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*Article*

# Generalized Right *e*-Core Inverse in Banach *-Algebras

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**Abstract**

In this paper, we introduce the concept of the generalized right weighted core inverse within the framework of a Banach *-algebra. We provide a characterization of this novel generalized inverse based on a unique type of decomposition that involves right weighted core-inverses and quasinilpotent elements. The relationships between the right weighted core inverse and the right g-Drazin inverse of an element in a Banach *-algebra are explored. We also establish representations for the generalized right weighted core inverse. As an application, we demonstrate new characteristics of the pseudo right $e$-core inverse and $\textcircled{m}$-core-EP inverse in Minkowski spaces.

**Keywords:** right $e$-core inverse; right generalized Drazin inverse; generalized right $e$-core inverse; $\textcircled{m}$-core-EP inverse; Banach algebra

**MSC:** 15A09; 16U90; 46H05

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## 1. Introduction

An involution of a Banach algebra $\mathcal{A}$ is an anti-automorphism whose square is the identity map 1. A Banach algebra $\mathcal{A}$ with involution $*$ is called a Banach *-algebra. Let $\mathcal{A}$ be a Banach *-algebra with an identity. An element $a \in \mathcal{A}$ has core inverse if there exists some $x \in \mathcal{A}$ such that

$$ax^2 = x, (ax)^* = ax, xa^2 = a.$$

If such $x$ exists, it is unique, and denote it by $a^{\#\!\!\!\bigcirc}$. An element $a \in \mathcal{A}$ has core-EP inverse (i.e., pseudo core inverse) if there exist $x \in \mathcal{A}$ and $k \in \mathbb{N}$ such that

$$ax^2 = x, (ax)^* = ax, xa^{k+1} = a^k.$$

If such $x$ exists, it is unique, and denote it by $a^{\textcircled{D}}$. Core and core-EP inverses are extensively studied by many authors from different views, e.g., [1,6,8,10,11,16–19,24,28,29,31].

Wang et al. generalized the core inverse to the right core inverse (see [27]). An element $a \in \mathcal{A}$ has right core inverse if there exist $x \in \mathcal{A}$ such that

$$ax^2 = x, (ax)^* = ax, axa = a.$$

If such $x$ exists, it is unique, and denote it by $a_r^{\#\!\!\!\bigcirc}$. In [3], the authors introduced and studied generalized right core inverse. An element $a \in \mathcal{A}$ has generalized right core decomposition there exist unique a $x \in \mathcal{A}$ such that

$$ax^2 = x, (ax)^* = ax, \lim_{n\to\infty} ||a^n - axa^n||^{\frac{1}{n}} = 0.$$

The preceding $x$ is called generalized right core inverse of $a$ and we denote it by $a_r^{\textcircled{d}}$. We refer the reader more properties of right core and generalized right core inverses in [3,7,27].

Mosić et al. introduced and studied weighted core inverse (see [21]). Let $a \in \mathcal{A}$ and $e \in \mathcal{A}$ is an invertible Hermitian element (i.e., $e$ is invertible and $e^* = e$). An element $a \in \mathcal{A}$ has $e$-core inverse if there exist $x \in \mathcal{A}$ and $k \in \mathbb{N}$ such that

$$ax^2 = x, (eax)^* = eax, xa^2 = a.$$

If such $x$ exists, it is unique, and denote it by $a^{e,\textcircled{\#}}$. As a natural generalization of weighted core and core-EP inverses, the authors introduced and studied generalized weighted core inverse in a Banach *-algebra. An element $a \in \mathcal{A}$ has generalized $e$-core decomposition if there exists $x \in \mathcal{A}$ such that

$$x = ax^2, (eax)^* = eax, \lim_{n\to\infty} ||a^n - xa^{n+1}||^{\frac{1}{n}} = 0.$$

The preceding $x$ is called generalized $e$-core inverse of $a$ and we denote it by $a^{e,\textcircled{d}}$. We refer the reader for weighted core and generalized weight core inverses in [2,9,13,14,20,32].

Recently, Ke et al. generalized the $e$-core inverse to the right $e$-core inverse (see [12]). An element $a \in \mathcal{A}$ has right $e$-core inverse if there exist $x \in \mathcal{A}$ such that

$$ax^2 = x, (eax)^* = eax, axa = a.$$

If such $x$ exists, it is unique, and denote it by $a_r^{e,\textcircled{\#}}$. Let $\mathcal{A}_r^{e,\textcircled{\#}}$ denote the set of all right $e$-core invertible elements in $\mathcal{A}$. Here we list some characterizations of right $e$-core inverse.

**Theorem 1.1** (see [12])**.** *Let $\mathcal{A}$ be a Banach *-algebra, and let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{e,\textcircled{\#}}$.
(2) *There exists $x \in \mathcal{A}$ such that $axa = a, ax^2 = x = xax, (eax)^* = eax$.*
(3) *There exists an idempotent $p \in \mathcal{A}$ such that $(ep)^* = ep, pa = 0, a + p \in \mathcal{A}_r^{-1}$.*
(4) *$a \in \mathcal{A}_e^{(1,3)}$ and $a\mathcal{A} = a^2\mathcal{A}$.*
(5) *$\mathcal{A}a = \mathcal{A}(a^*)^n ea$ for some $n \geq 2$.*

The motivation of this paper is to introduce and study a new kind of generalized inverse as a natural generalization of generalized inverses mentioned above. In Section 2, we introduce generalized right weighted core inverse in terms of a new kind of decomposition by using right weighted core-inverses and quasinilpotents. Many new properties of the right weighted (pesudo) core inverse and generalized weighted core inverse are thereby obtained.

**Definition 1.2.** *An element $a \in \mathcal{A}$ has generalized right e-core decomposition if there exist $x, y \in \mathcal{A}$ such that*

$$a = x + y, x^*ey = yx = 0, x \in \mathcal{A}_r^{e,\textcircled{\#}},\ y \in \mathcal{A}^{qnil}.$$

Let

$$\mathcal{A}^{qnil} = \{x \in \mathcal{A} \mid \lim_{n\to\infty} \parallel x^n \parallel^{\frac{1}{n}} = 0\}.$$

Evidently, $x \in \mathcal{A}^{qnil}$ if and only if $1 + \lambda x \in \mathcal{A}$ is invertible for any $\lambda \in \mathbb{C}$. We prove that $a \in \mathcal{A}$ has generalized right $e$-core decomposition if and only if there exists unique $x \in \mathcal{A}$ such that

$$x = ax^2, (eax)^* = eax, \lim_{n\to\infty} ||a^n - axa^n||^{\frac{1}{n}} = 0.$$

The polar-like properties of generalized right weighted core inverses are established.

In Section 3, we establish characterizations between generalized right weighted core inverse and right g-Drazin inverse for an element in a Banach *-algebra by using involved images. We prove that $a \in \mathcal{A}_r^{e,\textcircled{d}}$ if and only if $a$ has right g-Drazin inverse $x$ which has right $e$-core inverse.

In Section 4, we shift our focus to the study of representations for the generalized right weighted core inverse. We explore the generalized right weighted core inverse through an examination of diverse matrix conditions.

An element $a$ in $\mathcal{A}$ has pseudo right $e$-core inverse if there exists $x \in \mathcal{A}$ such that $x = ax^2, (eax)^* = eax, a^n = axa^n$. Such $x$ is unique, if exists, and denote it by $a_r^{e,\textcircled{D}}$. Finally, in Section 5, the pseudo right $e$-core inverse is characterized by certain new ways. As an application, $\textcircled{m}$-core-EP inverse in Minkowski spaces are studied.

Throughout the paper, all Banach *-algebras are complex with an identity. We use $\mathcal{A}_r^{-1}, \mathcal{A}_r^{\textcircled{d}}, \mathcal{A}_r^{e,\textcircled{\#}}, \mathcal{A}_r^{e,\textcircled{D}}$ and $\mathcal{A}_r^{e,\textcircled{d}}$ to denote the sets of all right invertible, generalized right core invertible, right $e$-core invertible, right $e$-core-EP invertible and generalized right $e$-core invertible elements in $\mathcal{A}$, respectively. If $a$ and $x$ satisfy the equations $a = axa$ and $(ax)^* = eax$, then $x$ is called $(1,3,e)$-inverse of $a$ and is denoted by $a_e^{(1,3)}$. We use $\mathcal{A}_e^{(1,3)}$ to stand for the set of all $(1,3,e)$-invertible elements in $\mathcal{A}$.

## 2. Generalized Right $e$-Core Decomposition

The aim of this section is to introduce the notion of the generalized weighted core inverse in a Banach *-algebra. We begin with

**Theorem 2.1.** *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}$ has generalized right $e$-core decomposition.
(2) There exists $x \in \mathcal{A}$ such that

$$x = ax^2, (eax)^* = eax, \lim_{n\to\infty} ||a^n - axa^n||^{\frac{1}{n}} = 0.$$

**Proof.** (1) $\Rightarrow$ (2) By hypothesis, there exist $z, y \in \mathcal{A}$ such that

$$a = z + y, z^*ey = yz = 0, z \in \mathcal{A}^{e,\textcircled{\#}}, y \in \mathcal{A}^{qnil}.$$

Set $x = z^{e,\textcircled{\#}}$. One easily checks that

$$\begin{array}{rcl} ax & = & (z+y)z^{e,\textcircled{\#}} = zz^{e,\textcircled{\#}} + yz(z^{e,\textcircled{\#}})^2 = zz^{e,\textcircled{\#}}, \\ ax^2 & = & (z+y)(z^{e,\textcircled{\#}})^2 = z(z^{e,\textcircled{\#}})^2 = z^{e,\textcircled{\#}} = x, \\ z^{e,\textcircled{\#}}y & = & xy = xzxy = xe^{-1}(ezx)y = xe^{-1}(ezx)^*y \\ & = & xe^{-1}x^*(z^*ey) = 0. \end{array}$$

Now by applying $z^{e,\textcircled{\#}}y = 0$ and Theorem 1.1, we deduce that

$$axa = (ax)a = zz^{e,\textcircled{\#}}(z+y) = zz^{e,\textcircled{\#}}z = z.$$

Then

$$\begin{array}{c} (eax)^* = (ezz^{e,\textcircled{\#}})^* = ezz^{e,\textcircled{\#}} = eax, \\ a(1-xa) = a - axa = a - z = y \in \mathcal{A}^{qnil}. \end{array}$$

Since $yz = 0$, we see that

$$(a - axa)z = (z + y - z)z = yz = 0.$$

Thus we have

$$\begin{array}{rcl} ||a^n - axa^n||^{\frac{1}{n}} & = & ||(a - axa)a^{n-1}||^{\frac{1}{n}} \\ & = & ||(a-axa)(z+y)^{n-1}||^{\frac{1}{n}} \\ & = & ||(a-axa)y^{n-1}||^{\frac{1}{n}} \\ & \leq & ||a - axa||^{\frac{1}{n}}[||y^{n-1}||^{\frac{1}{n-1}}]^{1-\frac{1}{n}}. \end{array}$$

Since $y \in \mathcal{A}^{qnil}$, we deduce that
$$\lim_{n\to\infty} ||a^n - axa^n||^{\frac{1}{n}} = 0,$$
as required.

(2) $\Rightarrow$ (1) By hypotheses, we have $z \in \mathcal{A}$ such that
$$z = az^2, (eaz)^* = eaz, \lim_{n\to\infty} ||a^n - aza^n||^{\frac{1}{n}} = 0.$$
For any $n \in \mathbb{N}$, we have
$$\begin{aligned} az &= a(az^2) = a^2z^2 = a^2(az^2)z \\ &= a^3z^3 = \cdots = a^nz^n. \end{aligned}$$
Hence
$$\begin{aligned} ||az - azaz|| &= ||(a^nz^n - aza^nz^n|| \\ &= ||(a^n - aza^n)z^n||. \end{aligned}$$
Then
$$||az - azaz||^{\frac{1}{n}} \leq ||(a^n - aza^n)||^{\frac{1}{n}}||z||.$$
We infer that
$$\lim_{n\to\infty} ||az - azaz||^{\frac{1}{n}} = 0,$$
hence, $az = azaz$.

Moreover, we check that
$$\begin{aligned} (a^2 - aza^2)z &= (a^2 - aza^2)az^2 \\ &= (a^2 - aza^2)a^2z^3 \\ &\vdots \\ &= (a^2 - aza^2)a^{n-2}z^{n-1} \\ &= (a^n - aza^n)z^{n-1}. \end{aligned}$$
Therefore
$$||(a^2 - aza^2)z||^{\frac{1}{n}} \leq ||a^n - aza^n|^{\frac{1}{n}}|||z^{n-1}||^{\frac{1}{n}}.$$
Since
$$\lim_{n\to\infty} ||a^n - aza^n||^{\frac{1}{n}} = 0,$$
we prove that
$$\lim_{n\to\infty} ||(a^2 - aza^2)z||^{\frac{1}{n}} = 0.$$
This implies that $(a^2 - aza^2)z = 0$. That is, $a(a - za^2)z = 0$.

Set $x = aza$ and $y = a - aza$. Then $a = x + y$. We claim that $x$ has right $e$-core inverse. Evidently, we verify that
$$\begin{aligned} zxz &= zazaz = zazz, \\ xz^2 &= azaz^2 = (azaz)z = az^2 = z, \\ (exz)^* &= (eazaz)^* = (eaz)^* = eaz = e(aza)z = exz. \end{aligned}$$
Therefore $x \in \mathcal{A}_r^{e,\textcircled{\#}}$ and $z = x_r^{e,\textcircled{\#}}$.

We verify that

$$\begin{aligned}
||(a-za^2)^{n+2}||^{\frac{1}{n+2}} &= ||(1-za)a(a-za^2)^n(a-za^2)||^{\frac{1}{n+2}} \\
&= ||(1-za)a(a-za^2)^{n-1}(a-za^2)a||^{\frac{1}{n+2}} \\
&= ||(1-za)a(a-za^2)^{n-1}a^2||^{\frac{1}{n+2}} \\
&\vdots \\
&= ||(1-za)a(a-za^2)a^n||^{\frac{1}{n+2}} \\
&\leq ||1-za||^{\frac{1}{n+2}}||a^{n+2}-aza^{n+2}||^{\frac{1}{n+2}}||a^n||^{\frac{1}{n+2}}.
\end{aligned}$$

Accordingly,

$$\lim_{n\to\infty} ||a^{n+2}-aza^{n+2}||^{\frac{1}{n+2}} = 0.$$

This implies that $a - za^2 \in \mathcal{A}^{qnil}$. By using Cline's formula (see [15, Theorem 2.1]), $y = a - aza \in \mathcal{A}^{qnil}$.

Moreover, we see that

$$\begin{aligned}
x^*ey &= (aza)^*e(1-az)a = a^*(az)^*e^*(1-az)a \\
&= a^*(eaz)^*(1-az)a = 0, \\
&= a^*(eaz)(1-az)a = 0, \\
yx &= (a-aza)aza = a(a-za^2)za = 0.
\end{aligned}$$

Then we have a generalized right $e$-core decomposition $a = x + y$, thus yielding the result. □

We denote $x$ in Theorem 2.1 by $a_r^{e,\textcircled{d}}$, and call it a generalized right $e$-core inverse of $a$. As an immediate consequence, we derive

**Corollary 2.2.** *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}$ has generalized right core decomposition.
(2) There exists $x \in \mathcal{A}$ such that

$$x = ax^2, (ax)^* = ax, \lim_{n\to\infty} ||a^n - axa^n||^{\frac{1}{n}} = 0.$$

**Theorem 2.3.** *Let $a = x + y$ be the generalized right e-core decomposition of $a \in \mathcal{A}$. Then $a_r^{e,\textcircled{d}} = x_r^{e,\textcircled{\#}}$.*

**Proof.** Let $a = x + y$ be the generalized right $e$-core decomposition of $a \in \mathcal{A}$. Analogously to the proof of Theorem 2.1, $x_r^{e,\textcircled{\#}}$ is the generalized right $e$-core inverse of $a$. This completes the proof. □

**Corollary 2.4.** *Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}^{e,\textcircled{d}}$ if and only if $a \in \mathcal{A}_r^{e,\textcircled{d}} \bigcap \mathcal{A}^d$.*

**Proof.** This is obvious by Theorem 2.3 and [4, Theorem 2.5] □

Let $\mathbb{C}^{n\times n}$ be the Banach algebra of all $n \times n$ complex matrices, with conjugate transpose as the involution. For a complex $A \in \mathbb{C}^{n\times n}$, it follows by Theorem 2.3 that the pseudo core inverse and generalized right core inverse coincide with each other for a complex matrix, i.e., $A^{\textcircled{D}} = A_r^{1,\textcircled{d}}$.

Next, we present a polar-like property for the generalized right $e$-core inverse in a Banach *-algebra and establish its related characterizations.

**Theorem 2.5.** *Let $a \in \mathcal{A}$ and $n \in \mathbb{N}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{e,\textcircled{d}}$.
(2) *There exists an idempotent $p \in \mathcal{A}$ such that*

$$a^n + p \in \mathcal{A}_r^{-1}, (ep)^* = ep, ap \in \mathcal{A}^{qnil}, (1-p)\mathcal{A} = a(1-p)\mathcal{A}.$$

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**Proof.** (1) $\Rightarrow$ (2) Since $a \in \mathcal{A}_r^{e,\textcircled{d}}$, by virtue of Theorem 2.1, there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*ey = yx = 0, x \in \mathcal{A}_r^{e,\textcircled{\#}}, y \in \mathcal{A}^{qnil}.$$

In view of Theorem 1.1, we have

$$x_r^{e,\textcircled{\#}} = x(x_r^{e,\textcircled{\#}})^2 = x_r^{e,\textcircled{\#}}xx_r^{e,\textcircled{\#}}, (exx_r^{e,\textcircled{\#}})^* = exx_r^{e,\textcircled{\#}}, x = xx_r^{e,\textcircled{\#}}x.$$

Let $p = 1 - xx_r^{e,\textcircled{\#}}$. Then $p^2 = p, (ep)^* = ep$ and $px = 0$. We directly check that

$$\begin{array}{rl} & [x^n + 1 - xx_r^{e,\textcircled{\#}}][(x_r^{e,\textcircled{\#}})^n + 1 - xx_r^{e,\textcircled{\#}}] \\ = & xx_r^{e,\textcircled{\#}} + x^n(1 - xx_r^{e,\textcircled{\#}}) + 1 - xx_r^{e,\textcircled{\#}} \\ = & 1 + x^n(1 - xx_r^{e,\textcircled{\#}}) \in \mathcal{A}^{-1}. \end{array}$$

Let $q = [(x_r^{e,\textcircled{\#}})^n + 1 - xx_r^{e,\textcircled{\#}}][1 + x^n(1 - xx_r^{e,\textcircled{\#}})]^{-1}$. Then $(x^n + p)q = 1$. We further verify that

$$\begin{array}{rl} & 1 + yq\sum\limits_{i=1}^{n} x^{n-i}y^{i-1} \\ = & 1 + [y(x_r^{e,\textcircled{\#}})^n + y - yxx_r^{e,\textcircled{\#}}][1 + x^n(1 - xx_r^{e,\textcircled{\#}})]^{-1}\sum\limits_{i=1}^{n} x^{n-i}y^{i-1} \\ = & 1 + y[1 + x^n(1 - xx_r^{e,\textcircled{\#}})]\sum\limits_{i=1}^{n} x^{n-i}y^{i-1} \\ = & 1 + y\sum\limits_{i=1}^{n} x^{n-i}y^{i-1} = 1 + y^n\mathcal{A}^{-1}. \end{array}$$

By using Cline's formula (see [15, Theorem 2.1]), $1 + q\sum\limits_{i=1}^{n} x^{n-i}y^{i} \in \mathcal{A}^{-1}$. Accordingly, we derive that

$$\begin{array}{rcl} pa & = & p(x + y) = py = (1 - xx_r^{e,\textcircled{\#}})y = y - e^{-1}(exx_r^{e,\textcircled{\#}})y \\ & = & y - e^{-1}(exx_r^{e,\textcircled{\#}})^*y = y - e^{-1}(x_r^{e,\textcircled{\#}})^*(x^*ey) = y \in \mathcal{A}^{qnil}, \\ pa(1 - p) & = & yxx_r^{e,\textcircled{\#}} = 0, \\ a^n + p & = & (x + y)^n + p = x^n + \sum\limits_{i=1}^{n} x^{n-i}y^i + p \\ & = & [x^n + p] + \sum\limits_{i=1}^{n} x^{n-i}y^i \\ & = & [x^n + p][1 + q\sum\limits_{i=1}^{n} x^{n-i}y^i] \in \mathcal{A}_r^{-1}. \end{array}$$

Moreover, we see that $1 - p = xx_r^{e,\textcircled{\#}} = [(x + y)xx_r^{e,\textcircled{\#}}]x_r^{e,\textcircled{\#}} \in a(1 - p)\mathcal{A}$. On the other hand, $a(1 - p) = (1 - p)a(1 - p) \in (1 - p)\mathcal{A}$. Then

$$(1 - p)\mathcal{A} = a(1 - p)\mathcal{A}.$$

(2) $\Rightarrow$ (1) By hypothesis, there exists an idempotent $p \in \mathcal{A}$ such that

$$a^n + p \in \mathcal{A}_r^{-1}, (ep)^* = ep, ap \in \mathcal{A}^{qnil}, (1 - p)\mathcal{A} = a(1 - p)\mathcal{A}.$$

Set $x = (1 - p)a$ and $y = pa$. Then

$$\begin{array}{rcl} x^*ey & = & [a^*(1 - p)^*]epa = [a^*(1 - p)^*](ep)^*a = a^*[(1 - p)^*p^*]e^*a^* = 0, \\ yx & = & pa(1 - p)a = 0, \\ y & = & pa \in \mathcal{A}^{qnil}. \end{array}$$

Write $(a^n + p)q = 1$ for some $q \in \mathcal{A}$. Then $(1 - p)a^nq = (1 - p)(a^n + p)q = 1 - p$. Set $z = a^{n-1}q$. Then $(1 - p)az(1 - p)a = (1 - p)a$ and $[(1 - p)az]^* = (1 - p)^* = 1 - p = (1 - p)az$. This implies that $(1 - p)a \in \mathcal{A}_e^{(1,3)}$.

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Since $(1-p)\mathcal{A} = a(1-p)\mathcal{A}$, we have $pa(1-p) = 0$. Write $1-p = a(1-p)r$ for some $r \in \mathcal{A}$. Then $1-p = (1-p)a(1-p)r$; hence,

$$\begin{array}{rcl} 1-p & = & (1-p)[a(1-p)r] = (1-p)a[(1-p)r] \\ & = & [(1-p)a][(1-p)a(1-p)r]r \\ & \in & [(1-p)a]^2\mathcal{A}. \end{array}$$

Then we have $(1-p)a\mathcal{A} = [(1-p)a]^2\mathcal{A}$. According to Theorem 1.1, $(1-p)a \in \mathcal{A}_r^{e,\textcircled{\#}}$. That is, $x \in \mathcal{A}_r^{e,\textcircled{\#}}$. Therefore $a \in \mathcal{A}_r^{e,\textcircled{d}}$. $\square$

**Corollary 2.6.** *Every power of a generalized right core invertible element in a Banach *-algebra is the sum of two invertible and a right invertible elements.*

**Proof.** Let $a \in \mathcal{A}_r^{e,\textcircled{d}}$ and $n \in \mathbb{N}$. In view of Theorem 2.5, we can find $p^2 = p \in \mathcal{A}$ such that $u := a^n + p \in \mathcal{A}_r^{-1}$. Then $a^n = u - p$. Obviously, we have $-p = \frac{1-2p}{2} - \frac{1}{2}$ and $(\frac{1-2p}{2})^2 = \frac{1}{4}$. Then

$$(\frac{1-2p}{2})^{-1} = 2(1-2p).$$

Accordingly, $a^n = u + \frac{1-2p}{2} - \frac{1}{2}$, as desired. $\square$

**Corollary 2.7.** *Let $a \in \mathcal{A}$ and $n \in \mathbb{N}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{e,\textcircled{d}}$.
(2) $a^n \in \mathcal{A}_r^{e,\textcircled{d}}$.

In this case, $a^{e,\textcircled{d}} = a^{n-1}(a^n)_r^{e,\textcircled{d}}$.

**Proof.** $(1) \Rightarrow (2)$ In light of Theorem 2.5, there exists an idempotent $p \in \mathcal{A}$ such that

$$a^n + p \in \mathcal{A}_r^{-1}, (ep)^* = ep, ap \in \mathcal{A}^{qnil}, (1-p)\mathcal{A} = a(1-p)\mathcal{A}.$$

By virtue of Cline's formula, $pap \in \mathcal{A}^{qnil}$. Hence $(pap)^n \in \mathcal{A}^{qnil}$. Clearly, $pa(1-p) = 0$, and so $pa = pap$. This implies that $pa^n = (pap)^n \in \mathcal{A}^{qnil}$. By using Cline's formula again, $a^n p \in \mathcal{A}^{qnil}$. Since $a^n(1-p) = (1-p)a^n(1-p) \in (1-p)\mathcal{A}$ and $1-p \in a(1-p)\mathcal{A} \subseteq a^n(1-p)\mathcal{A}$, we deduce that $(1-p)\mathcal{A} = a^n(1-p)\mathcal{A}$. By using Theorem 1.1, $a^n \in \mathcal{A}_r^{e,\textcircled{d}}$.

$(2) \Rightarrow (1)$ Let $x = a^{n-1}(a^n)_r^{e,\textcircled{d}}$. Then we directly verify that

$$\begin{array}{rcl} ax & = & a(a^{n-1}(a^n)_r^{e,\textcircled{d}}) = a^n(a^n)_r^{e,\textcircled{d}}, \\ ax^2 & = & [a^n(a^n)_r^{e,\textcircled{d}}]a^{n-1}(a^n)_r^{e,\textcircled{d}} = x, \\ (eax)^* & = & (a^n(a^n)_r^{e,\textcircled{d}})^* = a^n(a^n)_r^{e,\textcircled{d}} = eax, \\ ||a^m - axa^m||^{\frac{1}{m}} & \leq & ||(1-a^n(a^n)_r^{e,\textcircled{d}})a^m||^{\frac{1}{m}}|. \end{array}$$

For any $m \geq nk$, we see that

$$||a^m - axa^m||^{\frac{1}{m}} \leq ||(1-a^n(a^n)_r^{e,\textcircled{d}})(a^n)^k||^{\frac{1}{m}} = 0.$$

Since $\lim\limits_{k\to\infty} ||(1-a^n(a^n)_r^{e,\textcircled{d}})(a^n)^k||^{\frac{1}{k}} = 0$, we derive that

$$\lim_{m\to\infty} ||a^m - axa^m||^{\frac{1}{m}} = 0.$$

Therefore $a^{e,\textcircled{d}} = a^{n-1}(a^n)_r^{e,\textcircled{d}}$. $\square$

We are now ready to prove:

**Theorem 2.8.** *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{e,\textcircled{d}}$.
(2) There exists $b \in \mathcal{A}$ such that

$$bab = b, (eab)^* = eab, aba\mathcal{A} = a^2ba\mathcal{A}, a - a^2b \in \mathcal{A}^{qnil}.$$

**Proof.** (1) $\Rightarrow$ (2) By hypothesis, there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*ey = yx = 0, x \in \mathcal{A}_r^{e,\textcircled{\#}}, y \in \mathcal{A}^{qnil}.$$

It is easy to verify that

$$\begin{array}{rcl} x_r^{e,\textcircled{\#}}y & = & x_r^{e,\textcircled{\#}}xx_r^{e,\textcircled{\#}}y \\ & = & [x_r^{e,\textcircled{\#}}]e^{-1}[exx_r^{e,\textcircled{\#}}]y \\ & = & [x_r^{e,\textcircled{\#}}e^{-1}](exx_r^{e,\textcircled{\#}})^*y \\ & = & [x_r^{e,\textcircled{\#}}e^{-1}](x_r^{e,\textcircled{\#}})^*(x^*ey) \\ & = & 0. \end{array}$$

Set $b = x_r^{e,\textcircled{\#}}$. Then $ab = (x+y)x_r^{e,\textcircled{\#}} = xx_r^{e,\textcircled{\#}} + yx(x_r^{e,\textcircled{\#}})^2 = xx_r^{e,\textcircled{\#}}$. Hence, $(eab)^* = (exx_r^{e,\textcircled{\#}})^* = exx_r^{e,\textcircled{\#}} = eab$. We easily verify that

$$\begin{array}{rcl} ab^2 & = & (ab)b = (xx_r^{e,\textcircled{\#}})x_r^{e,\textcircled{\#}} = x_r^{e,\textcircled{\#}} = b, \\ b(1-ab) & = & x_r^{e,\textcircled{\#}}[1 - xx_r^{\textcircled{\#}}] = 0, \\ a - a^2b & = & a(1-ab) = a(1 - xx^{\textcircled{\#}}). \end{array}$$

Thus $b = bab$, and so $ab^2 = bab$.

Moreover, we see that

$$\begin{array}{rcl} aba & = & (xx_r^{e,\textcircled{\#}})(x+y) \\ & = & xx_r^{e,\textcircled{\#}}x = x; \\ a^2ba & = & a(aba) = (x+y)x = x^2. \end{array}$$

Since $x \in \mathcal{A}_r^{e,\textcircled{\#}}$, it follows by Theorem 1.1 that $x\mathcal{A} = x^2\mathcal{A}$. Thus, $aba\mathcal{A} = a^2ba\mathcal{A}$. Since $(1 - xx^{\textcircled{\#}})a = (1 - xx^{\textcircled{\#}})(x+y) = y \in \mathcal{A}^{qnil}$, by using Cline's formula, $a - a^2b = a(1 - xx^{\textcircled{\#}}) \in \mathcal{A}^{qnil}$.

(2) $\Rightarrow$ (1) By hypothesis, there exists $b \in \mathcal{A}$ such that

$$bab = b, (eab)^* = eab, aba\mathcal{A} = a^2ba\mathcal{A}, a - a^2b \in \mathcal{A}^{qnil}.$$

Let $x = aba$ and $y = a - aba$. Then

$$\begin{array}{rcl} a & = & x + y, \\ x^*ey & = & (aba)^*e^*(a - aba) = a^*(eab)^*(1-ab)a = a^*eab(1-ab)a = 0, \\ yx & = & (a - aba)aba = (1-ab)a^2ba = (1-ab)abar = 0 \text{ for a } r \in \mathcal{A}. \end{array}$$

Since $a - a^2b \in \mathcal{A}^{qnil}$. By using Cline's formula, we have $y = (1-ab)a \in \mathcal{A}^{qnil}$. Clearly, we have $xb = (aba)b = a(bab) = ab$, and so $xbx = ab(aba) = a(bab)a = aba = x$ and $(exb)^* = (eab)^* = eab = exb$. This implies that $x \in \mathcal{A}_e^{(1,3)}$. We easily verify that

$$aba = ababa \in aba^2ba\mathcal{A} = (aba)^2\mathcal{A}.$$

Hence, $x\mathcal{A} = x^2\mathcal{A}$. In view of Theorem 1.1, $x \in \mathcal{A}_r^{e,\textcircled{\#}}$. Therefore $a \in \mathcal{A}_r^{e,\textcircled{d}}$. $\square$

**Corollary 2.9.** *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{e,\textcircled{d}}$.

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(2) There exists $b \in \mathcal{A}$ such that

$$bab = b = ab^2, (eab)^* = eab, aba\mathcal{A} = a^2ba\mathcal{A}, a - a^2b \in \mathcal{A}^{qnil}.$$

**Proof.** (1) $\Rightarrow$ (2) Construct $x, y$ and $a, b$ as in the proof of Theorem 2.8, we have

$$bab = b, (eab)^* = eab, aba\mathcal{A} = a^2ba\mathcal{A}, a - a^2b \in \mathcal{A}^{qnil}.$$

Moreover, we verify that

$$\begin{array}{rcl} ab & = & (x+y)x_r^{e,\textcircled{\#}} = xx_r^{e,\textcircled{\#}} + yx(x_r^{e,\textcircled{\#}})^2 = xx_r^{e,\textcircled{\#}}, \\ ab^2 & = & (ab)b = (xx_r^{e,\textcircled{\#}})x_r^{e,\textcircled{\#}} = x_r^{e,\textcircled{\#}} = b, \end{array}$$

as desired.

(2) $\Rightarrow$ (1) This is obvious by Theorem Theorem 2.8. $\square$

## 3. Characterizations by Using Right g-Drazin Inverse

Let $a \in \mathcal{A}$. Set

$$\{a_r^d\} = \{x \in \mathcal{A} \mid ax^2 = x, a - xa^2 \in \mathcal{A}^{qnil}\}.$$

We now derive the following.

**Theorem 3.1.** *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{e,\textcircled{d}}$.
(2) $\{a_r^d\} \bigcap \mathcal{A}_r^{e,\textcircled{\#}} \neq \emptyset$.

In this case, $a_r^{e,\textcircled{d}} = z^2 z_r^{e,\textcircled{\#}}$ for $z \in \{a_r^d\} \bigcap \mathcal{A}_r^{e,\textcircled{\#}}$.

**Proof.** (1) $\Rightarrow$ (2) In view of Theorem 2.1, there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*ey = yx = 0, x \in \mathcal{A}_r^{e,\textcircled{\#}}, y \in \mathcal{A}^{qnil}.$$

Let $z = x_r^{e,\textcircled{\#}}$. Then

Claim 1. $z \in \{a_r^d\}$. We directly verify that

$$\begin{array}{rcl} az & = & (x+y)x_r^{e,\textcircled{\#}} = (x+y)x(x_r^{e,\textcircled{\#}})^2 = xx_r^{e,\textcircled{\#}}, \\ az^2 & = & [xx_r^{e,\textcircled{\#}}]x_r^{e,\textcircled{\#}} = x(x_r^{e,\textcircled{\#}})^2 = x_r^{e,\textcircled{\#}} = z, \\ aza & = & xx_r^{e,\textcircled{\#}}(x+y) = xx_r^{e,\textcircled{\#}}x + (x_r^{e,\textcircled{\#}})^*(x^*y) \\ & = & xx_r^{e,\textcircled{\#}}x + e^{-1}(exx_r^{e,\textcircled{\#}})^*y \\ & = & xx_r^{e,\textcircled{\#}}x + e^{-1}(x_r^{e,\textcircled{\#}})^*(x^*ey) = x, \\ a - aza & = & a - x = y \in \mathcal{A}^{qnil}. \end{array}$$

By using Cline's formula, we have $a - za^2 \in \mathcal{A}^{qnil}$. Therefore $z \in \{a_r^d\}$.

Claim 2. $z \in \mathcal{A}_r^{e,\textcircled{\#}}$. We verify that

$$\begin{array}{rcl} z[x^2z] & = & x_r^{e,\textcircled{\#}}[x^2x_r^{e,\textcircled{\#}}] = xx_r^{e,\textcircled{\#}}, \\ z[x^2z]^2 & = & [xx_r^{e,\textcircled{\#}}](x^2z) = x^2z, \\ \big(ez(x^2z)\big)^* & = & (exx_r^{e,\textcircled{\#}})^* = exx_r^{e,\textcircled{\#}} = ez(x^2z), \\ z(x^2z)z & = & [xx_r^{e,\textcircled{\#}}]x_r^{e,\textcircled{\#}} = x_r^{e,\textcircled{\#}} = z. \end{array}$$

Accordingly, $z \in \mathcal{A}_r^{e,\textcircled{\#}}$ and $z_r^{e,\textcircled{\#}} = x^2z$. Therefore $\{a_r^d\} \bigcap \mathcal{A}_r^{e,\textcircled{\#}} \neq \emptyset$.

(2) $\Rightarrow$ (1) Let $z \in \{a_r^d\} \bigcap \mathcal{A}_r^{e,\textcircled{\#}}$. Then

$$az^2 = z, a - za^2 \in \mathcal{A}^{qnil}.$$

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Set $x = z^2 z_r^{e,\textcircled{\#}}$. Then we check that

$$(zz_r^{e,\textcircled{\#}}a)x = zz_r^{e,\textcircled{\#}}(az^2)z_r^{e,\textcircled{\#}} = [zz_r^{e,\textcircled{\#}}]^2 = zz_r^{e,\textcircled{\#}},$$

hence, we see that

$$\begin{aligned}
[e(zz_r^{e,\textcircled{\#}}a)x]^* &= [ezz_r^{e,\textcircled{\#}}]^* = ezz_r^{e,\textcircled{\#}} = e(zz_r^{e,\textcircled{\#}}a)x,\\
zz_r^{e,\textcircled{\#}}ax^2 &= [zz_r^{e,\textcircled{\#}}][z^2z_r^{e,\textcircled{\#}}] = z^2z_r^{e,\textcircled{\#}} = x,\\
\big(zz_r^{e,\textcircled{\#}}a\big)x\big(zz_r^{e,\textcircled{\#}}a\big) &= \big(zz_r^{e,\textcircled{\#}}\big)\big(zz_r^{e,\textcircled{\#}}a\big) = zz_r^{e,\textcircled{\#}}a.
\end{aligned}$$

Then $zz_r^{e,\textcircled{\#}}a \in \mathcal{A}_r^{e,\textcircled{\#}}$ and $[zz_r^{e,\textcircled{\#}}a]_r^{e,\textcircled{\#}} = z^2z_r^{e,\textcircled{\#}}$.

Write $a = a_1 + a_2$, where $a_1 = zz_r^{e,\textcircled{\#}}a$ and $a_2 = a - zz_r^{e,\textcircled{\#}}a$. It is easy to verify that

$$\begin{aligned}
a_2a_1 &= [a - zz_r^{e,\textcircled{\#}}a]zz_r^{e,\textcircled{\#}}a\\
&= azz_r^{e,\textcircled{\#}}a - zz^{\textcircled{\#}}azz_r^{e,\textcircled{\#}}a\\
&= azz_r^{e,\textcircled{\#}}a - zz^{\textcircled{\#}}(az^2)(z_r^{e,\textcircled{\#}})^2a\\
&= azz_r^{e,\textcircled{\#}}a - zz^{\textcircled{\#}}z(z_r^{e,\textcircled{\#}})^2a\\
&= (az^2)(z_r^{e,\textcircled{\#}})^2a - z(z^{\textcircled{\#}})^2a\\
&= 0,\\
a_1^*a_2 &= a^*(zz_r^{e,\textcircled{\#}})^*[a - zz_r^{e,\textcircled{\#}}a]\\
&= a^*(zz_r^{e,\textcircled{\#}})[a - zz_r^{e,\textcircled{\#}}a]\\
&= a^*zz_r^{e,\textcircled{\#}}[1 - zz_r^{e,\textcircled{\#}}]a = 0.
\end{aligned}$$

Moreover, we check that

$$\begin{aligned}
[1 - zz_r^{e,\textcircled{\#}}]a &= [1 - zz_r^{e,\textcircled{\#}}]a - [1 - zz_r^{e,\textcircled{\#}}]za^2\\
&= [1 - zz_r^{e,\textcircled{\#}}](a - za^2).
\end{aligned}$$

Obviously,

$$az = azz_r^{e,\textcircled{\#}}z = az^2(z_r^{e,\textcircled{\#}})^2z = z(z_r^{e,\textcircled{\#}})^2z = z_r^{e,\textcircled{\#}}z.$$

It is easy to verify that

$$\begin{aligned}
(a - za^2)[1 - zz_r^{e,\textcircled{\#}}] &= a - za^2 - (1 - za)(az)z_r^{e,\textcircled{\#}}\\
&= a - za^2 - (1 - za)(z_r^{e,\textcircled{\#}}z)z_r^{e,\textcircled{\#}}\\
&= a - za^2 - (1 - za)z_r^{e,\textcircled{\#}}\\
&= a - za^2 - (1 - za)z^2(z_r^{e,\textcircled{\#}})^3\\
&= a - za^2 - [z_r^{e,\textcircled{\#}} - z(az^2)(z_r^{e,\textcircled{\#}})^3]\\
&= a - za^2 - [z_r^{e,\textcircled{\#}} - z^2(z_r^{e,\textcircled{\#}})^3]\\
&= a - za^2 \in \mathcal{A}^{qnil}.
\end{aligned}$$

By using Cline's formula again,

$$a_2 = [1 - zz_r^{e,\textcircled{\#}}]a = [1 - zz_r^{e,\textcircled{\#}}](a - za^2) \in \mathcal{A}^{qnil}.$$

Therefore $a = a_1 + a_2$ is the generalized right core decomposition of $a$. Therefore

$$a_r^{e,\textcircled{d}} = (a_1)_r^{e,\textcircled{\#}} = z^2z_r^{e,\textcircled{\#}},$$

as asserted. □

**Corollary 3.2.** *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{e,\textcircled{d}}$.

(2) $a \in \mathcal{A}^d$ and $a^d \in \mathcal{A}_r^{\#}$.

In this case, $a^{e,\textcircled{d}} = (a^d)^2(a^d)^{e,\#}$.

**Proof.** This is obvious by Theorem 3.1 and Corollary 2.4. □

**Lemma 3.3.** *Let $z \in \{a_r^d\}$. Then*
$$\lim_{n\to\infty} ||(a^n - aza^n)^*||^{\frac{1}{n}} = 0.$$

**Proof.** Let $x = a - aza$. Then $x \in \mathcal{A}^{qnil}$. For any $\lambda \in \mathbb{C}$, we have $1 - \overline{\lambda}x \in \mathcal{A}^{-1}$, and so $1 - \lambda x^* \in \mathcal{A}^{-1}$. This implies that $x^* \in \mathcal{A}^{qnil}$. We easily check that

$$\begin{aligned} ||(a^n - aza^n)^*||^{\frac{1}{n}} &= ||(a^n)^*(1-az)^*|| = ||(a^n)^*[(1-az)^n]^*|| \\ &= ||[(a-aza)^n]^*|| = ||(x^*)^n||. \end{aligned}$$

Since $x^* \in \mathcal{A}^{qnil}$, we have
$$\lim_{n\to\infty} ||(a^n - aza^n)^*||^{\frac{1}{n}} = 0.$$
□

**Lemma 3.4.** *Let $a \in \mathcal{A}_r^{e,\textcircled{d}}$. Then*
$$\lim_{n\to\infty} ||((aw)^n - awa_r^{e,\textcircled{d}}w(aw)^n)^*||^{\frac{1}{n}} = 0.$$

**Proof.** Construct $x, y, z$ as in the proof of Theorem 2.1. Then
$$\lim_{n\to\infty} ||(a^n - xa^{n+1})^*||^{\frac{1}{n}} = \lim_{n\to\infty} ||(y^*)^n||^{\frac{1}{n}} = 0.$$
Similarly to Corollary 2.2, we check that
$$||(a^n - a^nx^na^n)^*||^{\frac{1}{n}} \leq (1 + ||a^*||||x^*||)||(a^n - xa^{n+1})^*||^{\frac{1}{n}}.$$
Therefore
$$\lim_{n\to\infty} ||(a^n - a^n(a_r^{e,\textcircled{d}})^na^n)^*||^{\frac{1}{n}} = 0.$$
In view of Corollary 2.4, we have
$$\lim_{n\to\infty} ||(a^n - aa_r^{e,\textcircled{d}}a^n)^*||^{\frac{1}{n}} = 0,$$
as asserted. □

We are ready to prove:

**Theorem 3.5.** *Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}_r^{e,\textcircled{d}}$ if and only if there exist $x \in \mathcal{A}$ and $z \in \{a_r^d\}$ such that*
$$xax = x = ax^2, x\mathcal{A} = z\mathcal{A}, \mathcal{A}x = \mathcal{A}(az)^*e.$$
*In this case, $a_r^{e,\textcircled{d}} = x$.*

**Proof.** $\Longrightarrow$ Choose $x = a_r^{e,\textcircled{d}}$. In view of Theorem 1.1, $x = xax = ax^2$. By using Theorem 3.1, we can find $z \in \{a_r^d\} \bigcap \mathcal{A}_r^{e,\#}$ such that
$$x = z^2z_r^{e,\#}.$$
Then we have
$$az^2 = z, a - za^2 \in \mathcal{A}^{qnil}.$$

Obviously, $z = zz_r^{e,\circledast}z = (z^2 z_r^{e,\circledast})z_r^{e,\circledast}z = xz_r^{e,\circledast}z$. Accordingly, we have $x\mathcal{A} = z\mathcal{A}$.
Since $ax^2 = x$, we have $ax = a^n x^n$, and then

$$\begin{aligned} x &= xax = xe^{-1}(eax)^* = xe^{-1}(ea^n x^n)^* = xe^{-1}(x^n)^*(a^n)^* e \\ &= xe^{-1}(x^n)^*(a^n - aza^n)^* e + xe^{-1}(x^n)^*(a^n)^*(az)^* e \\ &= xe^{-1}(x^n)^*(a^n - aza^n)^* e + xe^{-1}(a^n x^n)^*(az)^* e \\ &= xe^{-1}(x^n)^*(a^n - aza^n)^* e + xe^{-1}(ax)^*(az)^* e. \end{aligned}$$

Hence,

$$\begin{aligned} ||x - xe^{-1}(ax)^*(az)^* e||^{\frac{1}{n}} &= ||xe^{-1}(x^n)^*(a^n - aza^n)^* e||^{\frac{1}{n}} \\ &\leq ||xe^{-1}(x^n)^*||^{\frac{1}{n}} ||(a^n - aza^n)^*||^{\frac{1}{n}} ||e||^{\frac{1}{n}}. \end{aligned}$$

In view of Lemma 3.4,

$$\lim_{n\to\infty} ||(a^n - aza^n)^*||^{\frac{1}{n}} = 0,$$

we derive that

$$\lim_{n\to\infty} ||x - xe^{-1}(ax)^*(az)^* e||^{\frac{1}{n}} = 0;$$

hence, $x = xe^{-1}(ax)^*(az)^* e$. Then $\mathcal{A}x \subseteq \mathcal{A}(az)^* e$.
Since $az^2 = z$, we have $a^n z^n = az$, and then we derive that

$$\begin{aligned} ||(az)^* e - (az)^* eax||^{\frac{1}{n}} &= ||(z^n)^*(ea^n)^* - (z^n)^*(eaxa^n)^*||^{\frac{1}{n}} \\ &= ||((z)^n)^*(a^n - axa^n)^* e||^{\frac{1}{n}} \\ &\leq ||(z^n)^*||^{\frac{1}{n}} ||(a^n - axa^n)^*||^{\frac{1}{n}} ||e||^{\frac{1}{n}}. \end{aligned}$$

In light of Lemma 3.4, we see that

$$\lim_{n\to\infty} ||(a^n - axa^n)^*||^{\frac{1}{n}} = 0.$$

Then

$$\lim_{n\to\infty} ||(az)^* e - (az)^* eax||^{\frac{1}{n}} = 0,$$

and so $(az)^* e = (az)^* eax$. Hence $\mathcal{A}(az)^* e \subseteq \mathcal{A}x$. Therefore $\mathcal{A}x = \mathcal{A}(az)^* e$, as required.
$\Longleftarrow$ By hypothesis, there exists $x \in \mathcal{A}$ such that there exist $x \in \mathcal{A}$ and $z \in \{a_r^d\}$ such that

$$xax = x = ax^2, x\mathcal{A} = z\mathcal{A}, \mathcal{A}x = \mathcal{A}(az)^* e.$$

We claim that $a_r^{e,\circledast} = x$.
Claim 1.

$$\lim_{n\to\infty} ||a^n - axa^n||^{\frac{1}{n}} = 0.$$

Write $z = xy$ for some $y \in \mathcal{A}$. For any $n \in \mathbb{N}$, we have

$$\begin{aligned} a^n &= (a^n - aza^n) + aza^n, \\ axa^n &= ax(a^n - aza^n) + axaza^n \\ &= ax(a^n - aza^n) + a(xax)ya^n \\ &= ax(a^n - aza^n) + (axy)a^n \\ &= ax(a^n - aza^n) + aza^n. \end{aligned}$$

Hence,

$$a^n - axa^n = (1 - ax)(a^n - aza^n),$$

and so

$$||a^n - axa^n||^{\frac{1}{n}} \leq ||1 - ax||^{\frac{1}{n}} ||a^n - aza^{n+1}||^{\frac{1}{n}},$$

we have
$$\lim_{n\to\infty} ||a^n - axa^n||^{\frac{1}{n}} = 0.$$

Claim 2. $(eax)^* = eax$.

Since $\mathcal{A}x = \mathcal{A}(az)^*e$, we have $x^*\mathcal{A} = eaz\mathcal{A}$. Write $eaz = x^*y$ for some $y \in \mathcal{A}$. Since $xax = x$, we have $x^*a^*x^* = x^*$, and then $(ax)^*x^* = x^*$. This implies that $(ax)^*eaz = (ax)^*(x^*y) = [(ax)^*x^*]y = x^*y = eaz$. Since $x\mathcal{A} = z\mathcal{A}$, we can find $s \in \mathcal{A}$ such that $x = zs$. Then $(ax)^*e(ax) = (ax)^*ea(zs) = [(ax)^*eaz]s = (eaz)s = eax$. Hence $(eax)^* = [(ax)^*e(ax)]^* = (ax)^*e(ax) = eax$.

Therefore $a_r^{e,ⓓ} = x$, as asserted. □

**Corollary 3.6.** *Let $a \in \mathcal{A}$. Then $a$ has pseudo right e-core inverse if and only if*

(1) $a \in \mathcal{A}_r^{e,ⓓ}$;
(2) $a$ has right Drazin inverse.

**Proof.** $\Longrightarrow$ By virtue of Theorem 2.1, $a$ has generalized right $e$-core inverse. Therefore $a$ has right Drazin inverse by Theorem 3.5.

$\Longrightarrow$ Since $a$ has generalized right $e$-core inverse, by Theorem 3.5, there exists $x \in \mathcal{A}$ and $z \in \mathcal{A}_r^{e,\#}$ such that
$$xax = x = ax^2, x\mathcal{A} = z\mathcal{A}, \mathcal{A}x = \mathcal{A}(az)^*e.$$

Since $a$ has right Drazin inverse, we have $a_r^d = a_r^D$. Let $n = ind(a)$. Then $a^n = a^{n+1}a^d, aa^d = a^da$ and $a^d = a(a^d)^2$. Hence, $a^d = a^n[(a^d)^{n+1}]$ and $a^n = a^da^{n+1}$. Then $a^n\mathcal{A} = a^d\mathcal{A}$. On the other hand, we have
$$(a^d)^* = [(a^d)^{n+1}]^*(a^n)^*, (a^n)^* = (a^{n+1})^*(a^d)^*.$$

Therefore $\mathcal{A}(a^d)^*e = \mathcal{A}a^ne$, and so $\mathcal{A}x = \mathcal{A}(a^n)^*e$. This implies that $a$ has pseudo right $e$-core inverse, as asserted. □

## 4. Representations of Generalized Right $e$-Core Inverse

Let $T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathcal{A}^{2\times 2}$. Let $e \in \mathcal{A}$ be an Hermitan invertible element and $E = diag(e, e)$. Using a similar approach, we now extend the result in Proposition 4.4 of [7] to the right e-core inverse.

**Lemma 4.1.** *Let $s = d - ba^{-1}c \in \mathcal{A}^{e,\#}$, $p = 1 - ss_r^{e,\#}$ and $t = s_r^{e,\#} + 1 - s_r^{e,\#}s$. If $v = a + c(1 - ts)ba^{-1} \in \mathcal{A}_r^{-1}$, then $TM_2(\mathcal{A}) = T^2M_2(\mathcal{A})$.*

**Proof.** Obviously, we have
$$T = PAQ, P = \begin{pmatrix} 1 & 0 \\ ba^{-1} & 1 \end{pmatrix}, A = \begin{pmatrix} a & 0 \\ 0 & s \end{pmatrix}, Q = \begin{pmatrix} 1 & a^{-1}c \\ 0 & 1 \end{pmatrix}.$$

Since $s \in \mathcal{A}_r^{e,\#}$, it follows by Theorem 1.1 that $s \in \mathcal{A}_e^{(1,3)}$. Hence we verify that $A_E^{(1,3)} = \begin{pmatrix} a^{-1} & 0 \\ 0 & s_e^{(1,3)} \end{pmatrix}$. Set $U = AQP + I_2 - AA_E^{(1,3)}$. Then
$$U = \begin{pmatrix} a + cba^{-1} & c \\ sba^{-1} & s + 1 - ss_e^{(1,3)} \end{pmatrix}.$$

One easily checks that
$$\begin{aligned}(s + 1 - ss_e^{(1,3)})(s_e^{(1,3)} + 1 - ss_e^{(1,3)}) &= 1 + s(1 - ss_e^{(1,3)}) \\ &= [1 - s(1 - ss_e^{(1,3)})]^{-1}.\end{aligned}$$

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Then $U$ is right inverse and

$$U_r^{-1} = \begin{pmatrix} v_r^{-1} & -v_r^{-1}ct \\ -tsba^{-1}v_r^{-1} & t + tsba^{-1}v_r^{-1}ct \end{pmatrix}.$$

Thus, $UM_2(\mathcal{A}) = M_2(\mathcal{A})$. As $s \in \mathcal{A}$ is regular, so is $T \in M_2(\mathcal{A})$. In view of [7, Lemma 3.3], $TM_2(\mathcal{A}) = T^2M_2(\mathcal{A})$, as required. □

**Lemma 4.2.** *Let* $s = d - ba^{-1}c \in \mathcal{A}_r^{e,\#}, p = 1 - ss_r^{e,\#}, t = s_r^{e,\#} + 1 - s_r^{e,\#}s$ *and* $e(ba^{-1}) = (ba^{-1})e$. *If* $u = 1 + (ba^{-1})^*pba^{-1} \in \mathcal{A}^{-1}, v = a + c(1 - ts)ba^{-1} \in \mathcal{A}_r^{-1}$, *then* $T \in M_2(\mathcal{A})_r^{E,\#}$. *In this case,*

$$T_r^{E,\#} = \begin{pmatrix} \beta & \gamma \\ \delta & \epsilon \end{pmatrix},$$

*where*

$$\begin{aligned}
\beta &= v_r^{-1}u^{-1} + v_r^{-1}ctss_r^{e,\#}ba^{-1}u^{-1}, \\
\gamma &= v_r^{-1}u^{-1}(ba^{-1})^*p - v_r^{-1}ctss_r^{e,\#}[1 - ba^{-1}u^{-1}(ba^{-1})^*p], \\
\delta &= (1 - ts)ba^{-1}v_r^{-1}u^{-1} + [(1 - ts)ba^{-1}v_r^{-1}c - 1]tss_r^{e,\#}ba^{-1}u^{-1}, \\
\epsilon &= (1 - ts)ba^{-1}v_r^{-1}u^{-1}(ba^{-1})^*p - [(1 - ts)ba^{-1}v_r^{-1}c - 1]tss_r^{e,\#} \\
&\quad [1 - ba^{-1}u^{-1}(ba^{-1})^*p].
\end{aligned}$$

**Proof.** Set

$$S = \begin{pmatrix} \alpha u^{-1} & \alpha u^{-1}(ba^{-1})^*p - a^{-1}cs_r^{e,\#} \\ -s_r^{e,\#}ba^{-1}u^{-1} & s_r^{e,\#}[1 - ba^{-1}u^{-1}(ba^{-1})^*p] \end{pmatrix},$$

where $\alpha = [1 + a^{-1}cs_r^{e,\#}b]a^{-1}$.

Then we verify that

$$\begin{aligned}
& a\alpha u^{-1} - cs_r^{e,\#}ba^{-1}u^{-1} \\
=\ & (1 + cs_r^{e,\#}ba^{-1})u^{-1} - cs_r^{e,\#}ba^{-1}u^{-1} \\
=\ & u^{-1},
\end{aligned}$$

$$\begin{aligned}
& a[\alpha u^{-1}(ba^{-1})^*p - a^{-1}cs_r^{e,\#}] + cs_r^{e,\#}[1 - ba^{-1}u^{-1}(ba^{-1})^*p] \\
=\ & (1 + cs_r^{e,\#}ba^{-1})u^{-1}(ba^{-1})^*p - cs_r^{e,\#} + cs_r^{e,\#}[1 - ba^{-1}u^{-1}(ba^{-1})^*p] \\
=\ & (1 + cs_r^{e,\#}ba^{-1})u^{-1}(ba^{-1})^*p - cs_r^{e,\#}ba^{-1}u^{-1}(ba^{-1})^*p \\
=\ & u^{-1}(ba^{-1})^*p,
\end{aligned}$$

$$\begin{aligned}
b\alpha u^{-1} - ds_r^{e,\#}ba^{-1}u^{-1} &= [b\alpha a - ds_r^{e,\#}b]a^{-1}u^{-1} \\
&= [b(1 + a^{-1}cs_r^{e,\#}b) - ds_r^{e,\#}b]a^{-1}u^{-1} \\
&= [b - (d - ba^{-1}c)s_r^{e,\#}b]a^{-1}u^{-1} \\
&= pba^{-1}u^{-1}
\end{aligned}$$

and

$$\begin{aligned}
& b[\alpha u^{-1}(ba^{-1})^*p - a^{-1}cs_r^{e,\#}] + ds_r^{e,\#}[1 - ba^{-1}u^{-1}(ba^{-1})^*p] \\
=\ & ba^{-1}(1 + cs_r^{e,\#}ba^{-1})u^{-1}(ba^{-1})^*p - ba^{-1}cs_r^{e,\#} + ds_r^{e,\#} - ds_r^{e,\#}ba^{-1}u^{-1}(ba^{-1})^*p \\
=\ & ba^{-1}u^{-1}(ba^{-1})^*p - [d - ba^{-1}c]s_r^{e,\#}ba^{-1}u^{-1}(ba^{-1})^*p + [d - ba^{-1}c]s_r^{e,\#} \\
=\ & ba^{-1}u^{-1}(ba^{-1})^*p - ss_r^{e,\#}ba^{-1}u^{-1}(ba^{-1})^*p + ss_r^{e,\#} \\
=\ & ss_r^{e,\#} + pba^{-1}u^{-1}(ba^{-1})^*p.
\end{aligned}$$

Then we verify that

$$\begin{aligned}
TS &= \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} \alpha u^{-1} & \alpha u^{-1}(ba^{-1})^*p - a^{-1}cs_r^{e,\#} \\ -s_r^{e,\#}ba^{-1}u^{-1} & s_r^{e,\#}[1 - ba^{-1}u^{-1}(ba^{-1})^*p] \end{pmatrix} \\
&= \begin{pmatrix} u^{-1} & u^{-1}(ba^{-1})^*p \\ p(ba^{-1})u^{-1} & ss_{r,e}^{\#} + p(ba^{-1})u^{-1}(ba^{-1})^*p \end{pmatrix}.
\end{aligned}$$

We observe that

$$\begin{aligned} u^*e &= [1+(ba^{-1})^*p^*(ba^{-1})]e \\ &= e+(ba^{-1})^*p^*e(ba^{-1}) = e+(ba^{-1})^*(ep)^*(ba^{-1}) \\ &= e+(ba^{-1})^*ep(ba^{-1}) = e+e(ba^{-1})^*p(ba^{-1}) = eu, \\ (eu^{-1})^* &= eu^{-1}, \end{aligned}$$

$$\begin{aligned} [eu^{-1}(ba^{-1})^*p]^* &= p^*(ba^{-1})(u^{-1})^*e = p^*(ba^{-1})(eu^{-1})^* \\ &= p^*(ba^{-1})eu^{-1} = p^*e(ba^{-1})u^{-1} \\ &= (ep)^*(ba^{-1})u^{-1} = ep(ba^{-1})u^{-1} \end{aligned}$$

and

$$\begin{aligned} & [ep(ba^{-1})u^{-1}(ba^{-1})^*p]^* \\ = & [(ba^{-1})u^{-1}(ba^{-1})^*p]^*(ep)^* = [(ba^{-1})u^{-1}(ba^{-1})^*p]^*ep \\ = & [e(ba^{-1})u^{-1}(ba^{-1})^*p]^*p = [(ba^{-1})(eu^{-1})(ba^{-1})^*p]^*p \\ = & [p^*(ba^{-1})(eu^{-1})^*(ba^{-1})^*]p = [p^*(ba^{-1})eu^{-1})(ba^{-1})^*]p \\ = & [p^*e(ba^{-1})u^{-1})(ba^{-1})^*]p = [(ep)^*(ba^{-1})u^{-1})(ba^{-1})^*]p \\ = & epep(ba^{-1})u^{-1}(ba^{-1})^*p, \end{aligned}$$

and then

$$[e\big(ss_{r,e}^{\#} + p(ba^{-1})u^{-1}(ba^{-1})^*p\big)]^* = e\big(ss_{r,e}^{\#} + p(ba^{-1})u^{-1}(ba^{-1})^*p\big).$$

Thus $(ETS)^* = ETS$. Moreover, we see that

$$\begin{aligned} u^{-1}a+u^{-1}(ba^{-1})^*pb &= a, \\ u^{-1}c+u^{-1}(ba^{-1})^*pd &= c, \\ p(ba^{-1})u^{-1}a+[ss_{r,e}^{\#}+p(ba^{-1})u^{-1}(ba^{-1})^*p]b &= b, \\ p(ba^{-1})u^{-1}c+[ss_{r,e}^{\#}+p(ba^{-1})u^{-1}(ba^{-1})^*p]d &= d. \end{aligned}$$

Hence, we have

$$\begin{aligned} TST &= \begin{pmatrix} u^{-1} & u^{-1}(ba^{-1})^*p \\ p(ba^{-1})u^{-1} & ss_{r,e}^{\#}+p(ba^{-1})u^{-1}(ba^{-1})^*p \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} \\ &= T. \end{aligned}$$

Hence $T_E^{(1,3)} = S$.

In view of Lemma 4.1, $TM_2(\mathcal{A}) = T^2M_2(\mathcal{A})$. According to Theorem 1.1, $T \in M_2(\mathcal{A})_r^{E,\#}$. Moreover, we have

$$\begin{aligned} T_r^{E,\#} &= PU_r^{-1}AQT_E^{(1,3)} = (PU_r^{-1})(AQT_E^{(1,3)}) \\ &= \begin{pmatrix} v_r^{-1} & -v_r^{-1}ct \\ (1-ts)ba^{-1}v_r^{-1} & t-(1-ts)ba^{-1}v_r^{-1}ct \end{pmatrix} \\ & \quad \begin{pmatrix} u^{-1} & u^{-1}(ba^{-1})^*p \\ -ss_r^{e,\#}ba^{-1}u^{-1} & ss_r^{e,\#}[1-ba^{-1}u^{-1}(ba^{-1})^*p] \end{pmatrix} \\ &= \begin{pmatrix} \beta & \gamma \\ \delta & \epsilon \end{pmatrix}, \end{aligned}$$

where $\beta, \gamma, \delta$ and $\epsilon$ as mentioned before. □

We are ready to prove:

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**Theorem 4.3.** *Let* $s = d - ba^{-1}c \in \mathcal{A}_r^{e,@}, p = 1 - ss_r^{e,@}, t = s_r^{e,@} + 1 - s_r^{e,@}s$, *and let* $u = 1 + (ba^{-1})^*pba^{-1} \in \mathcal{A}^{-1}, v = a + c(1-ts)ba^{-1} \in \mathcal{A}_r^{-1}$. *If* $e(ba^{-1}) = (ba^{-1})e$ *and* $psb = (epb)^*s = 0$, *then* $T \in M_2(\mathcal{A})_r^{E,@}$. *In this case,*

$$T_r^{E,@} = \begin{pmatrix} \beta & \gamma \\ \delta & \epsilon \end{pmatrix},$$

*where*

$$\begin{aligned} \beta &= v_r^{-1}u^{-1} + v_r^{-1}ctss_r^{e,\textcircled{\#}}ba^{-1}u^{-1}, \\ \gamma &= v_r^{-1}u^{-1}(ba^{-1})^*p - v_r^{-1}ctss_r^{e,\textcircled{\#}}[1 - ba^{-1}u^{-1}(ba^{-1})^*p], \\ \delta &= (1-ts)ba^{-1}v_r^{-1}u^{-1} + [(1-ts)ba^{-1}v_r^{-1}c - 1]tss_r^{e,\textcircled{\#}}ba^{-1}u^{-1}, \\ \epsilon &= (1-ts)ba^{-1}v_r^{-1}u^{-1}(ba^{-1})^*p - [(1-ts)ba^{-1}v_r^{-1}c - 1]tss_r^{e,\textcircled{\#}} \\ &\quad [1 - ba^{-1}u^{-1}(ba^{-1})^*p]. \end{aligned}$$

**Proof.** By virtue of Theorem 2.1, we have

$$d - ba^{-1}c = x + y, x \in \mathcal{A}^{e,\#}, y \in \mathcal{A}^{qnil}, x^*ey = yx = 0.$$

Evidently,

$$x = ss_r^{e,\textcircled{\#}}s, y = s - ss_r^{e,\textcircled{\#}}s.$$

Then

$$M = A + B, A = \begin{pmatrix} a & c \\ b & d - y \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}.$$

Step 1. $A$ has right $E$-core inverse and $B$ is quasinilpotent.
Clearly, $(d-y) - ba^{-1}c = [d - ba^{-1}c] - y = (x+y) - y = x \in \mathcal{A}_r^{e,\textcircled{\#}}$.
It is easy to verify that

$$\begin{aligned} p &= 1 - ss_r^{e,@} = 1 - (x+y)x_r^{e,\textcircled{\#}} = 1 - xx_r^{e,\textcircled{\#}}, \\ t &= 1 + s_r^{e,@} - s_r^{e,@}s = 1 + x_r^{e,\textcircled{\#}} - x_r^{e,\textcircled{\#}}(x+y) \\ &= 1 + x_r^{e,\textcircled{\#}} - x_r^{e,\textcircled{\#}}x + x_r^{e,\textcircled{\#}}xx_r^{e,\textcircled{\#}}y \\ &= 1 + x_r^{e,\textcircled{\#}} - x_r^{e,\textcircled{\#}}x + [x_r^{e,\textcircled{\#}}e^{-1}](exx_r^{e,\textcircled{\#}})y \\ &= 1 + x_r^{e,\textcircled{\#}} - x_r^{e,\textcircled{\#}}x + [x_r^{e,\textcircled{\#}}e^{-1}](exx_r^{e,\textcircled{\#}})^*y \\ &= 1 + x_r^{e,\textcircled{\#}} - x_r^{e,\textcircled{\#}}x. \end{aligned}$$

By hypothesis, we have

$$\begin{aligned} u &= 1 + (ba^{-1})^*pba^{-1} \in \mathcal{A}^{-1}, \\ v &= a + c(1-tx)ba^{-1} \in \mathcal{A}_r^{-1}. \end{aligned}$$

By hypothesis, we have $e(ba^{-1}) = (ba^{-1})e$. In light of Lemma 4.2, $A \in M_2(\mathcal{A})_r^{E,\textcircled{\#}}$. Moreover, we have

$$A_r^{E,\textcircled{\#}} = \begin{pmatrix} \beta & \gamma \\ \delta & \epsilon \end{pmatrix},$$

where

$$\begin{aligned} \beta &= v_r^{-1}u^{-1} + v_r^{-1}ctss_r^{e,\textcircled{\#}}ba^{-1}u^{-1}, \\ \gamma &= v_r^{-1}u^{-1}(ba^{-1})^*p - v_r^{-1}ctss_r^{e,\textcircled{\#}}[1 - ba^{-1}u^{-1}(ba^{-1})^*p], \\ \delta &= (1-ts)ba^{-1}v_r^{-1}u^{-1} + [(1-ts)ba^{-1}v_r^{-1}c - 1]tss_r^{e,\textcircled{\#}}ba^{-1}u^{-1}, \\ \epsilon &= (1-ts)ba^{-1}v_r^{-1}u^{-1}(ba^{-1})^*p - [(1-ts)ba^{-1}v_r^{-1}c - 1]tss_r^{e,\textcircled{\#}} \\ &\quad [1 - ba^{-1}u^{-1}(ba^{-1})^*p]. \end{aligned}$$

Step 2. $M$ has generalized right $e$-core inverse.

Obviously, $d - y = x + ba^{-1}c$. Then we check that

$$\begin{aligned} A^*EB &= \begin{pmatrix} a^* & b^* \\ c^* & (d-y)^* \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & ey \end{pmatrix} \\ &= \begin{pmatrix} 0 & b^*ey \\ 0 & (d-y)^*ey \end{pmatrix} = 0, \\ BA &= \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} a & c \\ b & d-y \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ yb & y(d-y) \end{pmatrix} = 0. \end{aligned}$$

We verify that

$$\begin{aligned} b^*ey &= b^*e[s - ss_r^{e,\textcircled{\#}}s] \\ &= b^*[e(1 - ss_r^{e,\textcircled{\#}})]s = b^*[e(1 - ss_r^{e,\textcircled{\#}})]^*s \\ &= (epb)^*s = 0, \\ (d-y)^*ey &= (x + ba^{-1}c)^*ey = x^*ey + (a^{-1}c)^*(b^*ey) = 0; \\ yb &= [s - ss_r^{e,\textcircled{\#}}s]b = psb = 0, \\ y(d-y) &= y(x + ba^{-1}c) = (yb)a^{-1}c = 0. \end{aligned}$$

According to Theorem 2.1, $M$ has generalized right $E$-core inverse. In this case,

$$M_r^{E,\textcircled{d}} = A_r^{E,\textcircled{\#}} = \begin{pmatrix} \beta & \gamma \\ \delta & \epsilon \end{pmatrix},$$

where $\beta, \gamma, \delta$ and $\epsilon$ as mentioned before. □

**Corollary 4.4.** *Let $a \in \mathcal{A}^{-1}, d \in \mathcal{A}_r^{e,\textcircled{d}}$. Then $\begin{pmatrix} a & c \\ 0 & d \end{pmatrix} \in M_2(\mathcal{A})_r^{E,\textcircled{d}}$. In this case,*

$$\begin{pmatrix} a & c \\ 0 & d \end{pmatrix}_r^{E,\textcircled{d}} = \begin{pmatrix} a^{-1} & -a^{-1}cd_r^{e,\textcircled{d}} \\ 0 & d_r^{e,\textcircled{d}} \end{pmatrix}.$$

**Proof.** Since $(1 - d_r^{e,\textcircled{d}}d)dd_r^{e,\textcircled{d}} = 0$, we easily obtain the result by Theorem 4.3. □

We are now ready to prove:

**Theorem 4.5.** *Let $a, x \in \mathcal{A}$. Then the following are equivalent:*

(1) $a_r^{e,\textcircled{d}} = x$.
(2) $a \in \mathcal{A}_r^d$ and there exists an idempotent $p \in \mathcal{A}$ such that $(ep)^* = ep$ and

$$a = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}_p, x = \begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix}_p,$$

where $a_1 \in (p\mathcal{A}p)_r^{-1}, x_1 = (a_1)_r^{-1}, a_1x_2 = a_3x_1 = a_3x_2 = 0$ and $(1-p)a^m \in \ell(\{a_r^d\})$ for any $m \in \mathbb{N}$.
(3) $a \in \mathcal{A}_r^d$ and there exists an idempotent $q \in \mathcal{A}$ such that $(eq)^* = eq$ and

$$a = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}_q, x = \begin{pmatrix} 0 & 0 \\ x_1 & x_2 \end{pmatrix}_q,$$

where $a_4 \in ((1-q)\mathcal{A}(1-q))_r^{-1}, x_2 = (a_4)_r^{-1}, a_2x_1 = a_2x_2 = a_4x_1 = 0$ and $qa^m \in \ell(\{a_r^d\})$ for any $m \in \mathbb{N}$.

**Proof.** (1) $\Rightarrow$ (2) Let $p = aa_r^{e,ⓓ}$. Then $(1-p)a_r^{e,ⓓ} = 0$. Write

$$a = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}_q, x = \begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix}_q.$$

Since

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}_q \begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix}_q = ax = q = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}_q,$$

we deduce that $a_1 \in (p\mathcal{A}p)_r^{-1}, x_1 = (a_1)_r^{-1}, a_1x_2 = a_3x_1 = a_3x_2 = 0$.

Let $z \in \{a_r^d\}$ and $m \in \mathbb{N}$. For any $n \geq m$, we have $z = az^2 = a^2z^3 = \cdots = a^nz^n$ for any $n \in \mathbb{N}$. Hence,

$$\begin{aligned} ||(1-p)a^mz||^{\frac{1}{n}} &= ||(1-ax)a^nz^{n-m}||^{\frac{1}{n}} \\ &\leq ||a^n - axa^n||^{\frac{1}{n}}||z||^{1-\frac{m}{n}}. \end{aligned}$$

Since $\lim\limits_{n\to\infty} ||a^n - axa^n||^{\frac{1}{n}} = 0$, we derive that

$$\lim_{n\to\infty} ||(1-p)a^mz||^{\frac{1}{n}} = 0;$$

hence, $(1-p)a^mz = 0$, as required.

(2) $\Rightarrow$ (1) By hypothesis, there exists an idempotent $p \in \mathcal{A}$ such that $(ep)^* = ep$ and

$$a = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}_p, x = \begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix}_p,$$

where $a_1 \in (p\mathcal{A}p)_r^{-1}, x_1 = (a_1)_r^{-1}, a_1x_2 = a_3x_1 = a_3x_2 = 0$ and $(1-p)a^m \in \ell(\{a_r^d\})$ for any $m \in \mathbb{N}$. Then we check that

$$\begin{aligned} ax &= \begin{pmatrix} a_1x_1 & 0 \\ 0 & 0 \end{pmatrix}_p = p, \\ (eax)^* &= (ep)^* = ep = eax, \\ ax^2 &= (ax)x = px = \begin{pmatrix} px_1 & px_2 \\ 0 & 0 \end{pmatrix}_p = x. \end{aligned}$$

Let $z \in \ell(\{a_r^d\})$ and $m \in \mathbb{N}$. Then $az^2 = z, a - za^2 \in \mathcal{A}^{qnil}$ and $(1-p)a^mz = 0$. Hence, $(1-p)z = (1-p)az^2 = [(1-p)az]z = 0$. One easily checks that

$$\begin{aligned} ||a^n - axa^n||^{\frac{1}{n}} &= ||(1-p)a^n||^{\frac{1}{n}} = ||(1-p)(a-za^2)^n||^{\frac{1}{n}} \\ &\leq ||1-p||^{\frac{1}{n}}||(a-za^2)^n||^{\frac{1}{n}}. \end{aligned}$$

As $a - za^2 \in \mathcal{A}^{qnil}$, we see that

$$\lim_{n\to\infty} ||(a-za^2)^n||^{\frac{1}{n}} = 0,$$

and then

$$\lim_{n\to\infty} ||a^n - axa^n||^{\frac{1}{n}} = 0.$$

Therefore $a_r^{e,ⓓ} = x$, as desired.

(1) $\Leftrightarrow$ (3) This is proved as as the preceding discussion for $q = 1 - aa_r^{e,ⓓ}$. □

**Corollary 4.6.** *Let $a, x \in \mathcal{A}$. Then the following are equivalent:*

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(1) $a^{\textcircled{d}} = x$.
(2) $a \in \mathcal{A}^d$ and there exists a projection $p \in \mathcal{A}$ such that

$$a = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}_p, x = \begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix}_p,$$

where $a_1 \in (p\mathcal{A}p)^{-1}, x_1 = a_1^{-1}, a_1x_2 = a_3x_1a_3x_2 = 0$ and $(1-p)a^d = 0$.
(3) $a \in \mathcal{A}^d$ and there exists a projection $q \in \mathcal{A}$ such that

$$a = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}_q, x = \begin{pmatrix} 0 & 0 \\ x_1 & x_2 \end{pmatrix}_q,$$

where $a_4 \in ((1-q)\mathcal{A}(1-q))^{-1}, x_2 = a_4^{-1}, a_2x_1 = a_2x_2 = a_4x_1 = 0$ and $qa^d = 0$.

**Proof.** This is obvious by choosing $e = 1$ in Theorem 4.5. □

## 5. Pseudo Right $e$-Core Inverse

Recall that $a \in \mathcal{A}$ has pseudo right $e$-core inverse provided that there exists $x \in \mathcal{A}$ such that

$$x = ax^2, (eax)^* = eax, a^n = axa^n.$$

We denote $x$ by $a_r^{e,\textcircled{D}}$. The aim of this section is to investigate pseudo right $e$-core inverse in a Banach *-algebra. Let $a \in \mathcal{A}$. Set

$$\{a_r^D\} = \{x \in \mathcal{A} \mid ax^2 = x, a^n = axa^n\}.$$

We now derive the following.

**Lemma 5.1.** *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{e,\textcircled{D}}$.
(2) $a^n \in \mathcal{A}_r^{e,\textcircled{\#}}$ for some $n \in \mathbb{N}$.
(3) $a \in \mathcal{A}_r^D$ and $a^n \in \mathcal{A}_e^{(1,3)}$ for some $n \in \mathbb{N}$.
(4) $a \in \mathcal{A}_r^D \bigcap \mathcal{A}_r^{e,\textcircled{d}}$.

**Proof.** These are proved as in [27, Theorem 4.8 and Theorem 4.9]. □

**Theorem 5.2.** *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{e,\textcircled{D}}$.
(2) There exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*ey = yx = 0, x \in \mathcal{A}_r^{e,\textcircled{\#}},\ y \in \mathcal{A}^{nil}.$$

(3) There exists an idempotent $p \in \mathcal{A}$ such that

$$a^n + p \in \mathcal{A}_r^{-1}, (ep)^* = ep, ap \in \mathcal{A}^{nil}, (1-p)\mathcal{A} = a(1-p)\mathcal{A}.$$

(4) $\{a_r^D\} \bigcap \mathcal{A}_r^{e,\textcircled{\#}} \neq \emptyset$.

In this case, $a_r^{e,\textcircled{D}} = z^2 z_r^{e,\textcircled{\#}}$ for $z \in \{a_r^D\} \bigcap \mathcal{A}_r^{e,\textcircled{\#}}$.

**Proof.** This is obvious by Theorem 2.1, Theorem 2.5, Theorem 3.1 and Lemma 5.1. □

**Corollary 5.3.** *Let $a, b \in \mathcal{A}_r^{e,\textcircled{D}}$. If $ab = ba = a^*eb = 0$, then $a + b \in \mathcal{A}_r^{e,\textcircled{D}}$. In this case,*

$$(a+b)_r^{e,\textcircled{D}} = a_r^{e,\textcircled{D}} + b_r^{e,\textcircled{D}}.$$

**Proof.** In view of Theorem 5.2, we have decompositions:

$$a = x + y, x^*ey = yx = 0, x \in \mathcal{A}_r^{e,\textcircled{\#}}, y \in \mathcal{A}^{nil};$$
$$b = s + t, s^*et = ts = 0, s \in \mathcal{A}_r^{e,\textcircled{\#}}, t \in \mathcal{A}^{nil}.$$

Explicitly, we have $x = aa^{e,\textcircled{D}}aa$ and $s = bb^{e,\textcircled{D}}b$. Then $a + b = (x + s) + (y + t)$. We directly check that

$$\begin{aligned} x+s &= (x+s)(x_r^{e,\textcircled{\#}} + s_r^{e,\textcircled{\#}})(x+s), \\ x_r^{e,\textcircled{\#}} + s_r^{e,\textcircled{\#}} &= (x+s)(x_r^{e,\textcircled{\#}} + s_r^{e,\textcircled{\#}})^2, \\ \left((x+s)(x_r^{e,\textcircled{\#}} + s_r^{e,\textcircled{\#}})\right)^* &= (x+s)(x_r^{e,\textcircled{\#}} + s_r^{e,\textcircled{\#}}). \end{aligned}$$

Then $x + s \in \mathcal{A}_r^{e,\textcircled{\#}}$ and $(x+s)_r^{e,\textcircled{\#}} = x_r^{e,\textcircled{\#}} + s_r^{e,\textcircled{\#}}$. Since $yt = (a - aa^{e,\textcircled{D}}a)(b - bb^{e,\textcircled{D}}b) = 0$, it follows by ??? that $y + t \in \mathcal{A}^{nil}$.

Obviously, we check that

$$\begin{aligned} (x+s)^*e(y+t) &= x^*ey + x^*et + s^*ey + s^*et = x^*et + s^*ey \\ &= (a^{e,\textcircled{D}}a)^*(a^*eb)(1 - b^{e,\textcircled{D}}b) + (b^{e,\textcircled{D}}b)^*(b^*ea)(1 - a^{e,\textcircled{D}}a) \\ &= 0, \\ (y+t)(x+s) &= yx + ys + tx + ts = ys + tx \\ &= (a - aa^{e,\textcircled{D}}a)bb^{e,\textcircled{D}}b + (b - bb^{e,\textcircled{D}}b)aa^{e,\textcircled{D}}a = 0. \end{aligned}$$

By using Theorem 2.1,

$$\begin{aligned} (a+b)^{e,\textcircled{D}} &= (x+s)^{e,\textcircled{\#}} \\ &= x^{e,\textcircled{\#}} + s^{e,\textcircled{\#}} \\ &= a^{e,\textcircled{D}} + b^{e,\textcircled{D}}, \end{aligned}$$

as asserted. □

**Theorem 5.4.** *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{e,\textcircled{D}}$.
(2) There exists $b \in \mathcal{A}$ such that

$$bab = b, (eab)^* = eab, aba\mathcal{A} = a^2ba\mathcal{A}, a - aba \in \mathcal{A}^{nil}.$$

(3) There exists $b \in \mathcal{A}$ such that

$$bab = b, (eab)^* = eab, aba\mathcal{A} = a^2ba\mathcal{A}, a^n = aba^n$$

for some $n \in \mathbb{N}$.

**Proof.** This is proved by Theorem 2.8 and Lemma 5.1. □

**Corollary 5.5.** *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{\textcircled{D}}$.
(2) There exists $b \in \mathcal{A}$ such that

$$bab = b, (ab)^* = ab, aba\mathcal{A} = a^2ba\mathcal{A}, a - aba \in \mathcal{A}^{nil}.$$

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(3) There exists $b \in \mathcal{A}$ such that

$$bab = b, (ab)^* = eab, aba\mathcal{A} = a^2ba\mathcal{A}, a^n = aba^n$$

for some $n \in \mathbb{N}$.

**Proof.** This is obvious by choosing $e = 1$ in Theorem 5.4. $\square$

Let $A \in \mathbb{C}^{n\times n}$ and $G$ be the Minkowski matric matrix, that is, $G = diag(1, -I_{n-1})$. The Minkowski adjoint of the matrix $A$ is defined as $A^{\sim} = GA^*G$. The $\textcircled{m}$-core-EP inverse of $A$ is defined as the matrix $X \in \mathbb{C}^{n\times n}$ satisfying four conditions:

$$XAX = X, XA^{k+1} = A^k, (AX)^{\sim} = AX \text{ and } \mathcal{R}(X) \subseteq \mathcal{R}(A^k),$$

is called the $\textcircled{m}$-core-EP inverse of $A$, and denoted by $A^{\textcircled{E}}$ (see [26,30])

**Theorem 5.6.** *Let $A \in \mathbb{C}^{n\times n}$ and $G$ be the Minkowski matric matrix. Then*

$$A^{\textcircled{E}} = A_r^{G,\textcircled{D}} = A^{G,\textcircled{D}}.$$

**Proof.** Since $G = diag(1, -I_{n-1})$, we check that $G^* = G$ and $G^2 = I_n$. Thus, $G$ is an Hermitian invertible matrix. It is easy to verify that

$$\begin{array}{rl} & (AX)^{\sim} = AX \\ \Leftrightarrow & G(AX)^*G = AX \\ \Leftrightarrow & (GAX)^* = (AX)^*G^* = (AX)^*G = G^{-1}AX = GAX. \end{array}$$

Therefore $A_r^{G,\textcircled{D}} = A^{\textcircled{E}}$, as asserted. $\square$

The $\textcircled{m}$-core inverse of $A$ is defined as the matrix $X \in \mathbb{C}^{n\times n}$ satisfying four conditions:

$$XAX = X, XA^2 = A, (AX)^{\sim} = AX \text{ and } \mathcal{R}(X) \subseteq \mathcal{R}(A),$$

is called the $\textcircled{m}$-core inverse of $A$, and denoted by $A^{\textcircled{m}}$.

**Corollary 5.7.** *Let $A \in \mathbb{C}^{n\times n}$. Then*

$$A = X + Y, X^*EY = YX = 0, X \textit{ has } \textcircled{m}\textit{-core inverse}, Y \textit{ is nilpotent}.$$

*In this case, $A^{\textcircled{E}} = X^{\textcircled{m}}$.*

**Proof.** We obtain the result by Theorem 5.2 and Theorem 5.6. $\square$

**Remark 5.8.** *Generalized left e-core inverse can be defined dually. We can establish the corresponding results for generalized left e-core inverse in a similar way.*

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