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Conformally Compactified Minkowski Space: A Re-Examination with Emphasis on the Double Cover and Conformal Infinity

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Conformally Compactified Minkowski Space: A Re-Examination with Emphasis on the Double Cover and Conformal Infinity

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Abstract

This paper presents a detailed re-examination of the conformal compactification of Minkowski space, $\overline{M}$, constructed as the projective null cone of the six-dimensional space $\mathbb{R}^{4,2}$. We provide an explicit and basis-independent formulation, emphasizing geometric clarity. A central result is the explicit identification of $\overline{M}$ with the unitary group $U(2)$ via a diffeomorphism, offering a clear matrix representation for points in the compactified space. We then systematically construct and analyze the action of the full conformal group $\mathrm{O}(4,2)$ and its connected component $\mathrm{SO}_0(4,2)$ on this manifold. A key contribution is the detailed study of the double cover, $\overline{\overline{M}}$, which is shown to be diffeomorphic to $S^3 \times S^1$. This construction resolves the non-effectiveness of the $\mathrm{SO}(4,2)$ action on $\overline{M}$, yielding an effective group action on the covering space. A significant portion of our analysis is devoted to a precise and novel geometric characterization of the conformal infinity. Moving beyond the often-misrepresented "double cone" description, we demonstrate that the infinity of the double cover, $\overline{\overline{M}}_\infty$, is a squeezed torus (specifically, a horn cyclide), while the simple infinity, $\overline{M}_\infty$, is a needle cyclide. We provide explicit parametrizations and graphical representations of these structures. Finally, we explore the embedding of five-dimensional constant-curvature spaces, whose boundary is the compactified Minkowski space, and discuss the interpretation of geodesics within these domains. The paper aims to clarify long-standing misconceptions in the literature and provides a robust, coordinate-free geometric foundation for conformal compactification, with potential implications for cosmology and conformal field theory.

Keywords: conformal compactification; Minkowski space; Einstein universe; conformal group $\mathrm{O}(4,2)$; conformal infinity; double cover; Dupin cyclide; unitary group $U(2)$

MSC: Primary 53B30, 53A30, 83C20; Secondary 22E70, 83C05

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0. Introduction

The conformal compactification of Minkowski space has long played a central role in both mathematical relativity and conformal field theory. In its standard realization, the compactified space is obtained as the projective null cone of a six-dimensional real vector space of signature $(4,2)$, and is often described, following Penrose, as the "Einstein universe" [1]. Although this construction is classical and widely used, several geometric and conceptual aspects of the compactification and of its conformal infinity are still presented in a manner that can obscure their true structure. In particular, the topology of the compactified space, the role of the natural double cover, and the precise geometry of conformal infinity are sometimes treated in the literature in a way that leads to confusion or to oversimplified pictures, such as the ubiquitous but misleading "double cone at infinity".

The first aim of this paper is to provide a detailed and explicit re-examination of the conformal compactification of Minkowski space within the framework of the null cone in $\mathbb{R}^{4,2}$. The construction is carried out in a basis-independent manner whenever possible, with coordinates used only as a

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technical tool where they clarify the geometry. A central result is an explicit identification of the compactified Minkowski space $\overline{M}$ with the unitary group $U(2)$ by means of a diffeomorphism, which gives a clear matrix representation of points in the compactification. This identification goes back to work of Uhlmann [2], who used the Cayley transform to relate hermitian 2×2 matrices to elements of $U(2)$. Here, an explicit formula is derived that realizes this correspondence directly on the projective null cone and shows in detail how the Cayley transform fits into the general geometric picture.

A second main theme of the paper is the analysis of the full conformal group $O(4,2)$ and its connected component $SO_0(4,2)$ acting on the compactified space. After recalling the standard embedding of Minkowski space into the null cone and the induced conformal structure on $\overline{M}$, several important Lie subgroups of $SO(4,2)$ are described explicitly in terms of their 6×6 matrix representations. These include the Poincaré group, dilatations, conformal inversion, and special conformal transformations. Their action on $\overline{M}$ and, in particular, on conformal infinity is analyzed in detail. It is shown that the action of $SO_0(4,2)$ on $\overline{M}$ is transitive, but not effective, since the central element $-I_6$ acts trivially on the projective null cone.

This observation leads naturally to the consideration of the double cover $\overline{\overline{M}}$ of the compactified Minkowski space, obtained by quotienting the null cone by positive rescalings rather than by all nonzero real scalars. The resulting manifold can be identified with the Grassmannian of oriented null lines in $\mathbb{R}^{4,2}$ and is shown to be diffeomorphic to $S^3 \times S^1$. On $\overline{\overline{M}}$ the action of $SO(4,2)$ becomes effective, and two disjoint embeddings of Minkowski space are obtained, corresponding to two non-intersecting copies $\overline{\overline{M}}_+$ and $\overline{\overline{M}}_-$. The remaining part, $\overline{\overline{M}}_\infty$, plays the role of a "doubled" conformal infinity. This point of view clarifies statements in the literature where $\overline{M}$ is directly identified with $S^3 \times S^1$ (see, for example, [1,3]) and makes explicit the relationship between the simple compactification $\overline{M}$ and its double cover $\overline{\overline{M}}$.

A substantial part of the paper is devoted to a precise geometric description of conformal infinity itself. Instead of the frequently drawn "double null cone", the conformal boundary of the double cover, $\overline{\overline{M}}_\infty$, is shown to be a squeezed torus, namely a horn-type Dupin cyclide, while the conformal infinity $\overline{M}_\infty$ of the simple compactification is identified with a needle cyclide. These identifications are obtained by an explicit analysis of the intersection of appropriate quadrics in $\mathbb{R}^{4,2}$, followed by a careful projection to three-dimensional Euclidean space. Parametric representations are derived and used to produce graphical illustrations of the resulting surfaces, which make the global structure of conformal infinity transparent and highlight, in particular, the presence of additional two-sphere components that are suppressed in the usual double-cone picture (cf. [4**?** ,5]).

The final sections place these constructions into a broader geometric context. First, five-dimensional constant-curvature spaces are embedded so that their common boundary is the compactified Minkowski space, and the interpretation of geodesics in these ambient spaces is discussed. Second, the entire theory is reformulated in a coordinate-free language, following the approach of Kopczyński and Woronowicz [7]. In this formulation the conformal compactification, its double cover, the conformal structure on the tangent bundle, and the null geodesics of the Einstein universe are described purely in terms of the quadratic form Q on a six-dimensional vector space and its isotropic subspaces. This geometric reformulation not only clarifies the role of the tautological bundle and its orthogonal complement over the projective null cone, but also provides a natural setting in which to understand the conformal structure as an equivalence class of scalar products on tangent spaces and to see how null geodesics depend only on the conformal class of the metric.

The paper is organized as follows. Section 4 recalls the basic construction of the compactified Minkowski space as the projective null cone and introduces the identification with $U(2)$ as well as the embedding of Minkowski space by means of the map τ and the Cayley transform. The conformal group $O(4,2)$, its connected component $SO_0(4,2)$, and their important subgroups are then discussed, together with their action on $\overline{M}$ and on conformal infinity. The double cover $\overline{\overline{M}}$ and its decomposition into two copies of Minkowski space and the doubled conformal infinity $\overline{\overline{M}}_\infty$ are introduced and studied in detail, including the description of $\overline{\overline{M}}_\infty$ as a horn cyclide and the corresponding structure of

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$\overline{M}_\infty$. Subsequent sections address the embedding into five-dimensional constant-curvature spaces, the induced conformal structure and its relation to null geodesics, and finally the coordinate-free reformulation of the theory in terms of the projective null cone and its tautological and orthogonal bundles.

1. Definitions

We denote by M the standard Minkowski space, that is $\mathbb{R}^{3,1} = \mathbb{R}^3 \oplus \mathbb{R}^1$, with coordinates $x = (x^1, \ldots, x^4)$, endowed with the quadratic form

$$q(x) = (x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2. \tag{1}$$

Let $\mathbb{R}^{1,1}$ be $\mathbb{R}^2$ endowed with the quadratic form q_2 defined by

$$q_2(x^5, x^6) = (x^5)^2 - (x^6)^2,\ (x^5, x^6) \in \mathbb{R}^2. \tag{2}$$

We denote by $\mathbb{R}^{4,2}$ the 6–dimensional space $\mathbb{R}^{3,1} \oplus \mathbb{R}^{1,1}$, with coordinates $(X^\alpha) = (x, x^5, x^6)$, and endowed with the quadratic form

$$Q(X) = q(x) + q_2(x^5, x^6) = (x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2 + (x^5)^2 - (x^6)^2. \tag{3}$$

Let $\mathcal{N}$ be the null cone of $\mathbb{R}^{4,2}$ minus the origin:

$$\mathcal{N} = \{X \in \mathbb{R}^{4,2} :\ X \neq 0 \text{ and } Q(X) = 0\}, \tag{4}$$

and let $P\mathcal{N}$ be the set of its generators, that is the set of straight lines through the origin in the directions nullifying $Q(X)$. In other words $P\mathcal{N} = \mathcal{N}/\sim$, where, for $X, X' \in \mathcal{N}$, $X \sim X'$ if and only if there exists a nonzero $\mu \in \mathbb{R}$ such that $X' = \mu X$. We denote by π the natural projection

$$\pi : \mathcal{N} \to P\mathcal{N}. \tag{5}$$

Then $P\mathcal{N}$, with its projective topology, is a compact projective quadric. $P\mathcal{N}$ is called the *compactified Minkowski space, denoted also by* $\overline{M}$.

A. Uhlmann [2] used the Cayley transform to identify the compactified Minkowski space with the group $U(2)$ of complex unitary 2×2 matrices. The following proposition provides the identification of $\overline{M}$ with $U(2)$ in an explicit form.

Proposition 1. *For each* $X = (x^1, \ldots, x^6) \in \mathcal{N}$ *the matrix*

$$u(X) = \frac{1}{x^6 + ix^4} \begin{bmatrix} x^5 + ix^3 & x^2 + ix^1 \\ -x^2 + ix^1 & x^5 - ix^3 \end{bmatrix} \tag{6}$$

is unitary and depends only on the equivalence class $[X]$ *of* X*. The map* $X \mapsto u(X)$ *descends to a diffeomorphism* $\tilde{u}$ *from* $\overline{M}$ *onto the unitary group* $U(2)$*.*

Proof. Let $X = (x^1, \ldots, x^6) \in \mathcal{N}$. From the definition (4) of $\mathcal{N}$ we have that $X \neq 0$ and

$$(x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2 + (x^5)^2 - (x^6)^2 = 0. \tag{7}$$

These two conditions are equivalent to

$$\mu(X) \stackrel{df}{=} (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^5)^2 = (x^4)^2 + (x^6)^2 > 0. \tag{8}$$

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It follows, in particular, that $x^6 + ix^4 \neq 0$. Therefore the right hand side of Eq. (6) is well defined for all $X \in \mathcal{N}$, and is a smooth function of $(x^1, \ldots, x^6)$. A straightforward calculation gives us

$$u(X)u(X)^* = \frac{(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^5)^2}{(x^4)^2 + (x^6)^2} I_2, \tag{9}$$

where I_2 is the 2×2 identity matrix. Taking into account the condition (8) we deduce that, for $X \in \mathcal{N}$, the matrix $u(X)$ is in $U(2)$.
Notice that it follows from Eq. (6) that for every $X \in \mathcal{N}$ and $\lambda \in \mathbb{R}$, $\lambda \neq 0$, we have

$$u(\lambda X) = u(X), \tag{10}$$

therefore the function $X \mapsto u(X)$ is constant on the equivalence classes of the equivalence relation $\sim$ defining $P\mathcal{N}$, and so it defines the map

$$\tilde{u} : [X] \mapsto u(X) \tag{11}$$

from $P\mathcal{N}$ to $U(2)$. We will now show that the map $\tilde{u} : P\mathcal{N} \to U(2)$ is surjective.

Let U be an arbitrary matrix in $U(2)$. Since $UU^* = I_2$, it follows that $|\det(U)| = 1$. Let c be one of the two (complex) square roots of $1/\det(U)$, so that $c^2 = 1/\det(U)$. Since $|det(U)| = 1$, we have $|c| = 1$. Then $U_1 \overset{df}{=} cU$ is of determinant 1, i.e. U_1 is in the group $SU(2)$, and we also have $U = \bar{c}U_1$. Now, every element u_1 of $SU(2)$ can be uniquely written in the form

$$u_1 = \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}, \tag{12}$$

where α, β are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$, which, by the way, gives us the well known topological identification of $SU(2)$ with the sphere S^3. Defining

$$\begin{aligned} x^1 &= \mathrm{Im}(\beta), \\ x^2 &= \mathrm{Re}(\beta), \\ x^3 &= \mathrm{Im}(\alpha), \\ x^4 &= \mathrm{Im}(c), \\ x^5 &= \mathrm{Re}(\alpha), \\ x^6 &= \mathrm{Re}(c), \end{aligned}$$

that is writing the complex numbers c, α, β as

$$\begin{aligned} c &= x^6 + ix^4, \\ \alpha &= x^5 + ix^3, \\ \beta &= x^2 + ix^1, \end{aligned} \tag{13}$$

one can verify that $U = u(X)$, with $X = (x^1, \ldots, x^6)$. Therefore the map $X \mapsto u(X)$ from $\mathcal{N}$ to $U(2)$ is indeed surjective. Let us finally check the injectivity of the map $\tilde{u} : [X] \mapsto u([X])$. Let $X = (x^1, \ldots, x^6)$ and $X' = (y^1, \ldots, y^6)$ be in $\mathcal{N}$, and suppose $u(X) = u(X')$. We may assume that $\mu(X)$ and $\mu(X')$, as defined in (8), are both equal to 1, otherwise one can rescale X and X' using property (10) to make it so. In other words, we may assume:

$$\begin{aligned} (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^5)^2 &= 1, \; (x^4)^2 + (x^6)^2 = 1, \\ (y^1)^2 + (y^2)^2 + (y^3)^2 + (y^5)^2 &= 1, \; (y^4)^2 + (y^6)^2 = 1. \end{aligned} \tag{14}$$

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Using simple algebra and taking into account Eqs. (14) we find that the equality $\det(u(X)) = \det(u(X'))$ leads to

$$(x^4)^2 - (x^6)^2 = (y^4)^2 - (y^6)^2,\ x^4x^6 = y^4y^6, \tag{15}$$

which, together with $(x^4)^2 + (x^6)^2 = (y^4)^2 + (y^6)^2 = 1$, entails

$$(x^4, x^6) = \pm(y^4, y^6). \tag{16}$$

From $u(X) = u(X')$ it then immediately follows that $X = \pm X'$, therefore $[X] = [X']$. □

2. Embedding of M into $\overline{M}$

Here we adapt the standard methods of Möbius geometry of Lie spheres, as discussed, for instance, in [4, Ch. 2, Eq. (2.6)]. Consider the following smooth map between manifolds: $\tau : M = \mathbb{R}^{3,1} \to \mathbb{R}^{4,2}$ given by the formula:

$$\tau(x) = (x, \frac{1}{2}(1 - q(x)), -\frac{1}{2}(1 + q(x))). \tag{17}$$

The map τ is evidently injective. Let $\mathcal{Z}$ be the hyperplane in $\mathbb{R}^{4,2}$ defined by the formula

$$\mathcal{Z} = \{X \in \mathbb{R}^{4,2} : x^5 - x^6 = 1\}. \tag{18}$$

Lemma 1. *The image $\tau(M)$ of M in $\mathbb{R}^{4,2}$ coincides with the intersection $\mathcal{N} \cap \mathcal{Z}$ of the null cone $\mathcal{N}$ with the hyperplane $\mathcal{Z}$.*

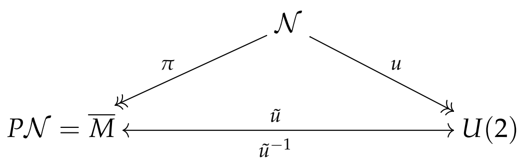
Proof. It is clear that, for all $x \in M$, $\tau(x) \neq 0$. It also follows by a straightforward calculation that $Q(\tau(x)) = 0$, thus $\tau(M) \subset \mathcal{N}$. From Eq. (17) we have that $\tau(x)$ is also in $\mathcal{Z}$. Thus $\tau(M)$ is a subset of $\mathcal{N} \cap \mathcal{Z}$. To show that $\tau(M) = \mathcal{N} \cap \mathcal{Z}$, let $X = (x, x^5, x^6)$ be in $\mathcal{N} \cap \mathcal{Z}$. From $Q(X) = Q(x, x^5, x^6) = 0$ we get $q(x) + (x^5)^2 - (x^6)^2 = 0$. But $(x^5)^2 - (x^6)^2 = (x^5 - x^6)(x^5 + x^6)$, so that, from $x^5 - x^6 = 1$, it follows that $q(x) + x^5 + x^6 = 0$. Together with $x^5 - x^6 = 1$ it implies $q(x) + 2x^5 = 1$ or $x^5 = \frac{1}{2}(1 - q(x))$ and $x^6 = -\frac{1}{2}(1 + q(x))$. It follows that $X = \tau(x)$. □

Remark 1. *Eqs. (17) and (18) take a simpler form when, instead of orthogonal coordinates x^5, x^6), null coordinates $H = x^5 - x^6$, $N = -(x^5 + x^6)$ are being used, as it is done e.g. in [6]. If we use these coordinates, and if we write X as $X = (x, H, N)$, then the quadratic form Q takes the form $Q(X) = q(x) - HN$, $\tau(x) = (x, 1, q(x))$, $\mathcal{Z} = \{X = (x, H, N) : H = 0\}$.*

We now define the embedding $M \to \overline{M}$ by

$$\tilde{\tau} = \pi \circ \tau. \tag{19}$$

We thus have the following (commutative) diagram, where doubled arrow ends denote surjections.



2.1. The Conformal Infinity

Now, if $X \in \mathcal{N}$ and $x^5 \neq x^6$, we can set $X' = X/(x^5 - x^6)$. Then $\pi(X) = \pi(X')$, so that X and X' define the same point in $\overline{M}$. But now $x'^5 - x'^6 = 1$. It follows that the points of $\overline{M}$ which are not in the image $\tilde{\tau}(M)$ of M are equivalence classes of those $X \in \mathcal{N}$ for which $x^5 - x^6 = 0$. These points define the *conformal infinity*. Another method of defining the same set is described in the next section

2.2. Embedding via the Cayley Transform

Denote by $H(2)$ the set of all complex hermitian 2×2 matrices. $H(2)$ is a real vector space of dimension 4. For $x = (x^1, \dots, x^4) \in M$ let $h(x) \in H(2)$ be defined as

$$h(x) = \begin{bmatrix} x^4 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^4 - x^3 \end{bmatrix}. \tag{20}$$

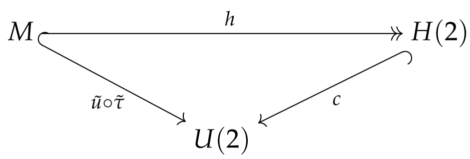
Then $h : x \mapsto h(x)$ is an isomorphism of real vector spaces M and $H(2)$.

For every $A \in H(2)$ let $c(A)$ denote the matrix

$$c(A) = \frac{A - iI_2}{A + iI_2}. \tag{21}$$

Then $c(A)$ is in $U(2)$, and the map $c : A \mapsto c(A)$ is injective. The map $c : H(2) \to U(2)$ is called the Cayley transform. Composing $c \circ h$ we get an embedding of M into $U(2)$, but the following proposition shows that it is the same as $\tilde{u} \circ \tilde{\tau}$.

Proposition 2. *The following diagram is commutative:*



Proof. Since all the maps in the diagram are given explicitly, checking of the commutativity is a matter of a simple algebra. □

2.3. Conformal Infinity Within $U(2)$.

Suppose we want to invert the Cayley transform

$$U = \frac{A - iI_2}{A + iI_2} \quad , \tag{22}$$

and express A in terms of U. We multiply both sides by $A + iI_2$ - it does not matter which side, since A and U commute. We get, after regrouping the terms:

$$(I_2 - U)A = i(I_2 + U). \tag{23}$$

We notice that we can express A in terms of U if and only if $I_2 - U$ is invertible (i.e. iff U does not have 1 as one of its eigenvalues). The set of those matrices for which $\det(I_2 - U) = 0$ is not in the image $c(h(M))$. We denote this set by $\mathcal{J}$

$$\mathcal{J} = \{U \in U(2) : \det(I_2 - U) = 0\} \tag{24}$$

and call it the *conformal infinity*. This is not a one-point compactification. We add to $\tau(M)$ a whole (closed) "cap" - a three -dimensional manifold $\mathcal{J}$.

Remark 2. *$\mathcal{J}$ is not exactly a smooth 3d manifold. As we will see it has a singular point, a 'corner'. Usually it is referred in the literature as 'the double cone at infinity', but it is not a double cone at all (Cf. [5]). It is a Dupin cyclide - Cf. Sec. (6.5).*

3. Action of the Conformal Group $O(4,2)$.

Let G be the diagonal matrix

$$G = \mathrm{diag}(1,1,1,-1,1,-1), \tag{25}$$

so that for $X \in \mathbb{R}^{4,2}$ we have

$$Q(X) = X^T G X. \tag{26}$$

The group $O(4,2)$ is defined as the group of all real 6×6 matrices L satisfying

$$L^T G L = G. \tag{27}$$

It is evident that, for every $L \in O(4,2)$, if X is in $\mathcal{N}$, i.e. if $X \neq 0$ and $Q(X) = 0$, then also LX is in $\mathcal{N}$. Moreover, L maps equivalence classes of $\sim$ into equivalence classes, therefore the action of $O(4,2)$ on $\mathcal{N}$ descends to its action on $\overline{M} = P\mathcal{N}$. We define $SO(4,2)$ as the subgroup of $O(4,2)$ consisting of matrices $L \in O(4,2)$ of determinant 1.

3.1. The Group $SO_0(4,2)$

In order to separate four space-time coordinates from additional two coordinates we have chosen the signature of Q as $(1,1,1,-1,1,-1)$. In this subsection it will be convenient to choose the signature $(1,1,1,1,-1,-1)$, the two signatures being equivalent by permutation of coordinates. Formally it can be achieved by switching the coordinates X^4 and X^5, that is by a similarity transformation $G \mapsto G' = RGR^{-1} = \mathrm{diag}(1,1,1,1,-1,-1)$, $L \mapsto L' = RLR^{-1}$, where R is the matrix

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{28}$$

Notice that $\det L = \det L'$, and that $L^T G L = G$ is equivalent to $L'^T G' L' = G'$. Now it is convenient to write L' in a block matrix form, using matrices A', B', C', D', of respective dimensions 4×4, 4×2, 2×4, 2×2:

$$L' = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}. \tag{29}$$

Then the condition $L'^T G' L' = G'$ translates to:

$$A'^T A' - C'^T C' = 1, \tag{30}$$
$$A'^T B' - C'^T D' = 0, \tag{31}$$
$$B'^T B' - D'^T D' = -1, \tag{32}$$

where we denoted by 1 the identity matrices of appropriate dimensions. In particular we have

$$A'^T A' = 1 + C'^T C' \geq 1,\ D'^T D' = 1 + B'^T B' \geq 1, \tag{33}$$

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which entails $(\det A')^2 \geq 1$, and $(\det D')^2 \geq 1$. Therefore either $\det A' \geq 1$ or $\det A' \leq -1$. Similarly, either $\det D' \geq 1$, or $\det D' \leq -1$. It can be shown (Cf. Refs. [8,9]) that, for $L' \in SO(4,2)$, $\det A' \geq 1$ if and only if $\det D' \geq 1$. We define

$$\begin{aligned} SO_0(4,2) &= \{L' \in SO(4,2) : \det A' \geq 1\} \\ &= \{L' \in SO(4,2) : \det D' \geq 1\} \\ &= \{L' \in SO(4,2) : \det A' \geq 1,\, \det D' \geq 1\} \end{aligned} .$$

Then (Cf. [10, p. 107]) $SO_0(4,2)$ is a connected subgroup (the connected component of identity) of $SO(4,2)$.

Proposition 3. *The action of $SO_0(4,2)$ on the compactified Minkowski space $\overline{M}$ is transitive.*

Proof. Let $[X]$ and $[Y]$ be any two points in $\overline{M}$. Let us choose the representatives of the equivalence classes so that $\mu(X) = \mu(Y) = 1$. That means $(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^5)^2 = (Y^1)^2 + (Y^2)^2 + (Y^3)^2 + (Y^5)^2 = 1$, and $(X^4)^2 + (X^6)^2 = (Y^4)^2 + (Y^6)^2 = 1$. The group $SO(4)$ acts transitively on the sphere S^3, therefore we can choose an orthogonal rotation from $SO(4) \subset SO(4,2)$ acting on the variables X^1, X^2, X^3, X^5, and leaving the variables X^4, X^6 fixed, that transforms (X^1, X^2, X^3, X^5) into (Y^1, Y^2, Y^3, Y^5). Similarly the group $SO(2)$ acts transitively on the sphere S^1, therefore we can choose an orthogonal rotation from $SO(2) \subset SO(4,2)$ acting on the variables X^4, X^6, and leaving the variables X^1, X^2, X^3, X^5 fixed, that transforms (X^4, X^6) into (Y^4, Y^6). The composition of these two transformations is in $SO_0(4,2)$ and maps X to Y. $\square$

Remark 3. *Notice that the transformation $X \mapsto -X$ is in $SO_0(4,2)$ and acts on $\overline{M}$ as the identity map. Thus the action of $SO_0(4,2)$ on $\overline{M}$ is not an effective action.*

4. The Double Cover $\overline{\overline{M}}$ of $\overline{M}$

For $X, X' \in \mathbb{R}^{4,2} \setminus \{0\}$ we define the equivalence relation $'\approx'$ as

$$X \approx X' \text{ if and only if } X = \lambda X',\, \lambda > 0. \tag{34}$$

Definition 1. *We will denote by $\hat{\pi}$ the canonical projection $\mathcal{N} \to \mathcal{N}/\approx$, $\hat{\pi} : X \mapsto [[X]]$.*

We denote by $[[X]]$ the equivalence classes of this relation. Notice that while $[X] = [-X]$, we have $[[X]] \neq [[-X]]$. In fact each equivalence class of $\sim$ contains two equivalence classes of $\approx$. The projective space $P\mathcal{N} = \mathcal{N}/\sim$ is known as the Grassmannian of null lines in $\mathbb{R}^{4,2}$. We define

$$\overline{\overline{M}} = \mathcal{N}/\approx, \tag{35}$$

i.e. as the Grassmannian of *oriented* null lines in $\mathbb{R}^{4,2}$.

Proposition 4. *$\overline{\overline{M}}$ is diffeomorphic to $S^3 \times S^1$.*

Proof. Each equivalence class $[[.]]$ has now a unique representative X with $\mu(X) = 1$. Namely, if $X \in \mathcal{N}$ is arbitrary, then $X/\mu(X)^{1/2}$ is this representative. Now, if $\mu(X) = 1$, we define $Z(X) = (X^1, X^2, X^3, X^5)$, $z(X) = (X^4, X^6)$, and then $Z \in S^3$ and $z \in S^1$. Conversely, any such pair of (Z, z) defines $X \in \mathcal{N}$ with $\mu(X) = 1$. $\square$

Remark 4. *Penrose and Rindler, in their classic monograph [1, p. 298-299] state (Eq. (9.2.1)) that $\overline{M}$ (denoted there $M^\#$) is homeomorphic to $S^3 \times S^1$. For a non-expert reader this can be somewhat confusing. As we have seen $S^3 \times S^1$ can be identified with $\overline{\overline{M}}$, which is a double cover of $\overline{M}$. In Ref. [3] $\overline{\overline{M}}$ is denoted $\widehat{\mathrm{Ein}}^{3,1}$ and called "the Einstein space".*

The action of $O(4,2)$ on $\overline{\overline{M}}$ is defined the same way as for $\overline{M}$. We define

$$L[[X]] = [[LX]]. \tag{36}$$

Proposition 5. *The action of $SO(4,2)$ on $\overline{\overline{M}}$ is transitive.*

Proof. The proof goes exactly the same way as for $\overline{M}$. □

5. Important Subgroups of $O(4,2)$

In Eq. (17) we have defined the embedding τ of M into $\mathbb{R}^{4,2}$ We will use this map to identify important Lie subgroups of $SO(4,2)$ (and discrete transformations in $O(4,2)$.)

5.1. Translation Subgroup

We start with translations $x \mapsto x' = x + a$, where $x, a \in \mathbb{R}^{3,1}$.

Let

$$X = \tau(x) = (x, \frac{1}{2}(1 - q(x)), -\frac{1}{2}(1 + q(x))), \tag{37}$$

and let

$$X' = \tau(x') = (x + a, \frac{1}{2}(1 - q(x + a)), -\frac{1}{2}(1 + q(x + a))). \tag{38}$$

We have

$$q(x + a) = q(x) + 2a \cdot x + a^2, \tag{39}$$

where $a \cdot x$ is the scalar product of vectors in $\mathbb{R}^{3,1}$, and $a \cdot a = a^2 = q(a)$. Thus

$$X' = (x + a, \frac{1}{2}(1 - q(x) - 2a \cdot x - a \cdot a), -\frac{1}{2}(1 + q(x) + 2a \cdot x + a \cdot a)), \tag{40}$$

or

$$X' = X + (a, -a \cdot x - a \cdot a/2, -a \cdot x - a \cdot a/2). \tag{41}$$

But we want the transformation to be linear in X. The trick is to multiply the right hand side by $1 = X^5 - X^6$. We get this way the desired formula:

$$X' = X + (a, -a \cdot x - a \cdot a/2, -a \cdot x - a \cdot a/2)(X^5 - X^6). \tag{42}$$

The transformation $X \mapsto X'$ is now implemented by the following matrix written in a block matrix form:

$$L(a) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{43}$$

where

$$A = I_4, B = \begin{bmatrix} a & -a \end{bmatrix}, C = \begin{bmatrix} -a^T G \\ -a^T G \end{bmatrix}, D = I_2 + \frac{1}{2}\begin{bmatrix} -q(a) & q(a) \\ -q(a) & q(a) \end{bmatrix}, \tag{44}$$

or explicitly:

$$L(a) = \begin{bmatrix} 1 & 0 & 0 & 0 & a^1 & -a^1 \\ 0 & 1 & 0 & 0 & a^2 & -a^2 \\ 0 & 0 & 1 & 0 & a^3 & -a^3 \\ 0 & 0 & 0 & 1 & a^4 & -a^4 \\ -a^1 & -a^2 & -a^3 & a^4 & 1 - q(a)/2 & q(a)/2 \\ -a^1 & -a^2 & -a^3 & a^4 & -q(a)/2 & 1 + q(a)/2 \end{bmatrix}. \tag{45}$$

One can easily check (for instance using a computer algebra software) that

$$L(a)^T G L(a) = G, \det(L(a)) = 1, \tag{46}$$

and that

$$L(a+b) = L(a)L(b). \tag{47}$$

5.2. Lorentz Rotations Subgroup

For any Lorentz rotation $\Lambda \in O(3,1)$ we have $q(\Lambda x) = q(x)$, therefore the way the Lorentz rotations are embedded in $O(4,2)$ (using our definition of G) is evident. We simply set, using block matrix form,

$$L(\Lambda) = \begin{bmatrix} \Lambda & 0 \\ 0 & I_2 \end{bmatrix}. \tag{48}$$

5.3. Dilation Subgroup

This time we choose a different way. There is a natural subgroup of $SO(4,2)$, similar to the Lorentz rotations case, but putting a "Lorentz rotation" in the lower right corner. This one-parameter Abelian subgroup is defined as follows:

$$L(\alpha) = \begin{bmatrix} I_4 & 0 \\ 0 & d(\alpha) \end{bmatrix}, \alpha \in \mathbb{R}, \tag{49}$$

where

$$d(\alpha) = \begin{bmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{bmatrix}. \tag{50}$$

Proposition 6. *For $x \in M$ we have*

$$L(\alpha)\tau(x) = e^{-\alpha}\tau(e^{\alpha}x). \tag{51}$$

Therefore the matrices $L(\alpha)$ implement dilations $x \mapsto e^{\alpha}x$ of the Minkowski space.

Proof. For $X = \tau(x)$ we have

$$X = (x, X^5, X^6),$$

where

$$X^5 = \frac{1}{2}(1 - q(x)),$$

$$X^6 = -\frac{1}{2}(1 + q(x)).$$

Let $X' = L(\alpha)X$. Then

$$X' = (x, X'^5, X'^6),$$

where

$$\begin{aligned} X'^5 &= \cosh(\alpha)x^5 + \sinh(\alpha)x^6, \\ X'^6 &= \sinh(\alpha)x^5 + \cosh(\alpha)x^6. \end{aligned}$$

Now $\cosh(\alpha) = (e^{\alpha} + e^{-\alpha})/2$, $\sinh(\alpha) = (e^{\alpha} - e^{-\alpha})/2$. Using also the definitions of X'^5 and X'^6 we obtain

$$\begin{aligned} X'^5 &= \frac{1}{2}\left(\frac{1}{2}(e^{\alpha} + e^{-\alpha})(1 - q(x)) - \frac{1}{2}(e^{\alpha} - e^{-\alpha})(1 + q(x))\right) \\ &= \frac{1}{2}(e^{-\alpha} - e^{\alpha}q(x)) \\ &= e^{-\alpha}\frac{1}{2}(1 - e^{2\alpha}q(x)) \\ &= e^{-\alpha}\frac{1}{2}(1 - q(e^{\alpha}x)) \end{aligned}$$

Similarly

$$X'^6 = -e^{-\alpha}\frac{1}{2}(1+q(e^\alpha x)).$$

Thus

$$X' = e^{-\alpha}\left(e^\alpha x, \frac{1}{2}(1-q(e^\alpha x), -\frac{1}{2}(1+q(e^\alpha x)) = e^{-\alpha}\tau(e^\alpha x)\right).$$

□

5.4. The Conformal Inversion

Conformal inversion is defined in M as the map

$$x \mapsto \frac{x}{q(x)}. \tag{52}$$

We assume that x is dimensionless, otherwise we would have to define it as $R^2x/q(x)$, where R is some "radius". Also notice that on M the transformation is singular on the null cone at the origin $\{x \in M : q(x) = 0\}$. Under conformal inversion the null cone is mapped into a set of points at the conformal infinity of $\overline{M}$. How? It will be clear after we derive the matrix $O(4,2)$ representation of the conformal inversion, which we will do right now.

Assuming $x \in M$, $q(x) \neq 0$, and with $x' = x/q(x)$, consider the following simple calculation:

$$\begin{aligned}
\tau(x') &= (x', \frac{1}{2}(1-q(x')), -\frac{1}{2}(1+q(x'))) \\
&= (\frac{x}{q(x)}, \frac{1}{2}(1-\frac{1}{q(x)}), -\frac{1}{2}(1+\frac{1}{q(x)})) \\
&= \frac{1}{q(x)}(x, \frac{1}{2}(q(x)-1), -\frac{1}{2}(q(x)+1)) \\
&= \frac{1}{q(x)}(x, -\frac{1}{2}(1-q(x)), -\frac{1}{2}(q(x)+1)).
\end{aligned} \tag{53}$$

But the end result is the same as acting on $\tau(x)$ with the following matrix $\mathcal{C}$ from the group $O(4,2)$:

$$\mathcal{C} = \begin{bmatrix} I_4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{54}$$

and then multiplying the resulting vector by $1/q(x)$ which is inessential when projecting on $\overline{M}$ using the projection π. In other words conformal inversion is implemented on $\overline{M}$ by the reflection in the variable X^5. Notice that

$$\mathcal{C}^2 = I, \text{ and } \det(\mathcal{C}) = -1, \tag{55}$$

so $\mathcal{C}$ is in $O(4,2)$, but not in $SO(4,2)$.

Finally let us address the question of the image of the null cone at the origin by the conformal inversion. It follows from Eq. (53) that this image is the 'double null cone at infinity': $\{(x,1,1) : q(x) = 0\}$. But that is not yet the whole conformal infinity. What remains is the set $\{[(x,0,0)] : q(x) = 0, x \neq 0\}$, which is homeomorphic to the sphere S^2. It is this sphere that connects the lower and upper branches of the null cone at infinity, and makes the conformal infinity into a compact set - Cf. Figure 2. Many published papers on the subject omit this two-sphere and incorrectly identify the conformal infinity with the 'double null cone', as e.g. in [6, p. 263].

5.5. Special Conformal Transformations

This four-parameter Abelian subgroup of $SO(4,2)$ is defined as:

$$K(a) = \mathcal{C}L(a)\mathcal{C},\ a \in \mathbb{R}^{3,1}. \tag{56}$$

The matrices representing $K(a)$ are similar to those representing $L(a)$, since the bracketing of any matrix A with $\mathcal{C}$, $A \mapsto \mathcal{C}A\mathcal{C}$, results only in changing signs of the elements A_{5i} in the fifth row and A_{i5} in the fifth column, the element A_{55} changes sign twice, so it stays unaltered. Thus we have:

$$K(a) = L(a) = I_6 + \begin{bmatrix} 0 & 0 & 0 & 0 & -a^1 & -a^1 \\ 0 & 0 & 0 & 0 & -a^2 & -a^2 \\ 0 & 0 & 0 & 0 & -a^3 & -a^3 \\ 0 & 0 & 0 & 0 & -a^4 & -a^4 \\ a^1 & a^2 & a^3 & -a^4 & -q(a)/2 & -q(a)/2 \\ -a^1 & -a^2 & -a^3 & a^4 & q(a)/2 & q(a)/2 \end{bmatrix}. \tag{57}$$

It is easy to get the action of the special conformal transformations on $x \in M$. From the definition we have

$$x \mapsto x' = \frac{\frac{x}{q(x)} + a}{q(\frac{x}{q(x)} + a)}, \tag{58}$$

which simplifies to

$$x' = \frac{x + q(x)a}{1 + 2a \cdot x + q(x)a \cdot a}. \tag{59}$$

Evidently, for a given a, the points x for which $1 + 2a \cdot x + q(x)a \cdot a = 0$ are transformed onto points in the conformal infinity.

5.6. Conformal Structure

Let p be a point in $\overline{M}$, and let ξ be a vector tangent to $\overline{M}$ at p. A vector tangent at p to $\overline{M}$ is a vector tangent to a smooth curve $\gamma(s)$, such that $\gamma(0) = p$. We will write it as

$$\xi = \dot{\gamma}(0). \tag{60}$$

We will use now one of the intuitively clear important properties of projective manifolds, such as $P\mathcal{N}$. Namely, given any $X \in \pi^{-1}(p)$, there always exists a smooth curve $\tilde{\gamma}_X(s)$, defined for s in a neighborhood of $s = 0$, such that $\tilde{\gamma}_X(0) = X$ and $\pi(\tilde{\gamma}_X(s)) = \gamma(s)$. We say that $\tilde{\gamma}_X(s)$ is a (local) lift of γ through X.

Of course such a lift is not unique. If $\lambda(s)$ is any smooth function such that $\lambda(0) = 1$, then, locally,

$$\tilde{\gamma}'_X(s) = \lambda(s)\tilde{\gamma}_X(s) \tag{61}$$

is another lift of γ through X. In fact any two lifts of γ through X are related that way.

Now, in coordinates, $\tilde{\gamma}_X(s)$ is described as a function $X(s) \in \mathcal{N} \subset \mathbb{R}^{4,2}$, with $X(0) = X$. It is convenient to use the notation $X \cdot Y$ for the scalar product $X^T G Y$ in $\mathbb{R}^{4,2}$. Since $X(s)$ is a null vector, we have $X(s) \cdot X(s) = 0$. Taking the derivative with respect to s at $s = 0$ we find that

$$X \cdot \dot{X}(0) = 0. \tag{62}$$

In other words $\dot{X}(0)$ is perpendicular to X.

Let now ξ_1, ξ_2 be any two tangent vectors at p, and let γ_1, γ_2 be two curves through p such that

$$\xi_1 = \dot{\gamma}_1(0),\ \xi_2 = \dot{\gamma}_2(0). \tag{63}$$

Let $X_1(s)$ and $X_2(s)$ be the coordinate expressions of lifts of $\gamma_1(s)$ and $\gamma_2(s)$ through the same X. We define the scalar product of tangent vectors by the formula

$$(\xi_1, \xi_2)_X = \dot{X}_1 \cdot \dot{X}_2. \tag{64}$$

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For this definition to make sense we must show that the scalar product defined in this way does not depend on the choice of lifts through X. To this end suppose

$$X_1'(s) = \lambda_1(s)X_1(s),\ X_2'(s) = \lambda_2(s)X_2(s) \tag{65}$$

are two other lifts. Differentiating with respect to s at $s = 0$ we get

$$\dot{X}_1'(0) = \dot{X}_1(0) + \dot{\lambda}_1(0)X,\ \dot{X}_2'(0) = \dot{X}_2(0) + \dot{\lambda}_2(0)X. \tag{66}$$

Now, taking into account the fact that $X \cdot X = 0$ and $X \cdot \dot{X}_1(0) = X \cdot \dot{X}_2(0) = 0$, we immediately find that

$$\dot{X}_1' \cdot \dot{X}_2{}' = \dot{X}_1 \cdot \dot{X}_2, \tag{67}$$

therefore the scalar product $(\xi_1, \xi_2)_X$ is indeed well defined. It is now immediate that for any $\lambda \in \mathbb{R}$, $\lambda \neq 0$, we have

$$(\xi_1, \xi_2)_{\lambda X} = \lambda^2(\xi_1, \xi_2)_X. \tag{68}$$

Therefore what is defined at p is not a scalar product, but an equivalence class of scalar products, where two scalar products are considered to be equivalent if they are proportional, with a proportionality constant that is strictly positive.

We say that we have defined this way a *conformal structure* on $\overline{M}$. To give a Riemannian (or pseudo-Riemannian) structure on a manifold is to specify a scalar product at each tangent space. When only a class of proportional scalar products is defined, we call it a conformal structure.

5.7. Calculating the Conformal Structure of $\overline{M}$ Explicitly

The map τ defined in Eq. (17) defines a coordinate system on an open dense subset of $\overline{M}$. We will now derive the expression of the $O(4,2)$ invariant conformal structure in the domain of this coordinate chart. Let p be a point in this domain, and let $x \in M$ be the point in the Minkowski space $M = \mathbb{R}^{3,1}$ for which $[\tau(x)] = p$. So p is represented by

$$X = \tau(x) = (x, \frac{1}{2}(1 - q(x)), -\frac{1}{2}(1 + q(x))). \tag{69}$$

Of course p can be represented as well by λX, $\lambda \neq 0$, but let us concentrate on X. In coordinates a tangent vector at x has coordinates ξ^μ, $(\mu = 1, \ldots, 4)$. Let $\gamma(s)$ be a curve through p, $\gamma(0) = p$. In coordinates it is represented by a curve $x(s)$, $x(0) = x$. Then $X(s)$ defined by

$$X(s) = (x(s), \frac{1}{2}(1 - q(x(s))), -\frac{1}{2}(1 + q(x(s)))), \tag{70}$$

is a lift of $\gamma(s)$ to $\mathcal{N}$. Differentiating the last equation with respect to s at $s = 0$, and taking into account the explicit form of $q(x) = x^T g x = x \cdot x$, we obtain

$$\dot{X}(0) = (\dot{x}(0), -x \cdot \dot{x}(0), -x \cdot \dot{x}(0)). \tag{71}$$

Let us calculate an explicit formula for $(\xi_1, \xi_2)_X$. With $\xi_1 = \dot{x}_1(0)$, $\xi_2 = \dot{x}_2(0)$, from Eq. (71) we easily get

$$(\xi_1, \xi_2)_X = \dot{X}_1(0) \cdot \dot{X}_2(0) = \xi_1 \cdot \xi_2 = \xi_1^T g \xi_2, \tag{72}$$

the other two terms cancel each other.

In other words: the $O(4,2)$ invariant conformal structure on $\overline{M}$ in a local chart of Minkowski space coordinates consists of scalar products proportional to the standard Minkowski space scalar product.

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6. The Case of the Double Covering Compactification $\overline{\overline{M}}$

Here we will find out what needs to be changed and how if we replace the simple compactification $\overline{M}$ with its double cover $\overline{\overline{M}}$.

6.1. The Action of $O(4,2)$ Becomes Effective

If G is a group acting on a set M, we say that the action is effective (resp. free) if, for $g \in G$ and $x \in M$, $gx = x$ for all (resp. for some) $x \in M$ implies that g is the identity element of G. In other words the action is effective if every different from the identity element of G does something nontrivial to something (to some element of M), and free if every different from the identity element of G does something nontrivial to everything (to every point of M). Thus every free action is effective, but there exist actions which are effective but not free.

Consider the matrix $-I_6$, which is in $SO(4,2)$. It acts on every $X \in \mathcal{N}$ replacing it with $-X$. But X and $-X$ are in the same equivalence class of $\sim$ so $\pi(X) = \pi(-X)$ - they define the same point of $\overline{M}$. The matrix $-I_6$ does nothing to all points of $\overline{M}$ - the action of the group $SO(4,2)$ on $\overline{M}$ is not effective - we are losing information. But, for $X \in \mathcal{N}$, X and $-X$ define two different points of $\overline{\overline{M}}$. Therefore $-I_6$ acts nontrivially on $\overline{\overline{M}}$. In fact, we have an effective action of $SO(4,2)$ on $\overline{\overline{M}}$. [1]

6.2. The two Embeddings $\tau_\pm$.

Eq. (17) defines the embedding of M into $\mathcal{N}$:

$$\tau(x) = (x, \frac{1}{2}(1 - q(x)), -\frac{1}{2}(1 + q(x))) \tag{73}$$

Then $-X$ becomes

$$-X = (-x, -\frac{1}{2}(1 - q(-x)), \frac{1}{2}(1 + q(-x))). \tag{74}$$

Replacing x by $-x$ and taking into account the fact that $q(-x) = q(x)$, we obtain another embedding, we call it τ_-:

$$\tau_-(x) = (x, -\frac{1}{2}(1 - q(-x)), \frac{1}{2}(1 + q(-x))), \tag{75}$$

while the first embedding we call τ_+

$$\tau_+(x) = (x, \frac{1}{2}(1 - q(x)), -\frac{1}{2}(1 + q(x))). \tag{76}$$

We define

$$\hat{\tau}_\pm = \hat{\pi} \circ \tau_\pm. \tag{77}$$

Remark 5. *We could as well define $\tau_-(x) = -\tau(x)$. It would also work. Which definition will prove to be more convenient will be seen only after a physical interpretation will be given to these constructions. As it is above the transformation $-I_6$ implements space and time inversion on M.*

6.3. The Doubled Conformal Infinity

$\hat{\tau}_+(M)$ and $\hat{\tau}_-(M)$ are two nonintersecting copies of M embedded in $\overline{\overline{M}}$. Their union $\hat{\tau}_+(M) \cup \hat{\tau}_-(M)$ forms an open dense set in $\overline{\overline{M}}$.

Let us define

$$\mathcal{N}_+ = \{X \in \mathcal{N} : X^5 - X^6 > 0\}, \tag{78}$$
$$\mathcal{N}_- = \{X \in \mathcal{N} : X^5 - X^6 < 0\}, \tag{79}$$
$$\mathcal{N}_0 = \{X \in \mathcal{N} : X^5 - X^6 = 0\}. \tag{80}$$

[1] Ref. [11, Proposition 8] ascertains the *existence* of the action of $SO(4,2)$ on $\overline{\overline{M}}$. Here we have its explicit realization

$$\overline{\overline{M}}_+ = \hat{\pi}(\mathcal{N}_+), \tag{81}$$
$$\overline{\overline{M}}_- = \hat{\pi}(\mathcal{N}_-), \tag{82}$$
$$\overline{\overline{M}}_\infty = \hat{\pi}(\mathcal{N}_0), \tag{83}$$

where $\hat{\pi}$ is as in the Definition 1, Sec. 4. Then $\overline{\overline{M}}$ is a sum of three disjoint sets

$$\overline{\overline{M}} = \overline{\overline{M}}_+ \cup \overline{\overline{M}}_- \cup \overline{\overline{M}}_\infty. \tag{84}$$

We have two copies of M: $\overline{\overline{M}}_+$ and $\overline{\overline{M}}_-$, and the *doubled conformal infinity* $\overline{\overline{M}}_\infty$.

The matrix $-I_6$ is in $SO(4,2)$ and it maps $\mathcal{N}$ into $\mathcal{N}$ and equivalence classes of $\approx$ into equivalence classes. Therefore it defines a unique mapping $\iota : \overline{\overline{M}} \to \overline{\overline{M}}$ such that:

$$\iota \circ \hat{\pi} = \hat{\pi} \circ (-I_6). \tag{85}$$

We evidently have $\iota^2 = \mathrm{Id}$.

Remark 6. *The action of ι on $\overline{\overline{M}}$ is free. Or more precisely: the action of the two-element group $\{Id, \iota\}$ on $\overline{\overline{M}}$ is free. The two-element group $\{Id, \iota\}$ is isomorphic to the group $\mathbb{Z}_2$ - the additive group of integers modulo 2. It acts freely on $\overline{\overline{M}}$. $\overline{M}$ can be identified with the quotient $\overline{\overline{M}}/\mathbb{Z}_2$.*

We will now analyze the structure of $\overline{\overline{M}}_\infty$ and, after skipping one space dimension, provide its graphical representation. To this end consider the two equations defining $\mathcal{N}_0$ written as

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^5)^2 = (x^4)^2 + (x^6)^2, \tag{86}$$

$$x^5 = x^6. \tag{87}$$

Clearly the number $(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^5)^2 = (x^4)^2 + (x^6)^2$ is positive, it cannot be zero because that would imply $X = 0$, and the origin is excluded. Therefore we can always choose a unique positive scaling factor and get two equations in $\mathbb{R}^6$: $(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^5)^2 = 1$, and $(x^4)^2 + (x^6)^2 = 1$. These are two intersecting cylinders. The infinity plane $x^6 = x^5$ cuts this intersection effectively reducing the number of dimensions to 3. We obtain:

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^5)^2 = 1, \tag{88}$$

$$(x^4)^2 + (x^5)^2 = 1. \tag{89}$$

Notice that $(x^4)^2 \leq 1$, therefore $(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 + (x^5)^2 \leq 2$. Thus the whole surface in $\mathbb{R}^5$ is contained within a ball of radius $\sqrt{2}$. In order to arrive at a graphics representation in $\mathbb{R}^3$ we suppress one space dimension, say x^3, so that two-spheres will be represented by circles. We are left now with four variables (x^1, x^2, x^4, x^5), and the intersection of two cylinders:

$$(x^1)^2 + (x^2)^2 + (x^5)^2 = 1, \tag{90}$$

$$(x^4)^2 + (x^5)^2 = 1, \tag{91}$$

6.4. Conceptual Structure of $\overline{\overline{M}}_\infty$

Let us start with considering the variable X^5 that plays a special role - it appears in both equations (90) and (91). It can vary in the closed interval $[-1, 1]$. Consider first the two endpoints of this interval. If $(X^5)^2 = 1$, it follows from Eqs. (90),(91) that the remaining variables cannot vary at all, they must all have value 0. These will be two special "endpoints" of $\overline{\overline{M}}_\infty$.

Assume now that $(X^5)^2 < 1$. In that case X^1, X^2 satisfy the equation

$$(X^1)^2 + (X^2)^2 = 1 - (X^5)^2 > 0. \tag{92}$$

This equation describes a circle of radius $\sqrt{1-(X^5)^2}$. The radius of this circle first grows from 0 to 1 when X^5 varies from -1 to 0, then the radius gets smaller and smaller when X^5 varies from 0 to 1. It becomes a point when $X^5 = 1$. For each value of $X^5 \in (-1, 1)$, the variable X^4 takes one of the two possible values $X^4 = \pm\sqrt{1-(X^5)^2}$.

The circles (in the plane (X^1, X^2)) corresponding to X^4 positive and those corresponding to X^4 negative are disjoint, as $[[X]]$ and $[[-X]]$ are always two different points in $\overline{\overline{M}}$.

It follows from the above discussion that we can safely conjecture that we do not really need four dimensions spanned by X^1, X^2, X^4, X^5. The whole $\overline{\overline{M}}_\infty$ should have a faithful representation of $\overline{\overline{M}}_\infty$ (with one space dimension skipped) in $\mathbb{R}^3$. We will now prove that this is indeed the case.

6.5. Graphical Representation of $\overline{\overline{M}}_\infty$.

We will provide a particular implementation, using mathematical formulas, the intuition provided by the reasoning above. It is a simple matter to see that we have the following:

Proposition 7. *Let a be a real number $a > 1$. For any X^1, X^2, X^4, X^5 satisfying Eqs. (90),(91) let $(x, y, z) \in \mathbb{R}^3$ be defined as*

$$x(X^1, X^2, X^4, X^5) = \frac{aX^4}{a - X^1} \tag{93}$$

$$y(X^1, X^2, X^4, X^5) = \frac{aX^2}{a - X^1} \tag{94}$$

$$z(X^1, X^2, X^4, X^5) = \frac{aX^5}{a - X^1}. \tag{95}$$

Then the mapping

$$(X^1, X^2, X^4, X^5) \to (x(X^1, X^2, X^4, X^5), y(X^1, X^2, X^4, X^5), z(X^1, X^2, X^4, X^5))$$

is injective. □

To obtain a parametrization of this two-dimensional surface embedded in $\mathbb{R}^3$ we first parametrize, using the parameter u, $(-\pi \le u < \pi)$, the circle given by Eq. (91):

$$X^4 = \cos(u), \tag{96}$$

$$X^5 = \sin(u). \tag{97}$$

From Eq. (90) we then have

$$(X^1)^2 + (X^2)^2 = 1 - (X^5)^2 = \cos^2(u). \tag{98}$$

This is a circle of radius $|\cos(u)|$. We will use parameter v, $-\pi \le v < \pi$ for this circle. Thus we can set

$$X^1 = |\cos(u)|\cos(v),$$

$$X^2 = |\cos(u)|\sin(v).$$

But it is convenient to reshuffle the sub-ranges of the parameters, and get rid of the absolute value setting simply

$$X^1 = \cos(u)\cos(v), \tag{99}$$
$$X^2 = \cos(u)\sin(v). \tag{100}$$

Eqs. (93)-(95) now become:

$$x(u,v) = \frac{a\cos(u)}{a-\cos(u)\cos(v)}, \tag{101}$$
$$y(u,v) = \frac{a\cos(u)\sin(v)}{a-\cos(u)\cos(v)}, \tag{102}$$
$$z(u,v) = \frac{a\sin(u)}{a-\cos(u)\cos(v)}. \tag{103}$$

The resulting parametric surface is represented in the following picture:

Essentially, what we have here is a torus, that has been squeezed to a point at two opposite ends.

6.6. *Action of the Poincaré Group and Dilations on Conformal Infinity*

With the notation as in Sec. **??**, the infinity is characterized by the condition $X^5 - X^6 = 0$, therefore

$$(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^5)^2 = (X^4)^2 + (X^5)^2 > 0. \tag{104}$$

If $X^4 = 0$, then $X^1 = X^2 = X^3 = 0$, and we get two possibilities $X^5 > 0$, and $X^5 < 0$. These are the two endpoints in Figure 1. One can easily check that each of these two endpoints is stable under the action of translations, Lorentz transformations and dilatations. Let us remove these two points from our considerations. Assuming now that $X^4 > 0$, we can chose a unique representative point of the corresponding open subset of $\overline{\overline{M}}$ with $\mu(X) = 1$.

We then denote by u the real parameter $u \in (-\infty, +\infty)$ and set $x = (\mathbf{x}, x^4)$. Then every point of the surface is uniquely represented in the form:

$$X(\mathbf{x}, u) = (\mathbf{x}, 1, u, u)/\sqrt{1+u^2}, \tag{105}$$

where $\|\mathbf{x}\| = 1$. We obtain an infinite cylinder $S^2 \times \mathbb{R}$. If we set $\alpha = \arctan(u)$, $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, we get

$$X(\mathbf{x}, \alpha) = ((\cos(\alpha)\mathbf{x}, \cos(\alpha), \sin(\alpha), \sin(\alpha)), \tag{106}$$

$\|\mathbf{x}\| = 1$, $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$, with $\mu(X) = 1$.

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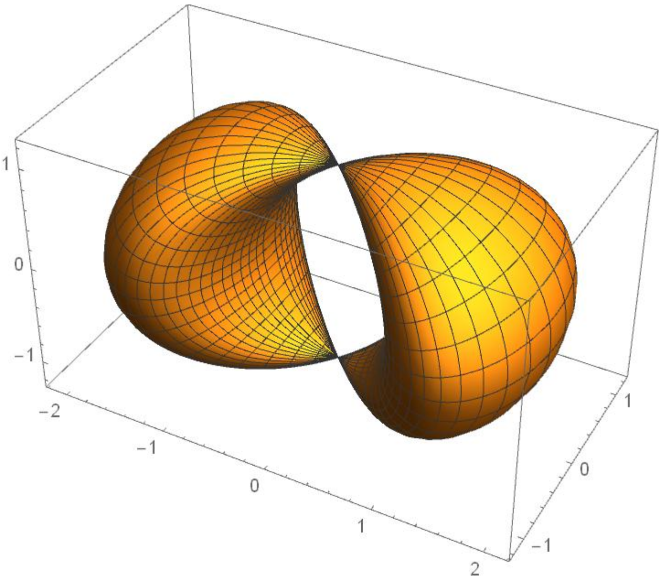

Figure 1. Double cover of the conformal infinity. A squeezed torus, known also as "Horn cyclide" (Cf. [4, Sec. 5.4 Cyclides of Dupin, p. 158]).

Lorentz rotations, translations and dilatations are acting on the conformal infinity leaving this cylinder invariant as a set. We will now calculate the corresponding actions and vector fields corresponding to one-parameter subgroups.

6.6.1. Translations

Eq. (45) gives us the action of translations subgroup. At the infinity $x^6 = x^5$ Skipping the third space coordinate, translation by $a, x \in \mathbb{R}^{2,1}$ is realized by

$$X = (x, x^5) \mapsto X' = (x + a, x^5 - a \cdot x). \tag{107}$$

In order to satisfy Eqs. (90),(91) we have to take the quotient by $\sqrt{(x'^4)^2 + (x'^5)^2}$. Therefore we get the following nonlinear translation transformation:

$$L(a, X) = \frac{(x, x^5 - a \cdot x)}{\sqrt{(x^4)^2 + (x^5 - a \cdot x)^2}}. \tag{108}$$

Taking partial derivatives of $L(a, x)$ with respect to a^i at $a = 0$ we get vector fields P_i – generators of translations:

$$P_i = \frac{\partial L(a, x)}{\partial a^i}|_{a=0}, \tag{109}$$

$$P_i = g_{ij} x^j x^5 \left(x^1 \frac{\partial}{\partial x^2} + x^2 \frac{\partial}{\partial x^2} + x^4 \frac{\partial}{\partial x^4} \right). \tag{110}$$

where $g_{ij} = \mathrm{diag}(1, 1, -1)$, $i, j = 1, 2, 4$. These vector fields are tangent to the infinity surface defined by Eqs. (90),(91) and can be expressed in terms of basic vector fields ∂_u, ∂_v on this surface. One can easily check that the following formulas hold:

$$P_1 \;=\; -\cos^2 u \cos v\, \partial_u, \tag{111}$$
$$P_2 \;=\; -\cos^2 u \sin v\, \partial_u, \tag{112}$$
$$P_4 \;=\; \cos^2 u\, \partial_u. \tag{113}$$

Evidently

$$P^2 = P_1^2 + P_2^2 - P_4^2 = 0. \tag{114}$$

6.6.2. Lorentz Transformations

Let us first consider a pure space rotation:

$$\begin{aligned} x'^1 &= x^1 \cos\alpha + x^2 \sin\alpha, \\ x'^2 &= -x^1 \sin\alpha + x^2 \cos\alpha, \end{aligned} \tag{115}$$

while $x'^4 = x^4$, $x'^5 = x^5$. The normalization in this case is unnecessary and differentiating with respect to α at $\alpha = 0$ we obtain the vector field M_{12} of pure space rotations:

$$M_{12} = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2} = -\partial_v. \tag{116}$$

Next let us consider Lorentz boosts in the direction of the x^1 axis:

$$\begin{aligned} x'^1 &= x^1 \cosh\beta + x^4 \sinh\beta, \\ x'^4 &= x^1 \sinh\beta + x^4 \cosh\beta, \end{aligned} \tag{117}$$

while $x'^2 = x^2$, $x'^5 = x^5$. This time we have to normalize as in Eq. (108). We get

$$L_{14}(\beta, X) = \frac{(x^1 \cosh\beta + x^4 \sinh\beta, x^2, x^1 \sinh\beta + x^4 \cosh\beta, x^5)}{\sqrt{(x^5)^2 + (x^4 \cosh\beta + x^1 \sinh\beta)^2}} \tag{118}$$

Differentiating with respect to β at $\beta = 0$, and using the fact that $(x^4)^2 + (x^5)^2 = 1$, we obtain the vector field M_{14}:

$$M_{14} = x^4(1 - (x^1)^2)\partial_1 - x^1x^2x^4\partial_2 + x^1(x^5)^2\partial_4 - x^1x^4x^5\partial_5. \tag{119}$$

Using Eqs. (99),(100),(96),(97) we can provide its expression in term of basic vector fields ∂_u, ∂_v as follows:

$$M_{14} = -\sin u \cos u \cos v\, \partial_u - \sin v\, \partial_v. \tag{120}$$

Similarly we get

$$M_{24} = -\sin u \cos u \sin v\, \partial_u + \cos v\, \partial_v. \tag{121}$$

Using the same method we obtain the expression for the generator D of the dilatations:

$$D = \sin u \cos u\, \partial_u. \tag{122}$$

A straightforward calculation leads to the Poincaré group (extended by dilatations) commutation relations

$$\begin{aligned} [M_{12}, P_1] &= P_2, \\ [M_{12}, P_2] &= -P_2, \\ [M_{12}, M_{1,4}] &= M_{24}, \\ [M_{12}, M_{24}] &= -M_{14}, \\ [M_{14}, M_{2,4}] &= -M_{12}, \\ [D, P_i] &= -P_i,\ (i = 1, 2, 4) \end{aligned} \tag{123}$$

6.7. Simple Conformal Infinity

By taking the quotient by $\sim$ rather than by $\approx$ we arrive at the same equations (88,89), but this time x and $-x$ describe the same point.

Jakob Steiner has faced a similar problem when studying the method of representing the projective plane in $\mathbb{R}^3$. One possible solution was to use quadratic expressions in the coordinates - Cf [12] and

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[13, p. 340]. Let us first follow a similar method. In order to represent the resulting variety graphically, we will need the following lemma:

Lemma 2. *With the notation as in section 6.5 introduce the following variables:*

$$y_\alpha = x^\alpha x^4. \tag{124}$$

Then, assuming that x^α, x'^α satisfy (90),(91), we have $y_\alpha = y'_\alpha$ if and only if either $x^\alpha = x'^\alpha$ or $x^\alpha = -x'^\alpha$, $\alpha = 1, ..., 5$.

Proof. The variables y being quadratic in x, it is clear that the 'if' part holds. Now suppose we have $y_\alpha = y'_\alpha$, $\alpha = 1, ..., 5$. If $x^4 = 0$, then $x'^4 = 0$, therefore from (91) we have that $x^5 = \pm 1$ and $x'^5 = \pm 1$. It follows then from (90) that $x^1 = x^2 = x^3 = 0$, and the same for x'. Therefore $x = (0, 0, 0, 0, \pm 1)$ and $x' = (0, 0, 0, 0, \pm 1)$, thus $x' = \pm x$. If $x^4 \neq 0$, then $x'^4/x^4 = \pm 1$ and $y'_\alpha = (x'^4/x^4) y_\alpha$. □

6.8. Graphic Representation of Simple Infinity

To obtain a graphic representation we proceed as before and arrive, after renaming of the variables, at the following set of parametric equations

$$x(u,v) = \frac{2\cos^2 u}{2 - \cos^2 u \cos v} \tag{125}$$

$$y(u,v) = \frac{2\cos^2 u \sin v}{2 - \cos^2 u \cos v} \tag{126}$$

$$z(u,v) = \frac{2\cos u \sin u}{2 - \cos^2 u \cos v} \tag{127}$$

The resulting surface has the shape of a simple elliptic supercyclide *needle (horn) cyclide* as in Fig. 6 - [14, Fig. 6, p. 80], [4, Fig. 5.7, p. 156], or, in French, *croissant simple* [15]. The surface is, in fact, made of closed null geodesics, all intersecting at the point with homogeneous coordinates $(\mathbf{0}, 0, 1, 1) \sim (\mathbf{0}, 0, -1, -1)$. Each od these geodesics is uniquely determined by a point on the 2-sphere $(\mathbf{n}, 1, 0, 0)$, $\mathbf{n}^2 = 1$. The geodesic is then given by the formula

$$\gamma(v) = [(\cos(v)\mathbf{n}, \cos v, \sin(v), \sin(v))], \quad v \in [0, \pi] \tag{128}$$

.

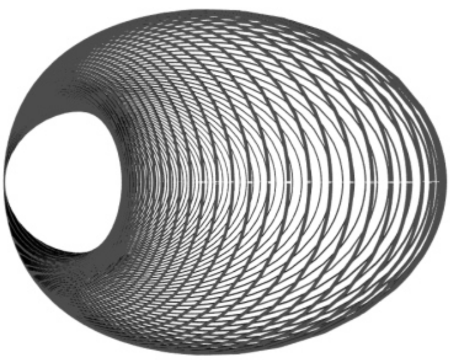

Figure 2. Pictorial representation of the simple conformal infinity with one dimension skipped - needle cyclide, made of a one parameter family of null geodesics trapped at infinity.

6.9. Conformal Structure of $\overline{\overline{M}}_\infty$

Let us begin with a simplified case, with a suppressed x^3 coordinate. As it was described above, in Sec. 6.5, we will use the u, v parametrization. We have the following equations parametrizing $\overline{\overline{M}}_\infty$:

$$x^1 = |\sin(u)|\cos(v), \tag{129}$$
$$x^2 = |\sin(u)|\sin(v), \tag{130}$$
$$x^4 = \sin(u), \tag{131}$$
$$x^5 = \cos(u), \tag{132}$$
$$x^6 = x^5. \tag{133}$$

Using this parametrization,

$$X(u,v) = (x^1(u,v), x^2(u,v), x^4(u,v), x^5(u,v), x^6(u,v)), \tag{134}$$

we calculate

$$ds^2 = (dx^1)^2 + (dx^2)^2 - (dx^4)^2 + (dx^5)^2 - (dx^6)^2. \tag{135}$$

Since on $\overline{\overline{M}}_\infty$ we have $x^6(u,v) = x^5(u,v)$, the last two terms cancel, so effectively

$$ds^2 = (dx^1)^2 + (dx^2)^2 - (dx^4)^2. \tag{136}$$

Assuming $\sin(u) \neq 0$, we easily get

$$ds^2 = \sin^2(u)\, dv^2. \tag{137}$$

Therefore the metric on our two-dimensional squeezed torus has the form

$$g_\infty(u,v) = \begin{bmatrix} 0 & 0 \\ 0 & \sin^2(u) \end{bmatrix}. \tag{138}$$

At the two squeezed points we have $\sin(u) = 0$, and the metric becomes totally degenerate. For other values of u we have a scaled (by $\sin^2(u)$), standard metric on the circles: dv^2. The metric along the u-lines, connecting the two end-points, is identically zero. Notice that the metric itself is defined up to a scale, since X and λX, $\lambda > 0$, describe the same point in $\overline{\overline{M}}_\infty$. So, in fact, we have a conformal structure, not a metric.

Let us now return to the full case, including the x^3 coordinate. The calculations become somewhat more complicated, but the end result is similar. Now instead of the coordinate v we introduce spherical coordinates θ, ϕ, $\theta \in (0, \pi]$, $\phi \in (0, 2\pi)$, and set

$$x^1 = |\sin(u)|\sin(\theta)\cos(\phi), \tag{139}$$
$$x^2 = |\sin(u)|\sin(\theta)\sin(\phi), \tag{140}$$
$$x^3 = |\sin(u)|\cos(\theta). \tag{141}$$

Calculating ds^2 we get

$$ds^2 = \sin^2(u)\left(d\theta^2 + \sin^2(\theta) d\phi^2\right), \tag{142}$$

which is also degenerate:

$$g_\infty(u,\theta,\phi) = \sin^2(u) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sin^2(\theta) \end{bmatrix}. \tag{143}$$

The θ, ϕ part of the metric is the standard metric on the unit sphere in $\mathbb{R}^3$.

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7. Geometrizing It All - Getting Rid of the Basis

In this section we will make use of the constructions in a paper [7] by W. Kopczyński and L.S. Woronowicz. They have built a really beautiful and solid foundation. Whenever there is a need, we will adapt their methods to our purposes (for instance in [7] the authors did not consider the double cover). We will use coordinates only when it is convenient to use them in order to prove some properties. Thus we are allowed to use a basis inside the proof, but not in definitions and in statements about the properties of objects and morphisms. Our reasonings will thus be more abstract, but it will be easier to grasp their geometrical meaning.

In a sense we will repeat all what has been done so far, rephrase it, but now, in definitions and in statements, applying only geometrical invariant concepts.

Let V be a six-dimensional real vector space endowed with a quadratic form of signature $(4,2)$. Let $(\cdot,\cdot)$ be the unique scalar product on V such that $Q(X) = (X,X)$ for all $X \in V$. For $X,Y \in V$ we will write $X \cdot Y$ for (X,Y).

Let $\mathcal{N}$ be the null cone of Q with removed origin:

$$\mathcal{N} = \{X \in V : X \neq 0 \text{ and } Q(X) = 0\}. \tag{144}$$

For $X, X' \in V \setminus \{0\}$ we define the equivalence relation $\approx$ as

$$X \approx X' \text{ if and only if } X = \lambda X', \lambda > 0. \tag{145}$$

We denote by $[[X]]$ the equivalence class of X with respect to $\approx$, and denote by $\hat{\pi}$ the natural projection.

From now on we define $P\mathcal{N}$ as

$$P\mathcal{N} = \hat{\pi}(\mathcal{N}). \tag{146}$$

We denote by $O(Q)$ the group of linear isometries of (V,Q), i.e. the group of all linear operators $A : V \to V$ for which $Q(AX) = Q(X)$ for all $X \in V$. We denote by $SO(Q)$ its subgroup of isometries of determinant one. Notice that $Q(AX) = Q(X)$ for all $X \in V$ implies $(AX) \cdot (AY) = X \cdot Y$ for all $X,Y \in V$. Also notice that the determinant of the matrix of a linear transformation does not depend on a choice of a basis.

Proposition 8. *The group $SO(Q)$ acts on $P\mathcal{N}$ transitively.*

Proof. The proof has been done choosing an orthonormal basis for Q. □

For $X \in \mathcal{N}$ and $A \in O(Q)$ we will write

$$A \cdot [[X]] = [[AX]]. \tag{147}$$

We will denote by ι the transformation

$$\iota(p) = (-I) \cdot p. \tag{148}$$

It is then immediate that the two-element subgroup $\{1, \iota\}$, isomorphic to $\mathbb{Z}_2$, acts freely on $P\mathcal{N}$.

If p and q are two points in $V/\approx$ then the property $X \cdot Y > 0$ does not depend of the choice of $X \in \hat{\pi}^{-1}(p)$, and $Y \in \hat{\pi}^{-1}(p)$. This fact justifies the following definition:

Definition 2. *For $p = [[X]]$, $q = [[Y]]$ we will write $p \perp q$ if $X \cdot Y = 0$. For any $p \in P\mathcal{N}$ we define $p^\perp$ as*

$$p^\perp = \{q \in P\mathcal{N} : q \perp p\}. \tag{149}$$

It is then evident that for $A \in O(Q)$ we have

$$(A \cdot p)^\perp = A \cdot p^\perp. \tag{150}$$

We also have

$$\iota(p^\perp) = p^\perp. \tag{151}$$

We will call $p^\perp$ *the conformal infinity at* p.

Exercise 1 (Why *the conformal infinity at* p**).** *Choose an orthonormal basis in* V *thus identifying* V *with* $\mathbb{R}^{4,2}$. *Choose* $X = (0,0,0,0,1,1)$ *and* $p = [[X]]$. *Find the explicit form of* $p^\perp$. *Identify the points* p *and* $\iota(p)$ *on Figure 1.*

Exercise 2. *Suppose we want to define the operation of addition on* $P\mathcal{N}$ *by defining*

$$[[X]] + [[Y]] \stackrel{def}{=} [[X+Y]]. \tag{152}$$

Can we do this?

Definition 3. *Given any* $p \in P\mathcal{N}$ *we define*

$$M_p = P\mathcal{N} \setminus p^\perp. \tag{153}$$

Definition 4. *For* $p = [[X]] \in P\mathcal{N}$ *let*

$$M_p^+ = \{[[Y]] \in P\mathcal{N} : X\cdot Y > 0\}, \tag{154}$$
$$M_p^- = \{[[Y]] \in P\mathcal{N} : X\cdot Y < 0\}. \tag{155}$$

Exercise 3. *Show that the sets* $M_p^\pm$ *are open in* $P\mathcal{N}$ *(this require some knowledge of topology).*

Exercise 4. *Show that*

$$M_p = M_p^+ \dot\cup M_p^-. \tag{156}$$

and that

$$P\mathcal{N} = p^\perp \dot\cup M_p, \tag{157}$$

where the dot in $\dot\cup$ *denotes the fact that we are dealing with a union of disjoint sets.*

Exercise 5. *Show that*

$$\iota(M_p^+) = M_p^-,\ \iota(M_p^-) = M_p^+. \tag{158}$$

7.1. Conformal Structure on the Tangent Bundle

The section 5.6 can be repeated here without any changes. We did not use coordinates there at all. However we will now discuss in some detail the tangent bundle $TP\mathcal{N}$ of $P\mathcal{N}$, that is the (disjoint) union of all tangent spaces

$$TP\mathcal{N} = \bigcup_{p\in P\mathcal{N}} T_pP\mathcal{N}. \tag{159}$$

As a manifold, $TP\mathcal{N}$ is eight-dimensional.

7.1.1. The Tautological Bundle

Every projective manifold comes with a gratis bundle - the tautological bundle. In our case we have the bundle $\mathcal{T}P\mathcal{N}$ with one-dimensional fibers $\mathcal{T}_pP\mathcal{N}$

$$\mathcal{T}P\mathcal{N} = \bigcup_{p\in P\mathcal{N}} \mathcal{T}_pP\mathcal{N}, \tag{160}$$

where, for $p = [[X]]$

$$\mathcal{T}_pP\mathcal{N} = \{\lambda X : \lambda \in \mathbb{R}\}. \tag{161}$$

Thus over each point $p = [[X]]$ we have one-dimensional fiber - the subspace spanned by X. Notice that the zero vector of V is in each of these fibers, nevertheless the fibers over different points are disjoint, because, strictly speaking, these "zeros" are ordered pairs $(p, 0)$, and $(p, 0) \neq (q, 0)$ if $q \neq p$.

Now, since V is equipped with a scalar product, we also have the orthogonal bundle $\mathcal{T}^{\perp}P\mathcal{N}$

$$\mathcal{T}^{\perp}P\mathcal{N} = \dot{\bigcup_{p \in P\mathcal{N}}} \mathcal{T}_p^{\perp}P\mathcal{N}, \tag{162}$$

where, for $p = [[X]]$, we have

$$\mathcal{T}_p^{\perp}P\mathcal{N} = \{Y \in V : Y \cdot X = 0\}. \tag{163}$$

The fibers of $\mathcal{T}_p^{\perp}P\mathcal{N}$ are five-dimensional. Notice that since $X \cdot X = 0$, we have

$$\mathcal{T}_pP\mathcal{N} \subset \mathcal{T}_p^{\perp}P\mathcal{N}. \tag{164}$$

Therefore we can take the quotient bundle $\mathcal{T}^{\perp}P\mathcal{N}/\mathcal{T}P\mathcal{N}$ whose fibers are the quotient spaces $\mathcal{T}_p^{\perp}P\mathcal{N}/\mathcal{T}_pP\mathcal{N}$.

Using the reasoning in section 7.1 about the conformal structure it is clear that the tangent bundle $TP\mathcal{N}$ can be naturally identified with the quotient bundle $\mathcal{T}^{\perp}P\mathcal{N}/\mathcal{T}P\mathcal{N}$.

Let us choose any $q = [[Y]] \in M_p$, such that $X \cdot Y = 1$.

Exercise 6. *Show that such a q always exists.*

Definition 5. *With $p = [[X]], q = [[Y]], X, Y \in \mathcal{N}, X \cdot Y = 1$ we define $M_{p,q}$ as follows*

$$M_{p,q} = \{Z \in V : X \cdot Z = Y \cdot Z = 0\}. \tag{165}$$

Remark 7. *Notice that $M_{p,q}$ is four-dimensional and that, in this definition, Z is assumed to be only in V, not necessarily in $\mathcal{N}$. In fact no such Z exists in $\mathcal{N}$. Proving this is another good exercise.*

It is clear from the definition that $M_{p,q}$ is a vector space. We can even easily argue that it is four-dimensional - we have two linear conditions on a vector in a six-dimensional space.

We will now construct a vector space isomorphism between $\mathcal{T}_p^{\perp}P\mathcal{N}/\mathcal{T}_pP\mathcal{N}$ and $M_{p,q}$.

Proposition 9. *With the notation as above the following map $\pi_{p,q} : M_{p,q} \to \mathcal{T}_p^{\perp}P\mathcal{N}/\mathcal{T}_pP\mathcal{N}$ is a vector space isomorphism*

$$\pi_{p,q}(Z) = [Z]_p, \tag{166}$$

where $[Z]_p$ denotes the equivalence class defining the quotient of vector spaces $\mathcal{T}_p^{\perp}P\mathcal{N}/\mathcal{T}_pP\mathcal{N}$.

Proof. Since Z is in $M_{p,q}$, we certainly have $X \cdot Z = 0$, therefore Z is in $\mathcal{T}_p^{\perp}P\mathcal{N}$. Let us first see that $\pi_{p,q}$ is injective. For this it is enough to show that if $\pi_{p,q}(Z) = 0$, then $Z = 0$. Now, from the very definition of the quotient of vector spaces it follows that $[Z]_p = 0$ means that $Z = \lambda X$ for some real λ. If $\lambda \neq 0$, then $Z \cdot Y = \lambda X \cdot Y = \lambda \neq 0$, which contradicts the assumption that Z is in $M_{p,q}$. Therefore $\lambda = 0$ and so $Z = 0$. To prove that τ_p is “onto" it would be enough to bring out the fact that both the domain and the range are vector spaces of the same dimension - four. But it is easy to show it explicitly. To this end, let $[Z_1]_p$ be an arbitrary element of the quotient space, with $Z_1 \cdot X = 0$. Now Z_1 is not necessarily in $M_{p,q}$ since $Z_1 \cdot Y$ is not necessarily zero. Let $Z_1 \cdot Y = \lambda$. Define $Z = Z_1 - \lambda X$. Then $[Z_1]_p = [Z]_p$, $Z \cdot X = 0$, and

$$Z \cdot Y = (Z_1 - \lambda X).Y = \lambda - \lambda = 0, \tag{167}$$

Therefore Z is in $M_{p,q}$. □

Now, since we have already identified the quotient $\mathcal{T}_p^{\perp}P\mathcal{N}/\mathcal{T}_pP\mathcal{N}$ with the tangent space $T_pP\mathcal{N}$, we have a vector space isomorphism between $T_pP\mathcal{N}$ and $M_{p,q}$ - different one for different choices of q.

Let us now recall that, for $q \in P\mathcal{N}$, we have defined M_q as $P\mathcal{N}$ with removed $q^{\perp}$ - we remove the infinity cap at q. If $q = [[Y]]$, then M_q consists of all those $[[X]] \in P\mathcal{N}$ for which $X \cdot Y \neq 0$. Then M_q splits into a disjoint union of M_q^+ and M_q^- according to whether $X \cdot Y$ is positive or negative. Let us concentrate now on the positive case. Then we can always choose Y so that $X \cdot Y = 1$. We can therefore construct $M_{p,q}$. We will now construct an embedding, which we will denote $\tau_{p,q}$, of the vector space $M_{p,q}$ into the projective null cone manifold $P\mathcal{N}$ such that the image $\tau_{p,q}(M_{p,q})$ is exactly M_p. It is similar to a stereographic map when we embed the plane - the tangent plane at the South Pole p into the sphere, so that the image of this tangent plane is the sphere with removed the North Pole q - the "infinity".

Proposition 10. *Choose a point $p_\infty = [[X_\infty]] \in P\mathcal{N}$ - an 'origin of infinity'. Choose another point $p_0^+ \in M_{p_\infty}^+$, and let X_0^+ be its representative in $\mathcal{N}$ such that $X_\infty \cdot X_0^+ = 1$. The point p_0^+ will be the "origin of the visible '+' universe". Define*

$$\hat{\tau}^+_{X^+_\infty, X_0^+} : M_{p_\infty, p_0^+} \to M^+_{p_\infty} \tag{168}$$

by the following formula:

$$\hat{\tau}^+_{X_\infty, X_0^+} : X \mapsto [[X']],\ X' = X - \frac{1}{2}(X \cdot X)X_\infty + X_0^+. \tag{169}$$

Then $\hat{\tau}^+_{X_\infty, X_0^+}$ is injective, and its image is exactly $M^+_{p_\infty}$.

Similarly, if we chose $p_0^- = [[X_0^-]] \in M^-_{p_\infty}$, for instance we can set $p_0^- = \iota(p_0^+)$, and define

$$\hat{\tau}^-_{X_\infty, X_0^-} : X \mapsto [[X']],\ X' = X + \frac{1}{2}(X \cdot X)X_\infty + X_0^-, \tag{170}$$

we obtain an onto diffeomorhism

$$\hat{\tau}^-_{X^+_\infty, X_0^-} : M_{p_\infty, p_0^-} \to M^-_{p_\infty} \tag{171}$$

Before stating the proof we first note that the mapping $X \mapsto X'$ is non-linear. It should not be a surprise, since we are wrapping a linear 4-dimensional space M_{p_∞, p_0} around a compact space M_{p_∞} diffeomorphic to $S^3 \times S^1$. Then we need to show that X' is indeed in $\mathcal{N}$, that is that $X' \cdot X' = 0$. Instead of doing just this, let us contemplate X' of a more general form

$$X' = X + \lambda X_\infty + X_0^+, \tag{172}$$

and find the condition on λ so that X' is a null vector. Using the facts that $X_0^+ \cdot X_0^+ = X_\infty \cdot X_\infty = 0$, $X_0^+ \cdot X_\infty = 1$, and $X \cdot X_0^+ = X.X_\infty = 0$, we instantly obtain

$$X' \cdot X' = X \cdot X + 2\lambda. \tag{173}$$

Therefore X' is a null vector if and only if $\lambda = -\frac{1}{2}X \cdot X$.

Next, it will be handy to use the following lemma:

Lemma 3. *Let us choose an arbitrary orthonormal basis in V, thus identifying V with $\mathbb{R}^{4,2}$. There exists a transformation $A \in SO_0(4,2)$ such that*

$$\begin{aligned} [[AX_\infty]] &= [[(0,0,0,0,1,1)]], \\ [[AX_0]] &= [[(0,0,0,1/2,-1/2)]]. \end{aligned} \tag{174}$$

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Proof. Since $SO_0(4,2)$ acts on $P\mathcal{N}$ transitively, we can always find a group element that transforms p_∞ into $[[Y_\infty]]$, where $Y_\infty = (0,0,0,0,1,1)$. After this transformation p_0 will have the form $p_0 = [Y_0]$, $Y_0 = (y^1,\ldots,y^6)$, with $Q(Y_0) = 0$ and $Y_0 \cdot Y_\infty = 1$. To transform further Y_0 into $(0,0,0,0,1/2,-1/2)$ we will use transformations form $SO(4,2)$ that do not affect the point p_∞

$$p_\infty = [[(0,0,0,0,1,1)]]. \tag{175}$$

These are translations, dilatations and 3D rotations. We will use a translation, rotations and dilatations will not be needed. From Eq. (41) it is clear that translating by $a = (-y^1,\ldots,-y^4)$ we assure that Y_0 takes the form $(0,0,0,0,y^5,y^6)$. But then we still have the two conditions: $Q(Y_0) = 0$ and $Y_0 \cdot Y_\infty = 1$. They give us two equations: $(y^5)^2 - (y^6)^2 = 0$, and $y^5 - y^6 = 1$, with a unique solution $y_5 = 1/2, y^6 = -1/2$. □

We will now use this lemma to prove Proposition 8.

Proof. (of Proposition 10) Using Lemma 3 without any lose of generality we may assume that $p_\infty = [[(0,0,0,0,1,1]]$ and $p_0 = [[(0,0,0,0,-1/2,1/2)]]$. But then our $\hat{\tau}^+_{X_\infty,X_0}$ is the same as $\hat{\tau}_+$ in Sec. 6.2, which we already know that it is an embedding. □

7.2. Another Take on Conformal Structure - Null Geodesics

We have identified $M_{p,q}$, defined in the Definition 5, with the tangent space $T_p\mathcal{N}$. Now $M_{p,q}$ is a vector subspace of V, therefore it inherits for V a scalar product. This induces a scalar product on $T_p\mathcal{N}$. It can be shown that this scalar product is compatible with the conformal structure already defined on $T_p\mathcal{N}$ - it is a good exercise. With a fixed p but different q these scalar products must thus be proportional (another exercise).

Null geodesics in General Relativity

In this paragraph let M be a four-dimensional manifold of general relativity. In local coordinates x^μ let $g_{\mu\nu}$ be the metric tensor of signature $(3,1)$. Metric tensor induces the Levi-Civita connection $\Gamma^\rho_{\mu\nu}$

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}). \tag{176}$$

Levi-Civita connection defines covariant derivative ∇_μ. For vector fields we have

$$\nabla_\mu X^\rho = \partial_\mu X^\rho + \Gamma^\rho_{\mu\nu}X^\nu. \tag{177}$$

A curve $x^\rho(\lambda)$ is called a geodesic (or autoparallel) if the tangent vector $\dot{x}^\rho = dx^\rho/d\lambda$ parallelly transported along the curve remains tangent to it. This condition entails the equation:

$$\frac{D\dot{x}^\rho}{d\lambda} = f(\lambda)\dot{x}^\rho(\lambda), \tag{178}$$

where, for a vector field X^ρ, $DX^\rho/d\lambda$ denotes the covariant derivative along the path:

$$\frac{D\dot{X}^\rho}{d\lambda} = \dot{x}^\mu\nabla_\mu X^\rho. \tag{179}$$

One can then always choose a parameter λ in such a way that $f(\lambda) \equiv 0$. The geodesic equations become then

$$\frac{d^2x^\rho}{d\lambda^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0. \tag{180}$$

Such a parameter is then-called an affine parameter for the geodesic.

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Let us see how the geodesic equations (180) change when we replace $g_{\mu\nu}$ by another metric

$$\bar{g}_{\mu\nu}(x) = e^{2\phi(x)} g_{\mu\nu}(x), \tag{181}$$

in the same conformal class? A straightforward calculation results in

$$\bar{\Gamma}^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} + \delta^{\rho}_{\mu}\phi_{\nu} + \delta^{\rho}_{\nu}\phi_{\mu} - g_{\mu\nu}\phi^{\rho}, \tag{182}$$

where

$$\phi_{\sigma} = \partial_{\sigma}\phi, \quad \phi^{\rho} = g^{\rho\mu}\phi_{\mu}. \tag{183}$$

If $\bar{\lambda}$ is an affine parameter for the geodesics of the metric $\bar{g}$, the equation of geodesics for the new metric become

$$\frac{d^2x^{\rho}}{d\bar{\lambda}^2} + \bar{\Gamma}^{\rho}_{\mu\nu}\frac{dx^{\mu}}{d\bar{\lambda}}\frac{dx^{\nu}}{d\bar{\lambda}} = 0, \tag{184}$$

or, using Eq. (182):

$$\frac{d^2x^{\rho}}{d\bar{\lambda}^2} + \Gamma^{\rho}_{\mu\nu}\frac{dx^{\mu}}{d\bar{\lambda}}\frac{dx^{\nu}}{d\bar{\lambda}} + 2\left(\frac{dx^{\mu}}{d\bar{\lambda}}\phi_{\mu}\right)\frac{dx^{\rho}}{d\bar{\lambda}} - \phi^{\rho}g_{\mu\nu}\frac{dx^{\mu}}{d\bar{\lambda}}\frac{dx^{\nu}}{d\bar{\lambda}} = 0. \tag{185}$$

Assume now that we dealing with a null geodesic, that is that the tangent vector is always a null vector, so that

$$g_{\mu\nu}\frac{dx^{\mu}}{d\bar{\lambda}}\frac{dx^{\nu}}{d\bar{\lambda}} = 0. \tag{186}$$

The last term in (185) vanishes. Setting $f(\bar{\lambda}) = -2\phi^{\rho}(x(\bar{\lambda})$, (185) takes the form of Eq.(178). Therefore we have the following result

Proposition 11. *While in general geodesics for two conformally equivalent metrics are different, null geodesics are the same. They depend only on the conformal structure and not on a particular choice of the representative of this structure.*

7.2.1. Coordinate Free Description of Null Geodesics in $P\mathcal{N}$

$P\mathcal{N}$ is a compact manifold diffeomorphic to $S^3 \times S^1$. It cannot be covered with one coordinate patch. When we describe null geodesics by geodesic equations, we usee coordinates. Suppose we find a particular solution, a path given in a coordinate system. Then the question arises: and what happens next? How this particular path is going to continue after the coordinates patch ends? In our case one can give an elegant answer to this question by providing a coordinate-free description of a null geodesic, the whole of it, without invoking differential equations. We follow here, [7, Theorem 5], modifying the proof to suit our needs.

Proposition 12. *Get Γ be a two-dimensional isotropic subspace of V. Then the submanifold*

$$\gamma = \hat{\pi}(\Gamma \setminus \{0\}) \tag{187}$$

is a null geodesic in $P\mathcal{N}$. Every null geodesic of $P\mathcal{N}$ is obtained this way. If $p = [[X]]$ is any point in $P\mathcal{N}$ then the set of all null geodesics through p corresponds to the set of all isotropic planes containing X.

Proof. In order to show that γ is a null geodesic it is enough to show that γ can be covered by open neighbourhood, and each of these neighbourhoods is an open segment of a null geodesic. Let $p_0^+ = [[X_0^+],]$ $p_0^+ \in \Gamma$, be an arbitrary point of γ. We will show that γ is a null geodesic in a certain neighbourhood of p_0^+. In order to analyze a neighborhood of p_0^+ we choose a point at infinity $p_\infty = [[X_\infty]]$ satisfying

$$X_\infty \cdot X_0^+ = 1. \tag{188}$$

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We will now apply the machinery developed in Proposition 10, and use Lemma 3 to introduce an orthonormal basis in V in which

$$X_\infty = (0,1,1),$$
$$X_0^+ = (0,\frac{1}{2},-\frac{1}{2}). \tag{189}$$

Here, and in what follows, we will use a shortcut notation, namely we will write $(x^1,x^2,x^3,x^4,x^5,x^6)$ as (x,x^5,x^6), understanding that x is a four-vector. We have now an open neighbourhood $M^+_{p_0,p^\perp}$ of p_0 that is diffeomorphic to the Minkowski space. Since Γ is two-dimensional, there exists in Γ another null vector, linearly independent from X_0. We call it X_1. If we choose X_1 close enough to X_0^+, the scalar product $X_\infty \cdot X_1$ will still be positive, and we can scale X_1 so that

$$X_\infty \cdot X_1 = 1. \tag{190}$$

From Eq. (76) we know that X_1 is of the form

$$X_1 = (a,\frac{1}{2}(1-a\cdot a),-\frac{1}{2}(1+a\cdot a)), \tag{191}$$

where a is Minkowski space four-vector, $a \in T_{p_0^+}\mathcal{N}$. The condition $X_0^+ \cdot X_1 =$ immediately implies $q(a) =$, thus a is a null vector in the Minkowski space, therefore

$$X_1 = (a,\frac{1}{2},-\frac{1}{2}), \tag{192}$$

Now every nonzero vector X in Γ can be uniquely represented as

$$X = \lambda(\cos(t)X_0^+ + \sin(t)X_1),\ \lambda > 0,\ -\pi \leq t < \pi. \tag{193}$$

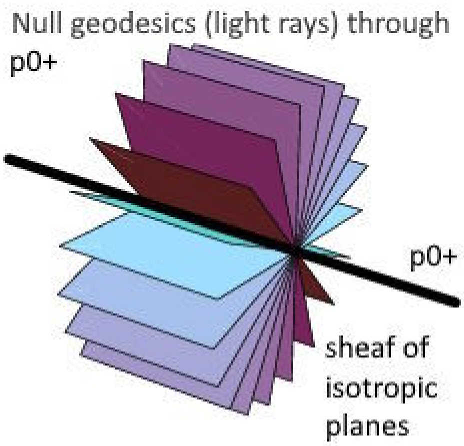

Figure 3. The pencil (sheaf) of light rays through a space-time point. Temporary illustration.

Using expressions (189) and (192) we get

$$X(t) = \left(\sin(t)\,a,\frac{1}{2}(\sin(t)+\cos(t)),-\frac{1}{2}(\sin(t)+\cos(t)\right). \tag{194}$$

The scalar product

$$X_\infty \cdot X(t) = \sin(t) + \cos(t) \tag{195}$$

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is positive for $-\pi/4 < t < 3\pi/4$, zero for $t = -\pi/4$ and $t = 3\pi/4$, and negative otherwise. That means the trajectory is in M_{p_∞,p_0^+} for $| -\pi/4 < t < 3pi/4$ and crosses the conformal infinity for $t = \pi/4$ and $= 3\pi/4$, and is M_{p_∞,p_0^-} for other values of t.

Now that we have a global picture, we concentrate on a neighborhood of the point p_0^+. Here it is enough to take

$$X(t) = X_0^+ + tX_1 = \left(ta, \frac{1}{2}(1+t), -\frac{1}{2}(1+t)\right) = (1+t)\left(\frac{t}{1+t}a, \frac{1}{2}, -\frac{1}{2}\right), \tag{196}$$

which, for t small enough, is a part of a null geodesic in Minkowski space through the origin, in the direction of the null vector a. □

7.3. Light Circling Forever at Infinity and Never Entering Minkowski Space

Given $p \in P\mathcal{N}$ we have the conformal infinity $p^\perp$ at p. If $X \in \hat{\pi}^{-1}(p)$, the image by $\hat{\pi}$ is contained in $p^\perp$. Therefore every null geodesic crossing p is totally contained in $p^\perp$. We will now see how it can be described in coordinates. To this end we will use the parametrization of $p^\perp$ as in Sec. 6.5. We will use an adapted orthonormal basis in which $p = [[X_0]]$, $X_0 = (0, 1, 1)$. Then any vector orthogonal to X_0 is of the form (x, α, α), where x is a null vector of the Minkowski space. As we want this vector to be linearly independent of X_0, we must have $x \neq 0$. In such a case the vectors X_0 and $X_1 = (x, 0, 0)$ span the same isotropic plane as X_0 and (x, α, α). Now, x being a null vector, we can choose it of the form $x = (\mathbf{a}, 1)$, $\mathbf{a} \in \mathbb{R}^3$, $\mathbf{a}^2 = 1$, - it will span, with X_0, the same isotropic plane. Now any vector in this plane is of the form

$$X(t) = \cos(t)X_0 + \sin(t)X_1 = (\cos(t)\mathbf{a}, \cos(t), \sin(t), \sin(t)). \tag{197}$$

Comparing the above formula for $X(t)$ with Eqs. (96), (97) we see that these are exactly the lines of the parameter u in Eqs. (101)-(103), which entails the following graphic representation:

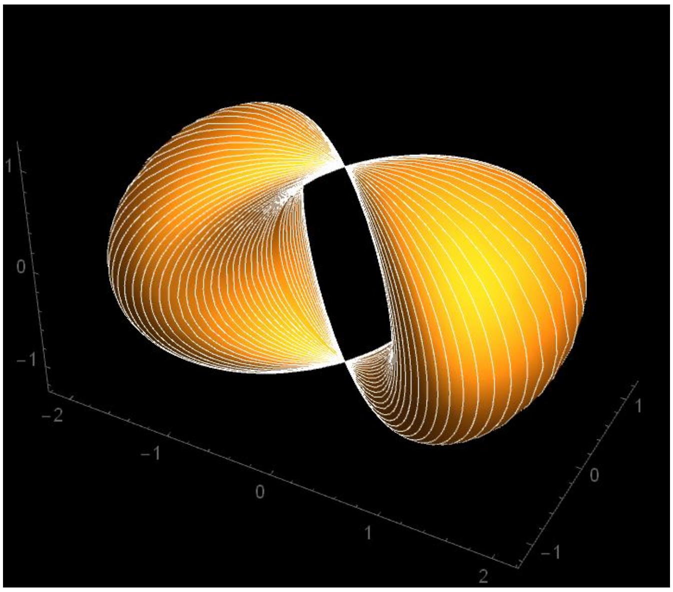

Figure 4. Null geodesics at conformal infinity

7.4. Segal's 'Unitime'

Segal's 'chronometric cosmology' model [16] differs from the standard cosmological models by the fact that it is based on a priori selected geometry, not Einstein's theory of General Relativity, even though it uses many of the concepts of General Relativity. The main **physical** idea at the foundations of Segal's cosmology is the **postulate** of existence of a distinguished cosmic **time flow** - he calls it

unitime. The prefix 'uni' comes probably from the word 'unitary', as it relates to the action of a specific subgroup of the conformal group SO(4,2), a subgroup isomorphic to the one-parameter Abelian group isomorphic to the group $U(1) \approx S^1$ of complex numbers of modulus 1. A theoretical physicist would probably ask: "Why should it be so? What is the mechanism of such a symmetry braking? Where is the Lagrangian?" But Segal, a mathematician, has an immediate answer to these questions: "Better check if it IS really so as I say, and if it is so, then you will certainly be motivated to find answers to your questions all by yourself."

As we already know $P\mathcal{N}$ is isomorphic to $U(1) \times SU(2) \approx S^1 \times S^3$. In an orthonormal basis, identifying V with $\mathbb{R}^{4,2}$ and $P\mathcal{N}$ with the manifold of all $X \in \mathbb{R}^{4,2}$ such that

$$\begin{aligned} (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^5)^2 = R^2, \\ (X^4)^2 + (X^6)^2 = R^2, \end{aligned} \tag{198}$$

The action of $U(1)$ corresponds to the Euclidean rotation of the X^4, X^6 coordinates, other variables being fixed:

$$\begin{aligned} X^4(s) &= \cos(s/R)X^4 - \sin(s/R)X^6, \\ X^6(s) &= \sin(s/)X^4 + \cos(s/R)X^6. \end{aligned} \tag{199}$$

Notice that now we pay attention to the physical dimensions and have replaced the parameter s with s/R, and have replaced 1 on the right-hand-side of the above equations by a dimensional constant R - the 'radius of the universe'.

Remark 8. *In such an approach the 'flow of time' has an objective reality. Which agrees with our observations of reality - if flow of time would be entirely subjective, it would be impossible to explain why most of the people agree on the duration of time sequences of events and respect their friends' birthday dates.*

Remark 9. *Speaking about 'physics': Segal allows s to be going from minus infinity to infinity, nor just from* 0 *to* $2\Pi R$. *In other words we replace* S^1 *by its universal cover, that is by* $\mathbb{R}$. *Cosmology becomes cyclic. We have an infinite number of cycles. There is still a question about what is the elementary cycle? Should we identify* X *with* $-X$, *or not? Is the Minkowski space simple or doubled? Usually it is assumed that it is a simple one, therefore already the interval* $[0, \pi R]$ *is assumed to be a cycle. I see no valid reason for such a postulate.*

Let us see how Segal's 'unitime' relates to the coordinate time of the Minkowski space. To this end let us analyze the 'unitime line' through the origin of the Minkowski space in adapted coordinates. We represent Minkowski space event with coordinates x by its image $\tau_+(X)$ as in Eq. (76). The origin is then represented by $X_0 \in \mathcal{N}$

$$X_0 = (\mathbf{0}, 0, R/2, -R/2), \tag{200}$$

therefore, using Eqs. (199), the unitime line becomes

$$X^4(s) = \frac{R}{2}\sin(s/R), \tag{201}$$

$$X^6(s) = -\frac{R}{2}\cos(s). \tag{202}$$

Thus

$$X(s) = (\mathbf{0}, \frac{R}{2}\sin(s/R), \frac{R}{2}, -\frac{R}{2}\cos(s)). \tag{203}$$

In order to find $x^4(s)$ we divide $X^4(s)$ by $X^5(s) - X^6(s)$ to obtain

$$x^4(s) = \frac{2R\sin(s/R)}{1+\cos(s/R)} = 2R\tan(\frac{s}{2R}). \tag{204}$$

$$s(x^4) = 2R \arctan(\frac{X^4}{2R}). \tag{205}$$

The space origin $\mathbf{X} = \mathbf{0}$ remains constant. We see that the unitime line is the same as the time line of the Minkowski space, but **the rate of time is different**. Developing into the Taylor series we get

$$x^4(s) = s + \frac{s^3}{12R^2} + O(s^4), \tag{206}$$

$$s(x^4) = x^4 - \frac{(x^4)^3}{12R^2} + O((x^4)^4), \tag{207}$$

The difference between the two times becomes significant only for x^4 big enough in the R-scale.

7.5. Guessing the Metric

In this section, to simplify the notation, let us call the x^4 coordinate simply t (assuming units in which $c = 1$). Inverting Eq. (205) we have

$$s(t) = 2R \arctan\left(\frac{t}{2R}\right). \tag{208}$$

Let us now try to find a metric in the conformal class of the flat Minkowski metric for which s would be the proper time on the unitime line. We look for a metric of the form

$$ds^2 = a(t)^2(d\mathbf{x}^2 - dt^2). \tag{209}$$

Along the time line $d\mathbf{x} = 0$, therefore, for the proper time parameter s we get

$$a(t) = \frac{ds}{dt}. \tag{210}$$

Using Eq. (208) we obtain

$$a(t) = \frac{1}{1 + \frac{t^2}{4R^2}}. \tag{211}$$

8. Compactified Minkowski Space as a Boundary of Five-Dimensional Domains

The null cone $\mathcal{N}$ of V separates two domains $D_5^{\pm}$ characterized as follows:

$$D_5^+ = \{X \in V : Q(X) > 0\}, \quad D_5^- = \{X \in V : Q(X) < 0\}. \tag{212}$$

Let us consider their projections $\mathcal{D}_5^+$ (resp. $\mathcal{D}_5^-$) obtained by taking the quotient by the equivalence relation $\approx$. For every point of $\mathcal{D}_5^+$ (resp. $\mathcal{D}_5^-$) there is a unique point X in D_5^+ (resp. D_5^-) for which $Q(X) = 1$ (resp. $Q(X) = -1$). Therefore, $\mathcal{D}_5^{\pm}$ can be, respectively, identified with a five-dimensional hyperboloid $\Sigma_{\pm}$ defined by

$$\Sigma^{\pm} = \{X \in V : Q(X) = \pm 1\}. \tag{213}$$

Now $P\mathcal{N}$ is a topological boundary of $\mathcal{D}_5^+$ and of $\mathcal{D}_5^-$, a four-dimensional boundary that separates these two five-dimensional domains.

Let us look now at the topology of the two five–dimensional domains $\Sigma_{\pm}$. For the domain Σ_+ we have the defining equation

$$(X^1)^2 + (X^2)^2 + (X^3)^2 - (X^4)^2 + (X^5)^2 - (X^6)^2 = 1. \tag{214}$$

We can write it as

$$(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^5)^2 = (X^4)^2 + (X^6)^2 + 1.$$

It is then clear that X^4 and X^6 can be arbitrary real numbers, and that introducing

$$Y^i = X^i/((X^4)^2 + (X^6)^2 + 1)^{1/2}, \ (i = 1, 2, 3, 5), \tag{215}$$

we have $(Y^1)^2 + \ldots (Y^3)^2 + (Y^5)^2 = 1$. Therefore, Σ_+ has the topology of $S^3 \times \mathbb{R}^2$.

Proceeding the same way with $\Sigma+$, we have

$$(X^1)^2 + (X^2)^2 + (X^3)^2 - (X^4)^2 + (X^5)^2 - (X^6)^2 = -1, \tag{216}$$

that is

$$(X^4)^2 + (X^6)^2 = 1 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^5))^2. \tag{217}$$

Introducing

$$Y^i = X^i/(1 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^6)^2)^{1/2}, \ (i = (1, 2, 3, 6), \tag{218}$$

we get

$$(Y^4)^2 + (Y^6)^2 = 1. \tag{219}$$

Therefore Σ_- has the topology of $S^1 \times \mathbb{R}^4$.

8.1. A Coordinate Description of Σ_-

In Σ_- we choose an open set defined by the condition $X^5 - X^6 > 0$. On this set we introduce five coordinates $(x^\mu, \lambda) \in \mathbb{R}^5$ defined by

$$x^\mu = \frac{X^\mu}{X^5 - X^6}, \quad \lambda = \frac{1}{X^5 - X^6} > 0. \tag{220}$$

On the other hand, given a point in $\mathbb{R}^5$ with coordinates $(x^\mu, \lambda > 0)$ we can embed it in Σ_- as follows:

$$(X^\mu) = \frac{x^\mu}{\lambda}, \quad X^5 = \frac{1 - q(x) - \lambda^2}{2\lambda}, \quad X^6 = -\frac{1 + q(x) + \lambda^2}{2\lambda}. \tag{221}$$

Reader is encouraged to verify by a straightforward calculation that with the above definition $Q(X(x^\mu, \lambda)) = -1$, and that applying formula (220) to $X(x^\mu, \lambda)$ we indeed recover (x^μ, λ).

We can now calculate a new metric. In general, when we are dealing with an embedded manifold parameterized by coordinates x^α, its metric $g_{\alpha\beta}$ is induced by a metric G_{AB} on a manifold into which our manifold is embedded, and it is given by the expression

$$g_{\alpha\beta} = \frac{\partial X^A}{\partial x^\alpha} \frac{\partial X^B}{\partial x^\beta} G_{AB}. \tag{222}$$

In our case, $(G_{AB}) = \text{diag}(1, 1, 1, -1, 1, -1)$ and it is easy to calculate $g_{\alpha\beta}$ using formula (221). The result of a straightforward calculation is:

$$(g_{\alpha\beta}) = \frac{1}{\lambda^2}\text{diag}(1, 1, 1, -1, 1). \tag{223}$$

Exactly the same method applies to the region $X^5 - X^6 < 0$. We get a five–dimensional pseudo–Riemannian, conformally flat manifold of constant curvature and signature $(4, 1)$. We have covered by coordinates two regions corresponding to different signs of the fifth coordinate. Physicists, when discussing representations of the conformal group with applications to elementary particle physics, often restrict their attention to these regions - Cf. for instance [22**?** ,23]. Yet, evidently the group $O(4, 2)$ acts on this part with singularities. Like in the case of Minkowski space, in order to avoid singularities one has to add "conformal infinity". In our case this is a region where $X^5 = X^6$. This conformal infinity of the five–dimensional domain has a simpler structure than the one for the Minkowski space. In fact, setting $X^5 = X^6$ in (216), we get

$$(X^1)^2 + \ldots + (X^3)^2 - (X^4)^2 = -1$$

with no scaling freedom. Therefore, the conformal infinity of our five–dimensional domain Σ_- is the Cartesian product of $\mathbb{R}$ ($X^5 - X^6 \in \mathbb{R}$) and the standard two-sheeted hyperboloid of the Minkowski space.

8.2. Christoffel Symbols and Geodesics

Given metric (223) it is easy (in our coordinate patch) to calculate the Christoffel symbols $\Gamma^{\mu}{}_{\nu\sigma}$ and geodesic equations - see e.g. [24, Mathematica Programs: Christoffel Symbols and Geodesic Equations]. The metric is conformally flat and the only non–vanishing Christoffel symbols are:

$$\Gamma^{\mu}{}_{5\sigma} = -\frac{1}{\lambda}\,\delta^{\mu}_{\sigma} \tag{224}$$

$$\Gamma^{5}{}_{\nu\sigma} = \frac{1}{\lambda}\,\eta_{\nu\sigma} \tag{225}$$

$$\Gamma^{5}{}_{55} = -\frac{1}{\lambda}. \tag{226}$$

The corresponding geodesic equations, when parameterized by an affine parameter s, are:

$$\frac{d^2x^{\mu}}{ds^2} = \frac{2}{\lambda}\,\frac{dx^{\mu}}{ds}\,\frac{d\lambda}{ds} \tag{227}$$

$$\frac{d^2\lambda}{ds^2} = -\frac{1}{\lambda}\left(\left(\frac{dx^1}{ds}\right)^2 + \left(\frac{dx^2}{ds}\right)^2 + \left(\frac{dx^3}{ds}\right)^2 - \left(\frac{dx^4}{ds}\right)^2 - \left(\frac{d\lambda}{ds}\right)^2\right), \tag{228}$$

where $\mu, \nu, \sigma = 1, \ldots, 4$, and $\eta_{\nu\sigma}$ is the flat Minkowski metric $\mathrm{diag}(1,1,1,-1)$. It is interesting to notice that, for $\lambda = \mathrm{const.}$, Minkowski's space null lines $x^{\mu}(s) = su^{\mu}$, where u^{μ}, $\mu = (1, \ldots, 4)$ is a fixed null vector, are geodesics of the five–dimensional space.

When λ is non–constant, it is convenient to choose λ as a (non–affine) parameter. The geodesic equations will read in such a case (adapted from [19, Appendix B, (B7)]) as

$$0 = \frac{d^2x^{\mu}}{d\lambda^2} - \frac{dx^{\mu}}{d\lambda}\left(\Gamma^{5}{}_{55} + 2\Gamma^{5}{}_{5\nu}\frac{dx^{\nu}}{d\lambda} + \Gamma^{5}{}_{\nu\sigma}\frac{dx^{\nu}}{d\lambda}\frac{dx^{\sigma}}{d\lambda}\right) + \Gamma^{\mu}{}_{55} + 2\Gamma^{\mu}{}_{5\nu}\frac{dx^{\nu}}{d\lambda} + \Gamma^{\mu}{}_{\nu\sigma}\frac{dx^{\nu}}{d\lambda}\frac{dx^{\sigma}}{d\lambda}, \tag{229}$$

which, in our case, reduces to:

$$x''^{\mu}(\lambda) = x'^{\mu}(\lambda)\frac{1 + x'^2(\lambda)}{\lambda}. \tag{230}$$

Here, we denote by a prime the derivative with respect to λ, and denote $x'^2(\lambda) = \eta_{\nu\sigma}\frac{dx^{\nu}}{d\lambda}\frac{dx^{\sigma}}{d\lambda}$. The direction of the vector x'^{μ} is kept constant along the geodesics. Thus, we need to consider three cases: $x'^2 = 0$, $x'^2 < 0$, and $x'^2 > 0$. If $x'^2 = 0$, we can use a Lorentz rotation (in the variables x^{μ}) to set the direction of x'^{μ} along the vector $(1,0,0,1)$. The differential equations reduce in this case to the following ones:

$$x^{1\prime\prime}(\lambda) = x^{1\prime}(\lambda)/\lambda, \quad x^{4\prime\prime}(\lambda) = x^{4\prime}(\lambda)/\lambda, \tag{231}$$

which, taking into account the constraint $x'^2 = 0$, solve to

$$x^1(\lambda) = a\lambda^2 + x^1_0, \quad x^4(\lambda) = a\lambda^2 + x^4_0, \tag{232}$$

with x^2 and x^3 constant.

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When $x'^2 < 0$, we can use a Lorentz rotation to rotate the geodesic into the (x^4, λ) plane. The relevant differential equation:

$$x^{4''}(\lambda) = (1 - x^{4'}(\lambda)^2)/\lambda \tag{233}$$

solves to $(x^4(\lambda) - x_0^4)^2 - \lambda^2 = a^2$ - a hyperbola.

When $x'^2 > 0$, we can Lorentz rotate the geodesic into the (x^1, λ) plane, and the differential equation

$$x^{1''}(\lambda) = (1 + x^{1'}(\lambda)^2)/\lambda \tag{234}$$

solves to $(x^1(\lambda) - x_0^1)^2 + \lambda^2 = a^2$ - a semi–circle.

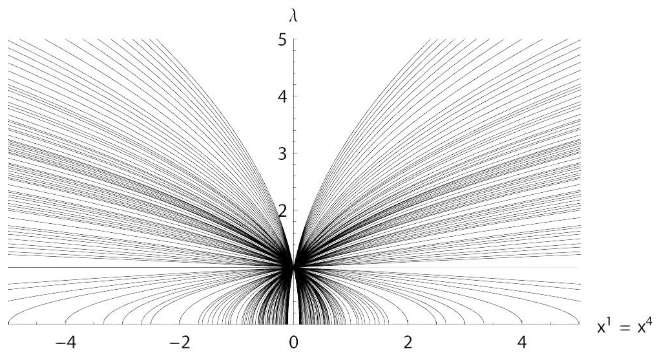

Figure 5. A family of geodesics in the $(x^1 = x^4, \lambda)$ plane through the point $(0, 1)$.

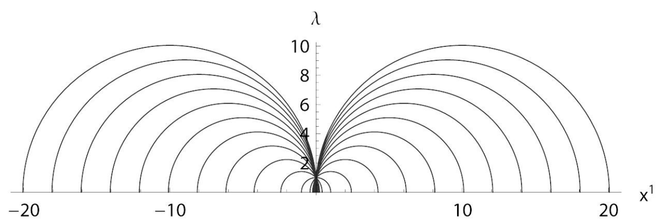

Figure 6. A family of geodesics in the (x^1, λ) plane through the point $(0, 1)$.

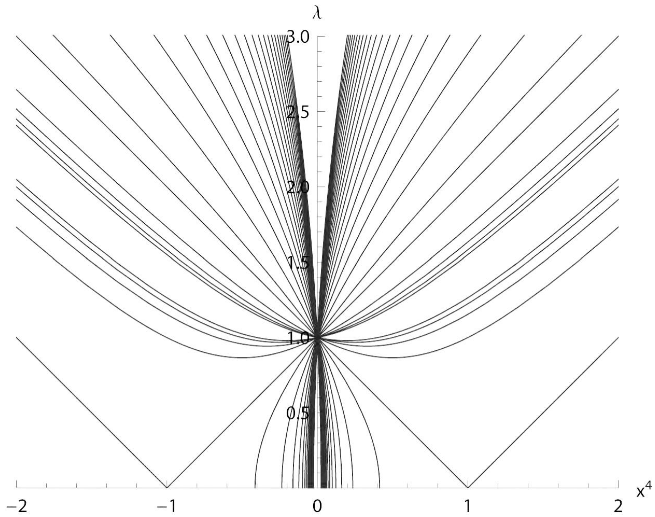

Figure 7. A family of geodesics in the (x^4, λ) plane through the point $(0, 1)$.

8.3. Σ_- as the Space of Hyperboloids

The five–dimensional homogeneous space Σ_- can be interpreted as a space of (unoriented) hyperboloids in the Minkowski space along the lines of a generalized Möbius geometry (Cf. e.g., [4, Ch. 1.2]). Let Y be in $\mathbb{R}^{4,2}$ with $Y^5 - Y^6 > 0$ and $Q(Y) < 0$. Consider a set of all $x \in M$ for which $(\tau_+(x), Y) = 0$. Normalizing Y so that $Y^5 - Y^6 = 1$ we can write it in the form $Y = \left(y^\mu, \frac{1-q(y)-\lambda^2}{2}, -\frac{1+q(y)+\lambda^2}{2}\right)$. A simple calculation shows that the condition $(\tau_+(x), Y) = 0$ translates then to $q(x-y) = -\lambda^2$. For $y = 0$ this is a double-sheeted hyperboloid with apex at $x^1 = x^2 = x^3 = 0, x^4 = \pm\lambda$. Each geodesic line in Σ_- can thus be interpreted as a particular one–parameter family of hyperboloids in the Minkowski space.

8.4. The Case of Σ_+

The same method as above applies in this case except that there is a change of signs in front of λ^2 in (221). The resulting metric is then

$$(g_{\alpha\beta}) = \frac{1}{\lambda^2}\,\mathrm{diag}(1,1,1,-1,-1), \tag{235}$$

with signature $(3,2)$. As in the case of Σ_-, the conformal infinity is the Cartesian product of $\mathbb{R}$ and, this time, the one-sheeted hyperboloid

$$(X^1)^2 + \ldots + (X^3)^2 - (X^4)^2 = 1.$$

The Minkowski space can be embedded in our five–dimensional manifold simply by putting $\lambda = 1$. It follows that the direction of the vector x' is constant along the geodesics.

Remark 10. *Following Wolf [25] we have considered in detail only the case of the equivalence relation $\approx$. In projective geometry one is using the weaker relation $\sim$. The standard projection can be discussed along the same lines as above. In that case the regions $X^5 - X^6 > 0$ and $X^5 - X^6 < 0$ are identified, so we can restrict our attention to $\lambda > 0$.[2] On the other hand, when discussing the topology - we have to additionally take the quotient by $\mathbb{Z}_2$.*

9. Conclusions

In this work the conformal compactification of Minkowski space has been revisited in an explicit and pedagogical manner, starting from the projective null cone in $\mathbb{R}^{4,2}$ and its identification with the unitary group $U(2)$. The role of the double cover $\overline{\overline{M}} \simeq S^3 \times S^1$ has been clarified, with particular emphasis on the effectiveness of the $SO(4,2)$ action and on the detailed geometry of conformal infinity, described in terms of horn and needle Dupin cyclides. By combining concrete coordinate constructions, explicit matrix formulas for the conformal group and its subgroups, and a coordinate-free reformulation in the spirit of Kopczyński and Woronowicz, the paper aims to provide a transparent geometric picture that can serve both as a reference and as a pedagogical introduction to conformal compactification in mathematical relativity and conformal field theory.

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2 That is why a similar coordinatization is often referred to as the "half–space model" in literature on hyperbolic geometry - see e.g. [26].

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