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Absence of Electric Field outside a Static Charged Black Hole – Something Awry in Reissner–Nordström Metric

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Using the principle of equivalence, it has recently been shown that the electrostatic field lines of a charge, stationary in the gravitational field, bend exactly like the trajectories of photons emitted isotropically from a source at the charge location and that the fraction of electric flux crossing a surface 'below' or 'above' the charge is exactly similar to the fraction of photon trajectories intersecting these surfaces, with more flux in the downward direction than upward. As one goes much deeper in the gravitational field, all electric field lines increasingly point in the vertically downward direction as is also the case for a stream of photons. Since photon trajectories as well as electric field lines, at any location in the gravitational field, are affected by the local space-time curvature an inference can be drawn that this parallel between the photon trajectories and the electric field lines is a general result. We could then apply these results in the external gravitational field of a black hole, where the trajectories of photons in the gravitational field are already well-known and the behaviour of electric field lines of a stationary charge could be inferred therefrom. Accordingly, we show that the electric field through an external spherical surface surrounding the black hole steadily reduces as the charge location approaches the event horizon (Schwarzschild radius), and like photons from a source inside the Schwarzschild radius cannot escape outside, the electric field lines of a charge within the black hole too remain trapped inside the event horizon. From this one arrives at a conclusion that, contrary to the conventional wisdom, the electric charge contained inside a static black hole cannot be detected or inferred by an external observer. A black hole, said to have no hair with the only external identifying characteristics being mass, electric charge, and angular momentum, is therefore all the more 'hairless', as even its charge cannot be ascertained. The derivation of the Reissner–Nordström metric, supposedly describing the gravitational field of a static charged black hole, presumes an external stress-energy tensor of the electrostatic field, as per Gauss law, even for the charges contained within the black hole. However, the absence of electric flux external to a static charged black hole implies that such a charged black hole is not described correctly by the Reissner–Nordström metric and the consequential peculiarities of the space-time geometry, leading in specific cases to the idea of a naked intrinsic singularity and a need for the "cosmic censorship" hypothesis, also do not arise here.

Keywords: Black Hole has no hair; Electromagnetic field of a charged black hole; electromagnetic field of a supported charge in gravitational field; electric fields follow photo trajectories in gravitational field

I. INTRODUCTION

Almost immediately after Schwarzschild arrived at an exact solution for a static spherical mass, in terms of (t, r, θ, ϕ), called Schwarzschild coordinates [1], Reissner and Nordström [2, 3], independently, came up with a solution for an electrically charged static spherical mass known as Reissner–Nordström (RN) metric, which supposedly describes the external gravitational field of a static charged black hole [4–9]. The derivation of the RN metric presumes a stress-energy tensor of the electrostatic field, in regions external to the the black hole, due to the electric charge within the black hole and the conventional wisdom is [4–9] that this electrostatic field obeys Gauss law, like in the case of any other electrically charged system, even when the region in question might be encompassing a space-time singularity.

Using the principle of equivalence, it has recently been shown [10] that in a uniform gravitational field not only the photon trajectories follow a curved path, even the electrostatic field lines of a charge follow exactly the bent trajectories of photons emitted isotropically from a source situated at the charge location. The fraction of electric field lines crossing a surface, above or below the charge stationary in the gravitational field, has been shown to be exactly similar to the fraction of photon trajectories intersecting that surface. As one goes much deeper in the gravitational field, the electric field lines, even if initially along horizontal directions while emanating from the charge, point progressively downward vertically, as is also the case for a stream of photons.

Maxwell equations can be transformed from a flat spacetime to a curved spacetime by replacing commas (partial derivatives) with semicolons (covariant derivatives), which is nothing but rephrasing the equivalence principle where a change from local flat spacetime to curved spaceetime takes place by such a replacement [4, 8, 11]. In the case of a black hole such an approach, which could be rather involved, has been made in literature to get a general, multipole solution in terms of Legendre polynomials for the electric potential of a charge stationary in the Schwarzschild space [12, 13].

However, our interest here is a much simpler problem of studying what fraction of electrostatic field lines reaches the horizon for a charge stationary at a certain 'height' in the gravitational field. This fraction has been

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shown to be similar for the trajectories of photons from a source at the charge position in a uniform gravitational field. Now, a uniform gravitational field and the horizon encountered there is an archetypal of a black hole field and the event horizon there [6, 14], and behaviour of many a phenomenon in the external gravitational field of a black hole field can be studied by analogy from a static 'uniform' gravitational field, which in turn is derived from a comoving uniformly accelerated frame [10], using the equivalence principle [4, 11]. It has been demonstrated from the equivalence principle applied to the charge stationary at different 'heights' in the gravitational field in Rindler space (of a uniformly accelerated charge) that the electric field through an external surface 'above' the charge reduces as the charge approaches the event horizon [10].

From the equivalence principle, the trajectories of the photons as well as the electric field lines, assumedly along straight lines in the local Lorentz frame, at any location in the gravitational field would be affected by the local space-time curvature, independent of the ultimate source of the gravitational field. Therefore we expect that this parallel between the trajectories of the photons and the electric field lines should be a general result. This actually has a genesis in the fact that both influences, viz. the electric field and the photons, start from their respective sources in radial directions, moving with the same speed c, therefore their transformations between different frames should be similar too. From that we infer the reduction of the electric flux external to a charged black hole, as the position of the stationary charge approaches the event horizon (Schwarzschild radius) in the case of a black hole, becoming zero when the charge position crosses the event horizon or is within the Schwarzschild radius of a black hole.

Actually no particle or non-gravitational field or influence will ever cross the Schwarzschild radius to go from inside to the outside of event horizon of the black hole. Event horizon around a black hole is a boundary through which matter and light can only pass inward towards the central black hole. Nothing, not even light, can escape out from inside the event horizon [4, 6–9]. As predicted by the general relativity, the presence of a mass deforms space-time in such a way that the paths taken by particles bend towards the mass. At the event horizon of a black hole, this deformation becomes so strong that there are no paths that lead away from the black hole. The photon sphere is a spherical boundary of zero thickness such that photons moving along tangents to the sphere will be trapped in a circular orbit [4, 6, 7]. For non-rotating black holes, the photon sphere has a radius 1.5 times the Schwarzschild radius. While light can still escape from inside the photon sphere, any light that crosses the photon sphere on an inbound trajectory will be captured by the black hole. Any light reaching an outside observer from inside the photon sphere must have been emitted within a certain critical (acute) angle with respect to the outward radial direction, by objects inside the photon sphere but still outside of the event horizon [4, 6, 9].

Thus, in the case of a black hole, the trajectories of photons in the gravitational field are well-known, and is a text-book material [4, 6, 7]. Therefore, in order to study the behaviour of electric field lines, we do not need to perform separate calculations because the configuration of the electric field of a supported charge can be obtained from the already-known trajectories of photons. If the photons from a source within certain radial coordinate, the Schwarzschild radius, cannot escape out to infinity and remain trapped inside the event horizon of the black hole, the same has to be true for the electric fields of a charge too. We shall thence demonstrate the reduction of the electric flux external to a charged black hole, as the position of the stationary charge approaches the event horizon (Schwarzschild radius), becoming zero when the charge position crosses the Schwarzschild radius or is within the black hole event horizon.

Here we shall be analysing the situation for a collapsed charged black hole and would not be concerned with the possibility that for an external observer any further electric charge falling into the already collapsed charged black hole may need infinite amount of external observer's time to pass through the event horizon, and all perceived black holes might for ever have their infalling charge ever outside the event horizon as far as an external observer is concerned.

According to the no-hair theorem [4], a collapsed black hole has no other distinguishing properties except its mass, angular momentum and the electric charge. Apart from these, there is no external characteristic distinguishing one black hole from another. Any two black holes that share the same values for these properties, or parameters, are indistinguishable according to classical physics. Black holes, therefore, are classified commonly according to their mass M, angular momentum J or/and electric charge Q. In particular, it is thought that a static charged black hole, with charges inside the event horizon, has, in addition to its gravitational field, an external electromagnetic stress-energy tensor, quite like around any other charged object. However, as we shall demonstrate the electric field lines too would not extend outside the event horizon, from the electric charges inside the black hole. Therefore one infers that the external electric field of a charged black hole is zero and that the electric charge contained inside a static black hole cannot be detected or inferred by an external observer. Accordingly, as we shall argue, a static charged black hole is not correctly described by the Reissner–Nordström metric and the consequential peculiarities of its space-time geometry. Moreover, a black hole, with only two external identifying characteristics, viz. its mass and angular momentum, can be said to be even more 'hairless', as even its charge cannot be ascertained.

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II. THE CRITERIA FOR PHOTONS OR ELECTRIC FIELD LINES REACHING THE HORIZON IN A SCHWARZSCHILD GEOMETRY

The metric for a static spherical mass M in Schwarzschild coordinates (t, r, θ, ϕ) is written as [1]

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2\mathrm{d}t^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right), \quad (1)$$

where we have adopted the sign convention of Misner et al [4]. The metric can be expressed alternately in terms of Schwarzschild radius, $r_\mathrm{s} = 2GM/c^2$, as

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s}{r}\right) c^2\mathrm{d}t^2 + \left(1 - \frac{r_s}{r}\right)^{-1} \mathrm{d}r^2 + r^2\mathrm{d}\omega^2, \quad (2)$$

where $\mathrm{d}\omega = \sqrt{\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2}$ is an element of solid angle.

Assuming it to be a "point" mass, the only intrinsic singularity in the Schwarzschild geometry occurs at $r = 0$. However, there is, in addition, a coordinate singularity at the Schwarzschild radius, r_s and it turns out that r_s presents a sort of barrier, called event horizon, from inside of which nothing, not even photons, can emerge out of the black hole. It is interesting to note that in Newtonian mechanics r_s corresponds to a radial distance where the escape velocity is equal to the speed of light.

From the trajectories of photons emitted from a source at different radial coordinate r in the external $(r > r_\mathrm{s})$ gravitational field of a black hole, we want to determine the fraction f of photons reaching the horizon in a Schwarzschild geometry. This would also tell us the fraction $1 - f$ of photons that could emerge out of the gravitational field of a black hole to ultimately escape to infinity, even if highly redshifted.

Suppose a source $\mathcal{S}$ of photons is supported at a radial coordinate r_0 in the gravitational field of a central spherical mass M with a corresponding Schwarzschild radius $r_\mathrm{s} = 2GM/c^2$ (Fig. 1). Trajectories of photons emitted by $\mathcal{S}$ at r_0 along directions in a plane perpendicular to the radial coordinate $(\psi_0 = \pi/2)$ will get bent due to gravity by an angle $\Delta\psi \approx \zeta^{-1}$ for $\zeta \gg 1$, where $\zeta = r_0/r_\mathrm{s}$ is the radial coordinate expressed in units of the Schwarzschild radius r_s [4, 6, 7]. This has been amply confirmed observationally where trajectories of photons from distant astronomical objects coming along a tangential direction to Sun's limb, get bent in Sun's gravitational field, with $2\zeta^{-1} \approx 1.75$ arcsec as the angle between the two asymptotes to the photon trajectories before and after the deflection [4, 6, 7, 15, 16]. This is consistent with $\zeta = r_0/r_\mathrm{s} \approx 2.36 \times 10^5$ for Sun's limb at $r_0 = 6.96 \times 10^5$ km and its Schwarzschild radius ≈ 2.95km, corresponding to Sun's mass $M = 1.99 \times 10^{30}$ kg. Now as r_0 becomes smaller, the bending would increase, and we might expect it to become so severe, as r_0 approaches r_s, that the photon might not be able to escape out to infinity and instead get captured by the central mass. This scenario is equally applicable to an electric field line that, starting horizontally from a charge held at ζ, would bend due to gravity by an angle ζ^{-1}, the bending increasing at lower ζ values, eventually ending in the central mass as $\zeta \to 1$.

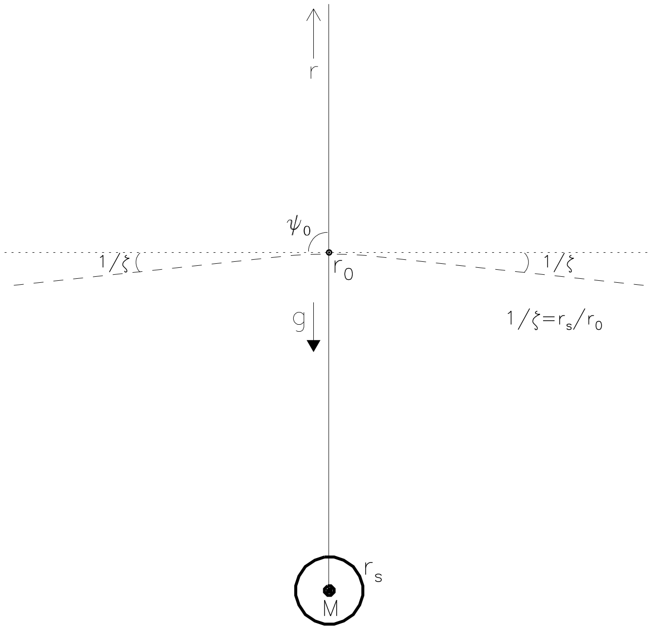

FIG. 1. A schematic of the bending of photon trajectories (electric field lines) in the gravitational field of a central spherical mass M. The source of photons (electric charge) is held stationary at a radial coordinate r_0, much larger than the Schwarzschild radius r_s of M. The initial photon trajectories (electric field lines), starting from the source (charge) along ψ_0 in the horizontal plane, are represented by the dotted line. The dashed curve represents the photon trajectories (field lines) bent due to gravity, with asymptotes at angle $\Delta\psi \approx \zeta^{-1} = r_\mathrm{s}/r_0$ with respect to the initial directions. The figure is drawn for $\zeta = r_0/r_\mathrm{s} = 10$.

Actually, in the case of a black hole, the trajectories of photons through the gravitational field are well-known for different $\zeta > 1$ values. Thus we do not need to do separate computations for the electric field lines because the configuration of the electric field of a supported charge can be obtained from the known trajectories of photons. If photons from a source at certain r_0 cannot escape out to infinity and do get trapped inside the event horizon of the black hole, the same has to be true for electric field lines from a charge located at that r_0 and these electric field lines too should also end up inside the event horizon and not extend outside to infinity. Thus in the case of a black hole, the electrostatic field lines following the path of photons emitted at the charge location, and if photons from within the Schwarzschild radius could not escape a black hole's event horizon, then the electric field lines too, from electric charges lying within the black hole, would not extend outside the event horizon.

Gravitational bending of electrostatic field lines, similar to the photon trajectories, is applicable at all radial coordinates beyond Schwarzschild radius, $\zeta > 1$, where the trajectories of photons through the gravitational field of a black hole are well understood and are now a textbook material [4, 6, 7, 9]. The ultimate fate of a photon emitted within the gravitational field at a given radial value $\zeta > 1$ from the central mass, depends on the initial angle ψ_0 (made with respect to the radial outward direction) compared to a critical value ψ_c at ζ [4, 6, 9], given by

$$\sin\psi_c = \frac{3\sqrt{3(1-1/\zeta)}}{2\zeta}\,. \tag{3}$$

Putting $\zeta = 3/2$, we get $\psi_c = \pi/2$. However, for any other $\zeta \neq 3/2$ but $\zeta > 1$ value, there can be an ambiguity in the ψ_c value, determined from Eq.(3), since $\sin(\pi - \psi_c) = \psi_c$. To resolve the ambiguity, we adopt the convention that for $\zeta < 3/2$, $\psi_c < \pi/2$, i.e., ψ_c lies between 0 and $< \pi/2$, while for $\zeta > 3/2$, $\psi_c > \pi/2$, implying ψ_c lies between $\pi/2$ and π. Then only photons emitted with $\psi_0 < \psi_c$ could escape to infinity.

Thus photons emitted even outward, i.e. $\psi_0 < \pi/2$ but along $\psi_0 > \psi_c$, which could happen for $\zeta < 3/2$, will not escape to infinity, instead attain a maximum distance and turn back to fall into the horizon [6]. On the other hand photons emitted inward but within the critical angle, i.e. $\pi/2 < \psi_0 < \psi_c$, which could happen for $\zeta > 3/2$, will escape to infinity.

III. THE FRACTION OF PHOTONS CAPTURED BY THE BLACK HOLE

For photons emitted from a source $\mathcal{S}$, stationed at various radial coordinates ζ in the gravitational field, we can compute the fraction of photon that would ultimately escape to infinity. Only photons emitted along $\psi_0 < \psi_c$ (Eq. 3) from the source at ζ, can escape the gravitational field and ultimately go to infinity. Let n be the temporal rate of number of photons emitted per unit solid angle by the source $\mathcal{S}$, assuming an isotropic initial distribution. If f is the fraction of the photons that falls in the horizon, then $4\pi n(1-f)$, the photon number-flux escaping the gravitational field, is calculated from

$$4\pi n(1-f) = n\int_0^{\psi_c} 2\pi\sin\psi\,\mathrm{d}\psi = 2\pi n(1-\cos\psi_c)\,, \tag{4}$$

implying, the fraction escaping the gravitation pull of the black hole to be

$$1-f = \frac{1-\cos\psi_c}{2}\,, \tag{5}$$

For very large initial distances, i.e. $\zeta \to \infty$, $\psi_c \to \pi$ and all photons would escape to infinity and photon number-flux through a spherical surface Σ, of large enough radius $r > r_0$ around the black hole and thus enclosing the source $\mathcal{S}$, would be $4\pi n$. However, for finite ζ, but with $\zeta \gg 1$, we can write $\sin\psi_c \approx 3\sqrt{3}/2\zeta \ll 1$, implying $\psi_c \approx \pi - 3\sqrt{3}/2\zeta$ and the number of photons that would escape to infinity or equivalently the photon number-flux through Σ, would be $2\pi n(1-\cos\psi_c) \approx 4\pi n(1 - 27/16\zeta^2)$. This is because photons that get immersed in the event horizon, do not emerge out again.

The fraction f of photons, reaching the event horizon and thus disappearing in the black hole, is calculated from Eq. (5) as

$$f = \frac{1+\cos\psi_c}{2}\,, \tag{6}$$

To get an idea of this fraction at different heights, for a source at $\zeta = 2.36\times10^5$ (the surface of Sun), the critical angle $\psi_c \sim \pi - 10^{-5}$ radians and $f \approx 3\times10^{-11}$, an insignificant fraction that may not escape the gravitational field. However, for decreasing values of ζ, the critical angle ψ_c also decreases and accordingly the fraction f steadily increases. For example, at $\zeta = 10$, $f = 0.015$, and the fraction falling on to the horizon may already be appreciable, but for $\zeta = 2$, it becomes a good fraction, $f \approx 0.3$. In fact, for $\zeta = 3/2$, $\psi_c = \pi/2$ and $f = 0.5$, photons emitted outward with $\psi_0 < \pi/2$, will ultimately travel to infinity, the other half number, with $\psi_0 > \pi/2$, will go through the horizon to disappear inside the black hole. The total photon number-flux through a large surface surrounding the black hole in this case would be $2\pi n$, with an equal number disappearing in the black hole. For the source at still closer distances to the event horizon, as $\zeta \to 1$, $\psi_c \to 0$, implying *all* photons will disappear through the horizon, with none reaching the distant observer, justifying the name 'black hole'.

TABLE I. Critical angle (ψ_c) for the source at some representative values of radial coordinate ($\zeta = r_0/r_s$) and the fraction f of photons that disappears into the horizon

ζ	$\psi_c(^\circ)$	f
2.36×10^5 (Sun)	180	3×10^{-11}
10^2	179	1.7×10^{-4}
10	166	1.5×10^{-2}
3	135	0.15
2	113	0.3
1.5	90	0.5
1.3	74	0.64
1.1	45	0.85
1	0	1.0

Table 1 shows for some representative values of radial coordinate ($\zeta = r_0/r_s$), where the source lies in the gravitation field, the critical angle (ψ_c) and the fraction f of photons that disappears into the event horizon of the black hole. The first row in Table 1 is for a source at the outer surface of Sun, where the critical angle $\psi_c \sim \pi - 10^{-5}$ radians and the fraction $f \approx 3\times10^{-11}$,

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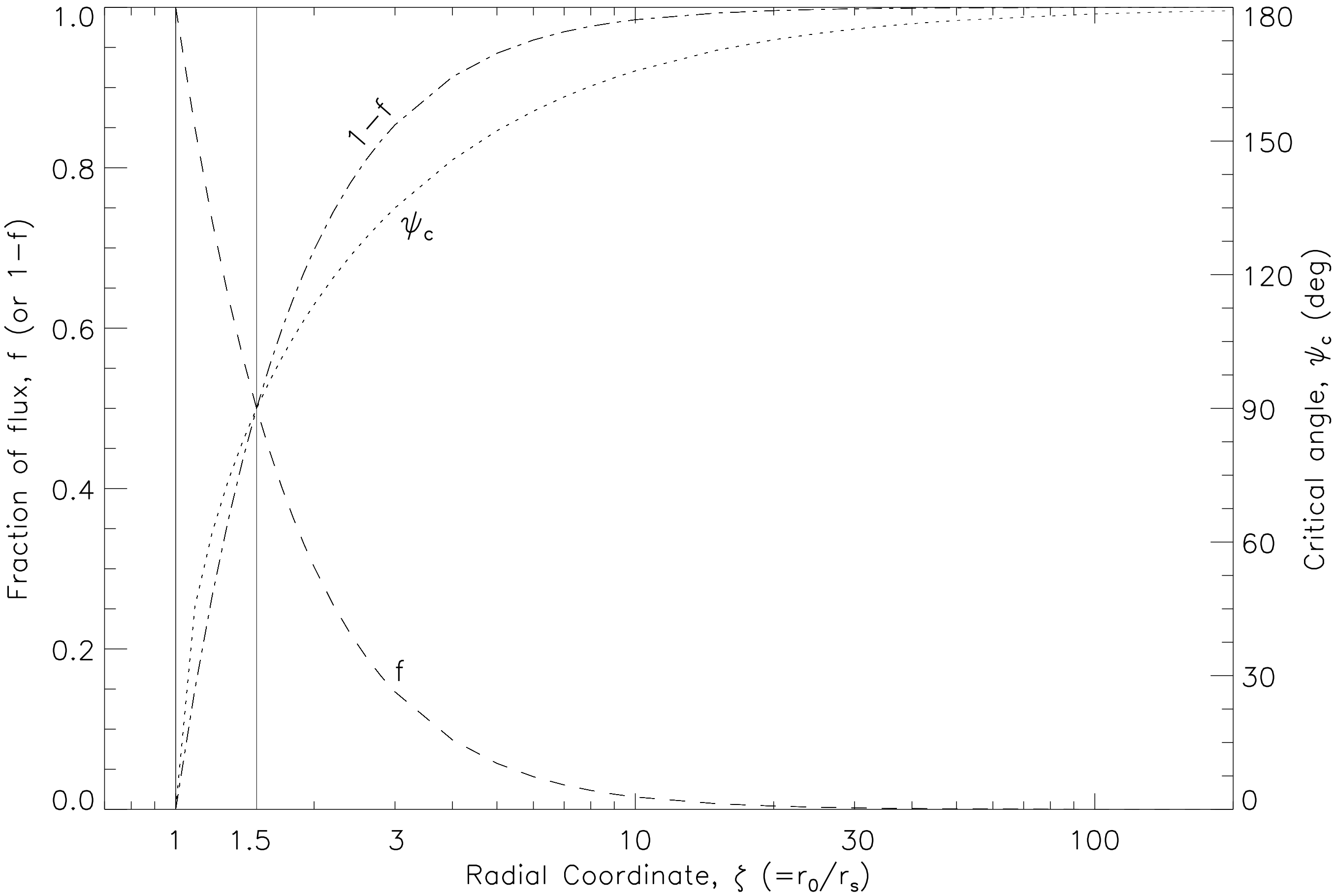

FIG. 2. For a source of photons held stationary at a radial coordinate $\zeta = r_0/r_s$, in the gravitational field of a black hole, the fraction f of the photon flux, that ultimately disappears in the horizon of the black hole, is shown by a dashed curve. The fraction $1 - f$, shown by a dash-dotted curve, escapes the gravitational field to reach infinity, and represents those photon trajectories that start from the source along initial angle $\psi_0 < \psi_c$, as seen by local observers in the gravitational field. Critical angle ψ_c, whose values can be read on the right hand vertical scale, is depicted by a dotted curve, as a function of ζ (Eq. 3).

quite negligible. However as can be seen from the table, for decreasing values of ζ, the critical angle ψ_c decreases steadily and accordingly the fraction f becomes appreciable. For $\zeta = 3$, $\psi_c = 135°$ with $f = 0.15$, while at $\zeta = 2$, about 30% of the total flux ends up in the horizon. Of course for $\zeta = 1.5$, $\psi_c = 90°$ with $f = 0.5$, but for $\zeta = 1.1$, $\psi_c = 45°$ with 85% of the flux lost in the horizon, and as ζ approaches unity, all photons, irrespective of the initial direction, reach the horizon and get captured by the black hole.

Figure 2 shows the fraction f of the photon flux, that ultimately disappears in the horizon of the black hole. Also shown in the figure is the fraction $1 - f$ that escapes the gravitational field to reach infinity. These are those photon trajectories that start from the source along an initial angle ψ_0, as measured by local observers in the gravitational field, lesser than a critical value, ψ_c, which depends upon ζ, the location of the source in the gravitational field, as shown in Fig. 2.

IV. THE ABSENCE OF EXTERNAL ELECTRIC FLUX OF A CHARGED BLACK HOLE

As discussed in section II, the behaviour of electric field lines in a gravitational field is similar to that of photons, in the sense that the field lines follow the photon trajectories and that the fraction of electric flux crossing a surface ‘below’ or ‘above’ the charge is exactly similar to the fraction of photon trajectories intersecting these surfaces. For any given position of the charge in the gravitational field, there may be less electric flux from the charge in the ‘upward’ direction and more in the ‘downward’ direction. This difference in the upward and downward flux increases as the charge position is brought nearer to the event horizon and the difference becomes acute as the charge approaches the horizon. The fraction of electric field lines that reach the horizon and may disappear in the black hole would exactly be similar as of photons that reach the horizon and get captured by the black hole, derived in section III.

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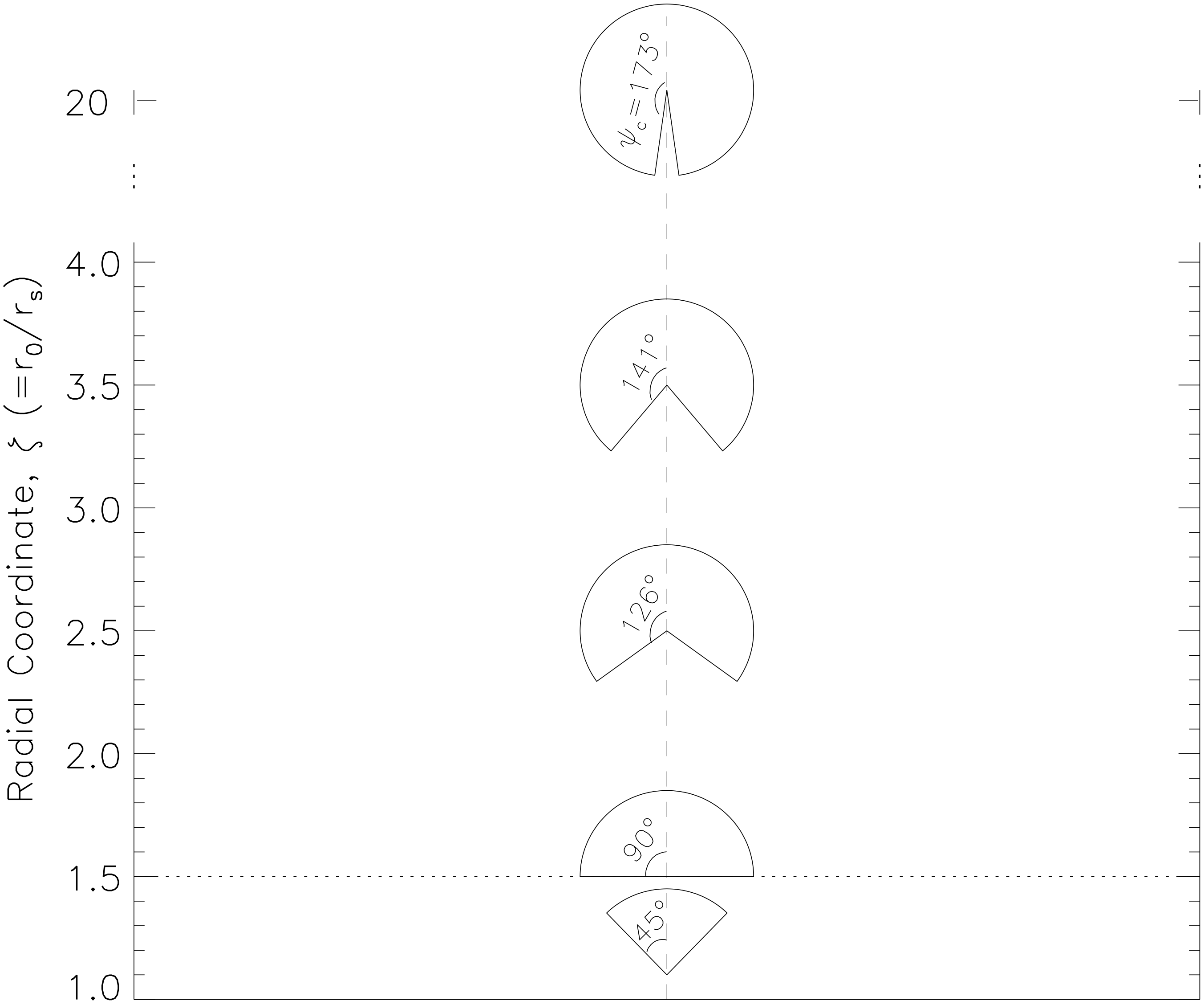

FIG. 3. A schematic presentation of the critical angle ψ_c as a function of the position of an electric charge in the gravitational field of a black hole. The electric field lines, starting along angle $\psi_0 < \psi_c$ from the charge at radial coordinate $\zeta = r_0/r_s$, where r_s is the Schwarzschild radius, can extend to infinity. The missing sector of a circle represents the initial directions of electric field lines, starting along $\psi_0 > \psi_c$ from the charge at that ζ, that intersect the horizon of the black hole and disappear into it. At large ζ, $\psi_c \approx 180°$, implying almost all electric field lines extend to infinity. However, as ζ decreases, ψ_c reduces and an increasing number of electric field lines do not make it to infinity and are thus trapped in the gravitational field. For instance, at $\zeta = 20$, $\psi_c = 173°$, while at $\zeta = 2.5$, $\psi_c = 126°$. At $\zeta = 1.5$ particularly, $\psi_c = 90°$ and only one half of the total number of electric field lines can reach the infinity. At still closer radial coordinates, majority of electric field lines get bent sufficiently to intersect the horizon, e.g. at $\zeta = 1.1$, $\psi_c \approx 45°$, with 85 percent of the total number of electric field lines reaching the horizon and disappearing into it; the remainder 15 percent only extends to infinity.

Figure 3 shows schematically the sectors drawn of circles for the initial angles $\psi_0 < \psi_c$ of electric field lines, that would extend to infinity, starting from a charge at radial coordinate $\zeta = r_0/r_s$. The missing sector of each circle in Fig. 3 represents the initial directions of electric field lines, starting along $\psi_0 > \psi_c$ from the charge at that ζ, that intersect the horizon of the black hole and disappear into it. At large ζ, $\psi_c \approx 180°$, implying almost all electric field lines extend to infinity. However, as ζ decreases, ψ_c reduces and an increasing number of field lines do not make it to infinity and are thus trapped in the gravitational field. For instance, at $\zeta = 3.5$, $\psi_c = 141°$ and 89 percent of electric field lines can reach the infinity, while at $\zeta = 1.5$, $\psi_c = 90°$, and only one half of the electric field lines could do so. At still closer radial coordinates, as the charge position nears the horizon, majority

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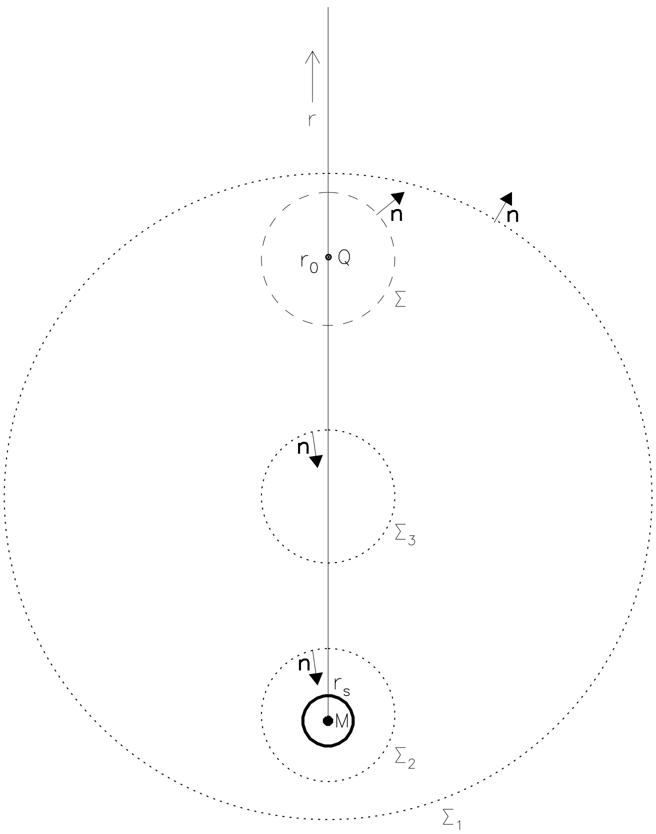

FIG. 4. A schematic of the computation of the electric flux in the gravitational field of a central spherical mass M of a Schwarzschild radius $r_{\rm s}$. The electric charge Q is held stationary at a radial coordinate r_0. The dashed-line circle represents the spherical surface Σ surrounding the charge Q. Various dotted-line circles represent spheres Σ_1, Σ_2, Σ_3, indicating the boundaries of various regions containing the charge Q and/or the spherical mass M. The unit vector $\mathbf{n}$ in each case shows the unit normal to the respective spherical surface, as seen from the perspective of the charge, located at r_0.

of electric field lines intersect the horizon, with hardly any electric field lines reaching the infinity for $\zeta \to 1$.

Thus the electric flux of a charge through an external spherical surface of large enough radius surrounding the black hole should be reducing as the charge location, enclosed within that spherical surface, is selected closer to the event horizon (Schwarzschild radius) becoming in limit zero as the event horizon is approached. For very large initial radial distance of the charge, i.e. for $\zeta = r_0/r_{\rm s} \to \infty$ in the gravitational field of a black hole, the electric flux through a spherical surface of large enough radius $r > r_0$ around the black hole and thus enclosing the charge Q, would be $4\pi Q$, as per Gauss law. However, for finite ζ, but with $\zeta \gg 1$ still, the electric flux through the spherical surface would be $2\pi Q(1-\cos\psi_{\rm c}) \approx 4\pi Q(1-27/16\zeta^2)$, an apparent violation of Gauss law, though the electric flux through a surface surrounding the charge within the gravitational field, that is when the spherical surface does not surround the central mass, would still be $4\pi Q$, thus in conformity with Gauss law. The fraction $1-f$ of the electric flux that escapes the black hole, for $\zeta = 10$ is 0.985, and for $\zeta = 2$, it is ≈ 0.7. For $\zeta = 3/2$, $\psi_{\rm c} = \pi/2$ and $1-f = 0.5$, and while one half of the total electric flux will reach the infinity, the other half, with $\psi_0 > \pi/2$, due to gravity will go through the horizon. The total electric flux through a large surface surrounding the black hole in this case would be $2\pi Q$, instead of the usual $4\pi Q$, calculated from Gauss law in the absence of a gravitational field. At still closer distances to the event horizon, as $\zeta \to 1$, $\psi_{\rm c} \to 0$, most of the electric flux from the charge will disappear through the horizon into the black hole.

As photons will not escape a black hole from the event horizon (Schwarzschild radius), and the electrostatic field lines following the path of photons emitted from the charge location, the electric field lines also, due to charges lying within the event horizon in the case of a black hole, will not extend outside the horizon. In fact, any electromagnetic influence that propagates with the velocity of light, may not cross the event horizon in the outward direction, therefore charges, if any, within the black hole may not get detected by external observers, i.e. outside the event horizon. Thus the electric flux through an external spherical surface surrounding the black hole will be nil due to the charges within the black hole. From that one can conclude the absence of electric flux external to a charged black hole. and the distant observers may not know about the presence of electric charges within the black hole.

Figure 4 shows schematically various spherical surfaces, Σ, Σ_1, Σ_2, Σ_3, for which we want to examine the electric flux passing through, from a charge Q, held stationary at a radial coordinate r_0, in the gravitational field of a central spherical mass M of a Schwarzschild radius $r_{\rm s}$. The unit vector $\mathbf{n}$ in the case of each spherical surface, shows a normal to the surface for computing the electric flux through the respective sphere, seen from the perspective of the charge Q, located at r_0. The electric flux passing through the surface Σ, that surrounds the charge at r_0 but does not enclose the black hole singularity, is $4\pi Q$, consistent with Gauss law. This will be true even if the charge is close to but still outside the event horizon, and the surface Σ surrounding the charge too lies wholly outside the event horizon, for instance, a tiny (infinitesimal!) spherical surface surrounding the charge. However, if the surface chosen is large enough to enclose the black hole singularity like Σ_1 in Fig. 4, then the field lines passing through will be $4\pi Q(1-f)$, where $1-f$ is given by Eq. (5), with $\psi_{\rm c}$ determined from Eq. (3), depending upon the radial position $\zeta = r_0/r_{\rm s}$ of the charge. It may be noted that the remaining flux, viz. $4\pi Qf$, will be passing through Σ_2 to disappear into the black hole horizon, even though the radial coordinate r_0 of the charge lies outside Σ_2. However, through Σ_3, it is nil, as it does not enclose any charge and whatever field lines may be entering Σ_3 the same must be leaving it.

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In an earlier investigation by Hani and Ruffini [13] of the electric field configuration of a charge, stationary at various radial coordinates in the gravitational field of a black hole, it was shown than field lines, bent due to gravity, are tangential to the horizon at some critical angle θ_c, with field lines within θ_c intersecting the horizon while those outside θ_c escaping the gravitation field. It should be noted that θ_c is not the same as our ψ_c, while θ_c is in terms of the Schwarzschild coordinate θ, angle ψ is around the charge position. Now taking the charge Q to be positive, the intersection of field lines with the horizon was interpreted in terms of negative charges induced on the horizon for $\theta < \theta_c$ and a further assumption made was that there are an equal amount of positive charges induced on the horizon for $\theta > \theta_c$. While the bending of electric field lines towards the central mass, resulting in these field lines intersecting the horizon could be expected, the field lines outside θ_c *bending away* from the central mass in its gravitational field looks rather strange, which is happening due to the assumed positive charges. In an equivalent picture for photon trajectories, it will seem as if photons were getting repelled by the gravitational field of the black hole. In fact, for field lines that intersected and disappeared into the horizon from one side of the black hole, an equal amount of field lines seem to be *emerging out* of the horizon on the opposite side of the black hole. Here it seems as if photons falling into the horizon on one side of the black hole were emerging out of the horizon on the opposite side of the black hole. In fact, as one approaches the event horizon, the electric field lines intersecting the horizon should do so orthogonally as their θ component should be reducing to zero near the horizon [13]. However, in the field line plots of Hani and Ruffini it does not seem to be so. This discrepancy actually stems from the assumed induced charge, especially the assumed positive charges induced outside $\theta > \theta_c$ [13], which cause additional outward field components. The assumption of equal and opposite charges induced on the different sides of the horizon of a black hole, and the electric field configuration derived therefrom does not seem be correct.

V. REISSNER–NORDSTRÖM METRIC FOR A CHARGED BLACK HOLE AND SOME PECULIARITIES OF ITS GEOMETRY

If a static spherical system of mass M, comprises in addition also an electric charge Q, then such a system is externally described usually by the Reissner–Nordström (RN) metric [2, 3, 5, 12, 17]

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{rc^2} + \frac{GQ^2}{r^2c^4}\right)c^2\mathrm{d}t^2 + \left(1 - \frac{2GM}{rc^2} + \frac{GQ^2}{r^2c^4}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\omega^2 . \quad (7)$$

We can express the metric in an alternate form

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s}{r} + \frac{r_q^2}{r^2}\right)c^2\mathrm{d}t^2 + \left(1 - \frac{r_s}{r} + \frac{r_q^2}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2 , \quad (8)$$

where $r_s = 2GM/c^2$ again is the Schwarzschild radius, while r_q, defined by $r_q^2 = GQ^2/c^4$, is a characteristic length scale in the presence of the electric charge Q. Here it is *presumed* that the exterior of the black hole is not a vacuum and is filled with a static electric field due to the charge contained within the black hole and accordingly a solution of the Einstein field equations is obtained for a static spherically symmetric system in the presence of a non-zero energy–momentum tensor, representing the electromagnetic field of charge Q inside the black hole [7, 18, 19]. This is the metric of a charged black hole, without a rotation or having nil angular momentum. In the limit that the charge Q goes to zero, one recovers the Schwarzschild metric (Eq. (1)).

The Schwarzschild metric has a long and somewhat tedious derivation, which is widely available in textbooks [4, 6–9, 11]. The derivation of the RN metric follows mostly on similar lines because of the assumed spherical symmetry, however, it involves some additional steps because of the presence of the electromagnetic stress-energy tensor due to the charge Q [7, 18, 19].

A. A heuristic derivation of the RN metric

It is possible to derive the RN metric in a heuristic manner, building on the Schwarzschild metric, incorporating in that the electric field of the charge Q. A rigorous derivation of the RN metric, starting from Einstein's field equations, employing the electromagnetic stress-energy tensor due to the charge Q, is given in Appendix.

The heuristic approach we employ here to arrive at the Reissner–Nordström metric emphasizes, in particular, the role of energy in the electrostatic field, in the process clarifying its relation to the charge-dependent terms in the Reissner–Nordström metric. For this we first look at the electric field of an isolated charge Q in ordinary space (devoid of gravity)

$$E = \frac{Q}{r^2} , \quad (9)$$

and compute the energy in the electric field in space exterior to a radial distance r from the charge Q at the center, as

$$\mathcal{E}_\mathrm{r} = \int_r^\infty \frac{E^2}{8\pi} 4\pi r^2 \mathrm{d}r = \frac{Q^2}{2r} , \quad (10)$$

We now superpose charge Q and its electric field on the Schwarzschild solution (Eq. (1)), treating r, the radial coordinate in the Schwarzschild geometry to be the radial *distance* from the center. In the Schwarzschild

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metric, M represents the total mass as inferred by a distant observer (at infinity!) from its gravitational influence, and it includes the electromagnetic mass-energy contribution of the charge Q as well. The quantity $M_{\rm q} = \mathcal{E}/c^2 = Q^2/(2rc^2)$ from Eq. (10) can be interpreted, from mass-energy equivalence, as the mass exterior to radius r [19], while $M' = M - Q^2/2rc^2$ then has the interpretation [19] as the mass interior to radius r. In other words, $M' = M - M_{\rm q}$ is the mass, M, including contributions from the charge Q, bereft $M_{\rm q}$, mass equivalent of energy $\mathcal{E}_{\rm r}/c^2$ in the electric field exterior to r. Then the Schwarzschild metric for M' (Eq. (1)) is

$$\begin{aligned} {\rm d}s^2 = &-\left(1-\frac{2GM'}{rc^2}\right)c^2{\rm d}t^2 \\ &+\left(1-\frac{2GM'}{rc^2}\right)^{-1}{\rm d}r^2 + r^2\,{\rm d}\omega^2 \\ =&-\left(1-\frac{2G(M-M_{\rm q})}{rc^2}\right)c^2{\rm d}t^2 \\ &+\left(1-\frac{2G(M-M_{\rm q})}{rc^2}\right)^{-1}{\rm d}r^2 + r^2\,{\rm d}\omega^2\,. \quad (11) \end{aligned}$$

With $M_{\rm q} = Q^2/2rc^2$, Eq. (11) is nothing but the RN metric (Eq. (7)).

B. Occurrence of naked singularities in RN geometry and the "cosmic censorship" hypothesis

The RN geometry leads to some interesting consequences. Assuming it to be a "point" mass, the only intrinsic singularity in the RN metric is at $r = 0$. However, from the RN metric (Eq. (8)) one finds that, in Schwarzschild coordinates (t, r, θ, ϕ), there also occurs a coordinate singularity wherever r satisfies

$$1 - \frac{r_{\rm s}}{r} + \frac{r_{\rm q}^2}{r^2} = 0\,, \quad (12)$$

or

$$r^2 - r_{\rm s} r + r_{\rm q}^2 = 0\,, \quad (13)$$

Accordingly, coordinate singularities occur at

$$r = \frac{r_{\rm s}}{2} \pm \left[\left(\frac{r_{\rm s}}{2}\right)^2 - r_{\rm q}^2\right]^{1/2}\,, \quad (14)$$

which for $Q = 0$ reduces of course to $r = r_{\rm s}$, the event horizon at the Schwarzschild radius, as would be expected. However, for $Q \neq 0$, and especially for $r_{\rm s} > 2r_{\rm q}$, there are two real values for r and thus two coordinate singularities, implying two event horizons [5, 7]. On the other hand, $r_{\rm s} < 2r_{\rm q}$ yields only imaginary solutions, implying the absence of a real event horizon. Without an event horizon, the intrinsic singularity at $r = 0$ is visible to the outside observers; such is termed a naked singularity. The idea of a naked singularity appears to be highly unacceptable and a "cosmic censorship" hypothesis has been proposed [20] that only singularities allowed in nature are the ones shrouded by an event horizon.

VI. PROBLEMS IN THE DERIVATION OF THE REISSNER–NORDSTRÖM METRIC AND THE CONSEQUENCES THERE OF

In the derivation of the Reissner–Nordström metric (Eq. (7) or (8)) one crucial step is the assumed expression for the electric field (Eq. (9) or (A.5)) for all r, employing the Gauss law for a spherically symmetric system, in the external regions of a black hole. However, as we discussed earlier, the electric field of a charged black hole does not exist in regions external to a black hole, then its influence should not affect the metric outside. Gauss law may not be applicable to calculate the electric field external to the black hole for an electric charge that is trapped within the black hole event horizon. Thus the presumption that the electromagnetic energy density $(E^2/8\pi)$, or the stress-energy tensor (Eq. (A.12)) exists for all r, i.e., even beyond $r_{\rm s}$, is not valid, then the RN metric cannot be correctly describing a charged black hole. In that case, with no external electric field, a charged black hole is still described by the Schwarzschild metric (Eq. (1) or (2)), of course the mass M, that enters in the expression for Schwarzschild radius $r_{\rm s} = 2GM/c^2$, encompasses contributions from the electromagnetic mass of the charge as well.

Since, Eq. 14 follows from the RN metric, which, as we pointed out, cannot be a valid description of a charged black hole, then the question of consequential two event horizons for $r_{\rm s} > 2r_{\rm q}$ or even the absence of singularity leading to the idea of a naked intrinsic singularity when $r_{\rm s} < 2r_{\rm q}$ and a need for the "cosmic censorship" hypothesis, do not even arise, at least in the case of a charged black hole *with no angular momentum.*

VII. CONCLUSIONS

We started from the result derived in the literature that the electrostatic field lines of a charge, stationary in the gravitational field, bend exactly like the trajectories of photons emitted isotropically from a source at the charge location, and that the fraction of electric flux crossing a surface 'below' or 'above' the charge, supported in the gravitational field, is exactly similar to the fraction of photon trajectories intersecting these surfaces. Applying these results in the external gravitational field of a black hole, it was shown that the electric field through an external spherical surface surrounding the black hole steadily reduces as the charge location approaches the event horizon (Schwarzschild radius). Further, from the well-known result that photons from a source inside the Schwarzschild radius cannot escape outside, it was argued that the electric field lines of a charge within the black hole too remain trapped inside the event horizon, which leads to the conclusion about the absence of electric flux external to a charged black hole. From this we arrived at a conclusion that, contrary to the conventional wisdom, the electric charge contained inside a

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static black hole cannot be detected or inferred by an external observer. A black hole, said to have no hair with the only external identifying characteristics being mass, electric charge, and angular momentum, is therefore all the more 'hairless', as even its charge cannot be ascertained. The derivation of the Reissner–Nordström metric, supposedly describing the gravitational field of a static charged black hole, presumes an external stress-energy tensor of the electrostatic field, as per Gauss law, even for the charges contained within the black hole. From the absence of electric flux external to a static charged black hole it was argued that such a charged black hole is not described correctly by the Reissner–Nordström metric and the consequential peculiarities of its space-time geometry, which lead to the idea of a naked intrinsic singularity in specific cases and a need for the "cosmic censorship" hypothesis, also do not arise, at least in the case of a charged black hole *with no angular momentum.*

DECLARATIONS

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Appendix: Derivation of Reissner–Nordström metric and the presumptions made therein

Einstein's field equations are [4, 6–9, 11]

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}\,, \qquad \text{(A.1)}$$

where $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ is the metric tensor, $R \equiv g^{\mu\nu} R_{\mu\nu}$ is the contraction of the Ricci tensor and $T_{\mu\nu}$ is the stress-energy tensor, the source of gravitation. We follow the sign conventions of Misner et al [4] for the curvature tensor as well as for the stress-energy tensor.

The electromagnetic stress-energy tensor in a gravitational field can be written as [4]

$$T_{\mu\nu} = \frac{1}{4\pi}[F_{\mu\alpha} g_{\nu\beta} F^{\beta\alpha} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}]\,, \qquad \text{(A.2)}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor [4, 8, 11]. The stress-energy tensor is, in general, symmetric ($T_{\mu\nu} = T_{\nu\mu}$) and for an electromagnetic field it is traceless ($T \equiv g^{\mu\nu} T_{\mu\nu} = 0$).

Using $g^{\mu\nu} g_{\mu\nu} = 4$, we get from Eq. (A.1)

$$g^{\mu\nu} G_{\mu\nu} = -R = \frac{8\pi G}{c^4} g^{\mu\nu} T_{\mu\nu} = \frac{8\pi G}{c^4} T\,. \qquad \text{(A.3)}$$

Therefore, for a traceless electromagnetic stress-energy tensor ($T = 0$), $R = 0$ and Einstein's field equations then become

$$R_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}\,. \qquad \text{(A.4)}$$

For a spherically symmetric system, which we assume to be the case for a simple electrostatic charge Q at the center, the electric field can have only a radial component, E_r, with no angular dependence. Because of the spherical symmetry, using Gauss theorem, we can write

$$E_\mathrm{r} = \frac{Q}{r^2} \qquad \text{(A.5)}$$

Unlike in Eq. (9), where r was taken to be a radial distance, here r is a radial coordinate, and not necessarily a radial proper distance in the gravitational field.

Then the electromagnetic field tensor, $F_{\mu\nu}$, which is antisymmetric, can be written as

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_\mathrm{r} & 0 & 0 \\ E_\mathrm{r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad \text{(A.6)}$$

The metric for our assumedly time-static, spherical symmetric system, can be written in a general form

$$\mathrm{d}s^2 = -Ac^2 \mathrm{d}t^2 + B\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2 \sin^2\theta\,\mathrm{d}\phi^2 \qquad \text{(A.7)}$$

where A and B are time-independent and have dependence at most on r.

Then the metric can be written as

$$g_{\mu\nu} = \begin{bmatrix} -A(r) & 0 & 0 & 0 \\ 0 & B(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{bmatrix}. \qquad \text{(A.8)}$$

Since $g_{\mu\nu}$ is diagonal, we also have

$$g^{\mu\nu} = \begin{bmatrix} -A^{-1}(r) & 0 & 0 & 0 \\ 0 & B^{-1}(r) & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2} \sin^{-2}\theta \end{bmatrix}. \qquad \text{(A.9)}$$

As the only non-zero components of the electromagnetic field tensor (Eq. (A.6) are $F_{01} = -F_{10} = -E_\mathrm{r}$, then using Eq. (A.9) in

$$F^{\mu\nu} = g^{\mu\alpha} F_{\alpha\beta} g^{\beta\nu}\,, \qquad \text{(A.10)}$$

we get the only non-vanishing components of $F^{\mu\nu}$ as

$$F^{01} = -F^{10} = g^{00} F_{01} g^{11} = -\frac{F_{01}}{AB} = \frac{E_\mathrm{r}}{AB}. \qquad \text{(A.11)}$$

Then from Eqs. (A.2), (A.6), (A.8) and (A.11), we obtain components of the stress-energy tensor as

$$T_{00} = \frac{1}{B(r)} \frac{E_\mathrm{r}^2}{8\pi} \qquad T_{11} = -\frac{1}{A(r)} \frac{E_\mathrm{r}^2}{8\pi}$$

$$T_{22} = \frac{r^2}{A(r)B(r)} \frac{E_\mathrm{r}^2}{8\pi} \qquad T_{33} = \frac{r^2 \sin^2\theta}{A(r)B(r)} \frac{E_\mathrm{r}^2}{8\pi}, \qquad \text{(A.12)}$$

with all non-diagonal components of $T_{\mu\nu}$ being zero. From Eqs. (A.9) and (A.12), it can be easily verified that the trace of the stress-energy tensor, $T \equiv g^{\mu\nu} T_{\mu\nu}$, is zero.

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This determines the right hand side of Eq. (A.4). In order to get components of the Ricci tensor on the left hand side of Eq. (A.4), we need to first compute Christoffel symbols or connection coefficients from $g_{\mu\nu}$ as [4, 8, 11]

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu}(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu})\,, \tag{A.13}$$

where $g_{\mu\beta,\gamma} \equiv \partial_\gamma g_{\mu\beta}$, is a partial derivative of metric coefficients $g_{\mu\beta}$ along the coordinate basis direction denoted by γ. Then from these we can form the symmetric Ricci tensor using the formulation [4, 8, 11]

$$R_{\mu\nu} \equiv R^{\alpha}_{\mu\alpha\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha}\,, \tag{A.14}$$

where $\Gamma^{\alpha}_{\mu\nu,\gamma} \equiv \partial_\gamma \Gamma^{\alpha}_{\mu\nu}$ is a partial derivative of $\Gamma^{\alpha}_{\mu\nu}$ along the direction denoted by γ. The computations involved, though straightforward, are yet lengthy due to their rather repetitive nature.

Substituting Eqs. (A.8) and (A.9) in (A.13), we get the only non-zero Christoffel symbols or connection coefficients as

$$\begin{aligned}
&\Gamma^0_{01} = \Gamma^0_{10} = \frac{A'}{2A} && \Gamma^1_{00} = \frac{A'}{2B} \\
&\Gamma^1_{11} = \frac{B'}{2B} \qquad \Gamma^1_{22} = -\frac{r}{B} && \Gamma^1_{33} = -\frac{r\sin^2\theta}{B} \\
&\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} && \Gamma^2_{33} = -\sin\theta\cos\theta \\
&\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} && \Gamma^3_{23} = \Gamma^3_{32} = \frac{\cos\theta}{\sin\theta}.
\end{aligned} \tag{A.15}$$

Here a prime ($'$) denotes a differentiation with respect to r. Of course, in our assumed time-static system, A and B are time-independent.

Using these in Eq. (A.14), and after some lengthy though straightforward computations, we arrive at the following non-vanishing components of the Ricci tensor

$$\begin{aligned}
R_{00} &= -\frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A''}{2B} + \frac{A'}{rB} \\
R_{11} &= \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A''}{2A} + \frac{B'}{rB} \\
R_{22} &= 1 - \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) - \frac{1}{B} \\
R_{33} &= R_{22}\sin^2\theta\,,
\end{aligned} \tag{A.16}$$

all non-diagonal components of $R_{\mu\nu}$ turn out to be zero.

Now, from Eq. (A.12) we have

$$\frac{T_{00}}{A} + \frac{T_{11}}{B} = 0\,, \tag{A.17}$$

which from Eq. (A.4) implies

$$\frac{R_{00}}{A} + \frac{R_{11}}{B} = 0\,. \tag{A.18}$$

Substituting Eq. (A.16) in Eq. (A.18), we get

$$\frac{A'}{A} + \frac{B'}{B} = 0\,, \tag{A.19}$$

implying $A(r)B(r) =$ constant, i.e. independent of r. Using the asymptotic requirement that the metric should be flat for $r \to \infty$, we could write $A(r)B(r) = 1$ or $A(r) = 1/B(r)$ [7, 18, 19].

Then from Eq. (A.5), we have

$$\frac{r^2}{A(r)B(r)} E_{\mathrm{r}}^2 = \frac{Q^2}{r^2} \tag{A.20}$$

Also from Eqs. (A.4), (A.12) and (A.20), we get

$$R_{22} = \frac{8\pi G}{c^4} T_{22} = \frac{G}{c^4}\frac{Q^2}{r^2}\,. \tag{A.21}$$

Then Eq. (A.16) gives

$$1 - \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) - \frac{1}{B} = \frac{G}{c^4}\frac{Q^2}{r^2}\,, \tag{A.22}$$

or

$$1 - \frac{G}{c^4}\frac{Q^2}{r^2} = rA' + A = \frac{\mathrm{d}(rA)}{\mathrm{d}r}\,. \tag{A.23}$$

Integrating and then dividing by r, we get

$$A(r) = \frac{\mathrm{a}}{r} + 1 + \frac{G}{c^4}\frac{Q^2}{r^2}\,. \tag{A.24}$$

The constant a can be determined from the limit $Q \to 0$, where the metric should reduce to the Schwarzschild metric, giving therefore $\mathrm{a} = -2GM/c^2$.

The upshot of these computations is a simple expression

$$A(r) = 1 - \frac{2GM}{rc^2} + \frac{GQ^2}{r^2c^4}\,, \tag{A.25}$$

as well as

$$B(r) = \frac{1}{A(r)} = \left(1 - \frac{2GM}{rc^2} + \frac{GQ^2}{r^2c^4}\right)^{-1}. \tag{A.26}$$

Writing $r_s = 2GM/c^2$ and $r_q^2 = GQ^2/c^4$, we have

$$A(r) = \frac{1}{B(r)} = 1 - \frac{r_s}{r} + \frac{r_{\mathrm{q}}^2}{r^2}\,, \tag{A.27}$$

Accordingly, substituting in Eq. (A.8), one obtains for the metric tensor

$$g_{\mu\nu} = \begin{bmatrix} -(1 - \frac{r_s}{r} + \frac{r_{\mathrm{q}}^2}{r^2}) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{r_s}{r} + \frac{r_{\mathrm{q}}^2}{r^2}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix}, \tag{A.28}$$

which of course is nothing but Eq. (8). Using Eqs. (A.25) and (A.26) in (A.16), we can cross-check that the trace of the Ricci tensor, $R \equiv g^{\mu\nu}R_{\mu\nu}$, is indeed zero in our case.

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