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Hybrid Functions Approach to Solving Nonlinear Integral Equations in Two Dimensions

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Abstract: This study presents the solution of the second type of a two-dimensional nonlinear integral equation in Banach space. Also, the existence and uniqueness of this equation's solution are discussed. We utilize a numerical approach involving hybrid and block-pulse functions to obtain the approximate solution of a two-dimensional nonlinear integral equation. Nonlinear integral equation in two dimensions is reduced numerically to a system of nonlinear algebraic equations that can be solved using numerical methods. This study focuses on showing the convergence analysis for the numerical approach and obtaining an error estimate. Some numerical examples have been provided to demonstrate the approach's viability and efficacy.

Keywords: two- dimensional nonlinear integral equation; Banach fixed point theorem; block-pulse function; hybrid functions; legendre polynomial

MSC: 41A30; 45G10; 46B45; 65R20

1. Introduction

Integral equations are used in many disciplines of applied mathematics to explore and solve problems. See [1,5,6,10,24,26,27] for more information on the topics of two-dimensional nonlinear integral equations, which have long been of growing interest in many fields, including medicine, biology, physics, geography, and fuzzy control. According to the references [2,3,7,9–12], many problems in engineering, applied mathematics, and mathematical physics can be reduced into two-dimensional nonlinear integral equations. The analytical solutions to these equations are typically difficult. Therefore, it is necessary to find approximations. For example, Bernstein polynomials hybrid with functions of block-pulse form [4,22] and Legendre hybrid with functions of block-pulse form [13,21] have both recently been examined as computational approaches for solving two-dimensional nonlinear integral equations. Electrical engineering was originally introduced to block-pulse functions by Harmuth, after which additional academics discussed the topic [8].

Recently, hybrid functions have been considered for solving numerous mathematical models, including [20,23,25]. Combining Legendre polynomials and block-pulse functions yields one of these functions. Using block-pulse functions and Legendre polynomials, [18] described a method for solving mixed-type Hammerstein integral equations, whereas [17] proposed a method for solving optimal control of Volterra integral systems. These hybrid functions have also been applied to solving nonlinear Fredholm-Hammerstein systems. [14,30] obtained a numerical solution of partial differential equations with nonlocal integral conditions; [19] solved Fredholm integral equation of the first kind; [28] discovered the optimal solution of linear time-delay systems; [29] discovered the numerical solutions of stochastic Volterra-Fredholm integral equations; and [13] includes the necessary definitions as well as some properties of Legendre polynomials and hybrid block-pulse functions.

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In this study, the second type of two-dimensional nonlinear integral is considered. Under special conditions, the Banach fixed point theorem is used to discuss and prove the existence of a unique solution to two-dimensional nonlinear integral equations. We discuss the properties of hybrid functions, which combine block-pulse functions and Legendre polynomials. These integral equations are solved based on some useful properties of hybrid functions. This technique's major characteristic is its ability to transform an integral problem into a set of algebraic equations; as a result, the solution processes are correspondingly either reduced or simplified.

The article's structure is as follows: : In section 2, the existence and unique solution of Eq. (1) are discussed. Section 3 describes a method for estimating a two-dimensional nonlinear integral equation's solution. The convergence analysis of the provided method is derived in section 4, Numerical results are shown in section 5, and conclusion and Remarks are presented in the last section 6.

This study aims to present a numerical approach for solving the following two-dimensional nonlinear integral equation approximatively:

$$\begin{aligned}\gamma\psi(x,y) &= f(x,y) + \lambda_1 \int_0^1 \int_0^1 \Phi(x,\tau;y,v)\mu(\tau,v,\psi(\tau,v))\mathrm{d}v\mathrm{d}\tau \\ &+ \lambda_2 \int_0^x \int_0^1 G(x,\tau;y,v)\nu(\tau,v,\psi(\tau,v))\mathrm{d}v\mathrm{d}\tau,\end{aligned} \tag{1}$$

where $\lambda_r, r = 1,2$ are constant scalers having several physical meanings, the function $\psi(x,y)$ is unknown in the Banach space $L_2[0,1] \times L_2[0,1]$. The kernels $\Phi(x,\tau;y,v), G(x,\tau;y,v)$ are continuous in the same space and the known function $f(x,y)$ is continuous in the space $L_2[0,1] \times L_2[0,1]$. In addition the constant γ defines the kind of two- dimensional nonlinear integral equations.

2. Existence of a unique solution for the integral equation

The existence of a unique solution of problem (1) will be discussed and proved in this section using the Banach fixed point theorem. For this, we write Eq. (1) in the form of an integral operator:

$$\overline{V}\psi(x,y) = \frac{1}{\gamma} f(x,y) + V\psi(x,y); \quad \gamma \neq 0, \tag{2}$$

where

$$\begin{aligned}V\psi(x,y) &= \frac{\lambda_1}{\gamma} \int_0^1 \int_0^1 \Phi(x,\tau;y,v)\mu(\tau,v,\psi(\tau,v))\mathrm{d}v\mathrm{d}\tau \\ &+ \frac{\lambda_2}{\gamma} \int_0^x \int_0^1 G(x,\tau;y,v)\nu(\tau,v,\psi(\tau,v))\mathrm{d}v\mathrm{d}\tau.\end{aligned} \tag{3}$$

Also, we assume the following conditions:

(i) The kernels $\Phi(x,\tau;y,v)$ and $G(x,\tau;y,v)$ satisfy the conditions: $\|\Phi(x,\tau;y,v)\| \leq A_1$, $\|G(x,\tau;y,v))\| \leq A_2$, where A_1 and A_2 are two constants, assume $A = \max\{A_1, A_2\}$.

(ii) $\|f(x,y)\| = \left[\int_0^1 \int_0^1 |f(x,y)|^2 \mathrm{d}x\mathrm{d}y\right]^{\frac{1}{2}} = D,\ D$ is a constant.

(iii) The function $\mu(x,y,\psi(x,y))$ satisfies the following conditions:

$$\|\mu(x,y,\psi(x,y))\| = \left[\int_0^1 \int_0^1 |\mu(x,y,\psi(x,y))|^2 \mathrm{d}x\mathrm{d}y\right]^{\frac{1}{2}} \leq M_1 \|\psi(x,y)\|, \tag{a}$$

$$\|\mu(x,y,\psi_1(x,y)) - \mu(x,y,\psi_2(x,y))\| \leq M_2 \|\psi_1(x,y) - \psi_2(x,y)\|. \tag{b}$$

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(iv) The function $\nu(x,y,\psi(x,y))$ is bounded and satisfy:

$$\|\nu(x,y,\psi(x,y))\| = \left[\int_0^1\int_0^1 |\nu(x,y,\psi(x,y))|^2 \mathrm{d}x\mathrm{d}y\right]^{\frac{1}{2}} \le N_1\|\psi(x,y)\|, \qquad (a)$$

$$\|\nu(x,y,\psi_1(x,y)) - \nu(x,y,\psi_2(x,y))\| \le N_2\|\psi_1(x,y) - \psi_2(x,y)\|. \qquad (b)$$

Theorem 1. *Assume that the conditions $(i) - iv)$ are satisfied. Eq. (1) has an unique solution $\psi(x,y)$ in the space, $L_2[0,1]\times L_2[0,1]$. If the condition*

$$\eta = A\left|\frac{\lambda}{\gamma}\right|[M+N] < 1;\ (\lambda = \max\{\lambda_1,\lambda_2\},\ M = \max\{M_1,M_2\},\ N = \max\{N_1,N_2\}) \tag{4}$$

is true.

The following two lemmas are necessary for the theorem's proof:

Lemma 1. *Under the conditions $(i),(ii),(iii-a)$, and $(iv-a)$, the operator $\overline{V}\psi(x,y)$ defined by Eq. (2) maps the space $L_2[0,1]\times L_2[0,1]$ into itself.*

Proof. In light of formulas (2) and (3), we obtain

$$\|\overline{V}\psi(x,y)\| \le \frac{1}{|\gamma|}\|f(x,y)\| + \left|\frac{\lambda_1}{\gamma}\right|\left\|\int_0^1\int_0^1 |\Phi(x,\tau;y,v)|\mu(\tau,v,\psi(\tau,v))\mathrm{d}v\mathrm{d}\tau\right\|$$
$$+\left|\frac{\lambda_2}{\gamma}\right|\left\|\int_0^x\int_0^1 |G(x,\tau;y,v)|\nu(\tau,v,\psi(\tau,v))\mathrm{d}v\mathrm{d}\tau\right\|.$$

Using conditions (i) and (ii), we get

$$\|\overline{V}\psi(x,y)\| \le \frac{D}{|\gamma|} + A\left|\frac{\lambda_1}{\gamma}\right|\left\|\int_0^1\int_0^1 |\mu(\tau,v,\psi(\tau,v))\mathrm{d}v\mathrm{d}\tau\right\|$$
$$+A\left|\frac{\lambda_2}{\gamma}\right|\left\|\int_0^x\int_0^1 \nu(\tau,v,\psi(\tau,v))\mathrm{d}v\mathrm{d}\tau\right\|.$$

Given conditions $(iii-a)$ and $(iv-a)$, the above inequality takes on the following form:

$$\|\overline{V}\psi(x,y)\| \le \frac{D}{|\gamma|} + AM\|\psi(x,y)\|\left|\frac{\lambda_1}{\gamma}\right|\left\|\int_0^1\int_0^1 \mathrm{d}v\mathrm{d}\tau\right\|$$
$$+AN\|\psi(x,y)\|\left|\frac{\lambda_2}{\gamma}\right|\left\|\int_0^x\int_0^1 \mathrm{d}v\mathrm{d}\tau\right\|,$$

where $\max\limits_{0\le x\le 1}|x| = 1$, so that last inequality becomes

$$\|\overline{V}\psi(x,y)\| \le \frac{D}{|\gamma|} + A\left|\frac{\lambda}{\gamma}\right|[M+N]\|\psi(x,y)\|,$$

since

$$\|\overline{V}\psi(x,y)\| \le \frac{D}{|\gamma|} + \eta\|\psi(x,y)\|; \qquad \eta = A\left|\frac{\lambda}{\gamma}\right|[M+N] < 1. \tag{5}$$

According to this inequality, the operator $\overline{V}$ maps the ball $B_r \subset L_2[0,1]\times L_2[0,1]$ into itself, where

$$r = \frac{D}{|\gamma|(1-\eta)},$$

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since, $r > 0, D > 0$, therefore we have $\eta < 1$. Furthermore, lower bounds for the operators V and $\overline{V}$ are involved in the inequality (5).

Lemma 2. *If the conditions $(i), (iii-b)$, and $(iv-b)$ are verified, then the operator $\overline{V}\psi(x,y)$ defined by Eq. (2) is continuous in the space $L_2[0,1] \times L_2[0,1]$.*

Proof. For the continuity, Given two functions $\Psi_1(x,y)$ and $\Psi_2(x,y)$ in the space $L_2[0,1] \times L_2[0,1]$ and satisfy Eq. (2), then

$$\overline{V}\psi_1(x,y) - \overline{V}\psi_2(x,y) = \frac{\lambda_1}{\gamma}\int_0^1\int_0^1 \Phi(x,\tau;y,v)[\mu(\tau,v,\psi_1(\tau,v)) - \mu(\tau,v,\psi_2(\tau,v))]\mathrm{d}v\mathrm{d}\tau + \frac{\lambda_2}{\gamma}\int_0^x\int_0^1 G(x,\tau;y,v)[\nu(\tau,v,\psi_1(\tau,v)) - \nu(\tau,v,\psi_2(\tau,v))]\mathrm{d}v\mathrm{d}\tau,$$

applying the properties of the norm, we obtain

$$\|\overline{V}\psi_1(x,y) - \overline{V}\psi_2(x,y)\| \le \left|\frac{\lambda_1}{\gamma}\right| \left\|\int_0^1\int_0^1 |\Phi(x,\tau;y,v)||\mu(\tau,v,\psi_1(\tau,v)) - \mu(\tau,v,\psi_2(\tau,v))|\mathrm{d}v\mathrm{d}\tau\right\| + \left|\frac{\lambda_2}{\gamma}\right| \left\|\int_0^x\int_0^1 |G(x,\tau;y,v)||\nu(\tau,v,\psi_1(\tau,v)) - \nu(\tau,v,\psi_2(\tau,v))|\mathrm{d}v\mathrm{d}\tau\right\|.$$

In view of the conditions (i), (iii-b), and (iv-b), the above inequality becomes

$$\|\overline{V}\psi_1(x,y) - \overline{V}\psi_2(x,y)\| \le A\left|\frac{\lambda}{\gamma}\right|[M+N]\|\psi_1(x,y) - \psi_2(x,y)\|,$$

since

$$\|\overline{V}\psi_1(x,y) - \overline{V}\psi_2(x,y)\| \le \eta\|\psi_1(x,y) - \psi_2(x,y)\|.$$

This inequality shows that, $\overline{V}$ is a continuous operator in $L_2[0,1] \times L_2[0,1]$. Moreover $\overline{V}$ is a contraction operator under the condition $\eta < 1$.

The previous two Lemmas 1 and 2 show that the operator $\overline{V}$ defined by (2) is a contraction operator in the space $L_2[0,1] \times L_2[0,1]$. Hence, from Banach fixed point theorem, $\overline{V}$ has a unique fixed point which is of course, the unique solution of Eq. (1).

3. Method of solution for the main problem

This section applies the collocation method, two-dimensional hybrid functions, and the Gauss quadrature formula to transform the integral equation (1) into nonlinear systems of equations. The following results are obtained by expanding the function $\Psi(x,y)$ in Eq. (1) in relation to two-dimensional hybrid functions:

$$\Psi(x,y) = \sum_{m_1=1}^{\infty}\sum_{n_1=0}^{\infty}\sum_{m_2=1}^{\infty}\sum_{n_2=0}^{\infty} c_{m_1n_1m_2n_2} h_{m_1n_1m_2n_2}(x,y), \tag{6}$$

where the finite series in equation (6) can be written as

$$\Psi_{S,K}(x,y) = \sum_{m_1=1}^{S}\sum_{n_1=0}^{K-1}\sum_{m_2=1}^{S}\sum_{n_2=0}^{K-1} c_{m_1n_1m_2n_2} h_{m_1n_1m_2n_2}(x,y). \tag{7}$$

where $c_{m_1n_1m_2n_2}$, $m_1, m_2 = 1,2,\ldots,S$, $n_1, n_2 = 1,2,\ldots,K-1$, and S, K are the unknown hybrid coefficients to be determined.

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Substituting Eq. (7) into Eq. (1) yields

$$\begin{aligned}\gamma\psi_{S,K}(x,y) &= f(x,y) + \lambda_1 \int_0^1 \int_0^1 \Phi(x,\tau;y,v)\mu(\tau,v,\psi_{S,K}(\tau,v))\mathrm{d}v\mathrm{d}\tau \\ &+ \lambda_2 \int_0^x \int_0^1 G(x,\tau;y,v)\nu(\tau,v,\psi_{S,K}(\tau,v))\mathrm{d}v\mathrm{d}\tau,\end{aligned} \tag{8}$$

Now, we discretize Eq. (8) at the set of collocation nodes (x_m, y_n) for $m, n = 1, 2, \ldots, SK$, as follows:

$$\begin{aligned}\gamma\psi_{S,K}(x_m,y_n) &= f(x_m,y_n) + \lambda_1 \int_0^1 \int_0^1 \Phi(x_m,\tau;y_n,v)\mu(\tau,v,\psi_{S,K}(\tau,v))\mathrm{d}v\mathrm{d}\tau \\ &+ \lambda_2 \int_0^{x_m} \int_0^1 G(x_m,\tau;y_n,v)\nu(\tau,v,\psi_{S,K}(\tau,v))\mathrm{d}v\mathrm{d}\tau,\end{aligned} \tag{9}$$

where

$$x_m = \frac{1}{2}\left(\cos\left(\frac{(2m-1)\pi}{2SK}\right) + 1\right),\ m = 1, 2, \ldots, SK,$$

and

$$y_n = \frac{1}{2}\left(\cos\left(\frac{(2n-1)\pi}{2SK}\right) + 1\right),\ n = 1, 2, \ldots, SK,$$

The integral operators in Eq. (9) are approximated using the Gauss-Legendre quadrature formula. For this, we use the following transformations to convert the integrals over $[0,1]$ into the integral over $[-1,1]$, respectively

$$\begin{aligned}\xi &= 2\tau - 1;\ \tau \in [0,1], \\ \varrho &= 2v - 1;\ v \in [0,1].\end{aligned}$$

The integral over $[0, x_m]$ must also be changed into the integral over $[-1,1]$, having the following form

$$\bar{\xi} = \frac{2}{x_m}\tau - 1;\ \tau \in [0, x_m].$$

Then Eq. (9) is converted to

$$\begin{aligned}&\gamma\psi_{S,K}(x_m,y_n) = f(x_m,y_n) \\ &+ \frac{\lambda_1}{4}\int_{-1}^1\int_{-1}^1 \Phi(x_m,\frac{1}{2}(\xi+1);y_n,\frac{1}{2}(\varrho+1))\mu(\frac{1}{2}(\xi+1),\frac{1}{2}(\varrho+1),\psi_{S,K}(\frac{1}{2}(\xi+1),\frac{1}{2}(\varrho+1)))\mathrm{d}\varrho\mathrm{d}\xi \\ &+ \frac{\lambda_2 x_m}{4}\int_{-1}^1\int_{-1}^1 G(x_m,\frac{x_m}{2}(\bar{\xi}+1);y_n,\frac{1}{2}(\varrho+1)) \\ &\times \nu(\frac{x_m}{2}(\bar{\xi}+1),\frac{1}{2}(\varrho+1),\psi_{S,K}(\frac{x_m}{2}(\bar{\xi}+1),\frac{1}{2}(\varrho+1)))\mathrm{d}\varrho\mathrm{d}\bar{\xi}.\end{aligned}$$

The above equation can be expressed as follows using Gauss-Legendre quadrature:

$$
\begin{aligned}
&\gamma\psi_{S,K}(x_m,y_n) = f(x_m,y_n)\\
&+\frac{\lambda_1}{4}\sum_{j=1}^{\ell_1}\sum_{i=1}^{\ell_2} w_j w_i \Phi(x_m,\frac{1}{2}(\xi_i+1);y_n,\frac{1}{2}(\varrho_j+1))\\
&\times\mu(\frac{1}{2}(\xi_i+1),\frac{1}{2}(\varrho_j+1),\psi_{S,K}(\frac{1}{2}(\xi_i+1),\frac{1}{2}(\varrho_j+1)))\\
&+\frac{\lambda_2 x_m}{4}\sum_{j=1}^{\ell_1}\sum_{i=1}^{\bar{\ell}_2} w_j\bar{w}_i G(x_m,\frac{x_m}{2}(\bar{\xi}_i+1);y_n,\frac{1}{2}(\varrho_j+1))\\
&\times\nu(\frac{x_m}{2}(\bar{\xi}_i+1),\frac{1}{2}(\varrho_j+1),\psi_{S,K}(\frac{x_m}{2}(\bar{\xi}_i+1),\frac{1}{2}(\varrho_j+1))),\\
&m=1,2,\ldots,SK,\ n=1,2,\ldots,SK,
\end{aligned}
\tag{10}
$$

and w_j, w_i and $\bar{w}_i$ are the corresponding weights.

This technique can be used to transform the two-dimensional nonlinear integral problem (1) into a solvable nonlinear system of algebraic equations.

4. Convergence analysis

The aim of this section is to describe the uniform convergence of the hybrid functions expansion and to determine the maximum absolute truncation error of the function Ψ based on hybrid functions.

Theorem 2. *If $\Psi \in C^4[0,1]$, then the function $\Psi(x,y)$ converges uniformly to the infinite sum of the hybrid functions of $\Psi(x,y)$ described by (6)*

Proof. The hybrid coefficients are defined as

$$
\begin{aligned}
c_{m_1n_1m_2n_2} &= \frac{\int_0^1\int_0^1 \Psi(x,y)h_{m_1n_1m_2n_2}(x,y)\mathrm{d}x\mathrm{d}y}{\int_0^1\int_0^1 h^2_{m_1n_1m_2n_2}(x,y)\mathrm{d}x\mathrm{d}y}\\
&= \frac{\int_{\frac{m_2-1}{S}}^{\frac{m_2}{S}}\int_{\frac{m_1-1}{S}}^{\frac{m_1}{S}} \Psi(x,y)L_{n_1}(2Sx-2m_1+1)L_{n_2}(2Sy-2m_2+1)\mathrm{d}x\mathrm{d}y}{\int_{\frac{m_1-1}{S}}^{\frac{m_1}{S}} L^2_{n_1}(2Sx-2m_1+1)\mathrm{d}x\int_{\frac{m_2-1}{S}}^{\frac{m_2}{S}} L^2_{n_2}(2Sy-2m_2+1)\mathrm{d}y}.
\end{aligned}
$$

Now, suppose that $2m_1-1=\hat{m}_1$ and $2Sx-\hat{m}_1=\Im$, therefore

$$
\begin{aligned}
c_{m_1n_1m_2n_2} &= \frac{\int_{\frac{m_2-1}{S}}^{\frac{m_2}{S}}\left(\int_{-1}^{1}\Psi(\frac{\hat{m}_1+\Im}{2S},y)L_{n_1}(\Im)\mathrm{d}\Im\right)L_{n_2}(2Sy-2m_2+1)\mathrm{d}y}{\int_{-1}^{1}L^2_{n_1}(\Im)\mathrm{d}\Im\int_{\frac{m_2-1}{S}}^{\frac{m_2}{S}} L^2_{n_2}(2Sy-2m_2+1)\mathrm{d}y}\\
&= \frac{(2n_1+1)}{2}\frac{\int_{\frac{m_2-1}{S}}^{\frac{m_2}{S}}\left(\int_{-1}^{1}\Psi(\frac{\hat{m}_1+\Im}{2S},y)L_{n_1}(\Im)\mathrm{d}\Im\right)L_{n_2}(2Sy-2m_2+1)\mathrm{d}y}{\int_{\frac{m_2-1}{S}}^{\frac{m_2}{S}} L^2_{n_2}(2Sy-2m_2+1)\mathrm{d}y}.
\end{aligned}
$$

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Using the technique of integration by parts with regard to $\Im$ and $(2n+1)L_n(\Im) = L'_{n+1}(\Im) - L'_{n-1}(\Im)$, we obtain

$$c_{m_1n_1m_2n_2} = -\frac{1}{2}\frac{\int_{\frac{m_2-1}{S}}^{\frac{m_2}{S}} \left(\int_{-1}^{1} \frac{\partial}{\partial \Im}\Psi(\frac{\hat{m}_1+\Im}{2S}, y)(L_{n_1+1}(\Im) - L_{n_1-1}(\Im))d\Im\right) L_{n_2}(2Sy - 2m_2 + 1)dy}{\int_{\frac{m_2-1}{S}}^{\frac{m_2}{S}} L_{n_2}^2(2Sy - 2m_2 + 1)dy}.$$

Once again, an integration by parts of above relation, results that

$$c_{m_1n_1m_2n_2} = \frac{1}{2}\frac{\int_{\frac{m_2-1}{S}}^{\frac{m_2}{S}} \left(\int_{-1}^{1} \frac{\partial^2}{\partial \Im^2}\Psi(\frac{\hat{m}_1+\Im}{2S}, y)[\frac{-L_{n_1}(\Im)}{2n_1+3} - \frac{L_{n_1}(\Im)-L_{n_1-2}(\Im)}{2n_1-1}]d\Im\right) L_{n_2}(2Sy - 2m_2 + 1)dy}{\int_{\frac{m_2-1}{S}}^{\frac{m_2}{S}} L_{n_2}^2(2Sy - 2m_2 + 1)dy}.$$

Now, we have

$$c_{m_1n_1m_2n_2} = \frac{1}{2(2n_1+3)(2n_1-1)}\frac{\int_{\frac{m_2-1}{S}}^{\frac{m_2}{S}} \left(\int_{-1}^{1} \frac{\partial^2}{\partial \Im^2}\Psi(\frac{\hat{m}_1+\Im}{2S}, y)\aleph_{n_1}(\Im)d\Im\right) L_{n_2}(2Sy - 2m_2 + 1)dy}{\int_{\frac{m_2-1}{S}}^{\frac{m_2}{S}} L_{n_2}^2(2Sy - 2m_2 + 1)dy},$$

where

$$\aleph_{n_1}(\Im) = (2n_1-1)L_{n_1+2}(\Im) - (4n_1+2)L_{n_1}(\Im) + (2n_1+3)L_{n_1-2}(\Im).$$

Similarly, changing the variable for y as $2m_2 - 1 = \hat{m}_2$ where $2Sy - \hat{m}_2 = \wp$ and integrating by parts with respect to $\wp$, we get

$$c_{m_1n_1m_2n_2} = \frac{1}{4(2n_1+3)(2n_1-1)(2n_2+3)(2n_2-1)} \int_{-1}^{1}\int_{-1}^{1} \frac{\partial^4}{\partial\wp^2\partial\Im^2}\Psi(\frac{\hat{m}_1+\Im}{2S}, \frac{\hat{m}_2+\wp}{2S})\aleph_{n_1}(\Im)\aleph_{n_2}(\wp)d\Im d\wp,$$

where

$$\aleph_{n_2}(\wp) = (2n_2-1)L_{n_2+2}(\wp) - (4n_2+2)L_{n_2}(\wp) + (2n_2+3)L_{n_2-2}(\wp).$$

Using the chain derivatives and $\sigma = \max_{(x,y)\in[0,1]}\left|\frac{\partial^4\Psi(x,y)}{\partial x^2\partial y^2}\right|$, it follows that

$$\begin{aligned} c_{m_1n_1m_2n_2} &\le \frac{\sigma}{64S^4(2n_1+3)(2n_1-1)(2n_2+3)(2n_2-1)}\int_{-1}^{1}\int_{-1}^{1} |\aleph_{n_1}(\Im)||\aleph_{n_2}(\wp)|d\Im d\wp, \\ &\le \frac{\sigma}{64m_1^2m_2^2(2n_1+3)(2n_1-1)(2n_2+3)(2n_2-1)}\int_{-1}^{1}\int_{-1}^{1} |\aleph_{n_1}(\Im)||\aleph_{n_2}(\wp)|d\Im d\wp. \end{aligned} \tag{11}$$

However

$$\begin{aligned} \left(\int_{-1}^{1} |\aleph_{n_1}(\Im)|d\Im\right)^2 &= \left(\int_{-1}^{1} |(2n_1-1)L_{n_1+2}(\Im) - (4n_1+2)L_{n_1}(\Im) + (2n_1+3)L_{n_1-2}(\Im)|d\Im\right), \\ &\le 2\int_{-1}^{1} |(2n_1-1)^2L_{n_1+2}^2(\Im) + (4n_1+2)^2L_{n_1}^2(\Im) + (2n_1+3)^2L_{n_1-2}^2(\Im)|d\Im, \end{aligned}$$

Using the Legendre polynomials' orthogonality property, we determine that

$$\left(\int_{-1}^{1} |\aleph_{n_1}(\Im)|d\Im\right)^2 \le \frac{24(2n_1+3)^2}{2n_1-3}, \tag{12}$$

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thus

$$\int_{-1}^{1} |\aleph_{n_1}(\Im)| d\Im \leq \frac{2\sqrt{6}(2n_1+3)}{\sqrt{2n_1-3}}, \tag{13}$$

and

$$\int_{-1}^{1} |\aleph_{n_2}(\wp)| d\wp \leq \frac{2\sqrt{6}(2n_2+3)}{\sqrt{2n_2-3}}. \tag{14}$$

By substituting (13) and (14) into (11), we obtain

$$\begin{aligned} c_{m_1 n_1 m_2 n_2} &\leq \frac{24\sigma}{64 m_1^2 m_2^2 (2n_1-1)(2n_2-1)\sqrt{(2n_1-3)}\sqrt{(2n_2-3)}}, \\ &\leq \frac{3\sigma}{8 m_1^2 m_2^2 (2n_1-3)^{\frac{3}{2}}(2n_2-3)^{\frac{3}{2}}}. \end{aligned} \tag{15}$$

Therefore, the series $\sum_{m_1=1}^{\infty}\sum_{n_1=0}^{\infty}\sum_{m_2=1}^{\infty}\sum_{n_2=0}^{\infty}$ is absolutely convergent. Also,

$$\begin{aligned} |\Psi(x,y)| &= \left| \sum_{m_1=1}^{\infty}\sum_{n_1=0}^{\infty}\sum_{m_2=1}^{\infty}\sum_{n_2=0}^{\infty} c_{m_1 n_1 m_2 n_2} h_{m_1 n_1 m_2 n_2}(x,y) \right| \\ &\leq \sum_{m_1=1}^{\infty}\sum_{n_1=0}^{\infty}\sum_{m_2=1}^{\infty}\sum_{n_2=0}^{\infty} |c_{m_1 n_1 m_2 n_2}| \\ &\leq \infty, \end{aligned}$$

and the series (6) converges to the function $\Psi(x,y)$ uniformly.

Theorem 3. *The maximum absolute truncation error of the series solution (6) to two- dimensional nonlinear integral equation (1) is estimated to be,*

$$\|\Psi(x,y) - \Psi_{S,K}(x,y)\| \leq \frac{3\sigma}{8S}\left(\sum_{m_1=S+1}^{\infty}\frac{1}{m_1^4}\sum_{n_1=K}^{\infty}\frac{1}{(2n_1-3)^4}\sum_{m_2=S+1}^{\infty}\frac{1}{m_2^4}\sum_{n_2=K}^{\infty}\frac{1}{(2n_2-3)^4} \right)^{\frac{1}{2}}.$$

Proof.

$$\begin{aligned} &\|\Psi(x,y) - \Psi_{S,K}(x,y)\| \\ &= \left(\int_0^1 \int_0^1 \left(\Psi(x,y) - \sum_{m_1=1}^{S}\sum_{n_1=0}^{K-1}\sum_{m_2=1}^{S}\sum_{n_2=0}^{K-1} c_{m_1 n_1 m_2 n_2} h_{m_1 n_1 m_2 n_2}(x,y) \right)^2 dxdy \right)^{\frac{1}{2}}, \\ &\leq \left(\sum_{m_1=S+1}^{\infty}\sum_{n_1=K}^{\infty}\sum_{m_2=S+1}^{\infty}\sum_{n_2=K}^{\infty} c^2_{m_1 n_1 m_2 n_2} \int_0^1 \int_0^1 h^2_{m_1 n_1 m_2 n_2}(x,y) dxdy \right)^{\frac{1}{2}}. \end{aligned}$$

Using the hybrid functions' orthogonality property and taking relation (15) into consideration, we are able to

$$\begin{aligned} \|\Psi(x,y) - \Psi_{S,K}(x,y)\| &\leq \left(\sum_{m_1=S+1}^{\infty}\sum_{n_1=K}^{\infty}\sum_{m_2=S+1}^{\infty}\sum_{n_2=K}^{\infty} c^2_{m_1 n_1 m_2 n_2}\frac{1}{S^2(2n_1+1)(2n_2+1)} \right)^{\frac{1}{2}} \\ &\leq \frac{3\sigma}{8S}\left(\sum_{m_1=S+1}^{\infty}\frac{1}{m_1^4}\sum_{n_1=K}^{\infty}\frac{1}{(2n_1-3)^4}\sum_{m_2=S+1}^{\infty}\frac{1}{m_2^4}\sum_{n_2=K}^{\infty}\frac{1}{(2n_2-3)^4} \right)^{\frac{1}{2}}. \end{aligned}$$

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5. Application and numerical results

In order to show the accuracy and efficiency of the proposed method, some numerical examples are given in this section. We introduce the following notation to study the absolute values of this method's errors:

$$R_{S,K} = |\Psi(x,y) - \Psi_{S,K}(x,y)|,$$

where $\Psi(x,y)$ and $\Psi_{S,K}(x,y)$ are the exact solution and the approximate solution of the integral equations, respectively.

Example 5.1. Consider the following two- dimensional nonlinear integral equation:

$$16\psi(x,y) = f(x,y) + \int_0^1\int_0^1 (x\tau + yv)\psi^2(\tau,v)\mathrm{d}v\mathrm{d}\tau + \int_0^x\int_0^1 (x^2\tau^2 + yv)\psi^2(\tau,v)\mathrm{d}v\mathrm{d}\tau, \tag{16}$$

where

$$f(x,y) = \frac{-7}{24} - \frac{28yx}{45} + 16(x^2+y^2) - \frac{x^2y^2}{360}(30y^4 + 72y^5x + 45y^2x^2 + 80y^3x^3 + 30x^4 + 72yx^5).$$

The exact solution is $\psi(x,y) = x^2 + y^2$. Using the proposed numerical technique, where $S = 2$ and $K = 2,4,6,8$ in the interval $[0,1)$.

In Table 1, we presented the absolute error $|\Psi(x,y) - \Psi_{S,K}(x,y)|$, using the introduced numerical method with $S = 2$ and $K = 2,4,6,8$ in the interval $[0,1)$. Table 2, shows the maximum absolute errors of the given method.

Table 1. Absolute error of solution of Eq. (16) by using present method with $S = 2$ and $K = 2,4,6,8$.

(x_i,t_i)	$S=2,\ K=2$	$S=2,\ K=4$	$S=2,\ K=6$	$S=2,\ K=8$
(0,0)	5.62845×10^{-9}	3.25447×10^{-10}	2.36512×10^{-13}	1.32654×10^{-16}
(0.1,0.1)	2.51405×10^{-7}	2.36524×10^{-8}	1.36524×10^{-10}	6.32514×10^{-13}
(0.2,0.2)	5.62103×10^{-6}	2.36985×10^{-7}	5.36214×10^{-9}	8.22551×10^{-12}
(0.3,0.3)	2.02154×10^{-4}	3.58412×10^{-5}	8.32541×10^{-8}	6.32165×10^{-10}
(0.4,0.4)	4.58721×10^{-4}	3.65413×10^{-4}	2.21345×10^{-7}	1.32114×10^{-9}
(0.5,0.5)	7.36212×10^{-4}	2.23651×10^{-4}	3.65221×10^{-7}	2.36985×10^{-8}
(0.6,0.6)	1.36521×10^{-3}	1.65214×10^{-4}	7.32651×10^{-7}	2.92541×10^{-8}
(0.7,0.7)	5.26512×10^{-3}	1.36524×10^{-3}	6.32541×10^{-6}	6.32548×10^{-8}
(0.8,0.8)	5.62514×10^{-2}	4.36210×10^{-3}	8.36251×10^{-6}	7.32614×10^{-8}
(0.9,0.9)	5.65214×10^{-2}	6.25489×10^{-3}	5.32658×10^{-5}	1.36524×10^{-6}

Table 2. The maximum error $R_{max}(x,y)$ for different values of $K = 2,4,6,8$ and $S = 2$ for Eq (16).

	$S=2,\ K=2$	$S=2,\ K=4$	$S=2,\ K=6$	$S=2,\ K=8$
R_{max}	6.2103×10^{-2}	6.53210×10^{-3}	5.32658×10^{-5}	1.36524×10^{-6}

Moreover, in Figures 1–4, we showed a comparison between the exact solution and the approximate solution using the presented numerical technique with different values of $K = 2,4,6,8$ with $S = 2$ in the interval $[0,1)$.

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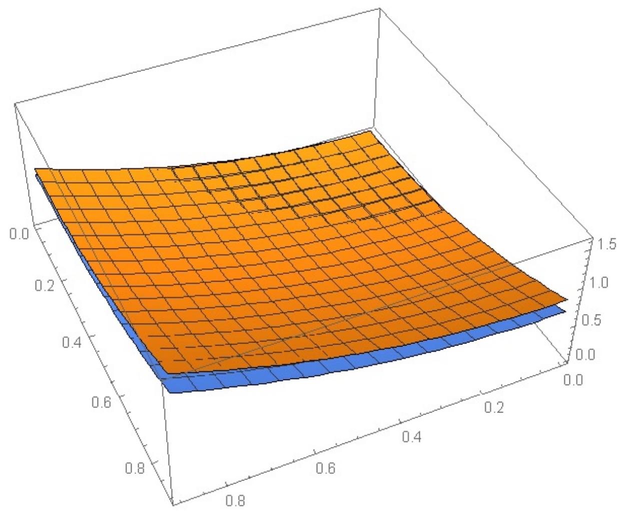

Figure 1. Exact and approximate solution of Eq. (16) with $S = 2$ and $K = 2$.

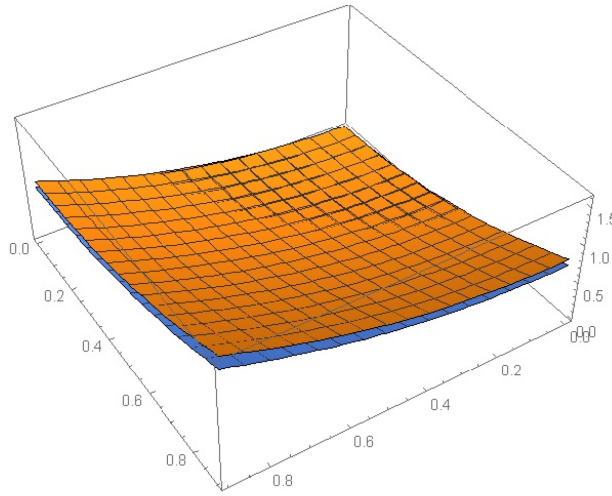

Figure 2. Exact and approximate solution of Eq. (16) with $S = 2$ and $K = 4$.

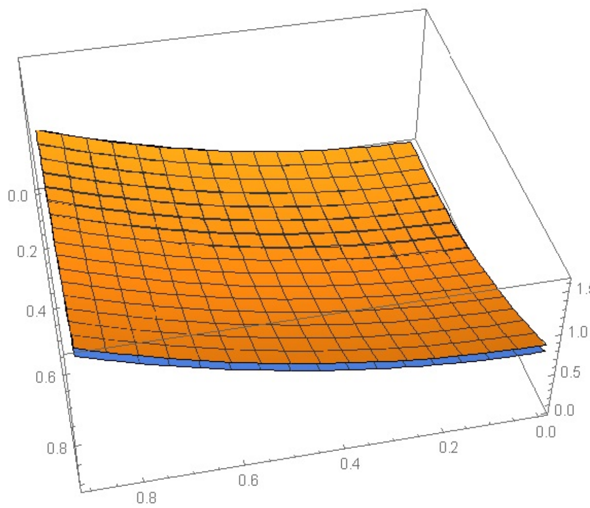

Figure 3. Exact and approximate solution of Eq. (16) with $S = 2$ and $K = 6$.

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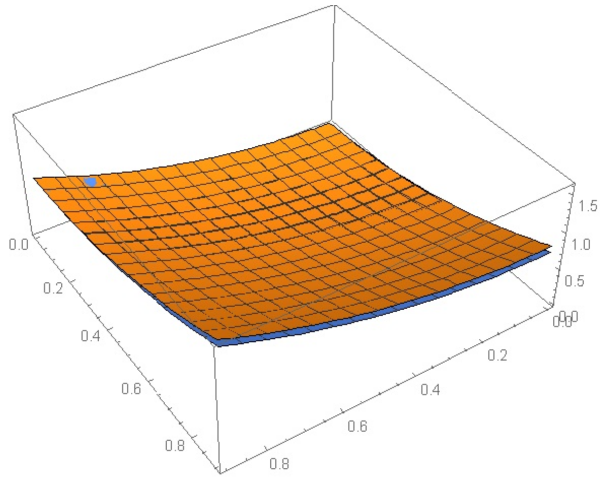

Figure 4. Exact and approximate solution of Eq. (16) with $S = 2$ and $K = 8$.

Example 5.2. Consider the nonlinear integral equation:

$$\psi(x,y) = f(x,y) + 3\int_0^1\int_0^1 (x\tau^2 + v\cos y)\psi^3(\tau,v)\mathrm{d}v\mathrm{d}\tau + 3\int_0^x\int_0^1 (x^2\tau + yv)\psi(\tau,v)\mathrm{d}v\mathrm{d}\tau, \quad (17)$$

where

$$f(x,y) = \frac{1}{12}\Big(27 + 16\cos 1 - 16\cos 2 - 18\cos 3 + 7\cos 4 - 12t^2x(2+\cos 1)\sin(\frac{1}{4})^4$$
$$-12\sin 1 - 36\sin 2 + 6\sin 3 + 6\sin 4) + x\sin y + \frac{1}{2}y^3\left(-3x^2 + x(2+3x)\cos x - 2\sin x\right).$$

The exact solution is $\psi(x,y) = x\sin y$. Using the presented numerical technique with $S = 2$ and $K = 3,5,7,9$ in the interval $[0,1)$.

In Table 3, we showed the absolute error $|\Psi(x,y) - \Psi_{S,K}(x,y)|$, using the introduced numerical method with $S = 2$ and $K = 3,5,7,9$ in the interval $[0,1)$. Table 4, the maximum absolute errors of the given method are obtained.

Table 3. Absolute error of solution of Eq. (17) by using present method with $S = 2$ and $K = 3,5,7,9$.

(x_i, t_i)	$S = 2,\ K = 3$	$S = 2,\ K = 5$	$S = 2,\ K = 7$	$S = 2,\ K = 9$
(0,0)	3.20514×10^{-5}	5.32641×10^{-6}	6.32141×10^{-9}	2.36541×10^{-11}
(0.1,0.1)	3.25481×10^{-4}	9.32541×10^{-5}	5.32187×10^{-7}	3.65874×10^{-8}
(0.2,0.2)	3.32541×10^{-3}	3.21554×10^{-4}	2.36414×10^{-6}	7.36584×10^{-8}
(0.3,0.3)	4.32641×10^{-3}	5.32654×10^{-4}	5.32684×10^{-6}	3.36241×10^{-7}
(0.4,0.4)	5.36854×10^{-3}	6.36524×10^{-4}	8.32546×10^{-6}	6.32584×10^{-7}
(0.5,0.5)	6.93154×10^{-3}	$7.1.365\times10^{-4}$	6.32541×10^{-5}	8.65241×10^{-7}
(0.6,0.6)	1.32511×10^{-2}	3.21547×10^{-3}	9.99215×10^{-5}	4.32516×10^{-6}
(0.7,0.7)	4.32658×10^{-2}	4.36561×10^{-3}	1.32154×10^{-4}	8.69854×10^{-6}
(0.8,0.8)	5.32666×10^{-2}	5.76524×10^{-3}	2.34541×10^{-4}	4.36215×10^{-5}
(0.9,0.9)	6.32541×10^{-2}	7.96525×10^{-3}	3.25456×10^{-4}	1.05214×10^{-4}

Furthermore, in Figures 5–8, we presented a comparison between the exact solution and the approximate solution using the introduced numerical method with different values of $K = 3,5,7,9$ with $S = 2$ in the interval $[0,1)$.

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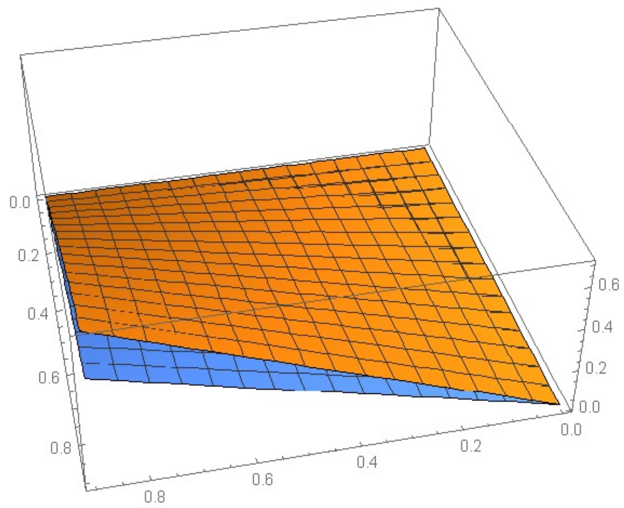

Figure 5. Exact and approximate solution of Eq. (17) with $S = 2$ and $K = 3$.

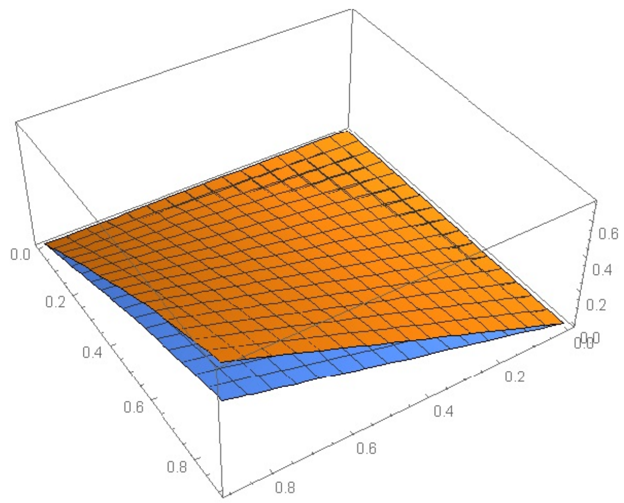

Figure 6. Exact and approximate solution of Eq. (17) with $S = 2$ and $K = 5$.

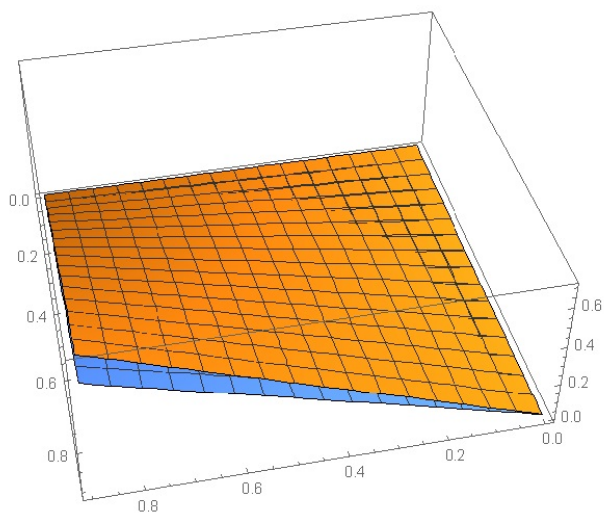

Figure 7. Exact and approximate solution of Eq. (17) with $S = 2$ and $K = 7$.

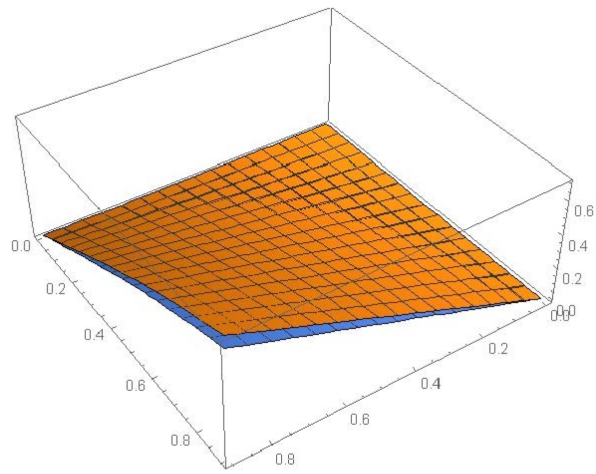

Figure 8. Exact and approximate solution of Eq. (17) with $S = 2$ and $K = 9$.

Table 4. The maximum error $R_{max}(x, y)$ for different values of $K = 3, 5, 7, 9$ and $S = 2$ for Eq (17).

	$S = 2,\ K = 3$	$S = 2,\ K = 5$	$S = 2,\ K = 7$	$S = 2,\ K = 9$
R_{max}	6.32541×10^{-2}	7.96525×10^{-3}	3.25456×10^{-4}	1.05214×10^{-4}

6. Conclusions and Remarks

The following can be deduced from the above analysis and discussion:

1. Under some conditions, the equation (1) has a unique solution $\Psi(x, y)$ in the space $L_2[0, 1] \times L_2[0, 1]$.
2. After applying the proposed method, a two-dimensional integral equation of the second kind, in time and position, tends to result in an algebraic system of equations.
3. A nonlinear system of algebraic equations has a solution.
4. Maximum error obtained by proposed method is decreasing when number of (K) is increasing.
5. Illustrative examples are provided to evaluate and validate the effectiveness and dependability of the proposed method.

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