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A Novel Types of Fuzzy Closed Sets, Separation Axioms, and Compactness via Double Fuzzy Topologies

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Abstract: In this article, we first defined a stronger form of (r,s)-generalized fuzzy semi-closed sets ⟨briefly, (r,s)-$gfsc$ sets⟩ called $(r,s)-g^{\circledast}fsc$ sets and investigated some of its features. Moreover, we showed that $(r,s)-fsc$ set $\Rightarrow$ $(r,s)-g^{\circledast}fsc$ set $\Rightarrow$ $(r,s)-gfsc$ set, but the converse may not be true. In addition, we explored novel types of fuzzy generalized mappings between double fuzzy topological spaces (U,τ,τ^*) and (V,η,η^*), and the relationships between these classes of mappings were examined with the help of some illustrative examples. Thereafter, we introduced novel types of higher separation axioms called (r,s)-$\mathcal{GFS}$-regular and (r,s)-$\mathcal{GFS}$-normal spaces with the help of (r,s)-$gfsc$ sets and discussed some topological properties of them. Finally, some novel types of compactness via (r,s)-$gfso$ sets were defined and the relationships between them were introduced.

Keywords: intuitionistic fuzzy set; double fuzzy topology; $(r,s)-gfsc$ set; $(r,s)-g^{\circledast}fsc$ set; continuity; (r,s)-$\mathcal{GFS}$-regular space; (r,s)-$\mathcal{GFS}$-normal space; compactness

MSC: 03E72; 54A05; 54A40; 54C08; 54D15

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1. Introduction and Preliminaries

The theory of fuzzy set was first presented by Zadeh [1]. Since then it has been improved and applied in most all the branches of technology and science, where theory of sets and mathematical logic play an important role. Also, many applications of these theory contributed to solving several practical problems in mathematics, social science, engineering, economics, etc. In recent years, many authors have contributed to fuzzy sets theory in the different directions in mathematics such as geometry, topology, algebra, operation research, see [2,3]. The notion of fuzzy sets was used to introduce fuzzy topological spaces in [4]. The study in [4] was particularly important in the development of the field of fuzzy topology, see [5–10]. The authors of [11–18] studied topological structures inspired by the hybridizations of soft sets [19] with fuzzy sets [1] and rough sets [20].

The concept of an intuitionistic fuzzy set was initiated by Atanassov [21,22], which is a generalization of a fuzzy set. Coker [23,24] introduced the concept of an intuitionistic fuzzy topological space based on the sense of Chang [4]. Later, Samanta and Mondal [25,26] gave the definition of an intuitionistic fuzzy topological space based on the sense of Šostak [27]. The name (intuitionistic) was replaced with the name (double) by Garcia and Rodabaugh [28]. The concept of $(r,s)-gfc$ sets was introduced and investigated by Abbas [29]. Thereafter, the concept of $(r,s)-sgfc$ sets was introduced by Zahran et al. [30] on double fuzzy topological space based on the sense of Šostak. Also, Taha [31] defined the concept of $(r,s)-gfsc$ sets and some characterizations were given. So far, lots of spectacular and creative studies about the theories of an intuitionistic fuzzy set have been considered by some scholars, see e. g. [32–36].

The organization of this article is as follows:

- Firstly, as a stronger form of $(r,s)-gfsc$ sets [31], the notion of $(r,s)-g^{\circledast}fsc$ sets is introduced and some properties are investigated. Moreover, we introduce new types of fuzzy mappings between double fuzzy topological spaces and relationships are obtained.

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- Secondly, we define new types of fuzzy separation axioms with the help of $(r,s)-gfsc$ sets and establish some of their properties.
- Finally, some new types of compactness in double fuzzy topological spaces are defined and the relationships between them are specified.
- In the end, we give some conclusions and make a plan for future works in Section 5.

Throughout this article, nonempty sets will be denoted by V, U, etc. The family of all fuzzy sets on U is denoted by I^U, and for $\mu \in I^U$, $\mu^c(u) = 1 - \mu(u)$, for all $u \in U$ (where $I = [0,1]$, $I_1 = [0,1)$, and $I_\circ = (0,1]$). Also, for $t \in I$, $\underline{t}(u) = t$, for all $u \in U$.

A fuzzy point u_t on U is a fuzzy set, defined as follows: $u_t(k) = t$ if $k = u$, and $u_t(k) = 0$ for all $k \in U - \{u\}$. u_t is said to belong to a fuzzy set μ, denoted by $u_t \in \mu$, if $t \leq \mu(u)$. The family of all fuzzy points on U is denoted by $P_t(U)$.

A fuzzy set μ is a quasi-coincident with λ, denoted by $\mu q \lambda$, if there is $u \in U$, such that $\mu(u) + \lambda(u) > 1$, if μ is not quasi-coincident with λ, we denote $\mu \overline{q} \lambda$.

The following results and notions will be used in the next sections:

Lemma 1.1 ([6]). *Let U be a nonempty set and $\nu, \mu \in I^U$. Then,*

(i) $\nu q \mu$ iff there is $u_t \in \nu$ such that $u_t q \mu$,

(ii) $\nu \wedge \mu \neq \underline{0}$ if $\nu q \mu$,

(iii) $\nu \overline{q} \mu$ iff $\nu \leq \mu^c$,

(iv) $\mu \leq \nu$ iff $u_t \in \mu$ implies $u_t \in \nu$ iff $u_t q \mu$ implies $u_t q \nu$ iff $u_t \overline{q} \nu$ implies $u_t \overline{q} \mu$,

(v) $u_t \overline{q} \bigvee_{\delta \in \Delta} \nu_\delta$ iff there is $\delta_0 \in \Delta$ such that $u_t \overline{q} \nu_{\delta_0}$.

Definition 1.1 ([25,30]). *A double fuzzy topology on U is a pair (η, η^*) of the mappings $\eta, \eta^* : I^U \to I$, which satisfy the following conditions.*

(i) $\eta(\nu) + \eta^*(\nu) \leq 1$, for each $\nu \in I^U$.

(ii) $\eta(\nu_1 \wedge \nu_2) \geq \eta(\nu_1) \wedge \eta(\nu_2)$ and $\eta^*(\nu_1 \wedge \nu_2) \leq \eta^*(\nu_1) \vee \eta^*(\nu_2)$, for each $\nu_1, \nu_2 \in I^U$.

(iii) $\eta(\bigvee_{\delta \in \Delta} \nu_\delta) \geq \bigwedge_{\delta \in \Delta} \eta(\nu_\delta)$ and $\eta^*(\bigvee_{\delta \in \Delta} \nu_\delta) \leq \bigvee_{\delta \in \Delta} \eta^*(\nu_\delta)$, for each $\{\nu_\delta\}_{\delta \in \Delta} \subset I^U$.

The triplet (U, η, η^*) is said to be a double fuzzy topological space $\langle$briefly, dfts$\rangle$ in the sense of Šostak. $\eta^*(\nu)$ and $\eta(\nu)$ may be interpreted as gradation of nonopenness and openness for $\nu \in I^U$, respectively.

In a dfts (U, η, η^*), the interior of $\nu \in I^U$, the closure of $\nu \in I^U$, the semi-closure of $\nu \in I^U$ and the semi-interior of $\nu \in I^U$ will be denoted by $I_{\eta,\eta^*}(\nu, r, s)$, $C_{\eta,\eta^*}(\nu, r, s)$, $SC_{\eta,\eta^*}(\nu, r, s)$ and $SI_{\eta,\eta^*}(\nu, r, s)$, respectively [26,32,37].

Definition 1.2 ([37,38]). *Let (U, η, η^*) be a dfts, $\nu \in I^U$, $r \in I_\circ$, and $s \in I_1$, then we have*

(i) ν is called an (r,s)-fsc $\langle$resp., (r,s)-fpc and (r,s)-$frc\rangle$ set if $\nu \geq I_{\eta,\eta^*}(C_{\eta,\eta^*}(\nu, r, s), r, s)$ $\langle$resp., $\nu \geq C_{\eta,\eta^*}(I_{\eta,\eta^*}(\nu, r, s), r, s)$ and $\nu = C_{\eta,\eta^*}(I_{\eta,\eta^*}(\nu, r, s), r, s)\rangle$.

(ii) ν is called an (r,s)-fso $\langle$resp., (r,s)-fpo and (r,s)-$fro\rangle$ set if $\nu \leq C_{\eta,\eta^*}(I_{\eta,\eta^*}(\nu, r, s), r, s)$ $\langle$resp., $\nu \leq I_{\eta,\eta^*}(C_{\eta,\eta^*}(\nu, r, s), r, s)$ and $\nu = I_{\eta,\eta^*}(C_{\eta,\eta^*}(\nu, r, s), r, s)\rangle$.

Definition 1.3 ([29–31]). *Let (U, η, η^*) be a dfts, $\mu, \nu \in I^U$, $r \in I_\circ$, and $s \in I_1$, then we have*

(i) μ is called an (r,s)-generalized fuzzy closed $\langle$briefly, (r,s)-$gfc\rangle$ set if $C_{\eta,\eta^*}(\mu, r, s) \leq \nu$ whenever $\mu \leq \nu$ and $\eta(\nu) \geq r$, $\eta^*(\nu) \leq s$.

(ii) μ is called an (r,s)-semi generalized fuzzy closed $\langle$briefly, (r,s)-$sgfc\rangle$ set if $SC_{\eta,\eta^*}(\mu, r, s) \leq \nu$ whenever $\mu \leq \nu$ and ν is (r,s)-fso set.

(iii) μ is called an (r,s)-generalized fuzzy semi-closed $\langle$briefly, (r,s)-$gfsc\rangle$ set if $SC_{\eta,\eta^*}(\mu, r, s) \leq \nu$ whenever $\mu \leq \nu$ and $\eta(\nu) \geq r$, $\eta^*(\nu) \leq s$.

Definition 1.4 ([26,30]). *Let $h : (U, \tau, \tau^*) \to (V, \eta, \eta^*)$ be a mapping, then h is said to be*

(i) $\mathcal{DF}$-continuous if $\tau(h^{-1}(\lambda)) \geq \eta(\lambda)$ and $\tau^*(h^{-1}(\lambda)) \leq \eta^*(\lambda)$ for each $\lambda \in I^V$.

(ii) $\mathcal{DF}$-open if $\eta(h(\nu)) \geq \tau(\nu)$ and $\eta^*(h(\nu)) \leq \tau^*(\nu)$ for each $\nu \in I^U$.

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(iii) $\mathcal{DF}$-closed if $\eta(h^c(\nu)) \geq \tau(\nu^c)$ and $\eta^*(h^c(\nu)) \leq \tau^*(\mu^c)$ for each $\nu \in I^U$.

Definition 1.5 (29,31,37)**.** *Let $h : (U, \tau, \tau^*) \to (V, \eta, \eta^*)$ be a mapping, $r \in I_\circ$, and $s \in I_1$, then h is said to be*

(i) $\mathcal{DFS}$-continuous $\langle$resp., $\mathcal{DFGS}$-continuous and $\mathcal{DFG}$-continuous$\rangle$ if $h^{-1}(\mu)$ is (r,s)-fso $\langle$resp., (r,s)-$gfso$ and (r,s)-$gfo\rangle$ set for each $\mu \in I^V$ with $\eta(\mu) \geq r$, $\eta^*(\mu) \leq s$.

(ii) $\mathcal{DFGS}$-irresolute $\langle$resp., $\mathcal{DF}$-irresolute$\rangle$ if $h^{-1}(\mu)$ is (r,s)-$gfso$ $\langle$resp., (r,s)-$fso\rangle$ set for each $\mu \in I^V$ is (r,s)-$gfso$ $\langle$resp., (r,s)-$fso\rangle$ set.

(iii) $\mathcal{DFS}$-open $\langle$resp., $\mathcal{DFGS}$-open and $\mathcal{DFG}$-open$\rangle$ if $h(\nu)$ is (r,s)-fso $\langle$resp., (r,s)-$gfso$ and (r,s)-$gfo\rangle$ set for each $\nu \in I^U$ with $\tau(\nu) \geq r$, $\tau^*(\nu) \leq s$.

(iv) $\mathcal{DFS}$-closed $\langle$resp., $\mathcal{DFGS}$-closed and $\mathcal{DFG}$-closed$\rangle$ if $h(\nu)$ is (r,s)-fsc $\langle$resp., (r,s)-$gfsc$ and (r,s)-$gfc\rangle$ set for each $\nu \in I^U$ with $\tau(\nu^c) \geq r$, $\tau^*(\nu^c) \leq s$.

The basic results and notions that we need in the next sections are found in [29–31,39–41].

2. A Stronger Novel form of $(r,s) - gfsc$ Sets

Here, we introduce and study a stronger form of $(r,s) - gfsc$ sets called $(r,s) - g^{\circledast}fsc$ sets. Also, we show that $(r,s) - fsc$ set [37] $\Rightarrow$ $(r,s) - g^{\circledast}fsc$ set $\Rightarrow$ $(r,s) - gfsc$ set [31], but the converse may not be true. After that, we introduce new types of fuzzy mappings between double fuzzy topological spaces and relationships are obtained.

Definition 2.1. *Let (V, η, η^*) be a $dfts$, $\nu, \rho \in I^V$, $r \in I_\circ$, and $s \in I_1$, then we have:*

(i) ρ is called an (r,s)-strongly generalized fuzzy semi-closed $\langle$briefly, $(r,s) - g^{\ominus}fsc\rangle$ if $SC_{\eta,\eta^*}(\rho, r, s) \leq \nu$ whenever $\rho \leq \nu$ and ν is $(r,s) - gfo$ set,

(ii) ρ is called an (r,s)-strongly* generalized fuzzy semi-closed $\langle$briefly, $(r,s) - g^{\circledast}fsc\rangle$ if $SC_{\eta,\eta^*}(\rho, r, s) \leq \nu$ whenever $\rho \leq \nu$ and ν is $(r,s) - gfso$ set.

Remark 2.1. *(i) A fuzzy set $\rho \in I^V$ is $(r,s) - g^{\ominus}fso$ if ρ^c is $(r,s) - g^{\ominus}fsc$ set.*

(ii) A fuzzy set $\rho \in I^V$ is $(r,s) - g^{\circledast}fso$ if ρ^c is $(r,s) - g^{\circledast}fsc$ set.

Remark 2.2. *From the previous definition, we can summarize the relationships among different types of fuzzy closed subsets as in the next diagram.*

$$\begin{array}{ccccc} (r,s) - fsc & \rightarrow & (r,s) - g^{\circledast}fsc & & \\ & & \downarrow & \downarrow & \\ & (r,s) - sgfc & & (r,s) - g^{\ominus}fsc & \\ & & \downarrow & \downarrow & \\ & & (r,s) - gfsc & & \end{array}$$

Remark 2.3. *The converses of the above implications may not be true, as shown by Examples 2.1, 2.2, 2.3 and 2.4.*

Example 2.1. *Let $V = \{v_1, v_2, v_3, v_4\}$ and $\rho, \nu \in I^V$ defined as follows: $\rho = \{\frac{v_1}{1.0}, \frac{v_2}{1.0}, \frac{v_3}{1.0}, \frac{v_4}{0.0}\}$ and $\nu = \{\frac{v_1}{0.0}, \frac{v_2}{0.0}, \frac{v_3}{1.0}, \frac{v_4}{1.0}\}$. Also, (η, η^*) defined on V as follows:*

$$\eta(\mu) = \begin{cases} 1, & \text{if} \quad \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if} \quad \mu = \nu, \\ 0, & \text{otherwise}, \end{cases}$$

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$$\eta^*(\mu) = \begin{cases} 0, & \textit{if} \quad \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \textit{if} \quad \mu = \nu, \\ 1, & \textit{otherwise}. \end{cases}$$

Thus, ρ is $(\frac{1}{2}, \frac{1}{2}) - g^{\circledast} fsc$ set, but it is not $(\frac{1}{2}, \frac{1}{2}) - fsc$ set.

Example 2.2. *Let $V = \{v_1, v_2, v_3, v_4\}$ and $\rho, \lambda_1, \lambda_2, \lambda_3 \in I^V$ defined as follows: $\rho = \{\frac{v_1}{1.0}, \frac{v_2}{0.0}, \frac{v_3}{1.0}, \frac{v_4}{0.0}\}$, $\lambda_1 = \{\frac{v_1}{0.0}, \frac{v_2}{1.0}, \frac{v_3}{1.0}, \frac{v_4}{1.0}\}$, $\lambda_2 = \{\frac{v_1}{0.0}, \frac{v_2}{1.0}, \frac{v_3}{1.0}, \frac{v_4}{0.0}\}$ and $\lambda_3 = \{\frac{v_1}{0.0}, \frac{v_2}{0.0}, \frac{v_3}{1.0}, \frac{v_4}{0.0}\}$. Also, (η, η^*) defined on V as follows:*

$$\eta(\mu) = \begin{cases} 1, & \textit{if} \quad \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{4}, & \textit{if} \quad \mu \in \{\lambda_1, \lambda_2, \lambda_3\}, \\ 0, & \textit{otherwise}, \end{cases}$$

$$\eta^*(\mu) = \begin{cases} 0, & \textit{if} \quad \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{4}, & \textit{if} \quad \mu \in \{\lambda_1, \lambda_2, \lambda_3\}, \\ 1, & \textit{otherwise}. \end{cases}$$

Thus, ρ is $(\frac{1}{4}, \frac{1}{4}) - g^{\ominus} fsc$ set, but it is not $(\frac{1}{4}, \frac{1}{4}) - g^{\circledast} fsc$ set.

Example 2.3. *Let $V = \{v_1, v_2, v_3\}$ and $\nu, \mu_1, \mu_2 \in I^V$ defined as follows: $\nu = \{\frac{v_1}{1.0}, \frac{v_2}{0.0}, \frac{v_3}{0.0}\}$, $\mu_1 = \{\frac{v_1}{0.0}, \frac{v_2}{0.0}, \frac{v_3}{1.0}\}$ and $\mu_2 = \{\frac{v_1}{1.0}, \frac{v_2}{1.0}, \frac{v_3}{0.0}\}$. Also, (η, η^*) defined on V as follows:*

$$\eta(\mu) = \begin{cases} 1, & \textit{if} \quad \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \textit{if} \quad \mu \in \{\mu_1, \mu_2\}, \\ 0, & \textit{otherwise}, \end{cases}$$

$$\eta^*(\mu) = \begin{cases} 0, & \textit{if} \quad \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \textit{if} \quad \mu \in \{\mu_1, \mu_2\}, \\ 1, & \textit{otherwise}. \end{cases}$$

Thus, ν is $(\frac{1}{2}, \frac{1}{2}) - sgfc$ set, but it is not $(\frac{1}{2}, \frac{1}{2}) - g^{\circledast} fsc$ set.

Example 2.4. *Let $V = \{v_1, v_2, v_3\}$ and $\nu, \mu_1, \mu_2 \in I^V$ defined as follows: $\nu = \{\frac{v_1}{1.0}, \frac{v_2}{0.0}, \frac{v_3}{1.0}\}$, $\mu_1 = \{\frac{v_1}{1.0}, \frac{v_2}{0.0}, \frac{v_3}{0.0}\}$ and $\mu_2 = \{\frac{v_1}{1.0}, \frac{v_2}{1.0}, \frac{v_3}{0.0}\}$. Also, (η, η^*) defined on V as follows:*

$$\eta(\mu) = \begin{cases} 1, & \textit{if} \quad \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{3}, & \textit{if} \quad \mu \in \{\mu_1, \mu_2\}, \\ 0, & \textit{otherwise}, \end{cases}$$

$$\eta^*(\mu) = \begin{cases} 0, & \textit{if} \quad \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{3}, & \textit{if} \quad \mu \in \{\mu_1, \mu_2\}, \\ 1, & \textit{otherwise}. \end{cases}$$

Thus, ν is $(\frac{1}{3}, \frac{1}{3}) - gfsc$ set, but it is not $(\frac{1}{3}, \frac{1}{3}) - g^{\circledast} fsc$ set.

Remark 2.4. *In general, $(r, s) - gfc$ sets [29] and $(r, s) - g^{\circledast} fsc$ sets are independent concepts, as shown by Example 2.5.*

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Example 2.5. *Let* $V = \{v_1, v_2, v_3, v_4\}$ *and* $\rho, \nu, \mu_1, \mu_2 \in I^V$ *defined as follows:* $\rho = \{\frac{v_1}{0.0}, \frac{v_2}{1.0}, \frac{v_3}{0.0}, \frac{v_4}{0.0}\}$, $\nu = \{\frac{v_1}{1.0}, \frac{v_2}{1.0}, \frac{v_3}{0.0}, \frac{v_4}{1.0}\}$, $\mu_1 = \{\frac{v_1}{1.0}, \frac{v_2}{0.0}, \frac{v_3}{0.0}, \frac{v_4}{0.0}\}$ *and* $\mu_2 = \{\frac{v_1}{1.0}, \frac{v_2}{1.0}, \frac{v_3}{0.0}, \frac{v_4}{0.0}\}$. *Also,* (η, η^*) *defined on* V *as follows:*

$$\eta(\mu) = \begin{cases} 1, & \text{if} \quad \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if} \quad \mu \in \{\mu_1, \mu_2\}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\eta^*(\mu) = \begin{cases} 0, & \text{if} \quad \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if} \quad \mu \in \{\mu_1, \mu_2\}, \\ 1, & \text{otherwise}. \end{cases}$$

Thus, ρ *is* $(\frac{1}{2}, \frac{1}{2}) - g^{\circledast}fsc$ *set, but it is not* $(\frac{1}{2}, \frac{1}{2}) - gfc$ *set. Also,* ν *is* $(\frac{1}{2}, \frac{1}{2}) - gfc$ *set, but it is not* $(\frac{1}{2}, \frac{1}{2}) - g^{\circledast}fsc$ *set.*

Remark 2.5. *In general, any intersection of* $(r,s) - g^{\circledast}fso$ *sets is not* $(r,s) - g^{\circledast}fso$, *and any union of* $(r,s) - g^{\circledast}fsc$ *sets is not* $(r,s) - g^{\circledast}fsc$, *as shown by Example 2.6.*

Example 2.6. *Let* $V = \{v_1, v_2, v_3, v_4\}$ *and* $\nu, \rho, \mu_1, \mu_2, \mu_3 \in I^V$ *defined as follows:* $\nu = \{\frac{v_1}{1.0}, \frac{v_2}{0.0}, \frac{v_3}{1.0}, \frac{v_4}{1.0}\}$, $\rho = \{\frac{v_1}{0.0}, \frac{v_2}{1.0}, \frac{v_3}{1.0}, \frac{v_4}{1.0}\}$, $\mu_1 = \{\frac{v_1}{1.0}, \frac{v_2}{0.0}, \frac{v_3}{0.0}, \frac{v_4}{0.0}\}$, $\mu_2 = \{\frac{v_1}{0.0}, \frac{v_2}{1.0}, \frac{v_3}{0.0}, \frac{v_4}{0.0}\}$ *and* $\mu_3 = \{\frac{v_1}{1.0}, \frac{v_2}{1.0}, \frac{v_3}{0.0}, \frac{v_4}{0.0}\}$. *Also,* (η, η^*) *defined on* V *as follows:*

$$\eta(\mu) = \begin{cases} 1, & \text{if} \quad \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{3}, & \text{if} \quad \mu \in \{\mu_1, \mu_2, \mu_3\}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\eta^*(\mu) = \begin{cases} 0, & \text{if} \quad \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{3}, & \text{if} \quad \mu \in \{\mu_1, \mu_2, \mu_3\}, \\ 1, & \text{otherwise}. \end{cases}$$

Thus, μ_1 *and* μ_2 *are* $(\frac{1}{3}, \frac{1}{3}) - g^{\circledast}fsc$ *sets, but* $\mu_1 \vee \mu_2$ *is not* $(\frac{1}{3}, \frac{1}{3}) - g^{\circledast}fsc$. *Also,* ρ *and* ν *are* $(\frac{1}{3}, \frac{1}{3}) - g^{\circledast}fso$ *sets, but* $\rho \wedge \nu$ *is not* $(\frac{1}{3}, \frac{1}{3}) - g^{\circledast}fso$.

Theorem 2.1. *Let* (V, η, η^*) *be a* $dfts$, $\mu, \lambda \in I^V$, $r \in I_\circ$, *and* $s \in I_1$, *then* λ *is* $(r,s) - g^{\circledast}fsc$ *set iff every* μ *is* $(r,s) - gfso$ *set and* $\lambda \leq \mu$, *there is* ρ *is* $(r,s) - fsc$ *set, such that* $\lambda \leq \rho \leq \mu$.

Proof. ($\Rightarrow$) Let λ be an $(r,s) - g^{\circledast}fsc$, $\lambda \leq \mu$ and μ be an $(r,s) - gfso$ set, then $SC_{\eta,\eta^*}(\lambda, r, s) \leq \mu$. Put $\rho = SC_{\eta,\eta^*}(\lambda, r, s)$, there is ρ is $(r,s) - fsc$ set such that $\lambda \leq \rho \leq \mu$.

($\Leftarrow$) Assume that $\lambda \leq \mu$ and μ is $(r,s) - gfso$ set, then by hypothesis, there is ρ is $(r,s) - fsc$ set such that $\lambda \leq \rho \leq \mu$, therefore, $SC_{\eta,\eta^*}(\lambda, r, s) \leq \mu$. So, λ is $(r,s) - g^{\circledast}fsc$ set. □

Proposition 2.1. *Let* (V, η, η^*) *be a* $dfts$, $\mu, \lambda \in I^V$, $r \in I_\circ$, *and* $s \in I_1$, *then the following properties hold.*

(i) If λ *is* $(r,s) - g^{\circledast}fsc$ *and* $\lambda \leq \mu \leq SC_{\eta,\eta^*}(\lambda, r, s)$, *then* μ *is* $(r,s) - g^{\circledast}fsc$ *set.*

(ii) If λ *is* $(r,s) - g^{\circledast}fso$ *and* $SI_{\eta,\eta^*}(\lambda, r, s) \leq \mu \leq \lambda$, *then* μ *is* $(r,s) - g^{\circledast}fso$ *set.*

(iii) If one of the following two cases hold:

(a) λ *is* $(r,s) - g^{\circledast}fsc$ *and* $(r,s) - gfso$.

(b) λ *is* $(r,s) - g^{\circledast}fsc$ *and* $\eta(\lambda) \geq r$, $\eta^*(\lambda) \leq s$.

Then, λ *is* $(r,s) - fsc$ *set.*

Proof. (i) Let ν be an $(r,s) - gfso$ set and $\mu \leq \nu$, then $\lambda \leq \nu$. Since λ is $(r,s) - g^{\circledast}fsc$ set, hence $SC_{\eta,\eta^*}(\lambda, r, s) \leq \nu$, but $\mu \leq SC_{\eta,\eta^*}(\lambda, r, s)$. Then, $SC_{\eta,\eta^*}(\mu, r, s) \leq \nu$. So, μ is $(r,s) - g^{\circledast}fsc$ set.

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(ii) and (iii) are easily proved by a similar way. □

Theorem 2.2. *Let (V, η, η^*) be a $dfts$, $\nu \in I^V$, $s \in I_1$, and $r \in I_\circ$, then the following statements are equivalent.*
(i) ν is $(r,s) - fro$ set.
(ii) ν is $(r,s) - g^{\circledast} fsc$ set and $\eta(\nu) \geq r$, $\eta^*(\nu) \leq s$.

Proof. (i) $\Rightarrow$ (ii) Let $\mu \in I^V$ be an $(r,s) - gfso$ set and $\nu \leq \mu$. Since ν is $(r,s) - fro$ set, then $\nu \vee I_{\eta,\eta^*}(C_{\eta,\eta^*}(\nu, r, s), r, s) = \nu \leq \mu$. So, $SC_{\eta,\eta^*}(\nu, r, s) \leq \mu$, and hence ν is $(r,s) - g^{\circledast} fsc$ set.
(ii) $\Rightarrow$ (i) Since ν is $(r,s) - g^{\circledast} fsc$ set and $\eta(\nu) \geq r$, $\eta^*(\nu) \leq s$, then by Proposition 2.1(iii), ν is $(r,s) - fsc$ set. But, ν is $(r,s) - fpo$ set. Therefore, ν is $(r,s) - fro$ set. □

Theorem 2.3. *Let (V, η, η^*) be a $dfts$, $\rho, \mu, \nu \in I^V$, $s \in I_1$, and $r \in I_\circ$, then the following statements are equivalent.*
(i) ν is $(r,s) - g^{\circledast} fso$ set.
(ii) For any μ is $(r,s) - gfsc$ set and $\mu \leq \nu$, then $\mu \leq SI_{\eta,\eta^*}(\nu, r, s)$.
(iii) For any μ is $(r,s) - gfsc$ set and $\mu \leq \nu$, there is ρ is $(r,s) - fso$ set such that $\mu \leq \rho \leq \nu$.

Proof. (i) $\Rightarrow$ (ii) Let μ be an $(r,s) - gfsc$ set and $\mu \leq \nu$. Then, $\nu^c \leq \mu^c$, which is $(r,s) - gfso$ set. Hence, $SC_{\eta,\eta^*}(\nu^c, r, s) \leq \mu^c$ implies $\mu \leq (SC_{\eta,\eta^*}(\nu^c, r, s))^c$. Then, $\mu \leq SI_{\eta,\eta^*}(\nu, r, s)$.
(ii) $\Rightarrow$ (iii) Let μ be an $(r,s) - gfsc$ set and $\mu \leq \nu$. Then, by hypothesis $\mu \leq SI_{\eta,\eta^*}(\nu, r, s)$. Put $SI_{\eta,\eta^*}(\nu, r, s) = \rho$. Hence, $\mu \leq \rho \leq \nu$.
(iii) $\Rightarrow$ (i) Let μ be an $(r,s) - gfso$ set and $\nu^c \leq \mu$. Then, $\mu^c \leq \nu$ and by hypothesis, there is ρ is $(r,s) - fso$ set such that $\mu^c \leq \rho \leq \nu$, that is, $\nu^c \leq \rho^c \leq \mu$. Therefore, by Theorem 2.1, ν^c is $(r,s) - g^{\circledast} fsc$ set. Hence, ν is $(r,s) - g^{\circledast} fso$ set. □

Definition 2.2. *Let $h : (U, \tau, \tau^*) \to (V, \eta, \eta^*)$ be a mapping, then h is said to be*
(i) Strongly double fuzzy generalized semi-continuous ⟨briefly, $\mathcal{S}^*\mathcal{DFGS}$-continuous⟩ if $h^{-1}(\nu)$ is $(r,s) - g^{\circledast} fso$ set for each $\nu \in I^V$ and $\eta(\nu) \geq r$, $\eta^*(\nu) \leq s$.*
(ii) $\mathcal{S}^*\mathcal{DFGS}$-irresolute if $h^{-1}(\nu)$ is $(r,s) - g^{\circledast} fso$ set for each $\nu \in I^V$ is $(r,s) - g^{\circledast} fso$ set.
(iii) $\mathcal{S}^*\mathcal{DFGS}$-open if $h(\rho)$ is $(r,s) - g^{\circledast} fso$ set for each $\rho \in I^U$ and $\tau(\rho) \geq r$, $\tau^*(\rho) \leq s$.
(iv) $\mathcal{S}^*\mathcal{DFGS}$-closed if $h(\rho)$ is $(r,s) - g^{\circledast} fsc$ set for $\rho \in I^U$ and $\tau(\rho^c) \geq r$, $\tau^*(\rho^c) \leq s$.

Remark 2.6. *From the previous definitions, we can summarize the relationships among different types of $\mathcal{DF}$-continuity as in the next diagram.*

$$\begin{array}{ccccc} & & \mathcal{DF}-\textit{continuity} & & \\ & \swarrow & & \searrow & \\ \mathcal{DFG}-\textit{continuity} & & & & \mathcal{DFS}-\textit{continuity} \\ \downarrow & & & & \downarrow \\ \mathcal{DFGS}-\textit{continuity} & & \longleftarrow & & \mathcal{S}^*\mathcal{DFGS}-\textit{continuity} \end{array}$$

Remark 2.7. *The converses of the above implications may not be true, as shown by Examples 2.7 and 2.8.*

Example 2.7. *Let $V = \{v_1, v_2, v_3, v_4\}$ and $\rho, \nu \in I^V$ defined as follows: $\rho = \{\frac{v_1}{0.0}, \frac{v_2}{0.0}, \frac{v_3}{1.0}, \frac{v_4}{1.0}\}$ and $\nu = \{\frac{v_1}{0.0}, \frac{v_2}{0.0}, \frac{v_3}{0.0}, \frac{v_4}{1.0}\}$. Define $\eta, \eta^*, \tau, \tau^* : I^V \longrightarrow I$ as follows:*

$$\eta(\mu) = \begin{cases} 1, & \text{if } \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \mu = \rho, \\ 0, & \text{otherwise}, \end{cases}$$

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$$\eta^*(\mu) = \begin{cases} 0, & \textit{if} \quad \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \textit{if} \quad \mu = \rho, \\ 1, & \textit{otherwise}, \end{cases}$$

$$\tau(\mu) = \begin{cases} 1, & \textit{if} \quad \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \textit{if} \quad \mu = \nu, \\ 0, & \textit{otherwise}, \end{cases}$$

$$\tau^*(\mu) = \begin{cases} 0, & \textit{if} \quad \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \textit{if} \quad \mu = \nu, \\ 1, & \textit{otherwise}. \end{cases}$$

Thus, the identity mapping $id_v : (V, \eta, \eta^*) \to (V, \tau, \tau^*)$ *is* $\mathcal{S}^*\mathcal{DFGS}$*-continuous, but it is not* $\mathcal{DFS}$*-continuous.*

Example 2.8. *Let* $V = \{v_1, v_2, v_3\}$ *and* $\mu_1, \mu_2, \mu_3 \in I^V$ *defined as follows:* $\mu_1 = \{\frac{v_1}{0.0}, \frac{v_2}{0.0}, \frac{v_3}{1.0}\}$, $\mu_2 = \{\frac{v_1}{1.0}, \frac{v_2}{1.0}, \frac{v_3}{0.0}\}$ *and* $\mu_3 = \{\frac{v_1}{0.0}, \frac{v_2}{1.0}, \frac{v_3}{1.0}\}$. *Define* $\eta, \eta^*, \tau, \tau^* : I^V \longrightarrow I$ *as follows:*

$$\eta(\mu) = \begin{cases} 1, & \textit{if} \quad \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \textit{if} \quad \mu \in \{\mu_1, \mu_2\}, \\ 0, & \textit{otherwise}, \end{cases}$$

$$\eta^*(\mu) = \begin{cases} 0, & \textit{if} \quad \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \textit{if} \quad \mu \in \{\mu_1, \mu_2\}, \\ 1, & \textit{otherwise}, \end{cases}$$

$$\tau(\mu) = \begin{cases} 1, & \textit{if} \quad \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \textit{if} \quad \mu = \mu_3, \\ 0, & \textit{otherwise}, \end{cases}$$

$$\tau^*(\mu) = \begin{cases} 0, & \textit{if} \quad \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \textit{if} \quad \mu = \mu_3, \\ 1, & \textit{otherwise}. \end{cases}$$

Thus, the identity mapping $id_v : (V, \eta, \eta^*) \to (V, \tau, \tau^*)$ *is* $\mathcal{DFGS}$*-continuous, but it is not* $S^*\mathcal{DFGS}$*-continuous.*

Lemma 2.1. *Every* $\mathcal{S}^*\mathcal{DFGS}$*-irresolute mapping is* $\mathcal{S}^*\mathcal{DFGS}$*-continuous.*

Remark 2.8. *The converse of Lemma 2.1 may not be true, as shown by Example 2.9.*

Example 2.9. *Let* $V = \{v_1, v_2\}$. *Define* $\eta, \eta^*, \tau, \tau^* : I^V \longrightarrow I$ *as follows:*

$$\eta(\rho) = \begin{cases} 1, & \textit{if} \quad \rho \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \textit{if} \quad \rho \in \{\underline{0.1}, \underline{0.3}\}, \\ 0, & \textit{otherwise}, \end{cases}$$

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$$\eta^*(\rho) = \begin{cases} 0, & \text{if} \quad \rho \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if} \quad \rho \in \{\underline{0.1}, \underline{0.3}\}, \\ 1, & \text{otherwise}, \end{cases}$$

$$\tau(\rho) = \begin{cases} 1, & \text{if} \quad \rho \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if} \quad \rho = \underline{0.1}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\tau^*(\rho) = \begin{cases} 0, & \text{if} \quad \rho \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if} \quad \rho = \underline{0.1}, \\ 1, & \text{otherwise}. \end{cases}$$

Thus, the identity mapping $id_v : (V, \eta, \eta^*) \to (V, \tau, \tau^*)$ is $\mathcal{S}^*\mathcal{DFGS}$-continuous, but it is not $\mathcal{S}^*\mathcal{DFGS}$-irresolute.

3. Some Novel Higher Separation Axioms

Here, we are going to give the definitions of two types of higher fuzzy separation axioms with the help of $(r,s) - gfsc$ sets [31] called (r,s)-$\mathcal{GFS}$-regular $\langle$resp., (r,s)-$\mathcal{GFS}$-normal$\rangle$ spaces and establish some of their properties.

Definition 3.1. *A dfts (U, η, η^*) is said to be*

(i) (r,s)-$\mathcal{GFS}$-regular iff $u_t \bar{q} \mu$ for each $\mu \in I^U$ is $(r,s) - gfsc$ set implies that, there is $\nu_\delta \in I^U$ with $\eta(\nu_\delta) \geq r$, $\eta^*(\nu_\delta) \leq s$ for $\delta \in \{1,2\}$, such that $u_t \in \nu_1$, $\mu \leq \nu_2$ and $\nu_1 \bar{q} \nu_2$.

(ii) (r,s)-$\mathcal{GFS}$-normal iff $\mu_1 \bar{q} \mu_2$ for each $(r,s) - gfsc$ sets $\mu_\delta \in I^U$ for $\delta \in \{1,2\}$ implies that, there is $\nu_\delta \in I^U$ with $\eta(\nu_\delta) \geq r$ and $\eta^*(\nu_\delta) \leq s$, such that $\mu_\delta \leq \nu_\delta$ and $\nu_1 \bar{q} \nu_2$.

Theorem 3.1. *Let (U, η, η^*) be a $dfts$, $r \in I_\circ$, and $s \in I_1$, then the following statements are equivalent.*

(i) (U, η, η^*) is (r,s)-$\mathcal{GFS}$-regular space.

(ii) If $u_t \in \lambda$ for each $\lambda \in I^U$ is $(r,s) - gfso$, there is $\mu \in I^U$ with $\eta(\mu) \geq r$ and $\eta^*(\mu) \leq s$, such that $u_t \in \mu \leq C_{\eta,\eta^*}(\mu, r, s) \leq \lambda$.

(iii) If $u_t \bar{q} \lambda$ for each $\lambda \in I^U$ is $(r,s) - gfsc$, there is $\mu_\delta \in I^U$ with $\eta(\mu_\delta) \geq r$, $\eta^*(\mu_\delta) \leq s$ for $\delta \in \{1,2\}$, such that $u_t \in \mu_1$, $\lambda \leq \mu_2$ and $C_{\eta,\eta^*}(\mu_1, r, s) \bar{q} C_{\eta,\eta^*}(\mu_2, r, s)$.

Proof. (i) $\Rightarrow$ (ii) Let $u_t \in \lambda$ for each $\lambda \in I^U$ is an $(r,s) - gfso$, then $u_t \bar{q} \lambda^c$ for $(r,s) - gfsc$ set λ^c. Since (U, η, η^*) is (r,s)-$\mathcal{GFS}$-regular, there is $\mu, \nu \in I^U$ with $\eta(\mu) \geq r$, $\eta^*(\mu) \leq s$ and $\eta(\nu) \geq r$, $\eta^*(\nu) \leq s$ such that $u_t \in \mu$, $\lambda^c \leq \nu$ and $\mu \bar{q} \nu$. It implies $u_t \in \mu \leq \nu^c \leq \lambda$. Since $\eta(\nu) \geq r$ and $\eta^*(\nu) \leq s$, $u_t \in \mu \leq C_{\eta,\eta^*}(\mu, r, s) \leq \lambda$.

(ii) $\Rightarrow$ (iii) Let $u_t \bar{q} \lambda$ for each $\lambda \in I^U$ is an $(r,s) - gfsc$, then $u_t \in \lambda^c$ for $(r,s) - gfso$ set λ^c. By (ii), there is $\mu \in I^U$ with $\eta(\mu) \geq r$, $\eta^*(\mu) \leq s$ such that $u_t \in \mu \leq C_{\eta,\eta^*}(\mu, r, s) \leq \lambda^c$. Since $\eta(\mu) \geq r$ and $\eta^*(\mu) \leq s$, then μ is $(r,s) - gfso$ and $u_t \in \mu$. Again, by (ii), there is $\mu_1 \in I^U$ with $\eta(\mu_1) \geq r$, $\eta^*(\mu_1) \leq s$ such that

$$u_t \in \mu_1 \leq C_{\eta,\eta^*}(\mu_1, r, s) \leq \mu \leq C_{\eta,\eta^*}(\mu, r, s) \leq \lambda^c.$$

It implies $\lambda \leq (C_{\eta,\eta^*}(\mu, r, s))^c = I_{\eta,\eta^*}(\mu^c, r, s) \leq \mu^c$. Put $\mu_2 = I_{\eta,\eta^*}(\mu^c, r, s)$, then $\eta(\mu_2) \geq r$, $\eta^*(\mu_2) \leq s$.

So, $C_{\eta,\eta^*}(\mu_2, r, s) \leq \mu^c \leq (C_{\eta,\eta^*}(\mu_1, r, s))^c$, that is, $C_{\eta,\eta^*}(\mu_1, r, s) \bar{q} C_{\eta,\eta^*}(\mu_2, r, s)$.

(iii) $\Rightarrow$ (i) It is trivial. □

In a similar way, we can prove Theorem 3.2.

Theorem 3.2. *Let (U, η, η^*) be a $dfts$, $r \in I_\circ$, and $s \in I_1$, then the following statements are equivalent.*

(i) (U, η, η^*) is (r,s)-$\mathcal{GFS}$-normal space.

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(ii) If $\nu \leq \lambda$ for each $\nu \in I^U$ is $(r,s)-gfsc$ and $\lambda \in I^U$ is $(r,s)-gfso$ set, there is $\mu \in I^U$ with $\eta(\mu) \geq r$ and $\eta^*(\mu) \leq s$, such that $\nu \leq \mu \leq C_{\eta,\eta^*}(\mu,r,s) \leq \lambda$.

(iii) If $\lambda_1 \bar{q} \lambda_2$ for each $(r,s)-gfsc$ sets $\lambda_\delta \in I^U$ for $\delta \in \{1,2\}$, there is $\mu_\delta \in I^U$ with $\eta(\mu_\delta) \geq r$ and $\eta^*(\mu_\delta) \leq s$, such that $\lambda_\delta \leq \mu_\delta$ and $C_{\eta,\eta^*}(\mu_1,r,s)\bar{q}C_{\eta,\eta^*}(\mu_2,r,s)$.

Theorem 3.3. *If $h:(U,\tau,\tau^*) \to (V,\eta,\eta^*)$ is $\mathcal{DF}$-irresolute, $\mathcal{DF}$-open and bijective map, and (U,τ,τ^*) is (r,s)-$\mathcal{GFS}$-regular $\langle$resp., (r,s)-$\mathcal{GFS}$-normal$\rangle$ space, then (V,η,η^*) is (r,s)-$\mathcal{GFS}$-regular $\langle$resp., (r,s)-$\mathcal{GFS}$-normal$\rangle$ space.*

Proof. Let $v_t \bar{q} \mu$ for each $\mu \in I^V$ is $(r,s)-gfsc$. Since h is $\mathcal{DF}$-irresolute, $\mathcal{DF}$-open and bijective map, then by Theorem 4.11 [31], h is $\mathcal{DFGS}$-irresolute. Hence, $h^{-1}(\mu)$ is $(r,s)-gfsc$ set. Put $v_t = h(u_t)$. Then, $u_t \bar{q} h^{-1}(\mu)$. Since (U,τ,τ^*) is (r,s)-$\mathcal{GFS}$-regular, there is $\mu_\delta \in I^U$ with $\tau(\mu_\delta) \geq r$, $\tau^*(\mu_\delta) \leq s$ and $\delta \in \{1,2\}$ such that $u_t \in \mu_1$, $h^{-1}(\mu) \leq \mu_2$ and $\mu_1 \bar{q} \mu_2$. Since h is $\mathcal{DF}$-open and bijective map, we have

$$v_t \in h(\mu_1),\ \mu = h(h^{-1}(\mu)) \leq h(\mu_2),\ h(\mu_1)\bar{q}h(\mu_2).$$

Hence, (V,η,η^*) is (r,s)-$\mathcal{GFS}$-regular space. The other case follows similar lines. □

Theorem 3.4. *If $h:(U,\tau,\tau^*) \to (V,\eta,\eta^*)$ is $\mathcal{DF}$-continuous, $\mathcal{DFGS}$-irresolute closed and injective map, and (V,η,η^*) is (r,s)-$\mathcal{GFS}$-regular $\langle$resp., (r,s)-$\mathcal{GFS}$-normal$\rangle$, then (U,τ,τ^*) is (r,s)-$\mathcal{GFS}$-regular $\langle$resp., (r,s)-$\mathcal{GFS}$-normal$\rangle$.*

Proof. Let $u_t \bar{q} \lambda$ for each $\lambda \in I^U$ is $(r,s)-gfsc$. Since h is $\mathcal{DFGS}$-irresolute closed, $h(\lambda)$ is $(r,s)-gfsc$. Since h is injective, $u_t \bar{q} \lambda$ implies $h(u_t)\bar{q}h(\lambda)$. Since (V,η,η^*) is (r,s)-$\mathcal{GFS}$-regular, there is $\mu_\delta \in I^U$ with $\eta(\mu_\delta) \geq r$, $\eta^*(\mu_\delta) \leq s$ and $\delta \in \{1,2\}$ such that $h(u_t) \in \mu_1$, $h(\lambda) \leq \mu_2$ and $\mu_1 \bar{q} \mu_2$. Since h is $\mathcal{DF}$-continuous, $u_t \in h^{-1}(\mu_1)$, $\lambda \leq h^{-1}(\mu_2)$ with $\eta(h^{-1}(\mu_\delta)) \geq r$, $\eta^*(h^{-1}(\mu_\delta)) \leq s$ and $\delta \in \{1,2\}$ and $h^{-1}(\mu_1)\bar{q}h^{-1}(\mu_2)$. Hence, (U,τ,τ^*) is (r,s)-$\mathcal{GFS}$-regular. The other case follows similar lines. □

Theorem 3.5. *If $h:(U,\tau,\tau^*) \to (V,\eta,\eta^*)$ is $\mathcal{DFGS}$-irresolute, $\mathcal{DF}$-open, $\mathcal{DF}$-closed and surjective map, and (U,τ,τ^*) is (r,s)-$\mathcal{GFS}$-regular $\langle$resp., (r,s)-$\mathcal{GFS}$-normal$\rangle$, then (V,η,η^*) is (r,s)-$\mathcal{GFS}$-regular $\langle$resp., (r,s)-$\mathcal{GFS}$-normal$\rangle$.*

Proof. Let $v_t \in \mu$ for each $\mu \in I^V$ is $(r,s)-gfso$. Since h is $\mathcal{DFGS}$-irresolute and surjective then, there is $u \in h^{-1}(\{v\})$ such that $u_t \in h^{-1}(\mu)$ with $(r,s)-gfso$ set $h^{-1}(\mu)$. Since (U,τ,τ^*) is (r,s)-$\mathcal{GFS}$-regular, by Theorem 3.1, there is $\nu \in I^U$ with $\tau(\nu) \geq r$, $\tau^*(\nu) \leq s$ such that $u_t \in \nu \leq C_{\tau,\tau^*}(\nu,r,s) \leq h^{-1}(\mu)$. It implies

$$v_t \in h(\nu) \leq h(C_{\tau,\tau^*}(\nu,r,s)) \leq \mu.$$

Since h is $\mathcal{DF}$-open and $\mathcal{DF}$-closed, then $\eta(h(\nu)) \geq r$, $\eta^*(h(\nu)) \leq s$ and $\eta(h^c(C_{\tau,\tau^*}(\nu,r,s))) \geq r$. Hence, $v_t \in h(\nu) \leq C_{\eta,\eta^*}(h(\nu),r,s) \leq C_{\eta,\eta^*}(h(C_{\tau,\tau^*}(\nu,r,s)),r,s) \leq \mu$. Thus, (V,η,η^*) is (r,s)-$\mathcal{GFS}$-regular. The other case follows similar lines. □

4. Novel Types of Compactness

Here, several types of compactness in double fuzzy topological spaces were introduced and the relationships between them were studied.

Definition 4.1. *Let (U,η,η^*) be a $dfts$, $r \in I_\circ$, and $s \in I_1$, then $\mu \in I^U$ is called an (r,s)-fuzzy compact iff for each family $\{\lambda_j \in I^U \mid \eta(\lambda_j) \geq r$ and $\eta^*(\lambda_j) \leq s\}_{j\in F}$, such that $\mu \leq \bigvee_{j\in F}\lambda_j$, there is a finite subset $F_\circ$ of F, such that $\mu \leq \bigvee_{j\in F_\circ}\lambda_j$.*

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Definition 4.2. *Let $(\mathcal{U}, \eta, \eta^*)$ be a dfts, $r \in I_\circ$, and $s \in I_1$, then $\mu \in I^{\mathcal{U}}$ is called an (r,s)-fuzzy $\mathcal{GS}$-compact iff for each family $\{\lambda_j \in I^{\mathcal{U}} \mid \lambda_j$ is $(r,s) - gfso\}_{j\in\digamma}$, such that $\mu \leq \bigvee_{j\in\digamma} \lambda_j$, there is a finite subset $\digamma_\circ$ of $\digamma$, such that $\mu \leq \bigvee_{j\in\digamma_\circ} \lambda_j$.*

Lemma 4.1. *Let $(\mathcal{U}, \eta, \eta^*)$ be a dfts, $r \in I_\circ$, and $s \in I_1$. If $\mu \in I^{\mathcal{U}}$ is (r,s)-fuzzy $\mathcal{GS}$-compact, then μ is (r,s)-fuzzy compact.*

Proof. Follows from Definitions 4.1 and 4.2. □

Theorem 4.1. *Let $h : (\mathcal{U}, \tau, \tau^*) \to (\mathcal{V}, \eta, \eta^*)$ be a $\mathcal{DFGS}$-continuous mapping, $r \in I_\circ$, and $s \in I_1$. If $\mu \in I^{\mathcal{U}}$ is (r,s)-fuzzy $\mathcal{GS}$-compact, then $h(\mu)$ is (r,s)-fuzzy compact.*

Proof. Let $\{\lambda_j \in I^{\mathcal{V}} \mid \eta(\lambda_j) \geq r$ and $\eta^*(\lambda_j) \leq s\}_{j\in\digamma}$ with $h(\mu) \leq \bigvee_{j\in\digamma} \lambda_j$, then $\{h^{-1}(\lambda_j) \in I^{\mathcal{U}} \mid h^{-1}(\lambda_j)$ is $(r,s) - gfso\}$ (by h is $\mathcal{DFGS}$-continuous), such that $\mu \leq \bigvee_{j\in\digamma} h^{-1}(\lambda_j)$. Since μ is (r,s)-fuzzy $\mathcal{GS}$-compact, there is a finite subset $\digamma_\circ$ of $\digamma$, such that $\mu \leq \bigvee_{j\in\digamma_\circ} h^{-1}(\lambda_j)$. Thus, $h(\mu) \leq \bigvee_{j\in\digamma_\circ} \lambda_j$. Hence, the proof is completed. □

Definition 4.3. *Let $(\mathcal{U}, \eta, \eta^*)$ be a dfts, $r \in I_\circ$, and $s \in I_1$, then $\mu \in I^{\mathcal{U}}$ is called an (r,s)-fuzzy almost compact iff for each family $\{\lambda_j \in I^{\mathcal{U}} \mid \eta(\lambda_j) \geq r$ and $\eta^*(\lambda_j) \leq s\}_{j\in\digamma}$, such that $\mu \leq \bigvee_{j\in\digamma} \lambda_j$, there is a finite subset $\digamma_\circ$ of $\digamma$, such that $\mu \leq \bigvee_{j\in\digamma_\circ} C_{\eta,\eta^*}(\lambda_j, r, s)$.*

Definition 4.4. *Let $(\mathcal{U}, \eta, \eta^*)$ be a dfts, $r \in I_\circ$, and $s \in I_1$, then $\mu \in I^{\mathcal{U}}$ is called an (r,s)-fuzzy almost $\mathcal{GS}$-compact iff for each family $\{\lambda_j \in I^{\mathcal{U}} \mid \lambda_j$ is $(r,s) - gfso\}_{j\in\digamma}$, such that $\mu \leq \bigvee_{j\in\digamma} \lambda_j$, there is a finite subset $\digamma_\circ$ of $\digamma$, such that $\mu \leq \bigvee_{j\in\digamma_\circ} C_{\eta,\eta^*}(\lambda_j, r, s)$.*

Lemma 4.2. *Let $(\mathcal{U}, \eta, \eta^*)$ be a dfts, $r \in I_\circ$, and $s \in I_1$. If $\mu \in I^{\mathcal{U}}$ is (r,s)-fuzzy almost $\mathcal{GS}$-compact, then μ is (r,s)-fuzzy almost compact.*

Proof. Follows from Definitions 4.3 and 4.4. □

Lemma 4.3. *Let $(\mathcal{U}, \eta, \eta^*)$ be a dfts, $r \in I_\circ$, and $s \in I_1$. If $\mu \in I^{\mathcal{U}}$ is (r,s)-fuzzy compact (resp., $\mathcal{GS}$-compact), then μ is (r,s)-fuzzy almost compact (resp., almost $\mathcal{GS}$-compact).*

Proof. Follows from Definitions 4.1, 4.2, 4.3 and 4.4. □

Remark 4.1. *The converse of Lemma 4.3 may not be true, as shown by Example 4.1.*

Example 4.1. *Let $\mathcal{V} = I$, $k \in N - \{1\}$, and $\rho, \lambda_k \in I^{\mathcal{V}}$ defined as follows:*

$$\rho(v) = \begin{cases} 1, & \text{if} \quad v = 0, \\ \frac{1}{2}, & \text{otherwise}, \end{cases}$$

$$\lambda_k(v) = \begin{cases} 0.8, & \text{if} \quad v = 0, \\ kv, & \text{if} \quad 0 < v \leq \frac{1}{k}, \\ 1\,, & \text{if} \quad \frac{1}{k} < v \leq 1. \end{cases}$$

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Also, (η, η^*) defined on V as follows:

$$\eta(\mu) = \begin{cases} 1, & \text{if} \quad \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{2}{3}, & \text{if} \quad \mu \leq \rho, \\ \frac{k}{k+1}, & \text{if} \quad \mu \leq \lambda_k, \\ 0, & \text{otherwise}, \end{cases}$$

$$\eta^*(\mu) = \begin{cases} 0, & \text{if} \quad \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{3}, & \text{if} \quad \mu \leq \rho, \\ \frac{1}{k+1}, & \text{if} \quad \mu \leq \lambda_k, \\ 1, & \text{otherwise}. \end{cases}$$

Thus, V is $(\frac{1}{2}, \frac{1}{2})$-fuzzy almost compact, but it is not $(\frac{1}{2}, \frac{1}{2})$-fuzzy compact.

Theorem 4.2. *Let $h : (U, \tau, \tau^*) \to (V, \eta, \eta^*)$ be a $\mathcal{DF}$-continuous mapping, $r \in I_\circ$, and $s \in I_1$. If $\mu \in I^U$ is (r,s)-fuzzy almost $\mathcal{GS}$-compact, then $h(\mu)$ is (r,s)-fuzzy almost compact.*

Proof. Let $\{\lambda_j \in I^V \mid \eta(\lambda_j) \geq r$ and $\eta^*(\lambda_j) \leq s\}_{j \in F}$ with $h(\mu) \leq \bigvee_{j \in F} \lambda_j$, then $\{h^{-1}(\lambda_j) \in I^U \mid h^{-1}(\lambda_j)$ is $(r,s) - gfso\}$ (by h is $\mathcal{DFGS}$-continuous), such that $\mu \leq \bigvee_{j \in F} h^{-1}(\lambda_j)$. Since μ is (r,s)-fuzzy almost $\mathcal{GS}$-compact, there is a finite subset $F_\circ$ of F, such that $\mu \leq \bigvee_{j \in F_\circ} C_{\tau,\tau^*}(h^{-1}(\lambda_j), r, s)$. Since h is $\mathcal{DF}$-continuous mapping, it follows

$$\begin{aligned} \mu &\leq \bigvee_{j \in F_\circ} C_{\tau,\tau^*}(h^{-1}(\lambda_j), r, s) \\ &\leq \bigvee_{j \in F_\circ} h^{-1}(C_{\eta,\eta^*}(\lambda_j, r, s)) \\ &= h^{-1}(\bigvee_{j \in F_\circ} C_{\eta,\eta^*}(\lambda_j, r, s)). \end{aligned}$$

Thus, $h(\mu) \leq \bigvee_{j \in F_\circ} C_{\eta,\eta^*}(\lambda_j, r, s)$. Hence, the proof is completed. □

Definition 4.5. *Let (U, η, η^*) be a $dfts$, $r \in I_\circ$, and $s \in I_1$, then $\mu \in I^U$ is called an (r,s)-fuzzy nearly compact iff for each family $\{\lambda_j \in I^U \mid \eta(\lambda_j) \geq r$ and $\eta^*(\lambda_j) \leq s\}_{j \in F}$, such that $\mu \leq \bigvee_{j \in F} \lambda_j$, there is a finite subset $F_\circ$ of F, such that $\mu \leq \bigvee_{j \in F_\circ} I_{\eta,\eta^*}(C_{\eta,\eta^*}(\lambda_j, r, s), r, s)$.*

Definition 4.6. *Let (U, η, η^*) be a $dfts$, $r \in I_\circ$, and $s \in I_1$, then $\mu \in I^U$ is called an (r,s)-fuzzy nearly $\mathcal{GS}$-compact iff for each family $\{\lambda_j \in I^U \mid \lambda_j$ is $(r,s) - gfso\}_{j \in F}$, such that $\mu \leq \bigvee_{j \in F} \lambda_j$, there is a finite subset $F_\circ$ of F, such that $\mu \leq \bigvee_{j \in F_\circ} I_{\eta,\eta^*}(C_{\eta,\eta^*}(\lambda_j, r, s), r, s)$.*

Lemma 4.4. *Let (U, η, η^*) be a $dfts$, $r \in I_\circ$, and $s \in I_1$. If $\mu \in I^U$ is (r,s)-fuzzy nearly $\mathcal{GS}$-compact, then μ is (r,s)-fuzzy nearly compact.*

Proof. Follows from Definitions 4.5 and 4.6. □

Lemma 4.5. *Let (U, η, η^*) be a $dfts$, $r \in I_\circ$, and $s \in I_1$. If $\mu \in I^U$ is (r,s)-fuzzy compact (resp., $\mathcal{GS}$-compact), then μ is (r,s)-fuzzy nearly compact (resp., nearly $\mathcal{GS}$-compact).*

Proof. Follows from Definitions 4.1, 4.2, 4.5 and 4.6. □

Remark 4.2. *The converse of Lemma 4.5 may not be true, as shown by Example 4.2.*

Example 4.2. *Let $V = I$, $0 < k < 1$, and $\nu, \rho, \lambda_k \in I^V$ defined as follows:*

$$\nu(v) = \begin{cases} \frac{1}{2}, & \text{if} \quad 0 \leq v < 1, \\ 1, & \text{if} \quad v = 1, \end{cases}$$

$$\rho(v) = \begin{cases} 1, & \text{if} \quad v = 0, \\ \frac{1}{2}, & \text{if} \quad 0 < v \leq 1, \end{cases}$$

$$\lambda_k(v) = \begin{cases} \frac{v}{k}, & \text{if} \quad 0 \leq v \leq k, \\ \frac{1-v}{1-k}, & \text{if} \quad k < v \leq 1. \end{cases}$$

Also, (η, η^*) defined on V as follows:

$$\eta(\mu) = \begin{cases} 1, & \text{if} \quad \mu \in \{\nu, \rho, \underline{0}, \underline{1}\}, \\ max(\{1-k, k\}), & \text{if} \quad \mu = \lambda_k, \\ 0, & otherwise, \end{cases}$$

$$\eta^*(\mu) = \begin{cases} 0, & \text{if} \quad \mu \in \{\nu, \rho, \underline{0}, \underline{1}\}, \\ min(\{k, 1-k\}), & \text{if} \quad \mu = \lambda_k, \\ 1, & otherwise. \end{cases}$$

Thus, V is $(\frac{1}{2}, \frac{1}{2})$-fuzzy nearly compact, but it is not $(\frac{1}{2}, \frac{1}{2})$-fuzzy compact.

Theorem 4.3. *Let $h : (U, \tau, \tau^*) \to (V, \eta, \eta^*)$ be a $\mathcal{DF}$-continuous and $\mathcal{DF}$-open mapping, $r \in I_\circ$, and $s \in I_1$. If $\mu \in I^U$ is (r,s)-fuzzy nearly $\mathcal{GS}$-compact, $h(\mu)$ is (r,s)-fuzzy nearly compact.*

Proof. Let $\{\lambda_j \in I^V \mid \eta(\lambda_j) \geq r$ and $\eta^*(\lambda_j) \leq s\}_{j \in \digamma}$ with $h(\mu) \leq \bigvee_{j \in \digamma} \lambda_j$, then $\{h^{-1}(\lambda_j) \in I^U \mid h^{-1}(\lambda_j)$ is $(r,s) - gfso\}$ (by h is $\mathcal{DFGS}$-continuous), such that $\mu \leq \bigvee_{j \in \digamma} h^{-1}(\lambda_j)$. Since μ is (r,s)-fuzzy nearly $\mathcal{GS}$-compact, there is a finite subset $\digamma_\circ$ of $\digamma$, such that $\mu \leq \bigvee_{j \in \digamma_\circ} I_{\tau,\tau^*}(C_{\tau,\tau^*}(h^{-1}(\lambda_j), r, s), r, s)$. Since h is $\mathcal{DF}$-continuous and $\mathcal{DF}$-open, it follows

$$h(\mu) \leq \bigvee_{j \in \digamma_\circ} h(I_{\tau,\tau^*}(C_{\tau,\tau^*}(h^{-1}(\lambda_j), r, s), r, s))$$

$$\leq \bigvee_{j \in \digamma_\circ} I_{\eta,\eta^*}(h(C_{\tau,\tau^*}(h^{-1}(\lambda_j), r, s)), r, s)$$

$$\leq \bigvee_{j \in \digamma_\circ} I_{\eta,\eta^*}(h(h^{-1}(C_{\eta,\eta^*}(\lambda_j, r, s))), r, s)$$

$$\leq \bigvee_{j \in \digamma_\circ} I_{\eta,\eta^*}(C_{\eta,\eta^*}(\lambda_j, r, s), r, s).$$

Hence, the proof is completed. □

Lemma 4.6. *Let (U, η, η^*) be a $dfts$, $r \in I_\circ$, and $s \in I_1$. If $\mu \in I^U$ is (r,s)-fuzzy soft nearly $\mathcal{GS}$-compact (resp., nearly compact), then μ is (r,s)-fuzzy soft almost $\mathcal{GS}$-compact (resp., almost compact).*

Proof. Follows from Definitions 4.3, 4.4, 4.5 and 4.6. □

Remark 4.3. *We can summarize the relationships among different types of fuzzy compactness as in the next diagram.*

$$\mathcal{GS}\text{-compactness} \quad \to \quad \text{compactness}$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \textit{nearly } \mathcal{GS}\textit{-compactness} & \rightarrow & \textit{nearly compactness} \\ \downarrow & & \downarrow \\ \textit{almost } \mathcal{GS}\textit{-compactness} & \rightarrow & \textit{almost compactness} \end{array}$$

5. Conclusion and Future Work

In this article, we have introduced a novel class of generalizations of fuzzy closed subsets called "$(r,s)-g^{\circledast}fsc$ sets" via double fuzzy topologies and some characterizations have been discussed. Moreover, we have defined novel types of fuzzy mappings and the relationship between these mappings have been introduced with the help of some problems. Also, we have shown that

$$\begin{array}{ccc} (r,s)-fsc & \Rightarrow & (r,s)-g^{\circledast}fsc \\ & \Downarrow & \Downarrow \\ & (r,s)-sgfc & (r,s)-g^{\ominus}fsc \\ & \Downarrow & \Downarrow \\ & & (r,s)-gfsc \end{array}$$

but in general, the converses of the above implications may not be true. Thereafter, "(r,s)-$\mathcal{GFS}$-regular" and "(r,s)-$\mathcal{GFS}$-normal" spaces have been defined as two new notions of higher fuzzy separation axioms and some characterizations of these separation axioms have been obtained. In the end, several novel types of fuzzy compactness in the frame of double fuzzy topologies have been introduced and some properties have been given. Also, the relationship between them have been explored.

In the upcoming papers, we shall discuss the concepts given here in the frames of a fuzzy idealization [42,43] and fuzzy soft r-minimal structures [44,45]. Moreover, we will study the main properties of classical compactness in the frame of double fuzzy topologies.

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