{C_k}{a}\ \frac{1}{2}\ \rho\ S\ V_{fluid}^2]\ a\ V_{fluid} = C_k\ \frac{1}{2}\ \rho\ S\ V_{fluid}^3$$

$$where \quad V \ = \ a\ V_{fluid} \quad and \quad C_k = 4a^2(1-a)$$

P_k the kinetic turbine power, (W)
C_k the kinetic power coefficient,
ρ the fluid density,$(\frac{kg}{m^3})$
V_{fluid} the fluid velocity.$(\frac{m}{s})$
V the fluid velocity at the position of the turbine.$(\frac{m}{s})$ S the swept area.(m^2)
According to the work of Betz, a kinetic energy approach shows that the maximum power coefficient C_T can not exceed a maximum of $\frac{16}{27}$

$$C_{T\ maxi} = C_{k\ Betz} = \frac{16}{27} \qquad C_k \leq C_{T\ maxi} \qquad P_{T\ maxi} = C_{k\ Betz}\ \frac{1}{2}\ \rho\ S\ V_{fluid}^3$$

High-speed wind turbines have to be stopped when the wind-speed is too high, not because they are producing too much but mainly because the stresses applied to their blades become too high and could cause damage.

The computation of these stresses is calculated starting from the forces defined according to the theory of Betz ([13] Chapter 5, [18]).
These constraints are the source of internal energy.

The theory behind Betz's limit is correct. However, it only takes into account the kinetic energy. In order to increase the efficiency of wind and hydrokinetic turbines, they should be designed to transform the potential energy into kinetic energy.

6 Extended theory

6.1 Continuity equation

The mass flow rate is the same everywhere along the streamtube and therefore

$$\rho S_{fluid} V_{fluid} = \rho S V = \rho S_{wake} V_{wake}$$

with
ρ the fluid density (constant),($\frac{kg}{m^3}$)
V_{fluid} the fluid velocity.($\frac{m}{s}$)
V the fluid velocity at the position of the turbine,($\frac{m}{s}$)
V_{wake} the streamwise velocity in the far wake.($\frac{m}{s}$)

6.2 Kinetic energy

The kinetic energy can be written like **(with k the index for 'kinetic')**

$$E_k = \frac{1}{2}\, m\, v^2$$

Considering a steady flow in the transversal section

$$\frac{dv}{dt} = 0$$

$$P_k = \frac{dE_k}{dt} = \frac{1}{2}\,\frac{dm}{dt}\, v^2 + \frac{1}{2}\, m\, \frac{dv^2}{dt}$$

$$P_k = \frac{dE_k}{dt} = \frac{1}{2}\,\dot{m}\, v^2 = \frac{1}{2} \rho\, S\, v^3$$

6.3 Potential energy

The potential energy can be written like **(with p the index for 'potential')**

$$E_p = m \frac{p}{\rho}$$

$$P_p = \frac{dE_p}{dt} = \frac{dm}{dt}\frac{p}{\rho} + m \frac{1}{\rho}\frac{dp}{dt} \qquad \frac{dp}{dt} = 0$$

$$P_p = \frac{dE_p}{dt} = \dot{m}\frac{p}{\rho} = Svp$$

7 Power coefficient calculation

7.1 Power calculation for kinetic energy

The force applied by the fluid on the rotor is given by

$$F_k = \frac{d(mV)}{dt} = m\frac{dV}{dt} + \dot{m}\Delta V \; = \rho S V^2 \quad as \quad \frac{dv}{dt} = 0$$

Then the power can be written like

$$P_k = F_k V \; = \rho S V^2 \, (V_{fluid} \, - \, V_{wake})$$

$$P_k = \frac{\Delta E_k}{\Delta t} = \frac{\frac{1}{2}mV_{fluid}^2 - \frac{1}{2}mV_{wake}^2}{\Delta t} = \frac{1}{2} \; \rho S V \, (V_{fluid}^2 \, - \, V_{wake}^2)$$

From theses equalities

$$V = \frac{V_{fluid} + V_{wake}}{2}$$

defining $a = \frac{V}{V_{fluid}}$ $\quad 0 \leq a \leq 1$

$$V_{wake} = V_{fluid}(2a - 1) \quad V_{wake} \geq 0 \quad a \geq \frac{1}{2}$$

Fluid forces and global power can be written like

$$F = \rho S V \, (V_{fluid} \, - \, V_{wake}) \; = \frac{1}{2} \rho S (V_{fluid}^2 \, - \, V_{wake}^2)$$

$$P = FV \; = \rho S V^2 \, (V_{fluid} \, - \, V_{wake})$$

With the corresponding velocities, and 'kinetic' power

$$V_{wake} \; = \; V_{fluid} \; (2 \; a \; - \; 1) \quad as \quad V_{wake} \geq 0 \quad a \geq \frac{1}{2}$$

$$P_k = \; 4 \; a^2 \; (1 - a) \; \frac{1}{2} \; \rho S V_{fluid}^3$$

defining the kinetic power coefficient $C_k = \frac{P_k}{\frac{1}{2}\rho S V_{fluid}^3} \; = \; 4 \; a^2 \; (1 - a)$
The search of the maximum power coefficient gives

$$\frac{dC_k}{da} = 0 \quad a \; (2 \; - \; 3 \; a) \; = \; 0 \qquad a = 0 \qquad or \qquad a = \frac{2}{3}$$

$$a = \frac{2}{3} \qquad C_k = \frac{16}{27} = 0.593$$

The maximum power coefficient $C_{k\,maxi}$ is defined by Betz like

$$C_{k\,maxi} \; = \; C_{k\,Betz} = \frac{16}{27} \approx 60\%$$

7.2 Power calculation for potential energy

The pressure difference is equal to

$$p_{fluid} - p_{wake} = \frac{1}{2}\,\rho\,(V_{fluid}^2 - V_{wake}^2)$$

This pressure difference creates a force on the blade's surface

$$F_p = (p_{fluid} - p_{wake})S$$

The power of this force is equal to

$$P_p = (p_{fluid} - p_{wake})SV = a(p_{fluid} - p_{wake})SV_{fluid} = \; 4\;a^2\;(1-a)\;\frac{1}{2}\;\rho S V_{fluid}^3$$

thus defining the potential power coefficient $C_p = \; 4\;a^2\;(1-a)$

For a Darrieus type turbine, the maximum mean torque defined the Betz power is

$$T_{mean\ Betz} \;=\; \frac{P_{k\ Betz}}{\omega} \;=\; \frac{C_{k\ Betz}}{\omega}\;\frac{1}{2}\;\rho\;S\;V_{fluid}^3$$

Defining λ as the ratio between rotational speed and fluid speed, the average axial force is equal to

$$F_{mean\ axial} \;=\; \frac{C_{k\ Betz}}{\lambda}\;\frac{1}{2}\;\rho\;S\;V_{fluid}^2$$

$$with \qquad \lambda \;=\; \frac{\omega\;R}{V_{fluid}} \qquad R\;:\;radius\ of\ Darrieus\ turbine$$

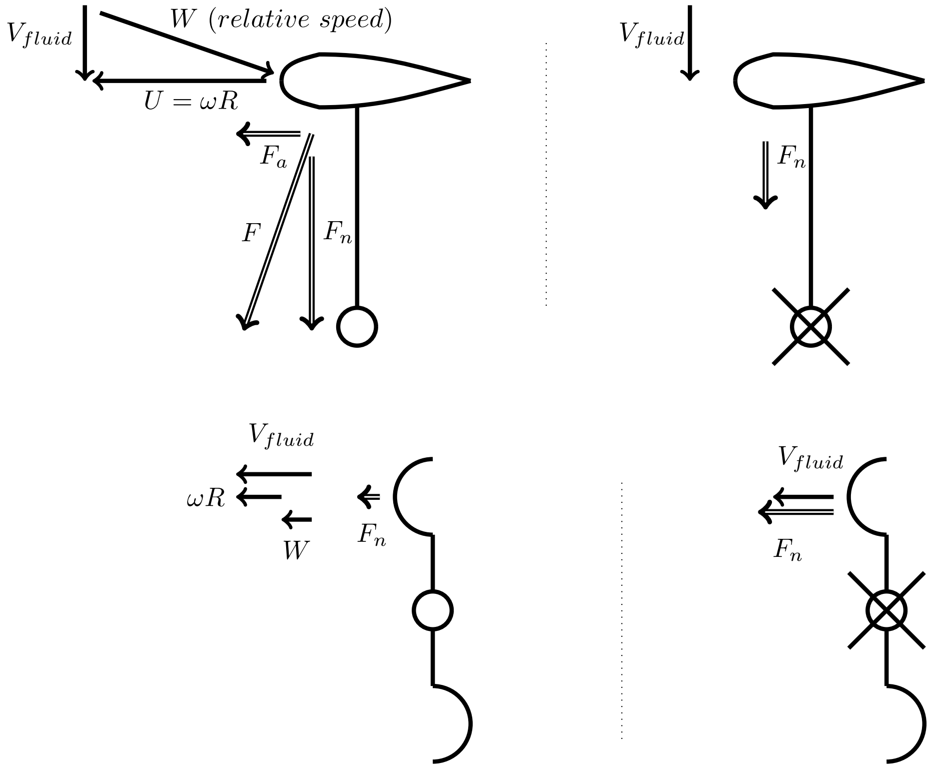
It was possible to create this mean axial force normal to the direction of the fluid to create a torque, because the force induced on the profile is not due directly to the speed of the fluid but to the combination of the speed of rotation and the speed of the fluid.

7.3 Main ojectives of the study and example

This pressure difference creates a force on the blade's surface. This pressure creates also stresses in the rotor. These stresses are the source of a potential energy because of the position of each internal particles that interact with each other ([12]). Using piezo-electric materials ([3]), it is possible to transform these stresses into electric energy. The objective is rather to transform this potential energy into kinetic mechanical energy.

For a sailboat, Betz's theory applies. The fluid flow due to the wind and the forward velocity of the sailboat creates an induced force on the sail. A small part of this force is used to move the sailboat forward, the other part causes the sailboat to heel and creates stresses on the keel. Instead of having static stresses on the keel, the 'trick' of using hydrofoils allows a kinetic flow. This flow lifts the sailboat, creates a reaction force against the tilt and considerably improves the performance of the sailboat. The principle of transforming potential energy into kinetic energy increases performance.

7.4 The advantage of using an airplane wing profile to recover additional energy



The cups of an anemometer are almost free of stress, if the anemometer is free of charge. If the rotation of the anemometer is blocked, the cups are subjected to stress.
In the case of a wing profile, it's completely different. The tangential rotational speed U of the profile is transverse to the air flow of the fluid.
The relative speed W due to the rotational speed U and the fluid speed V_{fluid} creates an induced force F on the profile. The axial component F_a of this force F, combined with the radius, creates a driving torque. For *horizontal axis wind turbines* (HAWT) this torque allows to improve the efficiency closed to that the one defined by Betz. Adding a transverse speed U will optimize the efficiency defined by Betz, but will add significant stress on the wind turbine due to the normal component F_n of force F. That's why HAWT wind turbines are stopped when the wind is too strong. They do not produce too much but they are subject to exceeding bending stress.
The subject of this article is to transform these constraints into additional energy recovery.

Anemometer

case	$U = \omega R$	W	F_n	$F_n R$
blocked	0	$= V_{fluid}$	*max*	0
free	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

case	$U = \omega R$	W	*stress*	*torque*
blocked	0	$= V_{fluid}$	*max*	*null*
free	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

case	$U = \omega R$	W	*potential energy*	*kinetic energy*
blocked	0	$= V_{fluid}$	*max*	*null*
free	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

Horizontal axis wind turbine **(HAWT)**

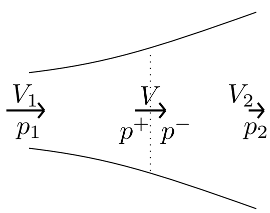
case	$U = \omega R$	W	F_n	$F_a R$
blocked	0	$= V_{fluid}$	*min*	0
free	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$

case	$U = \omega R$	W	*stress*	*torque*
blocked	0	$= V_{fluid}$	*min*	*null*
free	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$

case	$U = \omega R$	W	*potential energy*	*kinetic energy*
blocked	0	$= V_{fluid}$	*min*	*null*
free	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$

In the case of the anemometer, the kinetic energy varies in the opposite direction to the potential energy.
This is not the case for HAWT turbines. In the case of HAWT turbines, we must not consider only the speed of the fluid. The kinetic energy and potential energy are related to the fluid speed and the to the tangential rotation speed U. The direction of U is perpendicular to the direction of the fluid speed. The potential energy cannot grow more than the kinetic energy. The tangential rotation speed U is related to the fluid speed by this relation $\lambda = \frac{U}{V_{fluid}} \qquad U = \omega\ R.$

7.5 Energy conservation



Energy conservation in stationary state applied to the whole turbine (HAWT or VAWT) :

$$Ec_1 + Ep_1 = Ec_2 + Ep_2 + E_{Energy\ supplied\ to\ the\ turbine}$$

$$\rho\ m\ [\frac{V_1^2}{2} - \frac{V_2^2}{2} + \frac{p_1}{\rho} - \frac{p_2}{\rho}] = E_{Energy\ supplied\ to\ the\ turbine} \qquad with \qquad p_1 = p_2$$

$$\rho\ m\ [\frac{V_1^2}{2} - \frac{V_2^2}{2}] = E_{Energy\ supplied\ to\ the\ turbine}$$

Energy conservation in stationary state applied in the turbine area :

$$Ec_+ \; + \; Ep_+ \; = \; Ec_- \; + \; Ep_- \; + \; E_{potential}$$

$$\rho \; m \; [\frac{V_+^2}{2} \; - \; \frac{V_-^2}{2} \; + \; \frac{p_+}{\rho} \; - \; \frac{p_-}{\rho}] \; = \; E_{potential} \qquad with \qquad V_+ \; = \; V_- \; = \; V$$

$$\rho \; m \; [\frac{p_+}{\rho} \; - \; \frac{p_-}{\rho}] \; = \; E_{potential}$$

Potential energy is the source of stress in the turbine .

In the anemometer example, if the anemometer is free of charge, the kinetic energy is maximal and the potential energy is null. If the anemometer is subjected to a very resistant torque, the potential energy is maximal and the kinetic energy is null.

In the case of a turbine with wing profile, if the turbine is subjected to a very resistant torque, the kinetic energy and potential energy are maximal.

This is consistent with the variation of the total energy along the time is zero.

The difference is that in the case of the anemometer, the force induced is determined from the fluid velocity and the rotational speed.
In one case, the two speeds are collinear and the relative speed is lower than the speed of the fluid and in the other case, they are not collinear and the relative speed as a function of the angle of rotation can be higher than the speed of the fluid.

7.6 Stresses

In the case of horizontal wind turbines (HAWT, fast wind turbine type), the stresses in the blades for a defined wind speed, are constant.

$$\frac{d\sigma}{d\beta} = 0$$

σ Stress in turbine blade($\frac{N}{m^2}$)

β rotation angle of the blades (rad)

In fact, some variations of the stresses are existing due to gravitational forces and the differencial velocity within the boundary layer depending on the elevation.

In the case of vertical axis turbines (VAWT, Darrieus type) the blade and arm stresses are depending on the rotation angle of the blades (for a given wind speed).

$$\frac{d\sigma}{d\beta} \neq 0 \qquad \frac{1}{2\;\pi} \int_0^{2\;\pi} \sigma d\beta \;\; = \;\; \epsilon \;\; (\epsilon \; small)$$

During a half-turn, the arms are submitted to compression stresses whereas extending stresses are dominant during the next half-turn.

In the case of a HAWT, the conversion is not possible thanks to a dynamic mechanical system. The additionnal stresses are constant during a rotating for a given wind speed. Alternative stresses, encountered in a vertical axis wind turbine VAWT can allow the extraction of additional energy.

7.7 Power coefficient of a turbine with a conversion system

Moving fluid creates stresses on rotor and the conservation of energy can be applied. The total energy remains constant ([22]). Total energy is equal to the sum of kinetic energy and potential energy.

$$E_T = E_k + E_p$$

The variation of the total energy along the time is zero.

$$\frac{dE_{total}}{dt} = 0 \qquad \frac{dE_{kinetic}}{dt} = -\frac{dE_{potential}}{dt}$$

The variations of energy (kinetic and potential) vary simultaneously and in opposite sense.

$$\frac{dE_{kinetic}}{dt} = P_k = \frac{1}{2}\rho s v^3 \qquad \frac{dE_{potential}}{dt} = P_p = svp$$

$$\frac{dE_{kinetic}}{dt} = -\frac{dE_{potential}}{dt} \qquad P_k = -P_p \qquad p = -\frac{1}{2}\,\rho\, v^2$$

Pressure variations vary with speed variations. The pressure difference creates stresses on the turbine. The stresses are maximum when the forces due to the kinetic energy are maximum. The objective is to transform these constraints which is a potential energy source into kinetic energy.

The coefficient C_p can not be greater than or equal to C_k.

The fluid flow is the source of the rotor stresses. The potential energy due to the constraints cannot be higher than the kinetic energy due to the fluid flow.

7.8 Total recoverable power of the turbine

To determine the power, one selects the power defined from the kinetic energy and, in the case of conversion of stress into a mechanical movement, the power defined from the potential energy. Taking into account the differential of kinetic energy and the differential of the potential energy, the total recoverable power coefficient is

$$C_T \;=\; C_k \;+\; C_p$$

$C_p = 0$ when constraints are not converted

$$P_T = \; C_T\, \frac{1}{2}\rho S V_{fluid}^3 \;=\; (C_k \;+\; C_p)\, \frac{1}{2}\rho S V_{fluid}^3$$

In the case of horizontal wind turbines (HAWT, fast wind turbine type), the power coefficient $C_{T\ HAWT}$ is

$$C_{T\ HAWT} \;=\; C_k \;=\; 4\, a^2\, (1-a) \quad C_p = 0$$

In the case of vertical axis turbines (VAWT, Darrieus type) the power coefficient $C_{T\ VAWT\ Darrieus}$ is

$$C_{T\ VAWT\ Darrieus} = C_k = 4\,a^2\,(1-a) \quad C_p = 0$$

In the case of vertical axis turbines (VAWT with conversion), these stresses convert into additional energy, and the power coefficient $C_{T\ VAWT\ with\ conversion}$ is

$$C_{T\ VAWT\ with\ conversion} = C_k + C_p = 8\,a^2\,(1-a)$$

Various powers will then be defined by

$$P = C_T\ \frac{1}{2}\rho S V_{fluid}^3$$

$$P_{T\ HAWT} = 4\,a^2\,(1-a)\ \frac{1}{2}\rho S V_{fluid}^3$$

$$P_{T\ VAWT\ Darrieus} = 4\,a^2\,(1-a)\ \frac{1}{2}\rho S V_{fluid}^3$$

$$P_{T\ VAWT\ with\ conversion} = 8\,a^2\,(1-a)\ \frac{1}{2}\rho S V_{fluid}^3$$

7.9 HAWT-VAWT comparison

Following the work of [9], the power coefficient of different turbines is compared (a performance of 0.6 is applied for the supplementary energy recovery system).

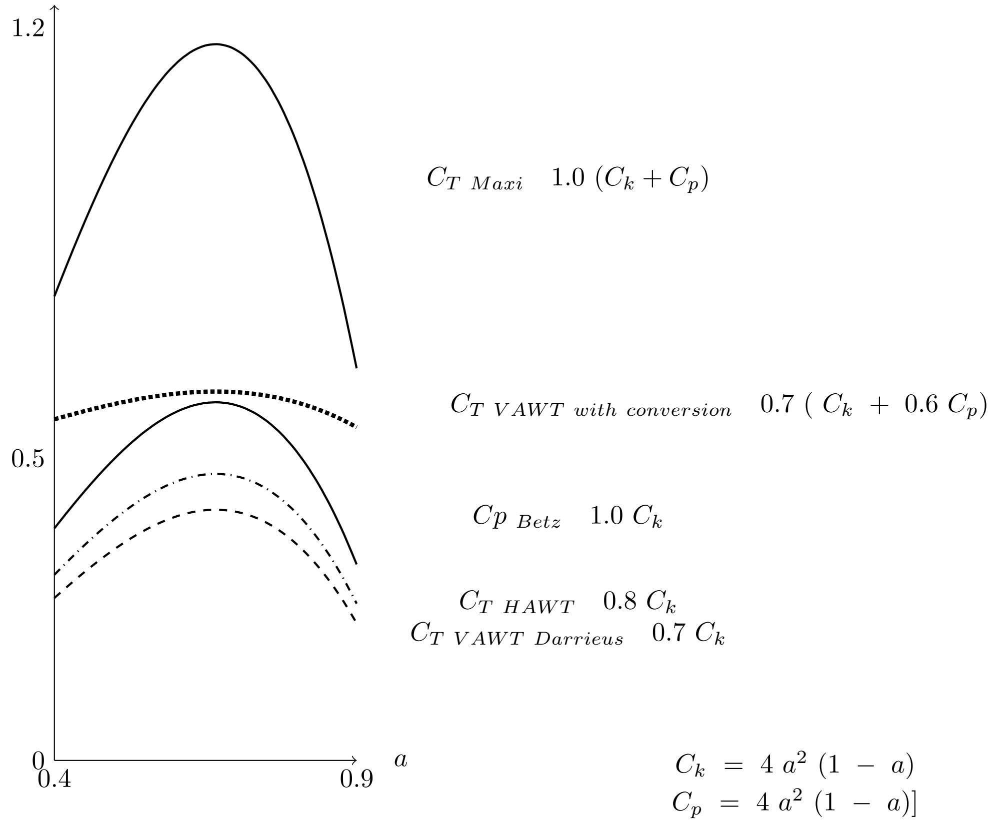
$$With\ a = \frac{2}{3} \qquad C_k \approx 60\% \qquad C_p \approx 60\%$$

notation $C_p\ w.c = C_p\ with\ conversion$

case	Coef.	HAWT	VAWT Darrieus	VAWT with conversion
perfect	C_k	60%	60%	60%
perfect	$C_p\ w.c$	0%	0%	60%
perfect	C_T	= 60%	= 60%	= 120%
in practice	C_k	$0.8 \times 60\% \approx 48\%$	$0.7 \times 60\% \approx 42\%$	$0.7 \times 60\% \approx 42\%$
in practice	$C_p\ w.c$	0%	0%	$0.6 \times 0.7 \times 60\% \approx 25\%$
in practice	C_T	= 48%	= 42%	= 67%
....				
gain/HAWT		+ 0%	−12 %	+ 39%
gain/$C_k(Betz)$		− 20%	−30 %	+ 11%

$$With\ a \approx 0.8 \qquad C_k \approx 50\% \qquad C_p \approx 50\%$$

case	Coef.	HAWT	VAWT Darrieus	VAWT with conversion
in practice	C_k	$0.8 \times 50\% \approx 40\%$	$0.7 \times 50\% \approx 35\%$	$0.7 \times 50\% \approx 35\%$
in practice	$C_p\ w.c$	0%	0%	$0.6 \times 0.7 \times 50\% \approx 21\%$
in practice	C_T	= 40%	= 35%	= 56%
....				
gain / HAWT		+ 0%	−12 %	+ 27%
gain / C_k		− 20%	−30 %	+ 12%



8 Active lift turbine project

The project "Active lift turbine" is a transformation example of potential energy into kinetic energy (see preprint : Simplified theory of an active lift turbine with controlled displacement ([14], [15]).

$$C_k = \frac{9\pi}{2^7}b^3 - \frac{2^3}{3}b^2 + \frac{\pi}{2}b$$

$$C_p = \frac{e}{R}\lambda(\frac{9\pi}{2^7}b^3 - \frac{2^2}{3}b^2 + \frac{\pi}{2}b)$$

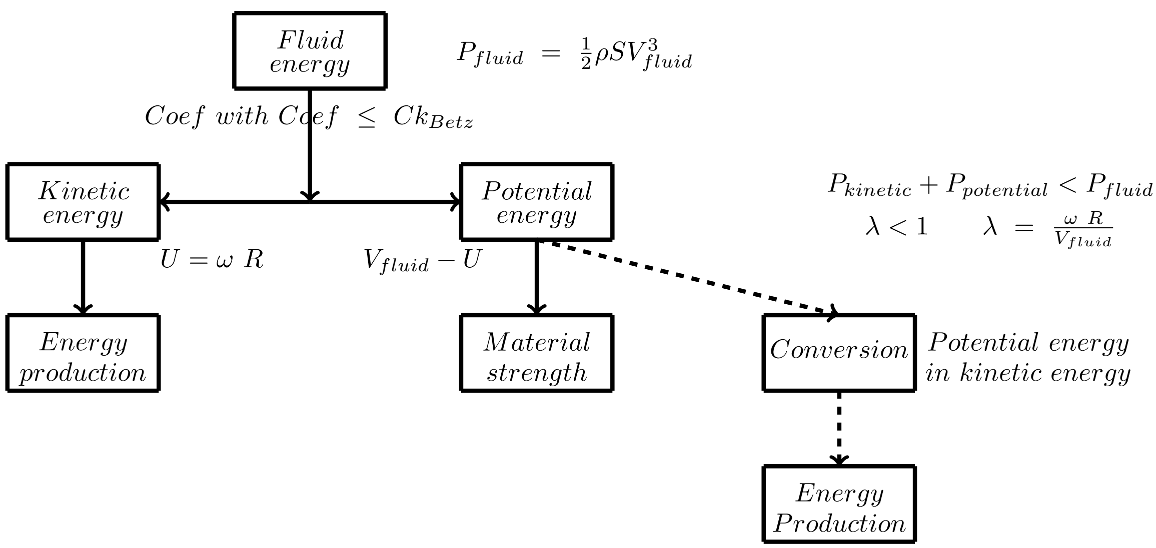
$$C_{active_lift_turbine} = (1 + \frac{e}{R}\lambda)(\frac{9\pi}{2^7}b^3 - \frac{2^2}{3}b^2 + \frac{\pi}{2}b)$$

$$with \quad b = \sigma\lambda$$

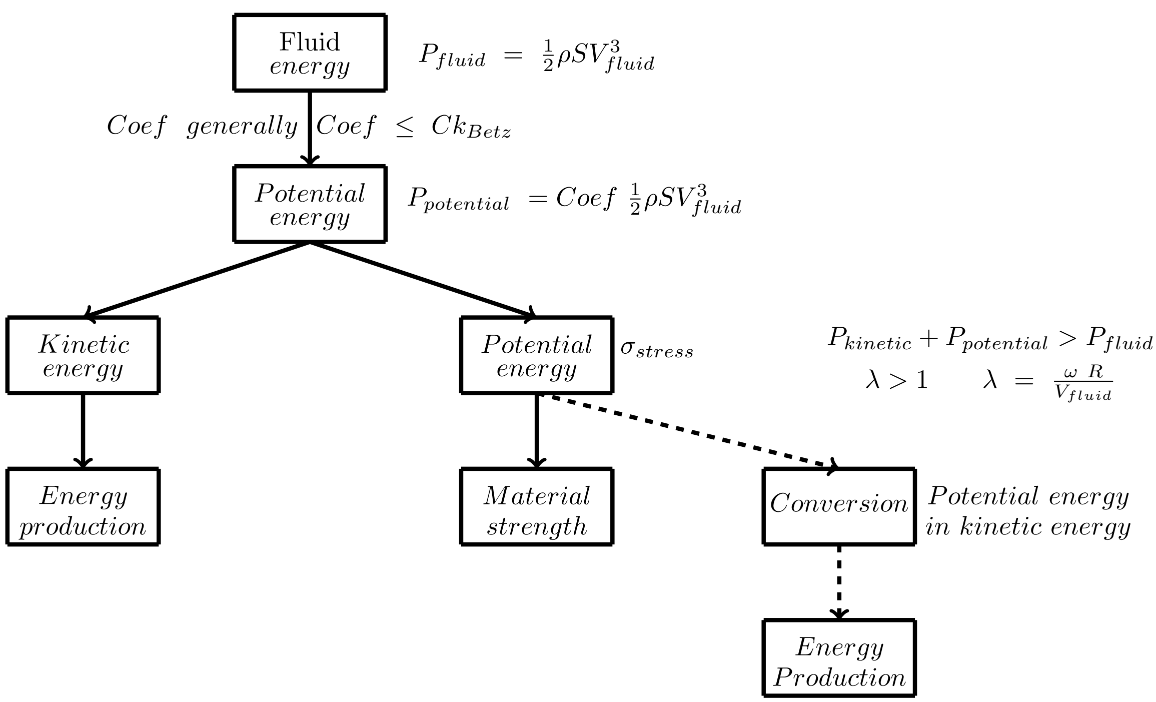
σ stiffness coefficient
λ Tip speed ratio
e eccentric distance
R Turbine radius

9 Synthesis scheme:

case : $\lambda \ < \ 1$ **— example : turbine type Anemometer with cups**



case : $\lambda \ > \ 1$ **— example : large wind turbines HAWT or VAWT**



In the case of a turbine with a lambda lower than 1, Of the recoverable energy defined by Betz, a part is transformed into kinetic energy and the other part into potential energy (stress in the turbine). Eventually a part of this potential energy could be transformed into kinetic energy.

In the case of a turbine with a lambda greater than 1, the recoverable energy defined by Betz, is transformed into potential energy. This energy creates kinetic energy and also creates potential energy (stress in the turbine). This is possible because we have created a speed of rotation higher than the speed of the fluid. A part of this potential energy (due to the stress) can be transformed into kinetic energy.

10 Synthesis on the extracted power:

With a mechanical conversion sytem, the powers for the different turbines are

$$P_{T\ HAWT} = 4\ a^2\ (1-a)\ \frac{1}{2}\rho S V_{fluid}^3$$

$$P_{T\ VAWT\ Darrieus} = 4\ a^2\ (1-a)\ \frac{1}{2}\rho S V_{fluid}^3$$

$$P_{T\ VAWT\ with\ conversion} = 8\ a^2\ (1-a)\ \frac{1}{2}\rho S V_{fluid}^3$$

Compared to a HAWT turbine, the gain of a VAWT Turbine with an energy recovery system is in practice from 20% to 50%.

Concerning a vertical axis turbine with a conversion system, the power factor is higher than the one defined by betz.

In the comparative table, a yield of 0.6 was chosen for the mechanical conversion system of the potential energy into mechanical energy. By choosing a powerful technology, this yield can be greatly increased, which will increase the performance of the turbine.

The definition of the maximum power coefficient is the one established by Betz which remains valid for the horizontal axis turbine HAWT and no longer makes sense for vertical axis turbines VAWT. The results given are examined a conversion of the potential energy into kinetic energy through a mechanical system which is not applicable for horizontal axis turbines HAWT. The calculation of the powers are the sum of the powers which one wants to consider.

The maximum power coefficient for a wind turbine or tidal turbine is

$$C_{T\ maxi} = \frac{32}{27}\ (\approx\ 118.5\%)$$

$C_{T\ maxi}$ is a limit value with $C_p\ \leq\ C_k$.

The coefficient is greater than 1 because the Betz limit only takes into account the kinetic power of the fluid.

Using piezo-electric materials would make it possible to transform potential energy into electrical energy. However, in order to achieve higher efficiency, the potential energy should be transformed into kinetic energy.

Thus, the power coefficient will be $C_T\ =\ C_k\ +\ C_p$ whatever the type of turbine.

Note : In the case of a turbine in a channel (see annexe[canal]), the maximum kinetic power coefficient is to 1. The maximum total power coefficient is then 200%.

11 Conclusion

Betz defined the maximum power coefficient $C_{k\ Betz}(=\frac{16}{27})$ of a wind turbine or tidal turbine from the calculation of kinetic energy.

HAWT fast speed turbines have an efficiency close to that the one defined by Betz. When the fluid speed is too high, they are stopped due to the excessive bending stresses on the blades.

The stresses on the blades are defined from this force.

$$F_{maxi} = Ck_{Betz}\ \frac{1}{2}\ \rho\ S_{swept\ area}\ V_{fluid}^2$$

Instead of creating stress, using this force to create additional torque significantly increases turbine efficiency.

Taking into account the kinetic energy and the potential energy, the coefficient of maximum power becomes $C_{T\ maxi}(=\frac{32}{27})$:

Transforming potential energy into kinetic energy greatly increases turbine performance.

The maximum power becomes

$$P_{T\ maxi} = C_{T\ maxi}\ \frac{1}{2}\ \rho\ S\ V_{fluid}^3 \quad with \quad C_{T\ maxi} = \frac{32}{27}$$

It is impossible to recover more than 100% kinetic energy from a fluid. However, $C_{T\ maxi} > 1$ due to the conversion of the fluids potential energy into kinetic energy.

The proposed evolution of VAWT can be compared to the evolution of sailboats. Square sails evolved to sails with a wing profile and now there are sailboats with hydrofoils. With the same source of wind, by using relative speed, the performance of the sailboats was improved and by transforming the stresses on the keel by flowing around the profile, we were able to improve again the performance of the sailboats.

The use of a fast speed wind turbine rather than a Savonius turbine means improved efficiency due to the use of relative speed.

Using an active lift turbine instead of a fast speed turbine improves efficiency due to the fact that stresses are transformed into extra torque.

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———————Annexe———————

Maximum wind power recovered from kinetic energy

$$E_k = \frac{1}{2}\, m\, V^2$$

$$\frac{dE_k}{dt} = \frac{1}{2}\,\frac{dm}{dt}\, V^2 + \frac{1}{2}\, m\, \frac{dV^2}{dt} \qquad \frac{dV}{dt} = 0$$

$$\frac{dE_k}{dt} = \frac{1}{2}\,\dot{m}\, V^2 = \frac{1}{2}\rho\, S\, V^3$$

V Wind speed at the turbine level
Force applied by the wind on the rotor

$$F = m\frac{dV}{dt} = \dot{m}\Delta V \; = \rho S V\,(V_{fluid} - V_{wake})$$

V_{wake} **streamwise velocity in the far wake**

$$P = FV \; = \rho S V^2\,(V_{fluid} - V_{wake})$$

$$P = \frac{\Delta E}{\Delta t} = \frac{\frac{1}{2} m V_{fluid}^2 - \frac{1}{2} m V_{wake}^2}{\Delta t}$$

$$P = \frac{\Delta E}{\Delta t} \; = \frac{1}{2}\dot{m}(V_{fluid}^2 - V_{wake}^2) = \frac{1}{2}\rho S V (V_{fluid}^2 - V_{wake}^2)$$

From theses equalities

$$V = \bar{V} = \frac{V_{fluid} + V_{wake}}{2}$$

$$F = \rho S V\,(V_{fluid} - V_{wake}) \; = \frac{1}{2}\rho S (V_{fluid}^2 - V_{wake}^2)$$

$$P = FV \; = \rho S V^2\,(V_{fluid} - V_{wake})$$

defining $a = \frac{V}{V_{fluid}}$

$$V_{wake} \; = V_{fluid}\,(2\ a \; - \; 1) \quad as \quad V_{wake} \geq 0 \quad a \geq \frac{1}{2}$$

$$P = \; 4\ a^2\ (1-a)\ \frac{1}{2}\ \rho S V_{fluid}^3$$

defining power coefficient $C_k = \frac{P}{\frac{1}{2}\rho S V_{fluid}^3} \; = \; 4\ a^2\ (1-a)$

Search of maximum power coefficient

$$\frac{dC_k}{da} = 0 \quad a\ (2\ -\ 3\ a)\ =\ 0 \qquad a = 0 \qquad or \qquad a = \frac{2}{3}$$

$$a = \frac{2}{3} \qquad C_k = \frac{16}{27} = 0.593$$

The maximum power coefficient C_{kmaxi} is defined by Betz

$$C_{kmaxi}\ =\ C_{p_{Betz}} = \frac{16}{27} \approx 60\%$$

The maximum power of the fluid is

$$P_{fluid} = \frac{1}{2}\rho S_{fluid} V_{fluid}^3$$

$S_{fluid}\ =\ \frac{S\ V}{V_{fluid}}\ =\ a\ S$
The power of the turbine is

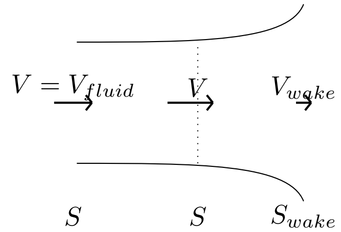
$$P = \frac{C_k}{a}\ P_{fluid}\ =\ C_k\ \frac{1}{2}\rho\frac{S_{fluid}}{a}V_{fluid}^3\ =\ C_k\ \frac{1}{2}\rho S V_{fluid}^3$$

The maximum power of the turbine is

$$P_{max} = \frac{C_{p\ Betz}}{\frac{2}{3}}\ P_{fluid}\ =\ \frac{8}{9}\ \ P_{fluid}\ =\ C_{p\ Betz}\ \frac{1}{2}\rho S V_{fluid}^3\ =\ \frac{16}{27}\ (\frac{1}{2}\rho S V_{fluid}^3)$$

$$P_{max}\ =\ C_{p\ Betz}\ \frac{1}{2}\rho S V_{fluid}^3 \tag{1}$$

Maximum wind power for a turbine in a channel (from kinetic energy)



$$E_k = \frac{1}{2}\, m\, V^2$$

$$\frac{dE_k}{dt} = \frac{1}{2}\,\frac{dm}{dt}\,V^2 + \frac{1}{2}\,m\,\frac{dV^2}{dt} \qquad \frac{dV}{dt} = 0$$

$$\frac{dE_k}{dt} = \frac{1}{2}\,\dot{m}\,V^2 = \frac{1}{2}\rho\, S\, V^3$$

V Wind speed at the turbine level
Force applied by the wind on the rotor

$$F = m\frac{dV}{dt} = \dot{m}\Delta V \;=\; \rho S V\,(V \;-\; V_{wake})$$

V_{wake} **streamwise velocity in the far wake**

$$P = FV \;=\; \rho S V^2\,(V \;-\; V_{wake})$$

$$P = \frac{\Delta E}{\Delta t} = \frac{\frac{1}{2}mV^2 - \frac{1}{2}mV_{wake}^2}{\Delta t}$$

$$P = \frac{\Delta E}{\Delta t} \;=\; \frac{1}{2}\dot{m}(V^2 \;-\; V_{wake}^2) = \frac{1}{2}\rho S V(V^2 \;-\; V_{wake}^2)$$

From theses equalities

$$V_{wake} \;=\; \frac{1}{2}\;V \qquad S_{wake} = \frac{S\;V}{V_{wake}} \;=\; 2\;S$$

$$F \;=\; \rho S V\,(V_{fluid} \;-\; V_{wake}) \;=\; \frac{1}{2}\rho S(V_{fluid}^2 \;-\; V_{wake}^2)$$

$$P = FV \;=\; \rho S V^2\,(V_{fluid} \;-\; V_{wake})$$

$$P = \;\frac{1}{2}\;\rho S V_{fluid}^3$$

defining power coefficient

$$C_k = \frac{P}{\frac{1}{2}\rho S V_{fluid}^3} \;=\; 1 \tag{2}$$

Additional recovery power (from potential energy)

The fluid creates stresses in the blade. They are due to thrust force. The energy of this force is

$$E_p \;=\; m\,\frac{f_s}{\rho}$$

f_s thrust force

For HAWT horizontal wind turbines (fast wind turbine type), the thrust force F_s are constant.

$$\frac{dE_p}{dt} \;=\; 0$$

For a VAWT, the thrust force depends on the time or the rotation angle $F_s(t)$ or $F_s(\beta)$ $\quad \beta = \omega\; t$

$\omega \;=\; \frac{d\dot{\beta}}{dt}$ angular frequency

$$\frac{dE_p}{dt} \;\neq\; 0 \quad and \quad \frac{1}{2\;\pi}\int_0^{2\;\pi} E_p(\beta)d\beta \;=\; \epsilon \;\;(\epsilon\; small)$$

As

$$F_s \;=\; f_s S \;=\; C_x \frac{1}{2}\;\rho\; S\; V_{fluid}^2$$

$$E_p \;=\; m\; C_x \frac{1}{2}\; V_{fluid}^2$$

V fluid speed at the level turbine

The power is

$$P_p \;=\; \frac{dE_p}{dt} \qquad dm = \rho S V dt$$

$$P_p \;=\; \frac{dE_p}{dt} \;=\; \frac{dm}{dt}\; C_x\; \frac{1}{2}\; S\; V_{fluid}^2 \;+\; m\; C_x \frac{1}{2}\; S\; \frac{dV_{fluid}^2}{dt} \qquad \frac{dV_{fluid}}{dt} \;=\; 0$$

$$P_p \;=\; a\; C_x\; \frac{1}{2}\; \rho\; S\; V_{fluid}^3 \quad with\; a = \frac{V}{V_{fluid}} \tag{3}$$

$$\frac{1}{2\;\pi}\int_0^{2\;\pi} E_p(\beta)d\beta \;=\; \epsilon \;\;(\epsilon\; small) \quad E_{p-max} \;\approx\; -\,E_{p-min}$$

the power depends on a potential energy difference

$$P_p \;=\; \frac{\Delta E_p}{\Delta t} \quad T \;=\; \frac{2\;\pi}{\omega} \quad P_p \;\leq\; \frac{E_{p-max} - E_{p-min}}{T} \quad P_p \;\leq\; \frac{E_{p-max}}{\pi}\;\omega$$

for a half-turn

$$E_p(\beta)\ R\ d\beta\ =\ dE_p\ \pi\ R$$

in particular

$$E_{p-max}\ =\ \frac{dE_p}{d\beta}\pi\ =\ \frac{dE_p}{dt}\ \frac{\pi}{\omega}$$

As

$$\frac{dE_p}{dt}\ =\ a\ C_x\ \frac{1}{2}\ \rho\ S\ V_{fluid}^3 \quad and \quad P_p\ \leq\ \frac{E_{p-max}}{\pi}\ \omega$$

$$P_p\ \leq\ a\ C_x\ \frac{1}{2}\ \rho\ S\ V_{fluid}^3$$

Variation of energy in opposite sense

Along a streamline, the Bernoulli's equation is see [1]

$$\frac{p}{\rho} \; + \; \frac{v^2}{2} \; = \; constant \quad with \;\; z \; = \; 0$$

By multiplying by m

$$m \; \frac{p}{\rho} \; + \; m \; \frac{v^2}{2} \; = \; constant$$

The differential of this equation is

$$d(\frac{1}{2} \; m \; v^2) \; = \; - \, d(m \; \frac{p}{\rho}) \tag{4}$$

the variations of energy vary simultaneously and in opposite sense.

$$As \quad \frac{dV}{dt} = 0 \quad and \quad \frac{dp}{dt} = 0 \quad p \; = \; - \, \frac{1}{2} \; \rho \; v^2$$