Option Pricing: Examples and Open Problems

Nikolaos Halidias

Department of Statistics and Actuarial-Financial Mathematics
University of the Aegean
Karlovassi 83200 Samos, Greece
email: nikoshalidias@hotmail.com

Abstract

The option pricing problem is equivalent with the hedging problem of the option, i.e. what the writer should do in order to hedge the risk that she undertakes selling a contract and moreover what is the probability of profit selling at a specific price. The probability of profit is also a useful information for the buyer. The hedging strategy should be practically possible for the writer otherwise has no meaning. In this note we will discuss the option pricing problem and in particular the effect of the volatility on the binomial model which is a way to hedge practically a specific option in contrast to every pricing model that assumes rebuilding of the replicating portfolio continuously in time. In order to use the binomial model we have to modified it accordingly as we have seen in a previous paper. We also point out three open problems regarding the binomial option pricing model.

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1 Introduction

The option pricing problem is equivalent with the hedging problem. One way to hedge the risk of selling an option is to built a replicating portfolio. The construction of this portfolio should be in discrete time therefore any pricing model that assumes rebuilding continuously in time can not be used in practice. In this direction the binomial option pricing model is a good candidate however we should update it in order to be practically useful as we did in [4]. We call the new modified binomial model as the realistic binomial (option pricing) model.

Let us recall here the binomial option pricing model for an asset S for the time period $[0, T]$.

Assumption 1 *Let the today price of the asset is S_0. We suppose that the possible values of the price at the time T are $S_T = uS_0$ with some probability p or $S_T = dS_0$ with probability $1 - p$, for some numbers u, d.*

We call u, d the upward and downward rate of the future price of the asset. There are several choices for the rates in the bibliography, see for example [2], [3] and [7] and the references therein.

For a specific contract, the binomial option pricing model computes the initial value of a replicating portfolio. The basic problem, however, of the binomial model is that the assumption 1 is not realistic, i.e. $\mathbb{P}(S_T = uS_0) = \mathbb{P}(S_T = dS_0) = 0$ in the real world! That

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is we are sure that the writer's guess will not come true so the question is: what happens in this case concerning the value of the replicating portfolio and the payoff?

Now let us recall the Black - Scholes option pricing model, see [1].

Assumption 2 *If the today price of the asset is S_0 then the price at time T is such that*

$$S_T = S_0 + m \int_0^T S_r dr + \sigma \int_0^T S_r dW_r$$

where $m, \sigma \in \mathbb{R}_+$ are some parameters specified by the writer.

The Black - Scholes model prices a contract by the initial value of a replicating portfolio that has to be reconstructed continuously in time. Assumption 2 maybe is not the best for a specific asset but is realistic. However, the problem of the Black - Scholes model is the reconstruction of the replicating portfolio which is not realistic because nobody can reconstruct such a replicating portfolio continuously in time in order to hedge the risk. Of course every pricing model via replication with reconstruction in continuous time has the same serious problem.

Note that the no arbitrage arguments (see for example [5]) does not give us a specific price for the call and put options but only a relation between them and some bounds for their prices. The situation is worse for more complicated contracts because even if we can prove some bounds for their prices the competition is maybe too low for them in order these bounds come true in the real world.

Therefore, in order to propose a pricing model via replication we should combine the realistic assumption 2 with a replication in discrete time. In [4] we have modified the binomial model in this direction.

2 The Realistic Binomial Model

As we have seen in the previous section, the writer's guess concerning the rates u, d will never come true in the real world! So what happens when the true upward rate becomes bigger or less than she expected?

In [4] we have show that for the usual call and put options the writer will have a profit if the future upward rate becomes smaller of what the writer predicted or if the downward rate becomes bigger than she expected. Denote the profit of the writer of a call or put option by Π when she is pricing by the binomial model. Let us denote the rates that she used to price the contract by u, d and by u^*, d^* the future rates. Then the profit of these contracts is a function of $u - u^*$ and $d^* - d$, that is $\Pi := \Pi(u - u^*, d^* - d)$, and moreover this function is increasing in both variables.

In [4] we have proposed a way of choosing d, u so as to have

$$\mathbb{P}\left(\frac{S_T}{S_0} \in (d, u)\right) = p$$

for a given probability p under assumption 2. We call this as the realistic binomial model because we assume a realistic assumption for the future prices of the underlying asset and we choose the rates d, u so as to fix the probability of the event $\{\frac{S_T}{S_0} \in (d, u)\}$. We have already see that this is not enough to have a practically useful pricing model via replication so we should also define a useful property of the replicating portfolio regarding the cases in which the writer has a profit.

Let a contract written on d assets and let the u_i, d_i are the rates guessed by the writer while by u_i^*, d_i^* we denote the future real rates where $i = 1, \cdots, d$. Let a replicating portfolio constructed by the (realistic) binomial model, that is let some $a_1, \cdots, a_d, b$ such that

$$V_T = a_1 S_1^T + \cdots + a_d S_d^T + b \geq X_T$$

assuming for simplicity the risk free rate equal to zero.

Definition 1 *We say that a replicating portfolio constructed by the (realistic) binomial model has the profit property for the contract X if the profit for the writer is a function of the $u_i - u_i^*$ and $d_i^* - d_i$ and is increasing in every variable.*

That is the profit property is a property of a replicating portfolio that has to be reconstructed in discrete time and not a property of the realistic binomial model.

Remark 1 *The replicating portfolios as constructed by the (realistic) binomial model has the profit property for both the call and the put options as we have seen in [4]. If we do not have this information then any guess of d, u is meaningless because the probability to guess right is zero in the real world. In addition, using the realistic binomial model, we know also the probability of profit for such a replicating portfolio.* □

So we have combined the realistic assumption 2 for the price of the asset with the realistic reconstruction of the replicating portfolio in discrete time.

What about the well known arbitrage theorem (see for example [6]) in the binomial model setting? Let us suppose that the writer chooses u, d such as $e^{rT} > u$. In this case the notion of arbitrage has no meaning because the probability of the event $\{u > e^{rT}\}$ is strictly positive!

3 Examples

Assumption 3 *We assume that all investors think rationally and according to their own benefit. We also assume that their decisions are based only on past data and not on future expectations.*

This assumption is obviously not that realistic, as most investors rely on their intuition about future market movement, so the reality is even more complicated. Anyway, the calculations we propose here using the historical data are certainly a good first estimate to which one can then add her/his intuition about the future evolution of the phenomenon. For example, a recent event has not left its mark on historical data, yet it is going to affect future ones. An investor should take this into account by increasing or decreasing the volatility appropriately which has been calculated based on historical data.

3.1 Prices of the call and put options

Let us suppose that a writer want to sell a put and a call option on an underlying asset that follows the following sde

$$S_T = 40 + 0.01 \int_0^T S_r dr + 0.4 \int_0^T S_r dW_r$$

where $T = 50$ days and the strike price is $K = 41$.

Let us recall here how the writer will compute the safe price for the put option as this was proposed in [4]. For a given probability p one has to find Y such that

$$\mathbb{P}(K - S_T \leq Y) = p$$

Under the assumption 2 about the price of the underlying asset is an easy problem to compute the safe price Y.

The price of the put option via replication (via the realistic binomial model) is about 2.8 with probability of profit 0.57 while the price without replication is 2.4 with the same probability of profit. Note that in order to construct the replicating portfolio the writer has to borrow some assets therefore under assumption 3 the writer will choose to sell without replication, i.e. at the price 2.4 or higher.

If she sell the put option at the price 2.4 then the price of the call option has to be 1.4 for the put - call parity to hold, assuming that the risk free rate is zero. At this price the probability of profit for the writer is about 0.677 for the call option without construction of a replicating portfolio. If she sell at this price the call option constructing a replicating portfolio via the realistic binomial model then the probability of profit is 0.49.

Let us suppose that the writer sell the put option at the price $Y = 3.1$. Then the probability of profit without replication is about 0.62. But in this case the writer can construct a replicating portfolio without borrowing assets i.e. with $a = 0.05$ and $b = 1$ assuming that $u = 1$ and $d = 0.95$. In this case the writer will have a (possible unbounded!) profit if $\frac{S_T}{S_0} \geq 0.95$ and the probability of this event is about 0.52.

However, the most likely price is about 1.4 for which both the buyer and the writer have the same probability of profit (without replication), i.e. 0.5, while for both of them the possible profit is bounded. That is the notion of the fair price exists only if both the buyer and the seller have bounded possible profits and does not come necessarily by a replicating portfolio!

Problem 1 *What is the probability of profit for a call option using the n period realistic binomial model?*

A partial answer is the following theorem. By q we denote the probability $q = \{\frac{S_{n+1}}{S_n} > 1\}$ which we have computed in [4].

Theorem 1 *Let a call option with strike price K. Suppose that the writer has priced it by using the n - period realistic binomial model under assumption 2 with*

$$u = e^{\sigma z_p \sqrt{\frac{T}{n}} + (m-\sigma^2/2)\frac{T}{n}}, \quad d = e^{-\sigma z_p \sqrt{\frac{T}{n}} + (m-\sigma^2/2)\frac{T}{n}}$$

Here z_p is such that $\frac{1}{\sqrt{2\pi\frac{T}{n}}}\int_{-z_p\frac{T}{n}}^{z_p\frac{T}{n}} e^{\frac{-t^2}{2\frac{T}{n}}}\,dt = p$ with p chosen by the writer and is such that $u > 1$ and $d < 1$. Suppose that the asset price has moved as the writer had predicted until the time $n-1$. Then the probability of profit $\mathbb{P}(\Pi \geq 0) \to 0$ as $n \to \infty$.

Proof. The profit at time T is

$$\Pi = a_{n-1}US_0 + b_{n-1} - H(US_0)$$

where $H(x) = (x-K)^+$ and a_{n-1}, b_{n-1} are such that $V_{n-1} = a_{n-1}S_{n-1} + b_{n-1}$ where V_{n-1} is the value of the replicating portfolio one period before the expiration. Finally $U = e^{\sigma W_T + (m-\sigma^2/2)T}$. From now on we will denote by a, b the a_{n-1} and b_{n-1}.

Suppose that $dS_{n-1} < US_0 < K$. Then the profit is

$$\begin{aligned}\Pi &= aUS_0 + b - H(US_0)\\ &= a(US_0 - dS_{n-1}) + H(dS_{n-1}) - H(US_0)\\ &= (US_0 - dS_{n-1})\left(a - \frac{H(US_0) - H(dS_{n-1})}{US_0 - dS_{n-1}}\right)\end{aligned}$$

and it follows that $\Pi \geq 0$. If $US_0 < sS_{n-1} < K$ then $\Pi \leq 0$ and if $US_0 < K < dS_{n-1}$ then $\Pi \leq 0$ because $a = 1$ and $\frac{H(US_0)-H(dS_{n-1})}{US_0-dS_{n-1}} \leq 1$. Similarly, if $K < US_0$, it follows that $\Pi \geq 0$ if $US_0 \leq uS_{n-1}$ and $\Pi \leq 0$ if $US_0 \geq uS_{n-1}$.

Therefore the profit is no-negative in the case where $dS_{n-1} \leq US_0 \leq uS_{n-1}$ and non positive otherwise.

The probability of profit is then

$$\mathbb{P}\left(dS_{n-1} \leq US_0 \leq uS_{n-1}\right) = \sum_{k=0}^{n-1} q^k(1-q)^{n-1-k}\mathbb{P}(u^k d^{n-1-k}(dS_0) \leq US_0 \leq u^k d^{n-1-k}(uS_0)) \leq \frac{C}{\sqrt{n}}$$

for some constant $C > 0$, independent of n. □

Problem 2 *At time k, knowing the actual path of the asset's price until that time, what the writer can do concerning the hedging strategy in order to increase the profit and the probability of profit? What about other types of options written on one asset, for example path dependent options?*

Intuitively speaking, the realistic binomial model can be used for one period without troubles while for n periods there are open questions concerning the hedging strategy and how should be modified by the writer given the actual path of the asset's price. Theorem 1 may be seen as a suggestion for static replication only.

Summing up, the well known binomial model has no meaning because the writer's guess will not come true in the real world and in addition the writer does not know anything about a possible profit and what is the probability of profit. On the other hand, concerning the realistic binomial model, given that the corresponding replicating portfolio has the profit property, the writer knows what is the probability of profit and in which cases will have a profit, at least for the one period model. Pricing by the Black - Scholes model the problem is that the writer can not built the replicating portfolio in order to hedge the risk and therefore nobody will price a contract in this way. Therefore the only practically useful way to construct a replicating portfolio is by the realistic binomial model assuming that this portfolio has the profit property.

3.2 An option written on two underlying assets

Let two assets S_1, S_2 that follows the following sdes

$$\begin{aligned} S_1(T) &= 0.98 + 0.001\int_0^T S_1(r)dr + 0.5\int_0^T S_1(r)dW_r \\ S_2(T) &= 0.99 - 0.02\int_0^T S_2(r)dr + 0.6\int_0^T S_2(r)dW_r \end{aligned}$$

where $T = 50$ days. Let $K = 0.99$ and consider the option that pays $P_T = \max\{S_1, S_2, K\}$ at the time T. The writer of the option computes a replicating portfolio via the realistic binomial model with probability of profit $p = 0.7$ and finds that she should buy $a_1 = 0.47$ shares of the S_1 asset, $a_2 = 0.49$ shares of the S_2 asset and $b = 0.23$ at the bank account assuming zero risk free rate. To be more precise the probability $p = 0.7$ is not the probability of profit but the probability of the event $\{\frac{S_i^T}{S_i^0} \in (d_i, u_i)\}$. The probability of profit in this case is not so easy to compute as in the call and put options. The initial value of this portfolio is $V_0 = 1.18$. It is easy to prove that this replicating portfolio has the profit property. In fact any replicating portfolio with $a_i \in [0,1]$ has the profit property concerning this type of contract. That is we can find the minimum replicating portfolio for $a_i \in [0,1]$ therefore this portfolio will have the profit property. The writer, if she wants to be more competitive, she will try to find the replicating portfolio with the profit property having the minimum initial value. At this example the notion of the fair value does not have any sense because the possible profit of the buyer is unbounded while the possible profit for the writer is bounded.

The writer computes also the safe price under the same hypotheses as above and finds that this price is $Y = 1.09$ with the same probability p but without replication. Let us recall here how to compute the safe price under the above assumptions as we have proposed in [4]. For a given probability p we find the prices S_1^Y and S_2^Y so that

$$\begin{aligned} \mathbb{P}(S_1^T \leq S_1^Y) &= p \\ \mathbb{P}(S_2^T \leq S_2^Y) &= p \end{aligned}$$

Then the safe price is $Y = \max\{S_1^Y, S_2^Y, K\}$.

The writer has to buy some call options in order to eliminate the risk of bankruptcy. At the first case, i.e. with the construction of the replicating portfolio, should buy $(1-a_1)$ calls with underlying asset S_1 for some strike price K_1 and $(1-a_2)$ calls with underlying asset S_2 for some strike price K_2. At the second case she should buy one call per asset.

The final price will be computed after the estimation of the transactions costs for the replication and the cost of the call options, i.e. in the first case the price will be $Y = 1.07 + (1-a_1)C(S_1, K_1) + (1-a_2)C(S_2, K_2) + T$ where $C(\cdot, \cdot)$ are the call options and T the transactions costs. At the second case the final price will be $U = 1.02 + C(S_1, K_1) + C(S_2, K_2) + T$.

As we have seen in [4] there is also another way to compute a price for some given probability of profit for the writer. The writer can assume that $P_T = \max\{X_T, K\}$ for a stochastic process X_t suitably chosen by her. In fact the same assumption can be done by the buyer in order to estimate the probability of profit for her, however, the way that the two parties estimates the probability of profit are in general different from each other.

Note that the writer's profit is always bounded while the buyer's possible profit in this case is unbounded. After the decision of the writer about the price (say U) of this option the buyer can compute also the probability of profit for her buying at this price. This probability is more likely to be under $1/2$ but this is acceptable by the buyer because the profit is unbounded from above. Recall that the call options will pay this extra difference.

Problem 3 *Does the (realistic) binomial model can produce a replicating portfolio with the profit property for any known contract? Let a replicating portfolio (a_1^*, a_2^*, b^*) such that*

$$V_0 = a_1^* S_1 + a_2^* S_2 + b^* = \min_{a_1, a_2, b \in \mathbb{R}} (a_1 S_1 + a_2 S_2 + b)$$

and the minimum is taken over all the replicating portfolios. The question is: does this portfolio has the profit property for a specific contract? If yes, what is the probability of profit for the writer?

3.3 A spread option

In this subsection we will study a spread option with payoff $P_T = \max\{S_1 - S_2, 0\}$. We will compute a replicating portfolio by using the realistic binomial model and this portfolio will have the profit property. Therefore the writer will know in which cases she will have a profit. Moreover, we will compute the safe price, i.e. a price without a replicating portfolio. The final price will be decided by the writer after the computation of the call options in order to eliminate the risk of bankruptcy and of course the transaction costs. On the other hand, the buyer has the ability to estimate the profit probability for her buying at this price.

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Let two assets S_1, S_2 that follows the following sdes

$$\begin{aligned} S_1(T) &= 0.98 + 0.001 \int_0^T S_1(r)dr + 0.5 \int_0^T S_1(r)dW_r \\ S_2(T) &= 0.99 - 0.02 \int_0^T S_2(r)dr + 0.6 \int_0^T S_2(r)dW_r \end{aligned}$$

where $T = 50$ days. The writer finds a replicating portfolio with $a_1 = 1.0$, $a_2 = -0.06$ and $b = -0.714$. This portfolio clearly has the profit property using the realistic binomial model with $p = 0.7$. The initial value of this portfolio is $V_0 = 0.19$. We compute also the safe price with probability $p = 0.7$ and the price is $Y = 0.19$.

The writer computes also another replicating portfolio with $a_1 = 0.09$, $a_2 = 0.07$ and $b = 0.058$ with initial value $V_0 = 0.22$ assuming $u_2 = 1.0$ and $d_2 = 0.95$. The advantage of this replicating portfolio is that the writer will have a profit if the price of $S_2(T)$ becomes bigger than $d_2 S_2(0)$ and that profit is unbounded from above on the event $\{\frac{S_1^T}{S_1^0} \in (d_1, u_1)\}$.

4 Conclusion

We have recalled the realistic binomial option pricing model as we had proposed it in [4] and we have point out three open problems. We have given some examples of option pricing using the realistic binomial model and the notion of the safe price. We compare them and give the advantages and their disadvantages in every situation so as the writer can decide how to hedge the option and consequently how to price the option. The notion of the safe price can be used also by the buyer in order to estimate the probability of profit.

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