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Article

Product Entropic Uncertainty Principle

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Abstract: Motivated from Deutsch entropic uncertainty principle and several product uncertainty principles, we derive an uncertainty principle for the product of entropies using functions.

Keywords: Uncertainty Principle; Frame

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1. Introduction

Let $\mathcal{H}$ be a finite dimensional Hilbert space. Given an orthonormal basis $\{\omega_j\}_{j=1}^n$ for $\mathcal{H}$, the **Shannon entropy** at a point $h \in \mathcal{H}_\tau$ is defined as

$$S_\tau(h) := \sum_{j=1}^n |\langle h, \tau_j \rangle|^2 \log \left(\frac{1}{|\langle h, \tau_j \rangle|^2} \right) \geq 0,$$

where $\mathcal{H}_\tau := \{h \in \mathcal{H} : \|h\| = 1, \langle h, \tau_j \rangle \neq 0, 1 \leq j \leq n\}$. In 1983, Deutsch derived following uncertainty principle for Shannon entropy [1].

Theorem 1.1 ((Deutsch Uncertainty Principle) [1]). Let $\{\tau_j\}_{j=1}^n, \{\omega_j\}_{j=1}^n$ be two orthonormal bases for a finite dimensional Hilbert space $\mathcal{H}$. Then

$$2 \log n \geq S_\tau(h) + S_\omega(h) \geq -2 \log \left(\frac{1 + \max_{1 \leq j, k \leq n} |\langle \tau_j, \omega_k \rangle|}{2} \right) \geq 0, \quad \forall h \in \mathcal{H}_\tau \cap \mathcal{H}_\omega. \quad (1)$$

Using Buzano inequality, it is easy to see that Theorem 1.1 holds for Parseval frames [2]. Note that Inequality (1) is for the sum of entropies and not for the product. It is best if we have uncertainty principles for the product because once we have an uncertainty principle for the product, then the uncertainty principle for the sum follows from the AM-GM inequality. Following is a small list of uncertainty principle (UP) for the product.

1. Heisenberg-Robertson-Schrodinger UP and its various generalizations [3–9].
2. Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani support size UP [10–12].
3. Smith UP and its generalizations [13–16].
4. Kuppinger-Durisi-Bölcskei support size UP and its generalization [17,18].
5. A. Wigderson-Y. Wigderson UP [19].
6. Maccone-Pati UP [20].
7. Goh-Goodman UP [21].
8. Jiang-Liu-Wu subfactor UP [22].
9. Bandeira-Lewis-Mixon numerical sparsity UP [23].
10. UP over finite fields [24–26].
11. Generalized UP [27–29].

It seems that for logarithm functions, we can not have uncertainty for the product of entropies using coherence. In this paper, we derive an uncertainty principle for the product of entropies coming from certain functions.

2. Product Entropic Uncertainty Principle

In the paper, $\mathbb{K}$ denotes $\mathbb{C}$ or $\mathbb{R}$ and $\mathcal{H}$ denotes a Hilbert space (need not be finite dimensional) over $\mathbb{K}$. We need the notion of continuous frame which is introduced independently by Ali, Antoine and Gazeau [30] and Kaiser [31].

Definition 2.1 ([30,31]). Let (Ω, μ) be a measure space. A collection $\{\tau_\alpha\}_{\alpha \in \Omega}$ in a Hilbert space $\mathcal{H}$ is said to be a **continuous Parseval frame** for $\mathcal{H}$ if the following conditions hold.

- (i) For each $h \in \mathcal{H}$, the map $\Omega \ni \alpha \mapsto \langle h, \tau_\alpha \rangle \in \mathbb{K}$ is measurable.
- (ii)

$$\|h\|^2 = \int_{\Omega} |\langle h, \tau_\alpha \rangle|^2 d\mu(\alpha), \quad \forall h \in \mathcal{H}.$$

Recall that a continuous Parseval frame $\{\tau_\alpha\}_{\alpha \in \Omega}$ for $\mathcal{H}$ is said to be **1-bounded** if

$$\|\tau_\alpha\| \leq 1, \quad \forall \alpha \in \Omega.$$

Let $\phi : (0, 1] \rightarrow (0, \infty)$ be a function satisfying following.

- 1. ϕ is continuous.
- 2. ϕ is decreasing.
- 3. $\phi(xy) \leq \phi(x)\phi(y)$, $\forall x, y \in (0, 1]$.

In the entire paper, we assume that ϕ satisfies above conditions. Given such a ϕ and a 1-bounded continuous Parseval frame $\{\tau_\alpha\}_{\alpha \in \Omega}$ for $\mathcal{H}$, we define entropy

$$S_{\tau, \phi}(h) = \int_{\Omega} |\langle h, \tau_\alpha \rangle|^2 \phi(|\langle h, \tau_\alpha \rangle|^2) d\mu(\alpha) \geq 0, \quad \forall h \in \mathcal{H}_\tau,$$

where $\mathcal{H}_\tau := \{h \in \mathcal{H} : \|h\| = 1, \langle h, \tau_\alpha \rangle \neq 0, \forall \alpha \in \Omega\}$. Since ϕ is continuous, the integral exists. Before deriving the theorem of this paper, we need a powerful inequality.

Lemma 2.1 ((Buzano Inequality) [32–34]). Let $\mathcal{H}$ be a Hilbert space. Then

$$|\langle u, h \rangle \langle h, v \rangle| \leq \|h\|^2 \frac{\|u\| \|v\| + |\langle u, v \rangle|}{2}, \quad \forall h, u, v \in \mathcal{H}.$$

Theorem 2.1. Let $(\Omega, \mu), (\Delta, \nu)$ be measure spaces. Let $\{\tau_\alpha\}_{\alpha \in \Omega}$ and $\{\omega_\beta\}_{\beta \in \Delta}$ be 1-bounded continuous Parseval frames for $\mathcal{H}$. Then

$$S_{\tau, \phi}(h) S_{\omega, \phi}(h) \geq \phi \left(\frac{\left(1 + \sup_{\alpha \in \Omega, \beta \in \Delta} |\langle \tau_\alpha, \omega_\beta \rangle|\right)^2}{4} \right) \geq 0, \quad \forall h \in \mathcal{H}_\tau \cap \mathcal{H}_\omega. \quad (2)$$

Proof. Let $h \in \mathcal{H}_\tau \cap \mathcal{H}_\omega$. Then using the properties of ϕ and Lemma 2.1, we get

$$\begin{aligned}
 S_{\tau,\phi}(h)S_{\omega,\phi}(h) &= \left(\int_{\Omega} |\langle h, \tau_\alpha \rangle|^2 \phi(|\langle h, \tau_\alpha \rangle|^2) d\mu(\alpha) \right) \left(\int_{\Delta} |\langle h, \omega_\beta \rangle|^2 \phi(|\langle h, \omega_\beta \rangle|^2) dv(\beta) \right) \\
 &= \int_{\Omega} \int_{\Delta} |\langle h, \tau_\alpha \rangle|^2 |\langle h, \omega_\beta \rangle|^2 \phi(|\langle h, \tau_\alpha \rangle|^2) \phi(|\langle h, \omega_\beta \rangle|^2) dv(\beta) d\mu(\alpha) \\
 &\geq \int_{\Omega} \int_{\Delta} |\langle h, \tau_\alpha \rangle|^2 |\langle h, \omega_\beta \rangle|^2 \phi(|\langle h, \tau_\alpha \rangle|^2) |\langle h, \omega_\beta \rangle|^2 dv(\beta) d\mu(\alpha) \\
 &\geq \int_{\Omega} \int_{\Delta} |\langle h, \tau_\alpha \rangle|^2 |\langle h, \omega_\beta \rangle|^2 \phi(|\langle h, \tau_\alpha \rangle|^2) |\langle h, \omega_\beta \rangle|^2 dv(\beta) d\mu(\alpha) \\
 &\geq \int_{\Omega} \int_{\Delta} |\langle h, \tau_\alpha \rangle|^2 |\langle h, \omega_\beta \rangle|^2 \phi \left(\frac{(\|\tau_\alpha\| \|\omega_\beta\| + |\langle \tau_\alpha, \omega_\beta \rangle|)^2}{4} \right) dv(\beta) d\mu(\alpha) \\
 &\geq \int_{\Omega} \int_{\Delta} |\langle h, \tau_\alpha \rangle|^2 |\langle h, \omega_\beta \rangle|^2 \phi \left(\frac{(1 + \sup_{\alpha \in \Omega, \beta \in \Delta} |\langle \tau_\alpha, \omega_\beta \rangle|)^2}{4} \right) dv(\beta) d\mu(\alpha) \\
 &= \phi \left(\frac{(1 + \sup_{\alpha \in \Omega, \beta \in \Delta} |\langle \tau_\alpha, \omega_\beta \rangle|)^2}{4} \right) \int_{\Omega} \int_{\Delta} |\langle h, \tau_\alpha \rangle|^2 |\langle h, \omega_\beta \rangle|^2 dv(\beta) d\mu(\alpha) \\
 &= \phi \left(\frac{(1 + \sup_{\alpha \in \Omega, \beta \in \Delta} |\langle \tau_\alpha, \omega_\beta \rangle|)^2}{4} \right).
 \end{aligned}$$

□

Theorem 2.1 gives the following question.

Question 2.1. Let (Ω, μ) , (Δ, ν) be measure spaces, $\mathcal{H}$ be a Hilbert space. For which pairs of 1-bounded continuous Parseval frames $\{\tau_\alpha\}_{\alpha \in \Omega}$ and $\{\omega_\beta\}_{\beta \in \Delta}$ for $\mathcal{H}$, we have equality in Inequality (2)?

In 1988, Maassen and Uffink (motivated from the conjecture by Kraus made in 1987 [35]) improved Deutsch entropic uncertainty principle.

Theorem 2.2 ([36]). (Maassen-Uffink Entropic Uncertainty Principle) Let $\{\tau_j\}_{j=1}^n$, $\{\omega_k\}_{k=1}^n$ be two orthonormal bases for a finite dimensional Hilbert space $\mathcal{H}$. Then

$$S_\tau(h) + S_\omega(h) \geq -2 \log \left(\max_{1 \leq j, k \leq n} |\langle \tau_j, \omega_k \rangle| \right), \quad \forall h \in \mathcal{H}_\tau \cap \mathcal{H}_\omega.$$

In 2013, Ricaud and Torr sani [11] showed that orthonormal bases in Theorem 2.2 can be improved to Parseval frames.

Theorem 2.3 ([11]). (Ricaud-Torr sani Entropic Uncertainty Principle) Let $\{\tau_j\}_{j=1}^n$, $\{\omega_k\}_{k=1}^m$ be two Parseval frames for a finite dimensional Hilbert space $\mathcal{H}$. Then

$$S_\tau(h) + S_\omega(h) \geq -2 \log \left(\max_{1 \leq j \leq n, 1 \leq k \leq m} |\langle \tau_j, \omega_k \rangle| \right), \quad \forall h \in \mathcal{H}_\tau \cap \mathcal{H}_\omega.$$

Proofs of Theorems 2.2 and 2.3 use Riesz-Thorin interpolation and the differentiability of logarithm function. We therefore end by formulating the following problem.

Problem 2.1. Let $\{\tau_\alpha\}_{\alpha \in \Omega}$ and $\{\omega_\beta\}_{\beta \in \Delta}$ be 1-bounded continuous Parseval frames for $\mathcal{H}$. For which functions ϕ , we have

$$S_{\tau, \phi}(h) S_{\omega, \phi}(h) \geq \phi \left(\left(\sup_{\alpha \in \Omega, \beta \in \Delta} |\langle \tau_\alpha, \omega_\beta \rangle| \right)^2 \right), \quad \forall h \in \mathcal{H}_\tau \cap \mathcal{H}_\omega.$$

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