Concept Paper

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Finding a Research Paper Which Meaningfully Averages Unbounded Sets

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Keywords: Expected Values; Sets; Hausdorff measure; Hausdorff dimension; Discretization; Segmentation; Partitions; Samples; Euclidean Distance; Choice Function

Concept Paper

Finding a Published Research Paper Which Meaningfully Averages Unbounded Sets (v3)

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Abstract: Suppose $n \in \mathbb{N}$. We wish to meaningfully average 'sophisticated' unbounded sets (i.e., sets with positive n-d Hausdorff measure, in any n-d box of the n-d plane, where the measures don't equal the area of the boxes). We do this by taking the most generalized, *satisfying* extension of the expected value, w.r.t the Hausdorff measure in its dimension, on bounded sets which takes finite values only. As of now, I'm unable to solve this due to limited knowledge of advanced math and most people are too busy to help. Therefore, I'm wondering if anyone knows a research paper which solves my doubts. (Unlike previous versions with similar names, we add examples, motivations, and explanations to this version.)

Keywords: expected values; sets; Hausdorff measure; Hausdorff dimension; discretization; segmentation; partitions; samples; Euclidean distance; choice function

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1. Intro

Let $n \in \mathbb{N}$. Suppose $A \subseteq \mathbb{R}^n$ is Borel and $\mathcal{U}$ is the set of all unbounded A. Also, $\dim_{\mathrm{H}}(\cdot)$ is the Hausdorff dimension and $\mathcal{H}^{\dim_{\mathrm{H}}(\cdot)}(\cdot)$ is the Hausdorff measure *in its dimension* on the Borel σ-algebra.

1.1. First Special Case of A

Suppose the n-dimensional box is B. How do we define **explicit** A, such that:

1. $\mathcal{H}^n(A \cap B) > 0$ for all B
2. $\mathcal{H}^n(A \cap B) \neq \mathcal{H}^n(B)$ for all B?

This case of A is called A'.

1.1.1. Potential Answer

If $n = 1$, using this reddit post [1], define A such that:

Find a set $A \subseteq \mathbb{R}$, where the cardinality is $|\cdot|$ and $0 < |A \cap I| < |I|$ for every interval I, taking its cross product with $\mathbb{R}$ to get the set desired.

Construct a subset of $[0,1]$ with this property (then copy and paste the set to get the desired subset A' of $\mathbb{R}$):

> Note, we do this by constructing a strange map from $[0,1] \to \mathbb{R}$. Take a real number $x \in [0,1]$, expand that number in binary as $0.b_0 b_1 b_2 \cdots$ and map the value to the series $\sum_{n=1}^{\infty}(2(b_n) - 1)/n$. It's possible using Khintchine's inequality [2, p.187-205] to show the sum converges for a.e. $x \in [0,1]$. Thus, our desired set A' will just consist of those x for which the sum is positive.
> The fact this set works is a little bit annoying to prove, but relies on Khintchine's inequality [2, p.187-205] and the divergence of the Harmonic series. Essentially, we want to show that for any initial seqeuence $b_0, b_1, \cdots, b_n$ of digits there is a positive probability that the final sum is positive and a positive probability that the final sum is negative.

Note, in case there is a research paper which can average A' into a finite number, consider the second example.

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1.2. Second Special Case of A

Suppose the n-dimensional box is B. Is there an explicit A, such that:

1. $\mathcal{H}^n(A \cap B) > 0$ for all B
2. $\mathcal{H}^n(A \cap B) \neq \mathcal{H}^n(B)$ for all B
3. For all n-d boxes $\beta \subseteq B$:

   (a) $\mathcal{H}^n(B \setminus \beta) > \mathcal{H}^n(\beta) \Rightarrow \mathcal{H}^n(A \cap (B \setminus \beta)) < \mathcal{H}^n(A \cap \beta)$
   (b) $\mathcal{H}^n(B \setminus \beta) < \mathcal{H}^n(\beta) \Rightarrow \mathcal{H}^n(A \cap (B \setminus \beta)) > \mathcal{H}^n(A \cap \beta)$
   (c) $\mathcal{H}^n(B \setminus \beta) = \mathcal{H}^n(\beta) \Rightarrow \mathcal{H}^n(A \cap (B \setminus \beta)) \neq \mathcal{H}^n(A \cap \beta)$?

If so, how do we define it? If not, how do we modify §1.2 so the answer is “yes"?

If such a set exists, this case of A is called A''. (Note, we haven't found an example of such a set yet.)

1.3. Attempting to Analyze/Average A

The expected value of A, w.r.t the Hausdorff measure in its dimension, is:

$$\text{Avg}(A) = \frac{1}{\mathcal{H}^{\dim_{\text{H}}(A)}(A)} \int_A (x_1, \cdots, x_n)\, d\mathcal{H}^{\dim_{\text{H}}(A)} \tag{1}$$

Nevertheless, we have the following problems:

Note 1 (Problem 1). When $A = A'$ (§1.1), $\text{Avg}(A')$ is undefined.

1.3.1. Explanation of Problem 1

When $n = 1$, using §1.1 and §1.1.1, *since* $\mathcal{H}^1(A' \cap B) > 0$ for all B, we can see that $\dim_{\text{H}}(A') = 1$ and $\mathcal{H}^{\dim_{\text{H}}(A')}(A') = +\infty$. This makes $\text{Avg}(A')$ undefined due to division by infinity. (Note, the most obvious extension of $\text{Avg}(A')$ or $\mathcal{A}(A')$ is also undefined: i.e., the following is *false*

$$\exists(\mathcal{A}(A') \in \mathbb{R})\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(t \in \mathbb{N})\big(t \geq N \Rightarrow \left|\text{Avg}(A' \cap [-t, t]) - \mathcal{A}(A')\right| < \epsilon\big)$$

since $\mathcal{H}^1(A' \cap B) \neq \mathcal{H}^1(B)$ for all B.)

Note 2 (Problem 2). When $\mathcal{U}' \subseteq \mathcal{U}$ is the set of all unbounded A with a finite $\text{Avg}(A)$, where the cardinality is $|\cdot|$, then $|\mathcal{U}'| < |\mathcal{U}|$ (i.e., $\mathcal{U}'$ is “measure zero" in $\mathcal{U}$)

1.3.2. Explanation of Problem 2

Note, unbounded A has a finite mean when all lines of symmetry of A intersect at the same point. Thus, for any sequence of bounded sets $(F_r)_{r \in \mathbb{N}}$ whose set theoretic limit is A in the former sentence (§5.1), observe $\text{Avg}((F_r), A)$ *always* exists:

$$\forall(\epsilon > 0)\exists(N \in \mathbb{N})\forall(r \in \mathbb{N})\left(r \geq N \Rightarrow \left|\frac{1}{\mathcal{H}^{\dim_{\text{H}}(F_r)}(F_r)} \int_{F_r} (x_1, \cdots, x_n)\, d\mathcal{H}^{\dim_{\text{H}}(F_r)} - \text{Avg}((F_r), A)\right| < \epsilon\right) \tag{2}$$

and has the same value.

Since the cardinality of all unbounded sets with a line of symmetry is $\mathfrak{c}$; the cardinality of all unbounded sets where all lines of symmetry intersect at one point is less than or equal to $\mathfrak{c}$. Hence, since the cardinality of all subsets of $\mathbb{R}^n$ is $2^{\mathfrak{c}}$; the cardinality of all unbounded sets where there are no lines of symmetry, or the lines of symmetry intersect at multiple points is $2^{\mathfrak{c}}$. Hence, since $|\mathcal{U}| = 2^{\mathfrak{c}}$ and $|\mathcal{U}'| = c$, therefore $c = |\mathcal{U}'| < |\mathcal{U}| = 2^{\mathfrak{c}}$, proving problem **??** is correct.

Next, consider the approach to solving problems **??** and **??**.

1.3.3. **Approach**

We want an unique, *satisfying* extension of the expected value of A, w.r.t the Hausdorff measure *in its dimension*, on bounded sets to A, which takes finite values only, such that when $\mathcal{U}'' \subseteq \mathcal{U}$ is the set of all unbounded A with this extension:

1. $|\mathcal{U} \setminus \mathcal{U}''| < |\mathcal{U}''|$ (i.e., $\mathcal{U}''$ is "almost everywhere" in $\mathcal{U}$)
2. If (1) isn't true, then $|\mathcal{U}''| = |\mathcal{U} \setminus \mathcal{U}''|$

1.3.4. Explanation of Approach

Using §1.3.2 eq. 2 on several A, depending on the bounded sequence of sets (i.e., $(F_r)_{r\in\mathbb{N}}$) chosen, we have multiple $\text{Avg}((F_r), A)$. Here is an example:

Example 1. *Suppose* $A = \mathbb{R}^2$*. We define bounded sequence of sets* $(F_r^{\star})_{r\in\mathbb{N}} = ([-r,r] \times [-r,r])_{r\in\mathbb{N}}$ *and* $(F_j^{\star\star})_{j\in\mathbb{N}} = ([-j+1, j+1] \times [-j-1, j-1])_{j\in\mathbb{N}}$ *whose set theoretic limit is* $\mathbb{R}^2$ *(§5.1). Note, the center of* $F_r^{\star}$*, for all* $r \in \mathbb{N}$*, is* $(0,0)$ *and the center of* $F_j^{\star\star}$*, for all* $j \in \mathbb{N}$*, is* $(1,-1)$*. Thus,* $Avg((F_r^{\star}), A) = (0,0)$ *and* $Avg((F_j^{\star\star}), A) = (1,-1)$*. What's more, depending on the* $(F_r^{\star})$ *chosen,* $Avg((F_r^{\star}), A)$ *can be any point in* $\mathbb{R}^2$ *(when it exists).*

1.4. Question

Is there a unique way to define "satisfying" & "extension" in the approach of §1.3.3, which solves problems 1 and 2 with *applications*?

For an example, keep reading.

2. Extending the Expected Value of A w.r.t the Hausdorff Measure

The following are two methods to finding the most *generalized*, *satisfying* extension of $\text{Avg}(A)$ in eq. 1 which we later use to answer the question in §1.4:

1. One way is defining a generalized, satisfying extension of the Hausdorff measure, on all A with positive & finite measure which takes positive, finite values for all Borel A. This can theoretically be done in the paper "A Multi-Fractal Formalism for New General Fractal Measures"[3], where in eq. 1 we replace the Hausdorff measure with the extended Hausdorff measure.
2. Another way is finding *generalized*, *satisfying* average of all A in the fractal setting. This can be done with the papers "Analogues of the Lebesgue Density Theorem for Fractal Sets of Reals and Integers" [4] and "Ratio Geometry, Rigidity and the Scenery Process for Hyperbolic Cantor Sets" [5] where we take the expected value of A w.r.t the densities in [4,5].

Note, neither approach answers §1.4 since they can't solve problems 1 and 2 (i.e., $\mathcal{U}$ is the set of all unbounded A). Therefore, we ask a leading question using §2 to guide the reader to an interesting solution to §1.4.

3. Attempt to Define "Unique and Satisfying" in The Approach of §1.3

3.1. Note

Before reading, when §3.2 is unclear, see §5 for clarity. In §5, we define:

1. "Set theoretic limit" (§5.1)
2. "Expected value on sequences of bounded sets" (§5.2)
3. "Equivelant sequences of bounded sets" (§5.3, def. 1)
4. "Nonequivelant sequences of bounded sets" (§5.3, def. 2)
5. The "measure" on a sequence of bounded sets which increases at a rate *linear* or *superlinear* to that of "non-equivelant" sequences of bounded sets (§5.4.1, §5.4.2)
6. The "actual" rate of expansion on a sequence of bounded sets (§5.5)

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3.2. Leading Question

To define *unique* and *satisfying* inside the approach of §1.3, we take the expected value of a sequence of bounded sets chosen by a choice function. To find the choice function, we ask the **leading question**...

If we make sure to:

(A) See §3.1 and (C)-(E) when something is unclear
(B) Take all sequences of bounded sets whose “set theoretic limit" is A
(C) Define C to be chosen center point of $\mathbb{R}^n$
(D) Define E to be the chosen, fixed rate of expansion of a sequence of bounded sets
(E) Define $\mathcal{E}$ to be actual rate of expansion of a sequence of bounded sets (§5.5)

Does there exist a unique choice function which chooses the set of all equivalent sequences of bounded sets where:

1. The chosen, equivelant sequences of bounded sets should satisfy (B).
2. The “measure" of all the chosen, equivalent sequences of bounded sets which satisfy (1) should increase at a rate *linear* or *superlinear* to that of non-equivalent sequences of bounded sets satisfying (B).
3. The expected values, defined in the papers of §2, for all equivalent sequences of bounded sets are equivalent and finite
4. For the chosen, equivalent sequences of bounded sets satisfying (1), (2), and (3).
   - The n-d Euclidean distance between criteria (3) and C is the *less than or equal* to that of all the non-equivalent sequences of bounded sets satisfying (1), (2), and (3)
   - The “rate of divergence" [6, p.275-322] of $||\mathcal{E} - E||$, using the absolute value $||\cdot||$, *is less than or equal* to that of all the non-equivalent sequences of bounded sets which satisfy (1), (2), and (3)
5. When set $\mathcal{U}'' \subseteq \mathcal{U}$ is the set of all unbounded A, where the choice function chooses the set of all equivalent sequences of bounded sets satisfying (1), (2), (3) and (4), then:
   - $|\mathcal{U} \setminus \mathcal{U}''| < |\mathcal{U}|$
   - When $|\mathcal{U} \setminus \mathcal{U}''| \not< |\mathcal{U}|$, then $|\mathcal{U}''| = |\mathcal{U} \setminus \mathcal{U}''|$
6. Out of all choice functions which satisfy (1), (2), (3), (4), and (5) we choose the one with the simplest form, meaning for each choice function fully expanded, we take the one with the fewest variables/numbers?

(In case this is unclear, see §5.)

3.2.1. Explaining Motivation Behind §3.2

1. When defining “the measure" (§5.4.1,§5.4.2) of an unbounded set, we want a set with a “high" entropic density (i.e., we aren't sure if this is infact what the “measure" measures.) For example, when $A = \mathbb{R}^2$, the “measure" mostly chooses bounded sequences of symmetrical shapes whose lines of symmetry intersect at one point rather than non-symmetrical shapes §5.4.1-§6, §8.
2. Using ex. 1, when $A = \mathbb{R}^2$, *depending on the bounded sequence of sets* $(F_r)_{r\in\mathbb{N}}$ *chosen with* a set theoretic limit of A: $\text{Avg}((F_r^\star), A)$ can be any point in $\mathbb{R}^2$ (when it exists). To fix this, we take all $(F_r^\star)_{r\in\mathbb{N}}$, where $\text{Avg}((F_r^\star), A)$ has the smallest n-d Euclidean distance from a reference point (i.e., the center point $C \in \mathbb{R}^n$). The problem is there exists A, where the average of non-equivalent sequences (§5.3, def. 2) of bounded sets have the same minimum Euclidean distance from C.
3. Hence, we take the sequence of sets whose actual rate of expansion $\mathcal{E}$ from C (§5.5) “diverges" [6, p.275-322] at the smallest rate from the chosen, fixed rate of expansion E from C (i.e., the “rate of divergence of $||\mathcal{E} - E||$, using the absolute value $||\cdot||$, *is less than or equal* to that of all the non-equivalent sequences of bounded sets which satisfy §3.2 criteria (1), (2), and (3)).

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4. Finally, since there might still be non-equivalent sequences (§5.3, def. 2) of bounded sets which satisfy §3.2.1 criteria (1), (2) and (3), but are congruent with different $\operatorname{Avg}((F_r^{\star}), A)$, we use equation T in §6.3 eq. 115 to choose a unique set of all equivalent sequences of bounded sets with the same expected value.

I'm convinced the expected values of the sequences of bounded sets chosen by a choice function which answers the *leading question* aren't *unique* nor *satisfying enough* to answer problems 1 and 2. Still, adjustments are possible by changing the criteria or by adding new criteria to §3.2.

4. Question Regarding My Work

Most don't have time to address everything in my research, hence I ask the following:

Is there a research paper which already solves the ideas I'm working on? (Non-published papers, such as mine [7], don't count.)

Using AI, papers that might answer this question are "Prediction of dynamical systems from time-delayed measurements with self-intersections" [8] and "A Hausdorff measure boundary element method for acoustic scattering by fractal screens" [9].

Does either of these papers solve problems 1 and 2 with *applications*?

5. Clarifying §3

Let $A \subseteq \mathbb{R}^n$ be an unbounded, Borel set. Suppose $\dim_{\mathrm{H}}(\cdot)$ be the Hausdorff dimension and $\mathcal{H}^{\dim_{\mathrm{H}}(\cdot)}(\cdot)$ be the Hausdorff measure *in its dimension* on the Borel σ-algebra. See §3.2, once reading this section, and consider the following:

Is there a simpler version of the definitions below?

5.1. Set Theoretic Limit of a Sequence of Bounded Sets

Note the set theoretic limit of a sequence of sets of bounded set $(F_r)_{r\in\mathbb{N}}$ is A when:

$$\limsup_{r\to\infty} F_r = \bigcap_{r\geq 1}\bigcup_{q\geq r} F_q$$
$$\liminf_{r\to\infty} F_r = \bigcup_{r\geq 1}\bigcap_{q\geq r} F_q$$

where:

$$\limsup_{r\to\infty} F_r = \liminf_{r\to\infty} F_r = A$$

Example 2. *Using example 1, when $A = \mathbb{R}$, define bounded sequence of sets $(F_r^{\star})_{r\in\mathbb{N}} = ([-r,r])_{r\in\mathbb{N}}$ and $(F_j^{\star\star})_{j\in\mathbb{N}} = ([-j-1, j-1])_{j\in\mathbb{N}}$. Notice, the set theoretic limit of the bounded of sequence of sets using §5.1 is $\mathbb{R}$. I don't know how to prove this, but I assume readers should be able to verify whether I am correct.*

5.2. Expected Value of Bounded Sequences of Sets

If $(F_r)_{r\in\mathbb{N}}$ is a sequence of bounded sets whose set-theoretic limit is A (§5.1), the expected value of A w.r.t $(F_r)_{r\in\mathbb{N}}$ is $\operatorname{Avg}((F_r), A)$ (when it exists) where:

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(r\in\mathbb{N})\left(r\geq N \Rightarrow \left|\frac{1}{\mathcal{H}^{\dim_{\mathrm{H}}(F_r)}(F_r)}\int_{F_r}(x_1,\cdots,x_n)\,d\mathcal{H}^{\dim_{\mathrm{H}}(F_r)} - \operatorname{Avg}((F_r),A)\right| < \epsilon\right) \quad (3)$$

Note, $\operatorname{Avg}((F_r), A)$ can be extended by using §2.

5.2.1. Example 1

Using example 2, suppose $A=\mathbb{R}$, $(F_r^{\star})_{r\in\mathbb{N}}=([-r,r])_{r\in\mathbb{N}}$, and $\text{Avg}((F_r^{\star}),A)=0$. Note:

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(r\in\mathbb{N})\left(r\geq N\Rightarrow\left|\frac{1}{\mathcal{H}^{\dim_{\text{H}}(F_r^{\star})}(F_r^{\star})}\int_{F_r^{\star}}(x_1,\cdots,x_n)\,d\mathcal{H}^{\dim_{\text{H}}(F_r^{\star})}-\text{Avg}((F_r^{\star}),A)\right|<\epsilon\right)= \quad (4)$$

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(r\in\mathbb{N})\left(r\geq N\Rightarrow\left|\frac{1}{\mathcal{H}^{\dim_{\text{H}}([-r,r])}([-r,r])}\int_{[-r,r]}x_1\,d\mathcal{H}^{\dim_{\text{H}}([-r,r])}-0\right|<\epsilon\right)= \quad (5)$$

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(r\in\mathbb{N})\left(r\geq N\Rightarrow\left|\frac{1}{\mathcal{H}^{1}([-r,r])}\int_{[-r,r]}x_1\,d\mathcal{H}^{1}\right|<\epsilon\right)= \quad (6)$$

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(r\in\mathbb{N})\left(r\geq N\Rightarrow\left|\frac{1}{2r}\int_{-r}^{r}x_1\,dx_1\right|<\epsilon\right)= \quad (7)$$

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(r\in\mathbb{N})\left(r\geq N\Rightarrow\left|\frac{1}{2r}\left(\frac{1}{2}{x_1}^2\Big|_{-r}^{r}\right)\right|<\epsilon\right)= \quad (8)$$

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(r\in\mathbb{N})\left(r\geq N\Rightarrow\left|\frac{1}{2r}\left(\frac{1}{2}r^2-\frac{1}{2}(-r)^2\right)\right|<\epsilon\right)= \quad (9)$$

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(r\in\mathbb{N})\left(r\geq N\Rightarrow\left|\frac{1}{2r}\left(\frac{1}{2}r^2-\frac{1}{2}r^2\right)\right|<\epsilon\right)= \quad (10)$$

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(r\in\mathbb{N})\left(r\geq N\Rightarrow\left|\frac{1}{2r}(0)\right|<\epsilon\right)= \quad (11)$$

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(r\in\mathbb{N})(r\geq N\Rightarrow 0<\epsilon) \quad (12)$$

Since eq. 12 is true, $\text{Avg}((F_r^{\star}),A)=0$

5.2.2. Example 2

Using example 2, suppose $A=\mathbb{R}$, $(F_j^{\star\star})_{j\in\mathbb{N}}=([-j-1,j-1])_{j\in\mathbb{N}}$, and $\text{Avg}\left(\left(F_j^{\star\star}\right),A\right)=-1$. Note:

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(j\in\mathbb{N})\left(j\geq N\Rightarrow\left|\frac{1}{\mathcal{H}^{\dim_{\text{H}}(F_j^{\star\star})}(F_j^{\star\star})}\int_{F_j^{\star\star}}(x_1,\cdots,x_n)\,d\mathcal{H}^{\dim_{\text{H}}(F_j^{\star\star})}-\text{Avg}((F_j^{\star\star}),A)\right|<\epsilon\right)= \quad (13)$$

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(j\in\mathbb{N})\left(j\geq N\Rightarrow\left|\frac{1}{\mathcal{H}^{\dim_{\text{H}}([-j-1,j-1])}([-j-1,j-1])}\int_{[-j-1,j-1]}x_1\,d\mathcal{H}^{\dim_{\text{H}}([-j-1,j-1])}-(-1)\right|<\epsilon\right)= \quad (14)$$

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(j\in\mathbb{N})\left(j\geq N\Rightarrow\left|\frac{1}{\mathcal{H}^{1}([-j-1,j-1])}\int_{[-j-1,j-1]}x_1\,d\mathcal{H}^{1}+1\right|<\epsilon\right)= \quad (15)$$

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(j\in\mathbb{N})\left(j\geq N\Rightarrow\left|\frac{1}{(j-1)-(-j-1)}\int_{-j-1}^{j-1}x_1\,dx_1+1\right|<\epsilon\right)= \quad (16)$$

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(j\in\mathbb{N})\left(j\geq N\Rightarrow\left|\frac{1}{j+j-1+1}\left(\frac{1}{2}{x_1}^2\Big|_{-j-1}^{j-1}\right)+1\right|<\epsilon\right)= \quad (17)$$

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(j\in\mathbb{N})\left(j\geq N\Rightarrow\left|\frac{1}{2j}\left(\frac{1}{2}(j-1)^2-\frac{1}{2}(-j-1)^2\right)+1\right|<\epsilon\right)= \quad (18)$$

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(j\in\mathbb{N})\left(j\geq N\Rightarrow\left|\frac{1}{2j}\left(\frac{1}{2}(j-1)^2-\frac{1}{2}(j+1)^2\right)+1\right|<\epsilon\right)= \quad (19)$$

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(j\in\mathbb{N})\left(j\geq N\Rightarrow\left|\frac{1}{2j}\left(\frac{1}{2}(j^2-2j+1)-\frac{1}{2}(j^2+2j+1)\right)+1\right|<\epsilon\right)= \quad (20)$$

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(j\in\mathbb{N})\left(j\geq N\Rightarrow\left|\frac{1}{2j}\left(\frac{1}{2}j^2-j+\frac{1}{2}-\frac{1}{2}j^2-j-\frac{1}{2}\right)+1\right|<\epsilon\right)= \quad (21)$$

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(j\in\mathbb{N})\left(j\geq N\Rightarrow\left|\frac{1}{2j}(-2j)+1\right|<\epsilon\right)= \quad (22)$$

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(j\in\mathbb{N})(j\geq N\Rightarrow|-1+1|<\epsilon)= \quad (23)$$

$$\forall(\epsilon>0)\exists(N\in\mathbb{N})\forall(j\in\mathbb{N})(j\geq N\Rightarrow 0<\epsilon)= \quad (24)$$

Since eq. 24 is true, $\text{Avg}((F_j^{\star\star}),A)=-1$.

5.3. Defining Equivelant and Non-Equivelant Sequences of Bounded Functions

The sequences below are sequences of sets with a set theoretic limit (§5.1) of A:

$$\overrightarrow{F_1}=(F_{r_1}^{(1)})_{r_1\in\mathbb{N}},\ \overrightarrow{F_2}=(F_{r_2}^{(2)})_{r_2\in\mathbb{N}},\cdots,\ \overrightarrow{F_s}=(F_{r_s}^{(s)})_{r_s\in\mathbb{N}}$$

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Thus, we define:

Definition 1 (Equivelant Sequences of Bounded Sets). *Suppose, $S \subseteq \mathbb{N}$ is an arbitrary set. Note, the sequences of bounded sets in:*

$$\left\{ \overrightarrow{F_s} : s \in S \right\}$$

are equivalent, if for all $k, v \in S$, where $k \neq v$, $\overrightarrow{F_k}$ and $\overrightarrow{F_v}$ are equivelant: i.e., there exists a $N' \in \mathbb{N}$, such for all $r_k \geq N'$, there is a $r_v \in \mathbb{N}$, where:

$$\mathcal{H}^{dim_H(F^{(k)}_{r_k})}(F^{(k)}_{r_k} \Delta F^{(v)}_{r_v}) = 0$$

and for all $r_v \geq N'$, there is a $r_k \in \mathbb{N}$, where:

$$\mathcal{H}^{dim_H(F^{(v)}_{r_v})}(F^{(k)}_{r_k} \Delta F^{(v)}_{r_v}) = 0$$

More, for each $s \in \mathbb{N}$, we denote all equivalent sequences of bounded functions to $(F^{(s)}_{r_s})_{r_s \in \mathbb{N}}$ using the notation

$$\sim (F^{(s)}_{r_s})_{r_s \in \mathbb{N}}$$

5.3.1. Explanation

We define $\overrightarrow{F_k}$ and $\overrightarrow{F_v}$ as equivalent, then for all sets $A \subseteq \mathbb{R}^n$, when either $\text{Avg}((F^{(k)}_{r_k}), A)$ or $\text{Avg}((F^{(v)}_{r_v}), A)$ exist:

$$\text{Avg}((F^{(k)}_{r_k}), A) = \text{Avg}((F^{(v)}_{r_v}), A) \quad (\S 5.2)$$

Hence, consider the following:

Theorem 1. *If the sequence of sets in:*

$$\left\{ \overrightarrow{F_s} : s \in S \right\}$$

are equivalent, then for all $k, v \in S$, where $k \neq v$:

$$Avg((F^{(k)}_{r_k}), A) = Avg((F^{(v)}_{r_v}), A) \quad (\S 5.2)$$

Below, is an example of an equivalent sequence of bounded sets:

5.3.2. Example of Equivalent Sequences of Bounded Sets

Suppose

$$\overrightarrow{F_1} = (F^{(1)}_{r_1})_{r_1 \in \mathbb{N}} = ([-r_1 - 2, r_1 + 2])_{r_1 \in \mathbb{N}}$$

$$\overrightarrow{F_2} = (F^{(2)}_{r_2})_{r_2 \in \mathbb{N}} = ([-r_2, r_2] \cup (\mathbb{Q} \cap [-r_2 - 1, r_2 + 1]))_{r_2 \in \mathbb{N}}$$

Note, using def. 1, $S = \{1, 2\}$, $k = 1$, and $v = 2$. In other words, $r_k = r_1$ and $r_v = r_2$. Now, suppose $N = 3$, where $r_1 \in \mathbb{N}$ and $r_2 \in \mathbb{N}$. Then,

1. For all $r_1 \geq N = 3$, there exists a $5 \leq r_2 =: r_1 + 2 \in \mathbb{N}$, where:

$$\mathcal{H}^{\text{dim}_\text{H}(F^{(1)}_{r_1})}(F^{(1)}_{r_1} \Delta F^{(2)}_{r_2}) = 0 \tag{25}$$

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We show this with the following:
In eq. 25, since $F_{r_1}^{(1)} = [-r_1 - 2, r_1 + 2]$ is a 1-d interval, $\dim_{\mathrm{H}}(F_{r_1}^{(1)}) = 1$. Hence,

$$\mathcal{H}^1([-r_1 - 2, r_1 + 2]\Delta([-r_2, r_2] \cup (\mathbb{Q} \cap [-r_2 - 1, r_2 + 1]))) = \tag{26}$$
$$\mathcal{H}^1([-r_1 - 2, r_1 + 2]\Delta([-(r_1 + 2), (r_1 + 2)] \cup (\mathbb{Q} \cap [-(r_1 + 2) - 1, (r_1 + 2) + 1])) = \tag{27}$$
$$\mathcal{H}^1([-r_1 - 2, r_1 + 2]\Delta([-(r_1 + 2), (r_1 + 2)] \cup (\mathbb{Q} \cap [-r_1 - 3, r_1 + 3])) = \tag{28}$$
$$\mathcal{H}^1(\mathbb{Q} \cap ([r_1 + 2, r_1 + 3]) \cup (\mathbb{Q} \cap [-r_1 - 3, r_1 - 2])) = \tag{29}$$
$$0 \tag{30}$$

We also show:

2. For all $r_2 \geq N = 3$, there exists a $1 \leq r_2 - 2 =: r_1 \in \mathbb{N}$, where:

$$\mathcal{H}^{\dim_{\mathrm{H}}(F_{r_2}^{(2)})}(F_{r_1}^{(1)} \Delta F_{r_2}^{(2)}) = 0 \tag{31}$$

We show this with the following:
In eq. 31, since $\dim_{\mathrm{H}}(F_{r_2}^{(2)}) = \dim_{\mathrm{H}}([-r_2, r_2] \cup (\mathbb{Q} \cap [-r_2 - 1, r_2 + 1])) = 1$:

$$\mathcal{H}^1([-r_1 - 2, r_1 + 2]\Delta([-r_2, r_2] \cup (\mathbb{Q} \cap [-r_2 - 1, r_2 + 1]))) = \tag{32}$$
$$\mathcal{H}^1([-(r_2 - 2) - 2, (r_2 - 2) + 2]\Delta([-r_2, r_2] \cup (\mathbb{Q} \cap [-r_2 - 1, r_2 + 1])) = \tag{33}$$
$$\mathcal{H}^1([-r_2, r_2]\Delta([-r_2, r_2] \cup (\mathbb{Q} \cap [-r_2 - 1, r_2 + 1])) = \tag{34}$$
$$\mathcal{H}^1((\mathbb{Q} \cap [-r_2 - 1, -r_2]) \cup (\mathbb{Q} \cap [r_2, r_2 + 1])) = \tag{35}$$
$$0 \tag{36}$$

Since crit. (1) and (2) is true, using def. 1, we have shown $\overrightarrow{F_1} = (F_{r_1}^{(1)})_{r_1 \in \mathbb{N}} = ([-r_1 - 2, r_1 + 2])_{r_1 \in \mathbb{N}}$ and $\overrightarrow{F_2} = (F_{r_2}^{(2)})_{r_2 \in \mathbb{N}} = ([-r_2, r_2] \cup (\mathbb{Q} \cap [-r_2 - 1, r_2 + 1]))_{r_2 \in \mathbb{N}}$ are equivalent.

Note, there exists zero sets $A \subseteq \mathbb{R}^n$, where $\mathrm{Avg}((F_{r_1}^{(1)}), A)$ and $\mathrm{Avg}((F_{r_2}^{(2)}), A)$ have different values.

Definition 2 (Non-Equivalent Sequences of Bounded Sets). *Again, suppose $S \subseteq \mathbb{N}$ is an arbitrary set. Then, the sequences of bounded sets in:*

$$\left\{ \overrightarrow{F_s} : s \in S \right\}$$

are non-equivalent, if def. 1 is false, meaning for some $k, v \in S$, where $k \neq v$, $\overrightarrow{F_k}$ and $\overrightarrow{F_v}$ are non-equivelant: there is a $N' \in \mathbb{N}$, where for all $r_k \geq N'$, there is either a $r_v \in \mathbb{N}$, where:

$$\mathcal{H}^{dim_H(F_{r_k}^{(k)})}(F_{r_k}^{(k)} \Delta F_{r_v}^{(v)}) \neq 0$$

or for all $r_v \geq N'$, there is a $r_k \in \mathbb{N}$, where

$$\mathcal{H}^{dim_H(F_{r_v}^{(v)})}(F_{r_k}^{(k)} \Delta F_{r_v}^{(v)}) \neq 0$$

5.3.3. Explanation

We define $\overrightarrow{F_k}$ and $\overrightarrow{F_v}$ as non-equivalent, then there exists a set $A \subseteq \mathbb{R}^n$, such that when $\mathrm{Avg}((F_{r_k}^{(k)}), A)$ or $\mathrm{Avg}((F_{r_v}^{(v)}), A)$ exist:

$$\mathrm{Avg}((F_{r_k}^{(k)}), A) \neq \mathrm{Avg}((F_{r_v}^{(v)}), A) \quad (\S 5.2)$$

Hence, consider the following:

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5.3.4. Example of Non-Equivalent Sequences of Bounded Sets

Suppose

$$\overrightarrow{F_1} = (F_{r_1}^{(1)})_{r_1 \in \mathbb{N}} = ([-r_1, r_1])_{r_1 \in \mathbb{N}}$$

$$\overrightarrow{F_2} = (F_{r_2}^{(2)})_{r_2 \in \mathbb{N}} = ([-2r_2, 2r_2])_{r_2 \in \mathbb{N}}$$

Note, using def. 1, $S = \{1, 2\}$, $k = 1$, and $v = 2$. In other words, $r_k = r_1$ and $r_v = r_2$. Now, suppose $N := 1$, where $r_1 \in \mathbb{N}$ and $r_2 \in \mathbb{N}$. Then,

1. For all $r_2 \geq N = 1$, there exists a $3 \leq 2r_2 + 1 =: r_2 \in \mathbb{N}$, where:

$$\mathcal{H}^{\dim_{\mathrm{H}}(F_{r_2}^{(2)})}(F_{r_1}^{(1)} \Delta F_{r_2}^{(2)}) \neq 0$$

   We show this with the following:

   Since $F_{r_2}^{(2)} = [-2r_2, 2r_2]$ is a 1-d interval, $\dim_{\mathrm{H}}(F_{r_1}^{(1)}) = 1$. Hence:

$$\mathcal{H}^1([-r_1, r_1] \Delta [-2r_2, 2r_2]) = \quad (37)$$

$$\mathcal{H}^1([-(2r_2+1), 2r_2+1] \Delta [-2r_2, 2r_2]) = \quad (38)$$

$$\mathcal{H}^1([-2r_2-1, 2r_2+1] \Delta [-2r_2, 2r_2]) = \quad (39)$$

$$\mathcal{H}^1([-2r_2-1, 2r_2+1] \Delta [-2r_2, 2r_2]) = \quad (40)$$

$$\mathcal{H}^1([-2r_2-1, -2r_2] \cup [2r_2, 2r_2+1]) = \quad (41)$$

$$1 + 1 \neq 0 \quad (42)$$

5.3.5. Question

When:

$$\overrightarrow{F_1} = (F_{r_1}^{(1)})_{r_1 \in \mathbb{N}} = ([-r_1, r_1])_{r_1 \in \mathbb{N}}$$

$$\overrightarrow{F_2} = (F_{r_2}^{(2)})_{r_2 \in \mathbb{N}} = ([-2r_2, 2r_2])_{r_2 \in \mathbb{N}}$$

How do we find A, where $\mathrm{Avg}((F_{r_1}^{(1)}), A) \neq \mathrm{Avg}((F_{r_2}^{(2)}), A)$ (§5.2)?

I remember finding such a set in one of my previous papers, but I'm now unable to find it.

5.4. Defining the "Measure"

5.4.1. Preliminaries

We define the "**measure**" of $(F_r)_{r \in \mathbb{N}}$ in §5.4.2. To understand this "measure", keep reading. (In case the steps are unclear, see §8 for examples.)

1. For every $r \in \mathbb{N}$, "over-cover" F_r with minimal, pairwise disjoint sets of equal $\mathcal{H}^{\dim_{\mathrm{H}}(F_r)}$ measure. (We denote the equal measures ε, where the former sentence is defined $\mathbf{C}(\varepsilon, F_r, \omega)$: i.e., $\omega \in \Omega_{\varepsilon,r}$ enumerates all collections of these sets covering F_r. In case this step is unclear, see §8.1.)
2. For every ε, r and ω, take a sample point from each set in $\mathbf{C}(\varepsilon, F_r, \omega)$. The set of these points is "the sample" which we define $\mathcal{S}(\mathbf{C}(\varepsilon, F_r, \omega), \psi)$: i.e., $\psi \in \Psi_{\varepsilon,r,\omega}$ enumerates all possible samples of $\mathbf{C}(\varepsilon, F_r, \omega)$. (In the case this is unclear, see §8.2.)
3. For every ε, r, ω and ψ,
   (a) Take a "pathway" of line segments: we start with a line segment from arbitrary point x_0 of $\mathcal{S}(\mathbf{C}(\varepsilon, F_r, \omega), \psi)$ to the sample point with smallest n-dimensional Euclidean distance to x_0 (i.e., when more than one sample point has smallest n-dimensional Euclidean distance to x_0, take either of those points). Next, repeat this process until the "pathway" intersects with every sample point once. (If this is unclear, see §8.3.1.)

(b) Take the set of the length of all segments in (3a), except for lengths that are outliers [10] (i.e., for any constant $C > 0$, the outliers are more than C times the interquartile range of the length of all line segments as $r \to \infty$). Define this $\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\varepsilon, F_r, \omega), \psi))$. (In the case this is unclear, see §8.3.2.)
(c) Multiply remaining lengths in the pathway by a constant so they add up to one (i.e., a probability distribution). This will be denoted $\mathbb{P}(\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\varepsilon, F_r, \omega), \psi)))$. (In case this step is unclear, see §8.3.3.)
(d) Take the shannon entropy [11][p.61-95] of step (3c). We define this:

$$\mathrm{E}(\mathbb{P}(\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\varepsilon, F_r, \omega), \psi)))) = \sum_{x \in \mathbb{P}(\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\varepsilon, F_r, \omega), \psi)))} -x \log_2 x$$

which will be *shortened* to $\mathrm{E}(\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\varepsilon, F_r, \omega), \psi)))$. (In case this is unclear, see §8.3.4)
(e) Maximize the entropy w.r.t all "pathways". This we will denote:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_r, \omega), \psi))) = \sup_{x_0 \in \mathcal{S}(\mathbf{C}(\varepsilon, F_r, \omega), \psi)} \mathrm{E}(\mathcal{L}(x_0, \mathcal{S}(\mathbf{C}(\varepsilon, F_r, \omega), \psi)))$$

(In case this step is unclear, see §8.3.5.)

4. Therefore, the **maximum entropy** of $(F_r)_{r \in \mathbb{N}}$ w.r.t ε, using (1) and (2) is:

$$\mathrm{E}_{\max}(\varepsilon, r) = \sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_r, \omega), \psi)))$$

5.4.2. What Am I Measuring?

Suppose we define two sequences of bounded functions which have a set theoretic limit of A: e.g., $(F_r^\star)_{r \in \mathbb{N}}$ and $(F_j^{\star\star})_{j \in \mathbb{N}}$, where for **constant** ε and *cardinality* $|\cdot|$

(a) Using (2) and (33e) of section 5.4.1, suppose:

$$\underline{|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^\star, \omega), \psi)|} =$$
$$\sup\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{\star\star}, \omega'), \psi')\right| : j \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,j},\ \psi' \in \Psi_{\varepsilon,j,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{\star\star}, \omega'), \psi'))) \le \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_r^\star, \omega), \psi)))\right\}$$

then (using $\underline{|\mathcal{S}(\mathbf{C}(\epsilon, F_r^\star, \omega), \psi)|}$) we get

$$\underline{\alpha}(\varepsilon, r, \omega, \psi) = \underline{|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^\star, \omega), \psi)|} / |\mathcal{S}(\mathbf{C}(\varepsilon, F_r^\star, \omega), \psi))|$$

(b) Also, using (2) and (33e) of section 5.4.1, suppose:

$$\overline{|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^\star, \omega), \psi)|} =$$
$$\inf\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{\star\star}, \omega'), \psi')\right| : j \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,j},\ \psi' \in \Psi_{\varepsilon,j,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{\star\star}, \omega'), \psi'))) \ge \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_r^\star, \omega), \psi)))\right\}$$

then (using $\overline{|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^\star, \omega), \psi)|}$) we also get:

$$\overline{\alpha}(\varepsilon, r, \omega, \psi) = \overline{|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^\star, \omega), \psi)|} / |\mathcal{S}(\mathbf{C}(\varepsilon, F_r^\star, \omega), \psi))|$$

1. If using $\overline{\alpha}(\epsilon, r, \omega, \psi)$ and $\underline{\alpha}(\epsilon, r, \omega, \psi)$ we have:

$$1 < \limsup_{\varepsilon \to 0} \limsup_{r \to \infty} \sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} \overline{\alpha}(\varepsilon, r, \omega, \psi),\ \liminf_{\varepsilon \to 0} \liminf_{r \to \infty} \inf_{\omega \in \Omega_{\varepsilon,r}} \inf_{\psi \in \Psi_{\varepsilon,r,\omega}} \underline{\alpha}(\varepsilon, r, \omega, \psi) < +\infty$$

then *what I'm measuring from* $(F_r^\star)_{r \in \mathbb{N}}$ increases at a rate **superlinear** to that of $(F_j^{\star\star})_{j \in \mathbb{N}}$.

2. If using equations $\overline{\alpha}(\varepsilon, j, \omega, \psi)$ and $\underline{\alpha}(\varepsilon, j, \omega, \psi)$ (where we swap $(F_r^\star)_{r \in \mathbb{N}}$ and $r \in \mathbb{N}$, in $\overline{\alpha}(\epsilon, r, \omega, \psi)$ and $\underline{\alpha}(\epsilon, r, \omega, \psi)$, with $(F_j^{\star\star})_{j \in \mathbb{N}}$ and $j \in \mathbb{N}$) we get:

$$1 < \limsup_{\varepsilon \to 0} \limsup_{j \to \infty} \sup_{\omega \in \Omega_{\varepsilon,j}} \sup_{\psi \in \Psi_{\varepsilon,j,\omega}} \overline{\alpha}(\varepsilon, j, \omega, \psi),\ \liminf_{\varepsilon \to 0} \liminf_{j \to \infty} \inf_{\omega \in \Omega_{\varepsilon,j}} \inf_{\psi \in \Psi_{\varepsilon,j,\omega}} \underline{\alpha}(\varepsilon, j, \omega, \psi) < +\infty$$

then *what I'm measuring from* $(F_r^\star)_{r \in \mathbb{N}}$ increases at a rate **sublinear** to that of $(F_j^{\star\star})_{j \in \mathbb{N}}$.

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3. If using equations $\overline{\alpha}(\varepsilon,r,\omega,\psi)$, $\underline{\alpha}(\varepsilon,r,\omega,\psi)$, $\overline{\alpha}(\varepsilon,j,\omega,\psi)$, and $\underline{\alpha}(\varepsilon,j,\omega,\psi)$, we **both** have:
   (a) $\limsup_{\varepsilon\to 0}\limsup_{r\to\infty}\sup_{\omega\in\Omega_{\varepsilon,r}}\sup_{\psi\in\Psi_{\varepsilon,r,\omega}}\overline{\alpha}(\varepsilon,r,\omega,\psi)$ or $\liminf_{\varepsilon\to 0}\liminf_{r\to\infty}\inf_{\omega\in\Omega_{\varepsilon,r}}\inf_{\psi\in\Psi_{\varepsilon,r,\omega}}\underline{\alpha}(\varepsilon,r,\omega,\psi)$ are equal to zero, one or $+\infty$
   (b) $\limsup_{\varepsilon\to 0}\limsup_{j\to\infty}\sup_{\omega\in\Omega_{\varepsilon,j}}\sup_{\psi\in\Psi_{\varepsilon,j,\omega}}\overline{\alpha}(\varepsilon,j,\omega,\psi)$ or $\liminf_{\varepsilon\to 0}\liminf_{j\to\infty}\inf_{\omega\in\Omega_{\varepsilon,j}}\inf_{\psi\in\Psi_{\varepsilon,j,\omega}}\underline{\alpha}(\varepsilon,j,\omega,\psi)$ are equal to zero, one or $+\infty$

   then *what I'm measuring from* $(F_r^\star)_{r\in\mathbb{N}}$ increases at a rate **linear** to that of $(F_j^{\star\star})_{j\in\mathbb{N}}$.

Now consider the following examples:

5.4.3. Example of the "Measure" of $(F_r^\star)$ Converging Super-Linearly to That of $(F_j^{\star\star})$

Using §5.4.2 consider the following:

Note 4. Recall, if $\dim_H(F_r^\star)=2$:

1. When $|\cdot|$ is the cardinality, for all $\omega\in\Omega_{\varepsilon,r}$ and $\psi\in\Psi_{\varepsilon,r,\omega}$, $|\mathcal{S}(\mathbf{C}(\varepsilon,F_r^\star,\omega),\psi)|=\lceil\text{Area}(F_r^\star)/\varepsilon\rceil$
2. For all $\omega\in\Omega_{\varepsilon,r}$ and $\psi\in\Psi_{\varepsilon,r,\omega}$, the largest $\text{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,F_r^\star,\omega),\psi)))$ can be is $\log_2(\lceil\text{Area}(F_r^\star)/\varepsilon\rceil-1)=\log_2(|\mathcal{S}(\mathbf{C}(\varepsilon,F_r^\star,\omega),\psi)|-1)$

Now consider:

1. $(F_r^\star)_{r\in\mathbb{N}}=\left(\{x^2+y^2=6r^2\}\right)_{r\in\mathbb{N}}$
2. $(F_j^{\star\star})_{j\in\mathbb{N}}=\left(\{x^2/(9j^2)+y^2/(4j^2)=1\}\right)_{j\in\mathbb{N}}$

where:

Note 5. The area of $F_r^\star$ and $F_j^{\star\star}$ are:

1. $\text{Area}(F_r^\star)=\pi(\sqrt{6}r)^2=6\pi r^2$
2. $\text{Area}(F_j^{\star\star})=3j\cdot 2j\cdot\pi=6\pi j^2$

Hence, we maximize $\text{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,F_r^\star,\omega),\psi)))$ by doing the following:

Note 6 (Procedure to Maximize $\mathbf{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,F_r^\star,\omega),\psi)))$) Consider the procedure below:

1. Cover the circle, with the same or larger-sized circle, which can be divided into minimum t "pie-slices" of equal area $\varepsilon>0$. Notice, $t=\lceil\text{Area}(F_r^\star)/\varepsilon\rceil=\lceil(6\pi r^2)/\varepsilon\rceil$.
2. Take the centroid of each slice
3. Out of all centroids in step 2, take the centroid with the largest x-coordinate: i.e., denote this point x_0 which is the start-point of the pathway of line segments in the resulting step
4. Take the distances between all pairs of consecutive centroids, starting with x_0, rotating counter-clockwise or clockwise. Either-way, the end result should change by only a negligible amount.
5. Multiply the distances by a constant so they add up to 1 (i.e., a probability distribution)
6. Take the shannon entropy of the distribution using log base 2 in §5.4.1(33*d*)-(33*e*). (Note, since the "pie-slices" of step 1 are congruent and the distances of step 4 are equal, the entropy of the distribution is the largest possible amount (i.e., note 4*crit*.2&*note*5 crit. 1):

$$\log_2(\lceil\text{Area}(F_r^\star)/\varepsilon\rceil-1)=\log_2(\lceil(6\pi r^2)/\varepsilon\rceil-1)$$

Repeat, steps 1-6 with $(F_j^{\star\star})$, where the circle is an ellipse (i.e., with this case the "pie-slices" of step 1 are non-congruent and the distances of step 4 are non-equivelant). Hence, we use programming:

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Listing 1: Note 6 Steps (1)-(6) on $(F_j^{\star\star})_{j\in\mathbb{N}}$

```
Clear["Global`*"]

(*The table below is note 6, steps 1-6 on F_j** *)
EntropyFj = Table[
  {eps = 3; (* ε > 0 *)
   n = Ceiling[6 (j^2) Pi/eps]; (* total number of "slices" with equal ε > 0 area*)

    (* 'p' is the scaling transform for the same or larger sized ellipse covering (F_j**)
which can be subdivided into minimum t pieces of equal area ε > 0, where t = ⌈(6πr^2)/ε⌉ *)
   p = Sqrt[(eps/(6 (j^2) Pi))*Ceiling[(6 (j^2) Pi)/eps]];

    (*'slices' is step 1 on F_j** *)
    slices =
     BoundaryDiscretizeRegion /@ (TransformedRegion[#,
          ScalingTransform[{3 j p, 2 j p}]] & /@ (Disk[{0, 0},
            1, #] & /@ Partition[Subdivide[0, 2 \[Pi], n], 2, 1]));

     (*'s' is step 2 on F_j**: take the centroid of each slice.*)
    s = RegionCentroid /@ slices;

    (*Below are all pathways for any x_0 in s*)
    ClearAll[pathway],
    pathway[path_, points_] := Block[{$RecursionLimit = 2000},
      Module[{next}, If[points == {}, Return[path]];
       next = Nearest[points, Last@path,
         DistanceFunction -> EuclideanDistance];
       Level[Map[
         pathway[Append[path, #],
           Delete[points, FirstPosition[points, #]]] &, next], {-3}]]],
    paths =
     Flatten[pathway[{s[[#]]}, Delete[s, #]] & /@ Range@Length@s, 1];
    steps = Partition[#, 2, 1] & /@ paths;
    dists = Apply[EuclideanDistance, steps, {2}];
    ClearAll[outlierLimit],
    outlierLimit[distances_] :=
     Module[{q = Quartiles[distances], iqr},
      iqr = First@Differences@q[[{1, 3}]]; q[[2]] + (3/2) iqr];

    (*'outlimits' remove any outliers as mentioned in §5.4.1 (33b)*)
    outLimits = outlierLimit /@ dists;
    nonoutliers =
     Select[#1, LessThan[#2]] & @@@ Thread[{dists, outLimits}];

    (*'normalize' is step 3, 4 & 5 on F_j**: Take a pathway of line segments starting from the centroid
    with the largest x-coordinate. Then, take the distances between pairs of consecutive centroids
    clockwise or counter-clockwise and multiply the distances by a constant so they add up to one. *)

    normalize = nonoutliers[[1]]/Total[nonoutliers[[1]]];

    (*'entropy' is step 6 on F_j** *)
    entropy = -Total[normalize Log[2, normalize]]}, {j, 1, 9}] (*The table for 'EntropyFj' ends here.*)

(*Table of all entropy values*)
Table[EntropyFj[[j, 4]], {j, 1, 10}]
```

where the output of the code is:

Listing 2: Ouput of Code 1

```
{2.56866, 4.63026, 5.79356, 6.6299, 7.28059, 7.8061, 8.24799, 8.63693, 8.97455, 9.28048}
```

Further note, adding the following code:

Listing 3: Extra Code for Code 1

```
(*Below is the entropy of circle (F_r*)_{r∈N} *)
Entropyofcircle[r_] :=
 Entropyofcircle[r] = Log[2, Ceiling[6 (r^2) Pi/eps] - 1]

(*'ratio' compares the rate of increase of the entropy of (F_j**)_{j∈N} with the entropy
of (F_r*)_{r∈N} *)
ratio[j_] :=
```

```
 ratio[j] =
  N[(Entropyofcircle[j] - EntropyFj[[j, 4]])/(Entropyofcircle[j] -
      Entropyofcircle[j - 1])]

(*Table of values of Function ratio*)
ratiotable=Table[ratio[j], {j, 2, 10}]

ratiodifferences = Differences[ratiotable]

(*Here is the evidence the difference of consecutive 'ratio' values is
almost constant. 'Evidence1' should be almost .005 and 'Evidence 2' should approach zero.*)
Evidence1 = Mean[ratiodifferences]
Evidence2 = N[Total[ratiodifferences - Mean[ratiodifferences]]/Mean[ratiodifferences]]
```

we get the output is:

Listing 4: Output of Code 3

```
(*Output of 'ratiotable' in code 3*)
{0.00660547, 0.0118564, 0.0166815, 0.021556, 0.0267818, 0.0319121, 0.0363022, 0.0418554, 0.0462194}

(*Output of 'ratiodifferences' in code 3. Notice, the outputs are around .005*)
{0.00525097, 0.00482508, 0.00487451, 0.00522582, 0.00513024, 0.00439012, 0.00555315, 0.00436408}

(*Output of 'Evidence1' in code 3. Note, this is close to .005*)
0.00495175

(*Output of 'Evidence2'. Note, this is extremely close to zero.*)
1.05098*10^-15
```

Note, the output of `ratiotable` in code 5 can be written as:

$$\approx .005(j-2)+.006 \tag{43}$$
$$\approx .005j-.01+.006 \tag{44}$$
$$\approx .005j-.004 \tag{45}$$

and is the same as:

$$\frac{\log_2\left(\left\lceil\frac{6j^2\pi}{\varepsilon}\right\rceil-1\right)-\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,F_j^{\star\star},\omega),\psi)))}{\log_2\left(\left\lceil\frac{6j^2\pi}{\varepsilon}\right\rceil-1\right)-\log_2\left(\left\lceil\frac{6(j-1)^2\pi}{\varepsilon}\right\rceil-1\right)} \tag{46}$$

Hence, with:

$$\frac{\log_2\left(\left\lceil\frac{6j^2\pi}{\varepsilon}\right\rceil-1\right)-\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,F_j^{\star\star},\omega),\psi)))}{\log_2\left(\left\lceil\frac{6j^2\pi}{\varepsilon}\right\rceil-1\right)-\log_2\left(\left\lceil\frac{6(j-1)^2\pi}{\varepsilon}\right\rceil-1\right)}\sim .005j-.004$$

we solve for $\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,F_j^{\star\star},\omega),\psi)))$:

$$\frac{\log_2(\lceil 6j^2\pi/\varepsilon\rceil-1)-\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,F_j^{\star\star},\omega),\psi)))}{\log_2\left(\frac{\lceil 6j^2\pi/\varepsilon\rceil-1}{\lceil 6(j-1)^2\pi/\varepsilon\rceil-1}\right)}\sim .005j-.004 \tag{47}$$

$$\log_2\left(\left\lceil 6j^2\pi/\varepsilon\right\rceil-1\right)-\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,F_j^{\star\star},\omega),\psi)))\sim(.005j-.004)\log_2\left(\frac{\lceil 6j^2\pi/\varepsilon\rceil-1}{\lceil 6(j-1)^2\pi/\varepsilon\rceil-1}\right) \tag{48}$$

$$-\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,F_j^{\star\star},\omega),\psi)))\sim(.005j-.004)\log_2\left(\frac{\lceil 6j^2\pi/\varepsilon\rceil-1}{\lceil 6(j-1)^2\pi/\varepsilon\rceil-1}\right)-\log_2\left(\left\lceil 6j^2\pi/\varepsilon\right\rceil-1\right) \tag{49}$$

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,F_j^{\star\star},\omega),\psi)))\sim\log_2\left(\left\lceil 6j^2\pi/\varepsilon\right\rceil-1\right)-(.005j-.004)\log_2\left(\frac{\lceil 6j^2\pi/\varepsilon\rceil-1}{\lceil 6(j-1)^2\pi/\varepsilon\rceil-1}\right) \tag{50}$$

since:

$$\lim_{j\to\infty}(.005j-.004)\log_2\left(\frac{\lceil 6j^2\pi/\varepsilon\rceil-1}{\lceil 6(j-1)^2\pi/\varepsilon\rceil-1}\right)\approx .014427\text{ for }\varepsilon>0 \tag{51}$$

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Listing 5: Limit of 51

```
Limit[(.005 j - .004) Log[
   2, (Ceiling[(6 (j^2) Pi)/eps] -
      1)/(Ceiling[(6 ((j - 1)^2 ) Pi)/eps] - 1)], j -> Infinity]

(*Output of code is .014427 if eps>0*)
```

Hence:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,F_j^{\star\star},\omega),\psi)))\sim\log_2\left(\left\lceil 6j^2\pi/\varepsilon\right\rceil-1\right)-(.005j-.004)\log_2\left(\frac{\lceil 6j^2\pi/\varepsilon\rceil-1}{\lceil 6(j-1)^2\pi/\varepsilon\rceil-1}\right) \tag{52}$$

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,F_j^{\star\star},\omega),\psi)))\sim\log_2\left(\left\lceil 6j^2\pi/\varepsilon\right\rceil-1\right)-.014427 \tag{53}$$

Note, using §5.4.2 (0a) and §5.4.2 (1), take $\left|\mathcal{S}(\mathbf{C}(\varepsilon,F_j^{\star\star},\omega'),\psi')\right|=\lceil(6\pi r^2)/\varepsilon\rceil$ to compute the following:

$$\begin{aligned}&\underline{|\mathcal{S}(\mathbf{C}(\varepsilon,F_r^{\star},\omega),\psi)|}= \qquad (54)\\ &\sup\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon,F_j^{\star\star},\omega'),\psi')\right|:j\in\mathbb{N},\ \omega'\in\Omega_{\varepsilon,j},\ \psi'\in\Psi_{\varepsilon,j,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,F_j^{\star\star},\omega'),\psi')))\le\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,F_r^{\star},\omega),\psi)))\right\}=\\ &\sup\left\{\lceil 6j^2\pi/\varepsilon\rceil:j\in\mathbb{N},\ \omega'\in\Omega_{\varepsilon,j},\ \psi'\in\Psi_{\varepsilon,j,\omega},\ \log_2(\lceil 6j^2\pi/\varepsilon\rceil-1)-.014427\le\log_2(\lceil 6r^2\pi/\varepsilon\rceil-1)\right\}=\end{aligned}$$

where:

1. For every $r\in\mathbb{N}$, we find a $j\in\mathbb{N}$, where $\log_2(\lceil 6j^2\pi/\varepsilon\rceil-1)-.014427\le\log_2(\lceil 6r^2\pi/\varepsilon-1)$, but the absolute value of $\log_2(\lceil 6r^2\pi/\varepsilon\rceil-1)-(\log_2(\lceil 6j^2\pi/\varepsilon\rceil-1)-.014427)$ is minimized. In other words, for every $r\in\mathbb{N}$, we want $j\in\mathbb{N}$ where:

$$\log_2(\lceil 6j^2\pi/\varepsilon\rceil-1)-.014427\le\log_2(\lceil 6r^2\pi/\varepsilon\rceil-1) \tag{55}$$
$$\log_2(\lceil 6j^2\pi/\varepsilon\rceil-1)\le\log_2(\lceil 6r^2\pi/\varepsilon\rceil-1)+.014427 \tag{56}$$
$$2^{\log_2(\lceil 6j^2\pi/\varepsilon\rceil-1)}\le 2^{\log_2(\lceil 6r^2\pi/\varepsilon\rceil-1)+.014427} \tag{57}$$
$$\lceil 6j^2\pi/\varepsilon\rceil-1\le\left(\lceil 6r^2\pi/\varepsilon\rceil-1\right)2^{.014427} \tag{58}$$
$$\lceil 6j^2\pi/\varepsilon\rceil\le 1.0101\left(\lceil 6r^2\pi/\varepsilon\rceil-1\right)+1\rceil \tag{59}$$
$$\lceil 6j^2\pi/\varepsilon\rceil=\lfloor 1.0101\left(\lceil 6r^2\pi/\varepsilon\rceil-1\right)+1\rfloor\sim\underline{|\mathcal{S}(\mathbf{C}(1,F_r^{\star},\omega),\psi)|} \tag{60}$$

Finally, since $|\mathcal{S}(\mathbf{C}(\varepsilon,F_r^{\star},\omega),\psi)|=\log_2(\lceil 6r^2\pi/\varepsilon\rceil-1)$, we wish to prove

$$1<\liminf_{\varepsilon\to0}\liminf_{r\to\infty}\inf_{\omega\in\Omega_{\varepsilon,r}}\inf_{\psi\in\Psi_{\varepsilon,r,\omega}}\underline{\alpha}(\varepsilon,r,\omega,\psi)<+\infty$$

within §5.4.2 crit. 1:

$$\liminf_{\varepsilon\to0}\liminf_{r\to\infty}\inf_{\omega\in\Omega_{\varepsilon,r}}\inf_{\psi\in\Psi_{\varepsilon,r,\omega}}\underline{\alpha}(\varepsilon,r,\omega,\psi)=\liminf_{\varepsilon\to0}\liminf_{r\to\infty}\inf_{\omega\in\Omega_{\varepsilon,r}}\inf_{\psi\in\Psi_{\varepsilon,r,\omega}}\frac{\underline{|\mathcal{S}(\mathbf{C}(\varepsilon,F_r^{\star},\omega),\psi)|}}{|\mathcal{S}(\mathbf{C}(1,F_r^{\star},\omega),\psi))|} \tag{61}$$
$$=\lim_{\varepsilon\to0}\lim_{r\to\infty}\frac{\lfloor 1.0101(\lceil 6r^2\pi/\varepsilon\rceil-1)+1\rfloor}{\lceil 6r^2\pi/\varepsilon\rceil} \tag{62}$$

where using mathematica, we get the limit is greater than one:

Listing 6: Limit of eq. 62

```
Clear["Global`*"]
(*We use a double limit since the iterated limit takes too long to compute.
If the double limit exists, it's the same as the iterated limit.*)

Limit[Floor[Rationalize[1.0101, 0]* (Ceiling[(6 (r^2) Pi)/eps] - 1) + 1]/
Ceiling[(6 (r^2) Pi)/eps], {eps, r} -> {0, Infinity}]
```

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```
(*The output is 1.0101*)
```

Also, using §5.4.2 (0b) and §5.4.2 (1), take $\left|\mathcal{S}(\mathbf{C}(\varepsilon, G_j^{\star\star}, \omega'), \psi')\right| = \lceil 6j^2\pi/\varepsilon\rceil$ and use 4 crit. 2 to compute the following:

$$\begin{aligned}
&\underline{|\mathcal{S}(\mathbf{C}(\varepsilon, G_r^{\star}, \omega), \psi)|} = \\
&\inf\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{\star\star}, \omega'), \psi')\right| : j \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,j},\ \psi' \in \Psi_{\varepsilon,j,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{\star\star}, \omega'), \psi'))) \geq \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_r^{\star}, \omega), \psi)))\right\} = \\
&\inf\left\{\frac{3}{\pi^2}j^2 : j \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,j},\ \psi' \in \Psi_{\varepsilon,j,\omega},\ \log_2(\lceil 6j^2\pi/\varepsilon\rceil - 1) - .014427 \geq \log_2(\lceil 6r^2\pi/\varepsilon\rceil)\right\} =
\end{aligned} \tag{63}$$

where:

1. For every $r \in \mathbb{N}$, we find a $j \in \mathbb{N}$, where $\log_2(\lceil 6j^2\pi/\varepsilon\rceil - 1) - .014427 \geq \log_2(\lceil 6r^2\pi/\varepsilon\rceil)$, but the absolute value of $\log_2(\lceil 6j^2\pi/\varepsilon\rceil - 1) - .014427 - \log_2(\lceil 6r^2\pi/\varepsilon\rceil)$ is minimized. In other words, for every $r \in \mathbb{N}$, we want $j \in \mathbb{N}$ where:

$$\log_2(\lceil 6j^2\pi/\varepsilon\rceil - 1) - .014427 \geq \log_2(\lceil 6r^2\pi/\varepsilon\rceil - 1) \tag{64}$$

$$\log_2(\lceil 6j^2\pi/\varepsilon\rceil - 1) \geq \log_2(\lceil 6r^2\pi/\varepsilon\rceil - 1) + .014427 \tag{65}$$

$$2^{\log_2(\lceil 6j^2\pi/\varepsilon\rceil - 1)} \geq 2^{\log_2(\lceil 6r^2\pi/\varepsilon\rceil - 1) + .014427} \tag{66}$$

$$\lceil 6j^2\pi/\varepsilon\rceil - 1 \geq \left(\lceil 6r^2\pi/\varepsilon\rceil - 1\right)2^{.014427} \tag{67}$$

$$\lceil 6j^2\pi/\varepsilon\rceil \geq 1.0101\left(\lceil 6r^2\pi/\varepsilon\rceil - 1\right) + 1\rceil \tag{68}$$

$$\lceil 6j^2\pi/\varepsilon\rceil = \lceil 1.0101\left(\lceil 6r^2\pi/\varepsilon\rceil - 1\right) + 1\rceil \sim \underline{|\mathcal{S}(\mathbf{C}(1, F_r^{\star}, \omega), \psi)|} \tag{69}$$

Finally, since $|\mathcal{S}(\mathbf{C}(1, F_r^{\star}, \omega), \psi)| = \lceil 6\pi r^2/\varepsilon\rceil$, we wish to prove

$$1 < \limsup_{\varepsilon\to 0}\limsup_{r\to\infty}\sup_{\omega\in\Omega_{\varepsilon,r}}\sup_{\psi\in\Psi_{\varepsilon,r,\omega}}\overline{\alpha}(\varepsilon, r, \omega, \psi) < +\infty$$

within §5.4.2 crit. 1:

$$\limsup_{\varepsilon\to 0}\limsup_{r\to\infty}\sup_{\omega\in\Omega_{\varepsilon,r}}\sup_{\psi\in\Psi_{\varepsilon,r,\omega}}\overline{\alpha}(\varepsilon, r, \omega, \psi) = \limsup_{\varepsilon\to 0}\limsup_{r\to\infty}\sup_{\omega\in\Omega_{\varepsilon,r}}\sup_{\psi\in\Psi_{\varepsilon,r,\omega}}\frac{\overline{|\mathcal{S}(\mathbf{C}(1, F_r^{\star}, \omega), \psi)|}}{|\mathcal{S}(\mathbf{C}(1, F_r^{\star}, \omega), \psi))|} \tag{70}$$

$$= \lim_{\varepsilon\to 0}\lim_{r\to\infty}\frac{\lceil 1.0101(\lceil 6r^2\pi/\varepsilon\rceil - 1) + 1\rceil}{\lceil 6r^2\pi/\varepsilon\rceil} \tag{71}$$

where using mathematica, we get the limit is greater than one:

Listing 7: Limit of eq. 71

```
Clear["Global`*"]
(*We use a double limit since the iterated limit takes too long to compute.
If the double limit exists, it's the same as the iterated limit.*)

Limit[Floor[Rationalize[1.0101, 0]* (Ceiling[(6 (r^2) Pi)/eps] - 1) + 1]/
Ceiling[(6 (r^2) Pi)/eps], {eps, r} -> {0, Infinity}]

(*The output is 1.0101*)
```

Hence, since the limits in eq. 62 and eq. 71 are greater than one and less than $+\infty$: i.e.,

$$1 < \liminf_{\varepsilon\to 0}\liminf_{r\to\infty}\inf_{\omega\in\Omega_{\varepsilon,r}}\inf_{\psi\in\Psi_{\varepsilon,r,\omega}}\underline{\alpha}(\varepsilon, r, \omega, \psi), \limsup_{\varepsilon\to 0}\limsup_{r\to\infty}\sup_{\omega\in\Omega_{\varepsilon,r}}\sup_{\psi\in\Psi_{\varepsilon,r,\omega}}\overline{\alpha}(\varepsilon, r, \omega, \psi) < +\infty \tag{72}$$

we have proven *what we're measuring from* $(G_r^{\star})_{r\in\mathbb{N}}$ increases at a rate **superlinear** to that of $(G_j^{\star\star})_{j\in\mathbb{N}}$ (i.e., 5.4.2 crit. 1).

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5.4.4. Example of The "Measure" from $(F_r^\star)_{r\in\mathbb{N}}$ Increasing at a Rate Sub-Linear to that of $(F_j^{\star\star})_{j\in\mathbb{N}}$

Using our previous example, we can use the following theorem:

Theorem 2. *If what we're measuring from $(F_r^\star)_{r\in\mathbb{N}}$ increases at a rate superlinear to that of $(F_j^{\star\star})_{j\in\mathbb{N}}$, then what we're measuring from $(F_j^{\star\star})_{r\in\mathbb{N}}$ increases at a rate* ***sublinear*** *to that of $(F_r^\star)_{r\in\mathbb{N}}$*

Hence, in our definition of super-linear (§5.4.2 crit. 1), swap $F_r^\star$ and $r\in\mathbb{N}$ for $(F_j^{\star\star})$ and $j\in\mathbb{N}$ within $\overline{\alpha}(\epsilon,r,\omega,\psi)$ and $\underline{\alpha}(\epsilon,r,\omega,\psi)$ (i.e., $\overline{\alpha}(\epsilon,j,\omega,\psi)$ and $\underline{\alpha}(\epsilon,j,\omega,\psi)$). Notice, thm. 2 is true when:

$$1<\limsup_{\varepsilon\to0}\limsup_{j\to\infty}\sup_{\omega\in\Omega_{\varepsilon,j}}\sup_{\psi\in\Psi_{\varepsilon,j,\omega}}\overline{\alpha}(\varepsilon,j,\omega,\psi),\ \liminf_{\varepsilon\to0}\liminf_{j\to\infty}\inf_{\omega\in\Omega_{\varepsilon,j}}\inf_{\psi\in\Psi_{\varepsilon,j,\omega}}\underline{\alpha}(\varepsilon,j,\omega,\psi)<+\infty$$

5.4.5. Example of the "measure" of $(F_r^\star)$ converging linearly to that of $(F_j^{\star\star})$

Using §5.4.2 consider the following:

Note 8. Recall, if $\dim_H(F_r^\star)=2$:

1. When $|\cdot|$ is the cardinality, for all $\omega\in\Omega_{\varepsilon,r}$ and $\psi\in\Psi_{\varepsilon,r,\omega}$, $|\mathcal{S}(\mathbf{C}(\varepsilon,F_r^\star,\omega),\psi)|=\lceil\text{Area}(F_r^\star)/\varepsilon\rceil$
2. For all $\omega\in\Omega_{\varepsilon,r}$ and $\psi\in\Psi_{\varepsilon,r,\omega}$, the largest $\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,F_r^\star,\omega),\psi)))$ can be is $\log_2(\lceil\text{Area}(F_r^\star)/\varepsilon\rceil-1)=\log_2(|\mathcal{S}(\mathbf{C}(\varepsilon,F_r^\star,\omega),\psi)|-1)$

Now consider:

1. $(F_r^\star)_{r\in\mathbb{N}}=([-r,r]\times[-r,r])_{r\in\mathbb{N}}$
2. $(F_j^{\star\star})_{j\in\mathbb{N}}=([-j^2,j^2]\times[-j^2,j^2])_{j\in\mathbb{N}}$

where:

Note 9. The area of $F_r^\star$ and $F_j^{\star\star}$ is:

1. $\text{Area}(F_r^\star)=(r-(-r))\cdot(r-(-r))=2r\cdot2r=4r^2$
2. $\text{Area}(F_j^{\star\star})=(j^2-(-j^2))\cdot(j^2-(-j^2))=2j^2\cdot2j^2=4j^4$

Notice, since $F_r^\star$ is a square, using note 8 crit. (2) and note 9 crit. (1), it's simple to see:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,F_r^\star,\omega),\psi)))=\log_2(\lceil\text{Area}(F_r^\star)/\varepsilon\rceil-1)=\log_2(\lceil4r^2/\varepsilon\rceil-1)$$

Hence, using §5.4.2 (0a) and note 8 crit. (2):

$$\begin{aligned}&\underline{|\mathcal{S}(\mathbf{C}(\varepsilon,F_r^\star,\omega),\psi)|}=\\&\sup\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon,F_j^{\star\star},\omega'),\psi')\right|:j\in\mathbb{N},\ \omega'\in\Omega_{\varepsilon,j},\ \psi'\in\Psi_{\varepsilon,j,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,F_j^{\star\star},\omega'),\psi')))\le\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,F_r^\star,\omega),\psi)))\right\}=\\&\sup\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon,F_j^{\star\star},\omega'),\psi')\right|:j\in\mathbb{N},\ \omega'\in\Omega_{\varepsilon,j},\ \psi'\in\Psi_{\varepsilon,j,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,F_j^{\star\star},\omega'),\psi')))\le\log_2(\lceil4r^2/\varepsilon\rceil-1)\right\}=\end{aligned}\tag{73}$$

Note, since $F_j^{\star\star}$ is also a square, for all $\omega'\in\Omega_{\varepsilon,j}$ and $\psi'\in\Psi_{\varepsilon,j,\omega}$

$$0<\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,F_j^{\star\star},\omega'),\psi')))\le\log_2(\lceil\text{Area}(F_j^{\star\star})/\varepsilon\rceil-1)=\log_2(\lceil4j^4/\varepsilon\rceil-1)\tag{74}$$

Thus, using eq. 73 and eq. 74:

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1. For every $r \in \mathbb{N}$, we find a $j \in \mathbb{N}$, where $\log_2(\lceil 4j^4/\varepsilon \rceil - 1) \leq \log_2(\lceil 4r^2/\varepsilon \rceil - 1)$, but the absolute value of $\log_2(\lceil 4r^2/\varepsilon \rceil - 1) - \log_2(\lceil 4j^4/\varepsilon \rceil - 1)$ is minimized. In other words, for every $r \in \mathbb{N}$, we want $j \in \mathbb{N}$ where:

$$\log_2(\lceil 4j^4/\varepsilon \rceil - 1) \leq \log_2(\lceil 4r^2/\varepsilon \rceil - 1) \tag{75}$$
$$4j^4 \leq 4r^2 \tag{76}$$
$$j^4 \leq r^2 \tag{77}$$
$$j \leq \sqrt{r} \tag{78}$$
$$j = \lfloor \sqrt{r} \rfloor \tag{79}$$
$$4j^4/\varepsilon = 4\lfloor \sqrt{r} \rfloor^4/\varepsilon \tag{80}$$
$$\lceil 4j^4/\varepsilon \rceil = \lceil 4\lfloor \sqrt{r} \rfloor^4/\varepsilon \rceil \tag{81}$$

which gives:

$$\lceil 4\lfloor \sqrt{r} \rfloor^4/\varepsilon \rceil \leq \sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} \underline{|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^\star, \omega), \psi)|} \leq \lceil 4r^2/\varepsilon \rceil \tag{82}$$

where using note 8 crit. 1: $|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^\star, \omega), \psi)| = \lceil \text{Area}(F_r^\star)/\varepsilon \rceil = \lceil 4r^2/\varepsilon \rceil$, we wish to prove §5.4.2 crit. 33a:

$$\begin{aligned} 1 = \limsup_{\varepsilon \to 0} \limsup_{r \to \infty} \lceil 4\lfloor \sqrt{r} \rfloor^4/\varepsilon \rceil / \lceil 4r^2/\varepsilon \rceil \leq \limsup_{\varepsilon \to 0} \limsup_{r \to \infty} \sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} \underline{\alpha}(\varepsilon, r, \omega, \psi) = \\ \limsup_{\varepsilon \to 0} \limsup_{r \to \infty} \sup_{\omega \in \Omega_{\varepsilon,r}} \sup_{\psi \in \Psi_{\varepsilon,r,\omega}} \underline{|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^\star, \omega), \psi)|} / |\mathcal{S}(\mathbf{C}(\varepsilon, F_r^\star, \omega), \psi))| \leq \\ \limsup_{\varepsilon \to 0} \limsup_{r \to \infty} \lceil 4r^2/\varepsilon \rceil / \lceil 4r^2/\varepsilon \rceil = 1 \end{aligned} \tag{83}$$

Hence §5.4.2 crit. 33a is true and then we prove §5.4.2 crit. 33b:
Note, since $F_j^{\star\star}$ is a square, using note 8 crit. (2) and note 9 crit. (2), it's simple to see:

$$\text{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{\star\star}, \omega), \psi))) = \log_2(\lceil \text{Area}(F_j^{\star\star})/\varepsilon \rceil - 1) = \log_2(\lceil 4j^4/\varepsilon \rceil - 1)$$

Thus, using §5.4.2 (0a) and note 8 crit. (2):

$$\begin{aligned} \overline{\left|\mathcal{S}(\mathbf{C}(\varepsilon, F_j^\star, \omega), \psi)\right|} = \\ \sup\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^\star, \omega'), \psi')\right| : r \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,r},\ \psi' \in \Psi_{\varepsilon,r,\omega},\ \text{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_r^\star, \omega'), \psi'))) \leq \text{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{\star\star}, \omega), \psi)))\right\} = \\ \sup\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{\star\star}, \omega'), \psi')\right| : r \in \mathbb{N},\ \omega' \in \Omega_{\varepsilon,r},\ \psi' \in \Psi_{\varepsilon,r,\omega},\ \text{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_r^{\star\star}, \omega'), \psi'))) \leq \log_2(\lceil 4j^4/\varepsilon \rceil - 1)\right\} = \end{aligned} \tag{84}$$

Note, since $F_r^\star$ is also a square, for all $\omega' \in \Omega_{\varepsilon,r}$ and $\psi' \in \Psi_{\varepsilon,r,\omega}$

$$0 < \text{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon, F_r^{\star\star}, \omega'), \psi'))) \leq \log_2(\lceil \text{Area}(F_r^\star)/\varepsilon \rceil - 1) = \log_2(\lceil 4r^2/\varepsilon \rceil - 1) \tag{85}$$

Therefore, using eq. 84 and eq. 85:

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1. For every $j \in \mathbb{N}$, we find a $r \in \mathbb{N}$, where $\log_2(\lceil 4r^2/\varepsilon \rceil - 1) \leq \log_2(\lceil 4j^4/\varepsilon \rceil - 1)$, but the absolute value of $\log_2(\lceil 4j^4/\varepsilon \rceil - 1) - \log_2(\lceil 4r^2/\varepsilon \rceil - 1)$ is minimized. In other words, for every $j \in \mathbb{N}$, we want a $r \in \mathbb{N}$, where:

$$\log_2(\lceil 4r^2/\varepsilon \rceil - 1) \leq \log_2(\lceil 4j^4/\varepsilon \rceil - 1) \tag{86}$$
$$4r^2 \leq 4j^4 \tag{87}$$
$$r^2 \leq j^4 \tag{88}$$
$$r \leq j^2 \tag{89}$$
$$r = \lfloor j^2 \rfloor = j^2 \tag{90}$$
$$4r^2/\varepsilon = 4(j^2)^2/\varepsilon = \tag{91}$$
$$\lceil 4r^2/\varepsilon \rceil = \lceil 4j^4/\varepsilon \rceil \tag{92}$$

Hence:

$$\overline{\left|\mathcal{S}(\mathbf{C}(\varepsilon, F_j^\star, \omega), \psi)\right|} = \lceil 4j^4/\varepsilon \rceil \tag{93}$$

and, using 8 crit. (1) and 8 crit. (1) with equation 92: $|\mathcal{S}(\mathbf{C}(\varepsilon, F_r^\star, \omega), \psi)| = \lceil \text{Area}(F_r^\star)/\varepsilon \rceil = \lceil 4r^2/\varepsilon \rceil = \lceil 4j^4/\varepsilon \rceil$. Thus, we wish to prove §5.4.2 crit. 33a using eq. 93:

$$\limsup_{\varepsilon\to 0} \limsup_{r\to\infty} \sup_{\omega\in\Omega_{\varepsilon,j}} \sup_{\psi\in\Psi_{\varepsilon,j,\omega}} \overline{\alpha}(\varepsilon, j, \omega, \psi) = \tag{94}$$
$$\limsup_{\varepsilon\to 0} \limsup_{r\to\infty} \sup_{\omega\in\Omega_{\varepsilon,r}} \sup_{\psi\in\Psi_{\varepsilon,j,\omega}} \overline{\left|\mathcal{S}(\mathbf{C}(\varepsilon, F_j^\star, \omega), \psi)\right|} / \left|\mathcal{S}(\mathbf{C}(\varepsilon, F_j^{\star\star}, \omega), \psi))\right| = \tag{95}$$
$$\limsup_{\varepsilon\to 0} \limsup_{j\to\infty} \lceil 4j^4/\varepsilon \rceil / \lceil 4j^4/\varepsilon \rceil = 1 \tag{96}$$

where §5.4.2 crit. 33b is true.

Therefore, since §5.4.2 crit. 3 is correct in this case, "the measure" of $(F_r^\star)$ increases at a rate linear to that of $(F_j^{\star\star})_{j\in\mathbb{N}}$

5.5. Defining The Actual Rate of Expansion of Sequence of Bounded Sets From C

In the next section, we will define the actual rate of expansion of sequence of bounded sets from C and give an example. (Note, the motivation for the definition is explained in §3.2.1.)

5.5.1. Definition of Actual Rate of Expansion of Sequence of Bounded Sets

Suppose $(F_r)_{r\in\mathbb{N}}$ is a bounded sequence of subsets of $\mathbb{R}^n$, and $d(Q, R)$ is the Euclidean distance between points $Q, R \in \mathbb{R}^n$. Therefore, using the "chosen" center point $C \in \mathbb{R}^n$, when:

$$\mathcal{G}(C, F_r) = \sup\{d(C, y) : y \in F_r\}$$

the **actual rate of expansion** is:

$$\mathcal{E}(C, F_r) = \mathcal{G}(C, F_{r+1}) - \mathcal{G}(C, F_r)$$

Note, there are cases of $(F_r)_{r\in\mathbb{N}}$ when $\mathcal{E}$ isn't fixed and $\mathcal{E} \neq E$ (i.e., **the chosen, fixed rate of expansion**).

5.5.2. Example

Suppose $A = \mathbb{R}^2$, $(F_r)_{r\in\mathbb{N}} = ([-r, r] \times [-r, r])_{r\in\mathbb{N}}$, and $C = (1, 1)$. One can clearly see $(-r, -r)$ is the point on F_r that is farthest from $(1, 1)$. Hence,

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$$\mathcal{G}(C,F_r) = \sup\{d(C,y) : y \in F_r\} = \tag{97}$$
$$d((-r,-r),(1,1)) = \tag{98}$$
$$\sqrt{(-r-1)^2 + (-r-1)^2} = \tag{99}$$
$$\sqrt{(-(r+1))^2 + (-(r+1))^2} = \tag{100}$$
$$\sqrt{(r+1)^2 + (r+1)^2} = \tag{101}$$
$$\sqrt{2(r+1)^2} = \tag{102}$$
$$\sqrt{2}(r+1) \tag{103}$$

and the actual rate of expansion is:

$$\mathcal{E}(C,F_r) = \mathcal{G}(C,F_{r+1}) - \mathcal{G}(C,F_r) = \tag{104}$$
$$\sqrt{2}((r+1)+1) - \sqrt{2}(r+1) = \tag{105}$$
$$\sqrt{2}(r+2) - \sqrt{2}(r+1) = \tag{106}$$
$$\sqrt{2}r + 2\sqrt{2} - \sqrt{2}r - \sqrt{2} = \tag{107}$$
$$\sqrt{2} \tag{108}$$

5.6. Reminder

See if §3.2 is easier to understand?

6. My Attempt At Answering The Approach of §1.3

6.1. Choice Function

Suppose we define the following:

1. $(F_k^{\star\star\star})_{k\in\mathbb{N}}$ is the sequence of bounded sets satisfying (1), (2), (3), (4), and (5) of the **leading question** in §3.2
2. $\mathbb{S}'(A)$ is all sequences of bounded sets which satisfy (1) and (2) of the **leading question**
3. $(F_j^{\star\star})_{j\in\mathbb{N}} \in \mathbb{S}'(A)$ but **not** in the set of equivelant sequences of bounded sets to $(F_k^{\star\star\star})_{k\in\mathbb{N}}$. Note, using the end of def. 1, we represent this criteria as:

$$(F_j^{\star\star})_{j\in\mathbb{N}} \in \mathbb{S}'(A) \setminus \sim (F_k^{\star\star\star})_{k\in\mathbb{N}}$$

Further note, from §5.4.2 (0b), if we take:

$$\overline{\left|\mathcal{S}(\mathbf{C}(\varepsilon,F_k^{\star\star\star},\omega),\psi)\right|} = \tag{109}$$
$$\inf\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon,F_j^{\star\star},\omega'),\psi')\right| : j\in\mathbb{N},\ \omega'\in\Omega_{\varepsilon,j},\ \psi'\in\Psi_{\varepsilon,j,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,F_j^{\star\star},\omega'),\psi'))) \geq \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,F_k^{\star\star\star},\omega),\psi)))\right\}$$

and from §5.4.2 (0a), we take:

$$\underline{\left|\mathcal{S}(\mathbf{C}(\varepsilon,F_k^{\star\star\star},\omega),\psi)\right|} = \tag{110}$$
$$\sup\left\{\left|\mathcal{S}(\mathbf{C}(\varepsilon,F_j^{\star\star},\omega'),\psi')\right| : j\in\mathbb{N},\ \omega'\in\Omega_{\varepsilon,j},\ \psi'\in\Psi_{\varepsilon,j,\omega},\ \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,F_j^{\star\star},\omega'),\psi'))) \leq \mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(\varepsilon,F_k^{\star\star\star},\omega),\psi)))\right\}$$

then, using §5.4.1 (2), eq. 109, and eq. 110:

$$\sup_{\omega\in\Omega_{\varepsilon,k}}\sup_{\psi\in\Psi_{\varepsilon,k,\omega}}|\mathcal{S}(\mathbf{C}(\varepsilon,F_k^{\star\star\star},\omega),\psi)| = \left|\mathcal{S}'(\varepsilon,F_k^{\star\star\star})\right| = \left|\mathcal{S}'\right| \tag{111}$$

$$\sup_{\omega\in\Omega_{\varepsilon,k}}\sup_{\psi\in\Psi_{\varepsilon,k,\omega}}\overline{|\mathcal{S}(\mathbf{C}(\varepsilon,F_k^{\star\star\star},\omega),\psi)|} = \overline{\left|\mathcal{S}'(\varepsilon,F_k^{\star\star\star})\right|} = \overline{\left|\mathcal{S}'\right|} \tag{112}$$

$$\sup_{\omega\in\Omega_{\varepsilon,k}}\sup_{\psi\in\Psi_{\varepsilon,k,\omega}}\underline{|\mathcal{S}(\mathbf{C}(\varepsilon,F_k^{\star\star\star},\omega),\psi)|} = \underline{\left|\mathcal{S}'(\varepsilon,F_k^{\star\star\star})\right|} = \underline{\left|\mathcal{S}'\right|} \tag{113}$$

6.2. Approach

We manipulate the definitions of §5.4.2 (0a), (0b) and §5.4.2 (1) to solve §3.2 (1)-(5) of the *leading question*.

6.3. Potential Answer

6.3.1. Preliminaries (Definition of T in case of §3.2.1 (4))

When the difference of point $X = (x_1, \cdots, x_n)$ and $Y = (y_1, \cdots, y_n)$ is:

$$X - Y = (x_1 - y_1, x_2 - y_2, \cdots, x_n - y_n)$$

the average of $F_r^\star$ for every $r \in \mathbb{N}$ is:

$$\text{Avg}(F_r^\star) = \frac{1}{\mathcal{H}^{\dim_{\text{H}}(F_r^\star)}(F_r^\star)}\int_{F_r^\star}(x_1, \cdots, x_n)\, d\mathcal{H}^{\dim_{\text{H}}(F_r^\star)} \tag{114}$$

and $d(P,Q)$ is the n-d Euclidean distance between points $P, Q \in \mathbb{R}^n$, we define an *explicit* injective $\mathscr{F} : \mathbb{R}^n \to \mathbb{R}$ such that:

1. If $d(\text{Avg}(F_r^\star), C) < d(\text{Avg}(F_j^{\star\star}), C)$, then $\mathscr{F}(\text{Avg}(F_r^\star) - C) < \mathscr{F}(\text{Avg}(F_j^{\star\star}) - C)$
2. If $d(\text{Avg}(F_r^\star), C) > d(\text{Avg}(F_j^{\star\star}), C)$, then $\mathscr{F}(\text{Avg}(F_r^\star) - C) > \mathscr{F}(\text{Avg}(F_j^{\star\star}) - C)$
3. If $d(\text{Avg}(F_r^\star), C) = d(\text{Avg}(F_j^{\star\star}), C)$, then $\mathscr{F}(\text{Avg}(F_r^\star) - C) \neq \mathscr{F}(\text{Avg}(F_j^{\star\star}) - C)$

where using “chosen" center point $C \in \mathbb{R}^n$:

$$T(C, F_r^\star) = \mathscr{F}(\text{Avg}(F_r^\star) - C) \tag{115}$$

6.3.2. Question

Does T exist? If so, how do we define it?

Thus, using $|\mathcal{S}'|$, $\overline{|\mathcal{S}'|}$, $\underline{|\mathcal{S}'|}$, E, $\mathcal{E}(C, F_k^{\star\star\star})$ (§5.5), and $T(C, F_k^{\star\star\star})$ with the absolute value function $||\cdot||$, ceiling function $\lceil\cdot\rceil$, and nearest integer function $[\cdot]$, we define:

$$K(\varepsilon, F_k^{\star\star\star}) =$$
$$(1 + ||\mathcal{E}(C, F_k^{\star\star\star}) - E||)\left(\left|\left|\frac{|\mathcal{S}'|\left(1 + \left[\frac{|\mathcal{S}'|\left(\underline{|\mathcal{S}'|}+2|\mathcal{S}'|\right)}{\left(\underline{|\mathcal{S}'|}+|\mathcal{S}'|\right)\left(\underline{|\mathcal{S}'|}+|\mathcal{S}'|+\overline{|\mathcal{S}'|}\right)}\right]\right)\left(1 + \left[\underline{|\mathcal{S}'|}/\,|\mathcal{S}'|\right]\right)}{\left(1 + \left[|\mathcal{S}'|\,/\overline{|\mathcal{S}'|}\right]\right)\left(1 + \left[\underline{|\mathcal{S}'|}/\overline{|\mathcal{S}'|}\right]\right)} - |\mathcal{S}'|\right|\right| + |\mathcal{S}'|\right) - T(C, F_k^{\star\star\star}) \tag{116}$$

the choice function, which answers the **leading question** in §3.2, could be the following, s.t. we explain the reason behind choosing the choice function in §6.4:

Theorem 3. *If we define:*

$$\mathcal{M}(\varepsilon, F_k^{\star\star\star}) = |\mathcal{S}'(\varepsilon, F_k^{\star\star\star})|\left(K(\varepsilon, F_k^{\star\star\star}) - |\mathcal{S}'(\varepsilon, F_k^{\star\star\star})|\right)$$

$$\mathcal{M}(\varepsilon, F_j^{\star\star}) = |\mathcal{S}'(\varepsilon, F_j^{\star\star})|\left(K(\varepsilon, F_j^{\star\star}) - |\mathcal{S}'(\varepsilon, F_j^{\star\star})|\right)$$

where for $\mathcal{M}(\varepsilon, F_k^{\star\star\star})$, we define $\mathcal{M}(\varepsilon, F_k^{\star\star\star})$ to be equivalent to $\mathcal{M}(\varepsilon, F_j^{\star\star})$ when swapping "$j \in \mathbb{N}$" with "$k \in \mathbb{N}$" (for eq. 109 & 110) and sets $F_k^{\star\star\star}$ with $F_j^{\star\star}$ (for eq. 109–116), then for constant $v > 0$ and variable $v^* > 0$, if:

$$\overline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star}) = \inf\left(\left\{|\mathcal{S}'(\varepsilon, F_j^{\star\star})| : j \in \mathbb{N}, \mathcal{M}(\varepsilon, F_j^{\star\star}) \geq \mathcal{M}(\varepsilon, F_k^{\star\star\star}) \geq v^*\right\} \cup \{v^*\}\right) + v \tag{117}$$

and:

$$\underline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star}) = \sup\left(\left\{|\mathcal{S}'(\varepsilon, F_j^{\star\star})| : j \in \mathbb{N}, v^* \leq \mathcal{M}(\varepsilon, F_j^{\star\star}) \leq \mathcal{M}(\varepsilon, F_k^{\star\star\star})\right\} \cup \{-v^*\}\right) + v \tag{118}$$

then for all $(F_j^{\star\star})_{j\in\mathbb{N}} \in \mathbb{S}'(A) \setminus \sim (F_k^{\star\star\star})_{k\in\mathbb{N}}$ (def. 1), if:

$$\inf\left\{ ||1-c|| : \forall(\epsilon > 0)\exists(c > 0)\forall(k \in \mathbb{N})\exists(j \in \mathbb{N})\left(\left|\left|\frac{|\mathcal{S}'(\varepsilon, F_k^{\star\star\star})|}{|\mathcal{S}'(\varepsilon, F_j^{\star\star})|} - c\right|\right| < \varepsilon\right)\right\} \tag{119}$$

where $\lceil \cdot \rceil$ is the ceiling function, E is the fixed rate of expansion, Γ is the gamma function, n is the dimension of $\mathbb{R}^n$, $\dim_H(F_k^{\star\star\star})$ is the Hausdorff dimension of set $F_k^{\star\star\star} \subseteq \mathbb{R}^n$, and $\mathbf{A}_k$ is area of the smallest n-dimensional box that contains $F_k^{\star\star\star}$, then

$$V(\varepsilon, G_k^{\star\star\star}, n) = \left\lceil \left(\mathbf{A_k}^{1-\operatorname{sign}(E)}(E - \operatorname{sign}(E)+1)\left(\frac{\exp(n\ln(\pi)/2)}{\Gamma(n/2+1)}\right)\left(k!^{(n-\dim_{\mathrm{H}}(F_k^{\star\star\star}))}\right)\left(k^{\operatorname{sign}(E)(\dim_{\mathrm{H}}(F_k^{\star\star\star})-\operatorname{sign}(\dim_{\mathrm{H}}(F_k^{\star\star\star}))+1)}\right) + (1-\operatorname{sign}(\dim_{\mathrm{H}}(F_k^{\star\star\star})))\right)/\varepsilon \right\rceil \Big/ |\mathcal{S}'(\varepsilon, G_k^{\star\star\star})| \tag{120}$$

the choice function is:

$$\limsup_{\varepsilon\to 0}\lim_{v^*\to\infty}\limsup_{k\to\infty}\left(\frac{\operatorname{sign}(\mathcal{M}(\varepsilon, F_k^{\star\star\star}))\overline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star})}{|\mathcal{S}'(\varepsilon, F_k^{\star\star\star})| + v} - c^{-V(\varepsilon, F_k^{\star\star\star}, n)}\right)\left(\frac{\operatorname{sign}(\mathcal{M}(\varepsilon, F_k^{\star\star\star}))\underline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star})}{|\mathcal{S}'(\varepsilon, F_k^{\star\star\star})| + v} - c^{-V(\varepsilon, F_k^{\star\star\star}, n)}\right) = \tag{121}$$

$$\liminf_{\varepsilon\to 0}\lim_{v^*\to\infty}\liminf_{k\to\infty}\left(\frac{\operatorname{sign}(\mathcal{M}(\varepsilon, F_k^{\star\star\star}))\overline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star})}{|\mathcal{S}'(\varepsilon, F_k^{\star\star\star})| + v} - c^{-V(\varepsilon, F_k^{\star\star\star}, n)}\right)\left(\frac{\operatorname{sign}(\mathcal{M}(\varepsilon, F_k^{\star\star\star}))\underline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star})}{|\mathcal{S}'(\varepsilon, F_k^{\star\star\star})| + v} - c^{-V(\varepsilon, F_k^{\star\star\star}, n)}\right) = 0 \tag{122}$$

such that $\sim (F_k^{\star\star\star})_{k\in\mathbb{N}}$ (def. 1) must satisfy eq. 121 & 122. (Note, we want $\sup\varnothing = -\infty$, $\inf\varnothing = +\infty$, and $(F_k^{\star\star\star})_{k\in\mathbb{N}}$ to answer the leading question of §3.2) where the answer to problems 1 and 2 & the approach of §1.3 is $Avg(F_k^{\star\star\star})$ (when it exists).

6.4. Explaining The Choice Function and Evidence The Choice Function Is Credible

Notice, before reading the programming in code 8, without the "c"-terms in eq. 121 and eq. 122:

1. The choice function in eq. 121 and eq. 122 is zero, when *what I'm measuring* from $(F_k^{\star\star\star})_{k\in\mathbb{N}}$ (§5.4.2 criteria 1) increases at a rate superlinear to that of $(F_j^{\star\star})_{j\in\mathbb{N}}$, where $\operatorname{sign}(\mathcal{M}(\varepsilon, F_k^{\star\star\star})) = 0$.
2. The choice function in eq. 121 and eq. 122 is zero, when for a given $(F_k^{\star\star\star})_{k\in\mathbb{N}}$ and $(F_j^{\star\star})_{j\in\mathbb{N}}$ there doesn't exist c where eq. 119 is satisfied or $c = 0$.

3. When c does exist, suppose:

$$\left\{\mathcal{J}(k) : k \in \mathbb{N}, \frac{|\mathcal{S}'(\varepsilon, F_k^{\star\star\star})|}{|\mathcal{S}'(\varepsilon, F_{\mathcal{J}(k)}^{\star\star})|} \approx c\right\} \tag{123}$$

(a) When $|\mathcal{S}'(\varepsilon, F_k^{\star\star\star})| < |\mathcal{S}'(\varepsilon, F_{\mathcal{J}(k)}^{\star\star})|$, then:

$$\limsup_{\varepsilon\to 0} \lim_{v^*\to\infty} \limsup_{k\to\infty} \frac{\operatorname{sign}(\mathcal{M}(\varepsilon, F_k^{\star\star\star}))\overline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star})}{|\mathcal{S}'(\varepsilon, F_k^{\star\star\star})| + v} = c \tag{124}$$

$$\liminf_{\varepsilon\to 0} \lim_{v^*\to\infty} \liminf_{k\to\infty} \frac{\operatorname{sign}(\mathcal{M}(\varepsilon, F_k^{\star\star\star}))\underline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star})}{|\mathcal{S}'(\varepsilon, F_k^{\star\star\star})| + v} = 0 \tag{125}$$

(b) When $|\mathcal{S}'(\varepsilon, F_k^{\star\star\star})| > |\mathcal{S}'(\varepsilon, F_{\mathcal{J}(k)}^{\star\star})|$, then:

$$\limsup_{\varepsilon\to 0} \lim_{v^*\to\infty} \limsup_{k\to\infty} \frac{\operatorname{sign}(\mathcal{M}(\varepsilon, F_k^{\star\star\star}))\overline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star})}{|\mathcal{S}'(\varepsilon, F_k^{\star\star\star})| + v} = +\infty \tag{126}$$

$$\liminf_{\varepsilon\to 0} \lim_{v^*\to\infty} \liminf_{k\to\infty} \frac{\operatorname{sign}(\mathcal{M}(\varepsilon, F_k^{\star\star\star}))\underline{\mathcal{S}}(\varepsilon, k, v^*, F_j^{\star\star})}{|\mathcal{S}'(\varepsilon, F_k^{\star\star\star})| + v} = 1/c \tag{127}$$

Hence, for each sub-criteria under crit. (3), if we subtract one of their limits by their limit value, then eq. 121 and eq. 122 is zero. (We do this by using the "c"-term in eq. 121 and 122). However, when the exponents of the "c"-terms aren't equal to -1, the limits of eq. 121 and 122 aren't equal to zero. We, infact, want this when we swap $|\mathcal{S}'(\varepsilon, F_k^{\star\star\star})|$ with $|\mathcal{S}'(\varepsilon, F_j^{\star\star})|$. Moreover, we define function $V(\varepsilon, F_k^{\star\star\star}, n)$ (i.e., eq. 120), where:

i. When $S'(\varepsilon, F_k^{\star\star\star}) \gg \operatorname{Numerator}\big(V(\varepsilon, F_k^{\star\star\star}, n)\big)$, then eq. 121 and 122 without the "c"-terms are zero. (The "c"-terms approach zero and still allow eq. 121 and 122 to equal zero.)

ii. When $S'(\varepsilon, F_k^{\star\star\star}) \ll \operatorname{Numerator}\big(V(\varepsilon, F_k^{\star\star\star}, n)\big)$ then $\operatorname{sign}(\mathcal{M}(\varepsilon, F_k^{\star\star\star}))$ is zero, which makes eq. 121 and 122 equal zero.

iii. Here are some examples of the numerator of $V(\varepsilon, F_k^{\star\star\star}, n)$ (eq. 120):

A. When $E = 0$, $n = 1$, and $\dim_H(A) = 0$, the numerator of $V(\varepsilon, F_k^{\star\star\star}, n)$ is $\big(\lceil \mathbf{A}k! + 1\big)/\varepsilon\rceil$

B. When $E = z$, $n = 1$, and $\dim_H(A) = 0$, the numerator of $V(\varepsilon, F_k^{\star\star\star}, n)$ is $\lceil(2zk \cdot k! + 1)/\varepsilon\rceil$

C. When $E = 0$, $n = z_2$, and $\dim_H(A) = z_2$, the numerator of $V(\varepsilon, F_k^{\star\star\star}, n)$ is ceiling of constant $\mathbf{A}$ times the volume of an n-dimensional ball with finite radius: i.e.,

$$\left\lceil \frac{\mathbf{A}z_1 \exp(z_2 \ln(\pi)/2)}{\Gamma(z_2/2+1)} \Big/ \varepsilon \right\rceil$$

D. When $E = z_1$, $n = z_2$, and $\dim_H(A) = z_2$, the numerator of $V(\varepsilon, F_k^{\star\star\star}, n)$ is ceiling of the volume of the n-dimensional ball: i.e.,

$$\left\lceil \frac{z_1 \exp(z_2 \ln(\pi)/2)}{\Gamma(z_2/2+1)} k^{z_2} \Big/ \varepsilon \right\rceil$$

Note 11. Now, consider the code for eq. 121 and eq. 122. (Note, the set theoretic limit $F_k^{\star\star\star}$ is set A. In this example, $A = \mathbb{R}$ and:

1. $(F_k^{\star\star\star})_{r\in\mathbb{N}} = \big(\{x^2 + y^2 = 6k^2\}\big)_{k\in\mathbb{N}}$
2. $(F_j^{\star\star})_{j\in\mathbb{N}} = \big(\{x^2/(9j^2) + y^2/(4j^2) = 1\}\big)_{j\in\mathbb{N}}$

We leave it to mathematicians to figure `LengthS1`, `LengthS2`, `Entropy1` and `Entropy2` for different A, $(F_k^{\star\star\star})_{k\in\mathbb{N}}$, and $(F_j^{\star\star})_{j\in\mathbb{N}}$.

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Listing 8: Code for eq. 121 and 122 on note 11

```
Clear["Global`*"]

(*'A' is an arbitrary subset of ℝ^n*)
A=Reals

eps=.001 (*Since 'A' doesn't have Hausdorff dimension zero, we set 'eps' or ε to a small value*)

(*'LengthS1' is |S'(ε,F_k^{***})|*)
LengthS1[k_] := LengthS1[k] = Ceiling[(6 Pi k^2)/eps]

(*'Entropy1' is the approximation of sup_{ω∈Ω_{ε,k}} sup_{ψ∈Ψ_{ε,k,ω}} E(𝓛(𝓢(C(ε,F_k^{***},ω),ψ))) using asymptotic analysis*)
Entropy1[k_] := Entropy1[k] = Log2[(6 Pi k^2)/eps] (*We don't use Ceiling due to the code below 'q':
'Ceiling' can't be used in 'Solve'*)

(*'LengthS2' is |S'(ε,F_j^{**})|*)
LengthS2[j_] := LengthS2[j] = Ceiling[(6 Pi k^2)/eps]

(*'Entropy2' is the approximation sup_{ω∈Ω_{ε,j}} sup_{ψ∈Ψ_{ε,j,ω}} E(𝓛(𝓢(C(ε,F_j^{**},ω),ψ))) using asmyptotic analysis *)
Entropy2[j_] := Entropy2[j] = Log2[(6 Pi j^2)/eps]-.014427

q = 35; (*We want 'q' as large as possible; however, this is limited by
computation time.*)

(*Below is the process of solving 'TableLowAlphk' which is underline(|S'(ε,F_k^{***})|)*)
LowAlphValuesk = Table[
  {sol1[k_] :=
    sol1[k] = Reduce[j > 0 && Entropy2[j] <= Entropy1[k], j, Integers],
   LowSamplek = Max[j /. Solve[sol1[k], {j}, Integers]],
   LowAlphk = N[LengthS2[LowSamplek]]}, {k, 3, q}];
TableLowAlphk = Table[LowAlphValuesk[[k - 3 + 1, 3]], {k, 3, q}]

(*Below is the process of solving 'TableUpAlphk' which is overline(|S'(ε,F_k^{***})|)*)
UpAlphValuesk = Table[
   {sol11[k_] :=
     sol11[k] =
      Reduce[j < 5000 && Entropy2[j] >= Entropy1[k], j, Integers],
    UpSamplek = Min[j /. Solve[sol11[k], {j}, Integers]],
    UpAlphk = N[LengthS2[UpSamplek]]}, {k, 3, q}];
TableUpAlphk = Table[UpAlphValuesk[[k - 3 + 1, 3]], {k, 3, q}]

(*Below is the process of solving 'TableLowAlphj' which is underline(|S'(ε,F_j^{**})|)*)
LowAlphValuesj = Table[
   {sol2[j_] :=
     sol2[j] =
      Reduce[k > 0 && Entropy1[k] <= Entropy2[j], k, Integers],
    LowSamplej = Max[k /. Solve[sol2[j], {k}, Integers]],
    LowAlphj = N[LengthS1[LowSamplej]]}, {j, 3, q}];
TableLowAlphj = Table[LowAlphValuesj[[j - 3 + 1, 3]], {j, 3, q}]

(*Below is the process of solving 'TableUpAlphj' which is overline(|S'(ε,F_j^{**})|)*)
UpAlphValuesj = Table[
   {sol21[j_] :=
     sol21[j] =
      Reduce[k < 5000 && Entropy1[k] >= Entropy2[j], k, Integers],
    UpSamplej = Min[k /. Solve[sol21[j], {k}, Integers]],
    UpAlphj = N[LengthS1[UpSamplej]]}, {j, 3, q}];
TableUpAlphj = Table[UpAlphValuesj[[j - 3 + 1, 3]], {j, 3, q}]

a[k_] := a[k] = TableUpAlphk[[k - 3 + 1]] (*This is underline(|S'(ε,F_k^{***})|)*)
b[k_] := b[k] = LengthS1[k] (*This is |S'(ε,F_k^{***})|*)
c[k_] := c[k] = TableLowAlphk[[k - 3 + 1]](*This is the same as overline(|S'(ε,F_k^{***})|)*)

(*'K1' is K(ε,F_k^{***})*)
K1[k_] :=
 K1[k] = N[
   RealAbs[(b[
          k] (1 + Ceiling[(b[
                k] (a[k] + 2 b[k]))/((a[k] + b[k]) (a[k] + b[k] +
                 c[k]))]) (1 + Round[a[k]/b[k]]))/((1 +
           Round[b[k]/c[k]]) (1 + Round[a[k]/c[k]])) - b[k]] + b[k]]
```

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```
a1[j_] :=
 a1[j] = TableUpAlphj[[j - 3 + 1]] (*This is |S'(ε,F_j^{**})| underline*)
b1[j_] := b1[j] = LengthS2[j] (*This is |S'(ε,F_j^{**})|*)
c1[j_] := c1[j] = TableLowAlphj[[j - 3 + 1]](*This is overline{|S'(ε,F_j^{**})|}*)

(*'K2' is K(ε,F_j^{**})*)
K2[j_] :=
 K2[j] = N[
   RealAbs[(b1[
         j] (1 + Ceiling[(b1[
                j] (a1[j] + 2 b1[j]))/((a1[j] + b1[j]) (a1[j] +
                b1[j] + c1[j]))]) (1 + Round[a1[j]/b1[j]]))/((1 +
          Round[b1[j]/c1[j]]) (1 + Round[a1[j]/c1[j]])) - b1[j]] +
    b1[j]]

(*'Mk' is M'(ε,F_k^{***})*)
Mk = Table[N[LengthS1[k] (K1[k] - LengthS1[k])], {k, 3, q - 1}]

(*'Mj' is M'(ε,F_j^{**})*)
Mj = Table[N[LengthS2[j] (K2[j] - LengthS2[j])], {j, 3, q - 1}]

(*'DownS' is underline{S}(ε,k,v*,F_j^{**})*)
DownS = Table[
  LengthS2[Flatten[
      Position[Mj, Max[Select[Mj, # <= Mk[[k - 4 + 2]] &]]]][[1]] +
    4 - 2], {k, 4, q - 3}]

(*'UpS' is overline{S}(ε,k,v*,F_j^{**})*)
UpS = Table[
  LengthS2[Flatten[
      Position[Mj, Min[Select[Mj, # >= Mk[[k - 4 + 2]] &]]]][[1]] +
    4 - 2], {k, 4, q - 3}]

E1 = 1 (*Constant rate of expansion*)
dimH = 2 (*Hausdorff Dimension of A*)

(*'Ak' is the smallest 2-dimensional box or [-√(6k),√(6k)] × [-√(6k),√(6k)]
covering F_k^{***} with an area of 24k^2*)
Ak[k_]:= Ak[k] = 24k^2

(*'V' is V(ε,F_k^{***}) or eq. 120. Note, n is the dimension
of n-Euclidean Plane for which A is a subset*)
V[k_, n_] :=
 V[k, n] =
  Ceiling[((Ak[k])^(1 - Sign[E1])) (E1 + (1 - Sign[E1])) ((Pi^(n/2))/
        Gamma[n/2 + 1]) (k!^(n -
          dimH)) (k^(Sign[E1] (dimH - Sign[dimH] + 1))) + (1 -
         Sign[dimH])
      Simplify[V[k]]/eps]/LengthS1[k]

(* We couldn't add v, v* or convert this to a limit due to
limitations of the programming *)
ChoiceFunction =
 Table[N[((Sign[Mk[[k - 5 + 2]]] UpS[[k - 5 + 2]])/(LengthS1[
          k]) - (LengthS1[k]/LengthS2[k])^(-V[k,2]))*((Sign[Mk[[k - 5 + 2]]]
          DownS[[k - 5 + 2]])/(LengthS1[k]) -
         (LengthS1[k]/LengthS2[k])^(-V[k,2]))], {k, 5, q - 3}]
```

7. Questions

1. Does §6 answer the **leading question** in §3.2
2. Using §1.1 and thm. 3, when $A = A'$ does $\text{Avg}((F_k^{\star\star\star}), A')$ have a finite value?
3. Using §1.2 and thm. 3, when $A = A''$ does $\text{Avg}((F_k^{\star\star\star}), A'')$ have a finite value?
4. If there's no time to check questions 1 and 3, see §4.

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8. Appendix of §5.4.1

8.1. Example of §5.4.1, step 1

Suppose

1. $A = \mathbb{R}^2$
2. $(F_r^\star)_{r\in\mathbb{N}} = ([-r,r]\times[-r,r])_{r\in\mathbb{N}}$

Then one example of $\mathbf{C}(13/6, F_2^\star, 1)$, using §5.4.1 step 1, (where $F_2^\star = [-2,2]\times[-2,2]$) is:

$$\begin{aligned}&\left\{\left[-\sqrt{\frac{13}{3}},\sqrt{\frac{13}{3}}\right]\times\left[-\sqrt{\frac{13}{3}},-\frac{3}{4}\sqrt{\frac{13}{3}}\right),\left[-\sqrt{\frac{13}{3}},\sqrt{\frac{13}{3}}\right]\times\left[-\frac{3}{4}\sqrt{\frac{13}{3}},-\frac{1}{2}\sqrt{\frac{13}{3}}\right),\left[-\sqrt{\frac{13}{3}},\sqrt{\frac{13}{3}}\right]\times\left[-\frac{1}{2}\sqrt{\frac{13}{3}},-\frac{1}{4}\sqrt{\frac{13}{3}}\right),\right.\\&\left[-\sqrt{\frac{13}{3}},\sqrt{\frac{13}{3}}\right]\times\left[-\frac{1}{4}\sqrt{\frac{13}{3}},0\right),\left[-\sqrt{\frac{13}{3}},\sqrt{\frac{13}{3}}\right]\times\left[0,\frac{1}{4}\sqrt{\frac{13}{3}}\right),\left[-\sqrt{\frac{13}{3}},\sqrt{\frac{13}{3}}\right]\times\left[\frac{1}{4}\sqrt{\frac{13}{3}},\frac{1}{2}\sqrt{\frac{13}{3}}\right),\\&\left.\left[-\sqrt{\frac{13}{3}},\sqrt{\frac{13}{3}}\right]\times\left[\frac{1}{2}\sqrt{\frac{13}{3}},\frac{3}{4}\sqrt{\frac{13}{3}}\right),\left[-\sqrt{\frac{13}{3}},\sqrt{\frac{13}{3}}\right]\times\left[\frac{3}{4}\sqrt{\frac{13}{3}},\sqrt{\frac{13}{3}}\right]\right\}\end{aligned} \tag{128}$$

Note, the area of each of the rectangles is 13/6, where the borders could be approximated as:

$$\begin{aligned}&\{[-2.082,2.082]\times[-2.082,-1.561),[-2.082,2.082]\times[-1.561,-1.041),[-2.082,2.082]\times[-1.041,-.520),\\&[-2.082,2.082]\times[-.520,0),[-2.082,2.082]\times[0,.520),[-2.082,2.082]\times[.520,1.041),\\&[-2.082,2.082]\times[1.041,1.561),[-2.082,2.082]\times[1.561,2.082]\}\end{aligned} \tag{129}$$

and we'll illustrate this as purple rectangles covering $F_2^\star$ (i.e., the red square).

(Note, the purple rectangles in Figure 1, satisfy step (1) of §5.4.1, since the Hausdorff measures *in its dimension* of the rectangles is 13/6 and there is a minimum 8 covers over-covering $F_2^\star$: i.e.,

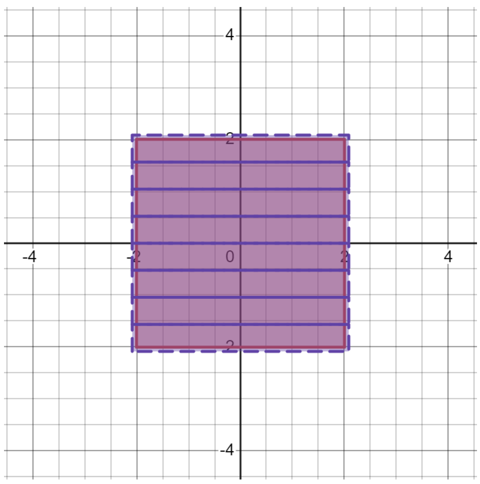

Figure 1. Purple rectangles are the "covers" and the red square is $F_2^\star$. (Ignore the boundaries).

Definition 3 (**Minimum Covers of Measure ε covering $F_r^\star$**)**.** *We can compute the minimum covers $\mathbf{C}(\varepsilon, F_r^\star, \omega)$, using the formula:*

$$\lceil \mathcal{H}^{\dim_H(F_r^\star)}(F_r^\star)/\varepsilon \rceil$$

where $\lceil \mathcal{H}^{\dim_H(F_2^\star)}(F_2^\star)/\varepsilon \rceil = \lceil \text{Area}([-2,2]^2)/(13/6) \rceil = \lceil 16/(13/6) \rceil = \lceil 96/13 \rceil = \lceil 7+(5/13) \rceil = 8$)

Note the covers in $\mathbf{C}(13/6, F_r^\star, \omega)$ need not be rectangles. In fact, they could be any set as long as the "area" of those sets is 13/6, and $F_2^\star$ is over-covered by the smallest number of sets possible. Here is an example:

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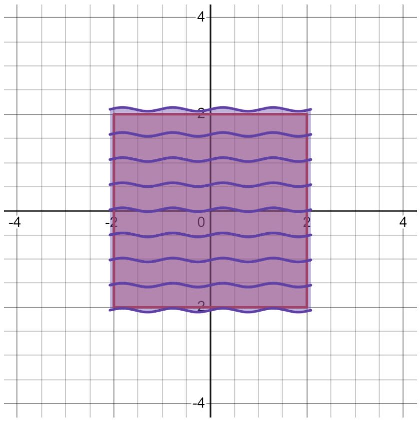

Figure 2. The eight purple sets are the "covers" and the red square is $F_2^{\star}$. (Ignore the boundaries).

To define this cover, start off with:

$$B_1(x) = -2.062 + .04\sin(6.037x) \tag{130}$$
$$B_2(x) = -1.541 + .04\sin(6.037x) \tag{131}$$
$$B_3(x) = -1.021 + .04\sin(6.037x) \tag{132}$$
$$B_4(x) = -0.500 + .04\sin(6.037x) \tag{133}$$
$$B_5(x) = 0.020 + .04\sin(6.037x) \tag{134}$$
$$B_6(x) = 0.540 + .04\sin(6.037x) \tag{135}$$
$$B_7(x) = 1.061 + .04\sin(6.037x) \tag{136}$$
$$B_8(x) = 1.581 + .04\sin(6.037x) \tag{137}$$
$$B_9(x) = 2.102 + .04\sin(6.037x) \tag{138}$$
$$\tag{139}$$

Then, for each $i \in \mathbb{N}$ in $1 \le i \le 7$, we define:

$$\mathcal{B}_i = \{(x,y) : -2.082 \le x \le 2.082, B_i(x) \le y < B_{i+1}(x)\}$$

except $i = 8$ where:

$$\mathcal{B}_8 = \{(x,y) : -2.082 \le x \le 2.082, B_8(x) \le y \le B_9(x)\}$$

such that an example of $\mathbf{C}(13/6, F_2^{\star}, 2)$ is:

$$\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8\} \tag{140}$$

In the case of $F_2^{\star}$, there are uncountably many $\mathbf{C}(13/6, F_2^{\star}, \omega)$ of *different shapes and sizes* which we can use. However, these examples were the ones taken.

8.2. Example of §5.4.1, step 2

Suppose

1. $A = \mathbb{R}^2$
2. $(F_r^{\star})_{r\in\mathbb{N}} = ([-r,r] \times [-r,r])_{r\in\mathbb{N}}$
3. $F_2^{\star} = [-2,2] \times [-2,2]$

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4. $\mathbf{C}(13/6, F_2^{\star}, 3)$, using eq. 129 and Figure 1, is

$$\{[-2.082, 2.082] \times [-2.082, -1.561), [-2.082, 2.082] \times [-1.561, -1.041), [-2.082, 2.082] \times [-1.041, -.520),$$
$$[-2.082, 2.082] \times [-.520, 0), [-2.082, 2.082] \times [0, .520), [-2.082, 2.082] \times [.520, 1.041), \quad (141)$$
$$[-2.082, 2.082] \times [1.041, 1.561), [-2.082, 2.082] \times [1.561, 2.082]\}$$

Then, an example of $\mathcal{S}(\mathbf{C}(13/6, F_2^{\star}, 3), 1)$ is:

$$\{(-1.5, -1.7), (1.5, -1.2), (-1.5, -.7), (-1.5, -.2), (-1.5, .3), (-1.5, .8), (-1.5, 1.3), (-1.5, 1.8)\} \quad (142)$$

Below, is an illustration of the sample: i.e., the set of all black points *in each purple rectangle of* $\mathbf{C}(13/6, F_2^{\star}, 3)$ covering $F_2^{\star} = [-2, 2]^2$:

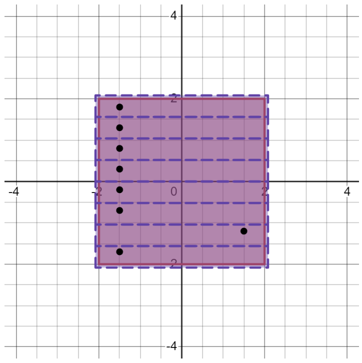

Figure 3. The black points are the "sample points", the eight purple rectangles are the "covers", and the red square is $F_2^{\star}$. (Ignore the boundaries).

Note there are multiple samples we can take, as long as one sample point is taken from each cover in $\mathbf{C}(13/6, F_2^{\star}, 1)$.

8.3. Example of §5.4.1, step 3

Suppose

1. $A = \mathbb{R}^2$
2. $(F_r^{\star})_{r \in \mathbb{N}} = ([-r, r] \times [-r, r])_{r \in \mathbb{N}}$
3. $F_2^{\star} = [-2, 2] \times [-2, 2]$
4. $\mathbf{C}(13/6, F_2^{\star}, 3)$, using eq. 129 and Figure 1, is

$$\{[-2.082, 2.082] \times [-2.082, -1.561), [-2.082, 2.082] \times [-1.561, -1.041), [-2.082, 2.082] \times [-1.041, -.520),$$
$$[-2.082, 2.082] \times [-.520, 0), [-2.082, 2.082] \times [0, .520), [-2.082, 2.082] \times [.520, 1.041),$$
$$[-2.082, 2.082] \times [1.041, 1.561), [-2.082, 2.082] \times [1.561, 2.082]\}$$

5. $\mathcal{S}(\mathbf{C}(13/6, F_2^{\star}, 3), 1)$, using eq. 142, is:

$$\{(-1.5, -1.7), (1.5, -1.2), (-1.5, -.7), (-1.5, -.2), (-1.5, .3), (-1.5, .8), (-1.5, 1.3), (-1.5, 1.8)\} \quad (143)$$

Therefore, consider the following process:

8.3.1. Step 33a

If $\mathcal{S}(\mathbf{C}(13/6, F_2^{\star}, 3), 1)$ is:

$$\{(-1.5, -1.7), (1.5, -1.2), (-1.5, -.7), (-1.5, -.2), (-1.5, .3), (-1.5, .8), (-1.5, 1.3), (-1.5, 1.8)\} \quad (144)$$

suppose $x_0 = (-1.5, -1.7)$. Note, the following:

1. $x_1 = (-1.5, -.7)$ is the next point in the "pathway" since it's a point in $\mathcal{S}(\mathbf{C}(13/6, F_2^\star, 3), 1)$ with the smallest 2-d Euclidean distance to x_0 instead of x_0.
2. $x_2 = (-1.5, -.2)$ is the third point since it's a point in $\mathcal{S}(\mathbf{C}(13/6, F_2^\star, 3), 1)$ with the smallest 2-d Euclidean distance to x_1 instead of x_0 and x_1.
3. $x_3 = (-1.5, .3)$ is the fourth point since it's a point in $\mathcal{S}(\mathbf{C}(13/6, F_2^\star, 3), 1)$ with the smallest 2-d Euclidean distance to x_2 instead of x_0, x_1, and x_2.
4. we continue this process, where the "**pathway**" of $\mathcal{S}(\mathbf{C}(13/6, F_2^\star, 3), 1)$ is:

$$(-1.5, -1.7) \to (-1.5, -.7) \to (-1.5, -.2) \to (-1.5, .3) \to (-1.5, .8) \to (-1.5, 1.3) \to (-1.5, 1.8) \to (1.5, -1.2) \tag{145}$$

Note 12. If more than one point has the minimum 2-d Euclidean distance from x_0, x_1, x_2, etc. take all potential pathways: e.g., using the sample in eq. 144, if $x_0 = (-1.5, -.2)$, then since $(-1.5, -.7)$ and $(-1.5, .3)$ have the smallest Euclidean distance from $(-1.5, -.2)$, and for point $x_1 = (-1.5, -.7)$, since $(-1.5, -1.7)$ and $(-1.5, .3)$ have the smallest Euclidean distance from $(-1.5, -.7)$, we take **three** pathways:

$$(-1.5, -.2) \to (-1.5, -.7) \to (-1.5, -1.7) \to (-1.5, .3) \to (-1.5, .8) \to (-1.5, 1.3) \to (-1.5, 1.8) \to (1.5, -1.2) \quad \text{(path 1)}$$
$$(-1.5, -.2) \to (-1.5, -.7) \to (-1.5, .3) \to (-1.5, .8) \to (-1.5, 1.3) \to (-1.5, 1.8) \to (-1.5, -1.7) \to (1.5, -1.2) \quad \text{(path 2)}$$
$$(-1.5, -.2) \to (-1.5, .3) \to (-1.5, .8) \to (-1.5, 1.3) \to (-1.5, 1.8) \to (-1.5, -.7) \to (-1.5, -1.7) \to (1.5, -1.2) \quad \text{(path 3)}$$

8.3.2. Step 33b

Take the length of all line segments in the pathway. In other words, suppose $d(P, Q)$ is the n-th dim. Euclidean distance between points $P, Q \in \mathbb{R}^n$. Using the pathway of eq. 145, we want:

$$\{d((-1.5, -1.7), (-1.5, -.7)), d((-1.5, -.7), (-1.5, -.2)), d((-1.5, -.2), (-1.5, .3)), d((-1.5, .3), (-1.5, .8)), d((-1.5, .8), (-1.5, 1.3)), d((-1.5, 1.3), (-1.5, 1.8)), d((-1.5, 1.8), (1.5, -1.2))\} \tag{146}$$

Whose distances can be approximated as:

$$\{1, .5, .5, .5, .5, .5, 4.25\} \tag{147}$$

As we can see, the outlier [10] is 4.25 (i.e., note these outliers become more prominent for $\varepsilon \ll 13/6$). Therefore, remove 4.25 from the set of distances:

$$\{1, .5, .5, .5, .5, .5\}$$

which we can illustrate with:

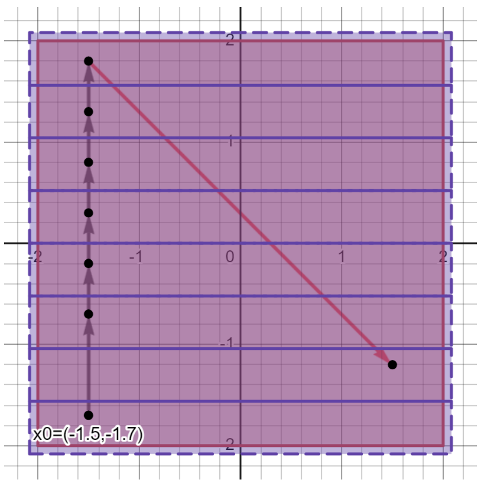

Figure 4. x_0 is the start point in the pathway. The black line segments in the "pathway" have lengths which aren't outliers. The length of the red line segment is an outlier.

Hence, when $x_0 = (-1.5, -1.7)$, using §5.4.1 step 33b & eq. 144, we note:

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$$\mathcal{L}((-1.5,-1.7),\mathcal{S}(\mathbf{C}(13/6,F_2^\star,3),1)) = \{1,.5,.5,.5,.5,.5\} \tag{148}$$

8.3.3. Step 33c

To convert the set of distances in eq. 148 into a probability distribution, we take:

$$\sum_{x\in\{1,.5,.5,.5,.5,.5\}} x = 1+.5+.5+.5+.5+.5 = 3.5 \tag{149}$$

Then divide each element in $\{1,.5,.5,.5,.5,.5\}$ by 3.5

$$\{1/(3.5),.5/(3.5),.5/(3.5),.5/(3.5),.5/(3.5),.5/(3.5)\}$$

which gives us the probability distribution:

$$\{2/7,1/7,1/7,1/7,1/7,1/7\}$$

Hence,

$$\mathbb{P}(\mathcal{L}((-1.5,-1.7),\mathcal{S}(\mathbf{C}(\varepsilon,F_2^\star,3),1))) = \{2/7,1/7,1/7,1/7,1/7,1/7\} \tag{150}$$

8.3.4. Step 33d

Take the shannon entropy of eq. 150:

$$\begin{aligned}&\mathrm{E}(\mathbb{P}(\mathcal{L}((-1.5,-1.7),\mathcal{S}(\mathbf{C}(13/6,F_2^\star,1),3)))) =\\&\sum_{x\in\mathbb{P}(\mathcal{L}((-1.5,-1.7),\mathcal{S}(\mathbf{C}(13/6,F_r,\omega),\psi)))} -x\log_2 x = \sum_{x\in\{2/7,1/7,1/7,1/7,1/7,1/7\}} -x\log_2 x =\\&-(2/7)\log_2(2/7)-(1/7)\log_2(1/7)-(1/7)\log_2(1/7)-(1/7)\log_2(1/7)-(1/7)\log_2(1/7)-(1/7)\log_2(1/7) =\\&-(2/7)\log_2(2/7)-(5/7)\log_2(1/7)\approx 2.5216\end{aligned}$$

We shorten $\mathrm{E}(\mathbb{P}(\mathcal{L}((-1.5,-1.7),\mathcal{S}(\mathbf{C}(13/6,F_2^\star,3),1))))$ to $\mathrm{E}(\mathcal{L}((-1.5,-1.7),\mathcal{S}(\mathbf{C}(13/6,F_2^\star,3),1)))$, giving us:

$$\mathrm{E}(\mathcal{L}((-1.5,-1.7),\mathcal{S}(\mathbf{C}(13/6,F_2^\star,3),1)))\approx 2.5216 \tag{151}$$

8.3.5. Step 33e

Take the entropy w.r.t all pathways of:

$$\begin{aligned}&\mathcal{S}(\mathbf{C}(13/6,F_2^\star,1),3))) =\\&\{(-1.5,-1.7),(1.5,-1.2),(-1.5,-.7),(-1.5,-.2),(-1.5,.3),(-1.5,.8),(-1.5,1.3),(-1.5,1.8)\}\end{aligned}$$

In other words, we'll compute:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(13/6,F_2^\star,3),1))) = \sup_{x_0\in\mathcal{S}(\mathbf{C}(13/6,F_2^\star,3),1)} \mathrm{E}(\mathcal{L}(x_0,\mathcal{S}(\mathbf{C}(13/6,F_2^\star,3),1)))$$

We do this by repeating §8.3.1-§8.3.4 for different $x_0\in\mathcal{S}(\mathbf{C}(13/6,F_2^\star,1),3)))$ (i.e., in the equations with multiple values, see note 12)

$$\mathrm{E}(\mathcal{L}((-1.5,-1.7),\mathcal{S}(\mathbf{C}(13/6,F_2^\star,3),1)))\approx 2.52164 \tag{152}$$
$$\mathrm{E}(\mathcal{L}((1.5,-1.2),\mathcal{S}(\mathbf{C}(13/6,F_2^\star,3),1)))\approx 2.34567 \tag{153}$$
$$\mathrm{E}(\mathcal{L}((-1.5,-0.7),\mathcal{S}(\mathbf{C}(13/6,F_2^\star,3),1)))\approx 2.2136 \tag{154}$$
$$\mathrm{E}(\mathcal{L}((-1.5,-0.2),\mathcal{S}(\mathbf{C}(13/6,F_2^\star,3),1)))\approx 2.32193, 2.11807, 2.3703 \tag{155}$$
$$\mathrm{E}(\mathcal{L}((-1.5,0.3),\mathcal{S}(\mathbf{C}(13/6,F_2^\star,3),1)))\approx 2.24244, 2.40051 \tag{156}$$
$$\mathrm{E}(\mathcal{L}((-1.5,0.8),\mathcal{S}(\mathbf{C}(13/6,F_2^\star,3),1)))\approx 2.22393, 2.41651 \tag{157}$$
$$\mathrm{E}(\mathcal{L}((-1.5,1.3),\mathcal{S}(\mathbf{C}(13/6,F_2^\star,3),1)))\approx 2.22208, 2.4076 \tag{158}$$
$$\mathrm{E}(\mathcal{L}((-1.5,1.8),\mathcal{S}(\mathbf{C}(13/6,F_2^\star,3),1)))\approx 2.52164 \tag{159}$$

Hence, since the largest value out of eq. 152-159 is 2.52164:

$$\mathrm{E}(\mathcal{L}(\mathcal{S}(\mathbf{C}(13/6,F_2^\star,3),1))) = \sup_{x_0\in\mathcal{S}(\mathbf{C}(\varepsilon,F_2^\star,3),1)} \mathrm{E}(\mathcal{L}(x_0,\mathcal{S}(\mathbf{C}(\varepsilon,F_2^\star,3),1)))\approx 2.52164$$

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