Article

On Generalized Fractional Operator and Related Fractional Integral Equations in Orlicz Spaces

Kottakkaran Sooppy Nisar *

Posted Date: 3 December 2024

doi: 10.20944/preprints202412.0168.v1

Keywords: Measure of noncompactness; g-fractional operator; Fixed point theorem; Orlicz spaces doi:10.20944/preprints202412.0168.v1

Article

On Generalized Fractional Operator and Related Fractional Integral Equations in Orlicz Spaces

Kottakkaran Sooppy Nisar

Department of Mathematics, College of Science and Humanities, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia; n.sooppy@psau.edu.sa

Abstract: This article aims to prove and explain novel properties of the g-fractional type operators, like boundedness, continuity, and monotonicity within Orlicz spaces L_ψ. We utilize such properties through Darbo's fixed-point theorem ($\mathcal{FPT}$) and the measure of noncompactness ($\mathcal{MNC}$) to study the existence in addition to the uniqueness of the solution to a quadratic integral equation in L_ψ. These results are new as the g-fractional operators are investigated for the first time in L_ψ. Our work generalizes and extends several fractional operators like the Riemann-Liouville, Hadamared, and Erdlyi–Kober and covers and unifies the results of many particular cases of classical and quadratic fractional problems studied in the former literature.

Keywords: measure of noncompactness ($\mathcal{MNC}$); g-fractional operator; fixed point theorem ($\mathcal{FPT}$); Orlicz spaces L_ψ

MSC: 47N20; 45G10; 47H30; 47H10

---

1. Introduction

Fractional calculus is an important tool for describing memory and hereditary properties related to various processes and materials. It is an important scientific field due to its applications in economics, biology, physics, viscoelasticity, engineering, fluid dynamics, electrical circuits, earthquakes, electrochemistry, and traffic models (cf. [2–6]).

The "g-fractional operator", which is the fractional integral of a Lebesgue function y concerning another function g was introduced in ([5, Sect. 18.2] and [2, Sect. 2.5.]). It combines and unifies the Hadamard, Riemann-Liouville, and Erdlyi–Kober fractional operators into one form (see [7–12]), and is better able to present memory properties connected to various types of materials and processes.

Quadratic integral equations using the g-fractional operator are more appropriate in the kinetic theory of gases [13], radiative transfer [16], neutron transport [14] etc. The primary objective of the work is to prove and describe the vital properties of the g-fractional type operator, encompassing boundedness, continuity, and monotonicity in Orlicz spaces L_ψ. We will apply these properties to illustrate and analyze the monotonic solutions of the equation

$$y(v) \;=\; f(v) + \frac{h_1(v, y(v))}{\Gamma(\alpha)} \cdot \int_0^v \frac{h_2(s, y(s))}{\Big(g(v) - g(s)\Big)^{1-\alpha}} g'(s)\, ds, \;\; v \in [0, d], \tag{1}$$

where $0 < \alpha < 1$ in the mentioned spaces.

This is inspired by statistical physics, physics models (cf. [17,18]), and various applications of partial differential equations or integral equations in Orlicz spaces L_ψ [31,32].

Our method covers and generalizes different types of fractional integrals that have been examined separately and encourages us to recall some of them. The author in [22] presented some basic properties of the Riemann-Liouville type fractional integral operator and explored equation solutions

$$y(v) \;=\; f(v) + G(y)(v) \int_0^v \frac{(v-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))\, ds, \;\; 0 < \alpha < 1, \; v \in [0, d]$$

in Orlicz spaces L_ψ.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 3 December 2024 doi:10.20944/preprints202412.0168.v1

The author in [23] demonstrated and studied fundamental features of the Hadamard type fractional operator within L_ψ-spaces and utilized them to solve the equation:

$$y(v) = G_2(y)(v) + \frac{G_1(y)(v)}{\Gamma(\alpha)} \int_1^v \left(\log \frac{v}{s}\right)^{\alpha-1} G_2(y)(s)\, ds, \;\; v \in [1,e],\ 0<\alpha<1.$$

The authors in [30] showed the basic characteristics of the Erdlyi–Kober fractional operators in Lebesgue and Orlicz spaces and used them to analyze the problem.

$$y(v) = f(v) + f_1(v, y(v)) + f_2\left(v, \frac{\beta h_1(v,y(v))}{\Gamma(\alpha)} \cdot \int_0^v \frac{t^{\beta-1} h_2(s,y(s))}{(v^\beta - s^\beta)^{1-\alpha}}\, ds\right), \;\; v \in [0,d],$$

where $0<\alpha<1$ & $\beta>0$ in the indicated spaces.

Furthermore, the noncompactness measure ($\mathcal{MNC}$) and Darboe's fixed point hypothesis ($\mathcal{FPT}$) were used to study different types of quadratic integral equations in Orlicz spaces L_ψ under various sets of assumptions (cf. [19–21]).

This article aims to illustrate and demonstrate some essential aspects of the g-fractional operator, including boundedness, action, continuity, and monotonicity within L_ψ-spaces. We utilize such properties through the measure of noncompactness ($\mathcal{MNC}$) and fixed-point theorem ($\mathcal{FPT}$) to investigate the existence and uniqueness of the solution for a quadratic integral equation (1) within the given spaces.

2. Preliminaries

Let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{I} = [0,d] \subset \mathbb{R}^+ = [0,\infty)$. Denote the Young function (YF) by $P : \mathbb{R}^+ \to \mathbb{R}^+$, where

$$P(\theta) = \int_0^\theta g(s) ds, \;\; \text{for } \theta \geq 0$$

and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is neither identically zero nor infinite and an increasing and left-continuous function on $\mathbb{R}^+$. The pair (P,Q) is said to be a complementary pair of YF if $Q(y) = \sup_{x\geq 0}(yx - P(y))$.

The function P is known as N-function when it is a finite-valued and verifies $\lim_{\theta\to\infty} \frac{P(\theta)}{\theta} = \infty$, $\lim_{\theta\to 0} \frac{P(\theta)}{\theta} = 0$, and $P(\theta) > 0$ if $\theta > 0$ ($P(\theta) = 0 \iff \theta = 0$).

The *Orlicz space* $L_P = L_P(\mathbb{I})$ is the space of all measurable functions $y : \mathbb{I} \to \mathbb{R}$ with the norm

$$\|y\|_P = \inf_{\lambda>0}\left\{ \int_{\mathbb{I}} P\left(\frac{y(s)}{\lambda}\right) ds \leq 1 \right\} < \infty.$$

It is important to recall that for any YF P, we have $P(\theta+s) \leq P(\theta) + P(s)$ and $P(\rho\theta) \leq \rho P(\theta)$, where $\theta, s \in \mathbb{R}$, and $\rho \in [0,1]$.

Assume that $E_P(\mathbb{I})$ is the set of all bounded functions in $L_P(\mathbb{I})$ contain absolutely continuous norms.

Moreover, we get $L_P = E_P$ if P satisfies the Δ_2 condition, i.e.:

$$(\Delta_2)\ \exists\, \omega,\ \theta_0 \geq 0 \;\text{ such that } P(2\theta) \leq \omega P(\theta),\ \theta \geq \theta_0.$$

It is noting that, the classical Lebesgue spaces $L_p(\mathbb{I})$ shall be considered as a particular case of Orlicz spaces $L_{P_p}(\mathbb{I})$ with corresponding N-function $P_p = s^p$, $p>1$ satisfies the above Δ_2 condition.

Lemma 1. [17] *Assume that, the function $h(v,y) : \mathbb{I}\times\mathbb{R} \to \mathbb{R}$ verifies Carathéodory conditions (i.e., it is continuous in the variable y for almost all $v \in \mathbb{I}$ and measurable in v for any $y \in \mathbb{R}$). The superposition operator $F_h = h(v,y) : E_{\psi_1} \to L_P = E_P$ is bounded and continuous if*

$$|h(v,y)| \leq a(v) + bP^{-1}(\psi_1(y)), \;\; y \in \mathbb{R}, \;\; v \in \mathbb{I},$$

where $b \geq 0$, $a \in L_P$ and the N-function $P(y)$ verifies the Δ_2 condition.

Lemma 2. *[24] Assume that ψ, ψ_1, and ψ_2 are arbitrary different N-functions. The following given conditions are equivalent:*

1. *For every functions $y_1 \in L_{\psi_1}$ and $y_2 \in L_{\psi_2}$, $y_1 \cdot y_2 \in L_\psi$.*
2. *$\exists\, k > 0$ s.t. for all measurable y_1, y_2 on $\mathbb{I}$, we have $\|y_1 y_2\|_\psi \leq k\|y_1\|_{\psi_1}\|y_1\|_{\psi_2}$.*
3. *$\exists\, C > 0$, $s_0 \geq 0$ s.t. for all $v, s \geq s_0$, $\psi\left(\frac{sv}{C}\right) \leq \psi_1(s) + \psi_2(v)$.*
4. $\limsup_{v\to\infty} \frac{\psi_1^{-1}(v)\psi_2^{-1}(v)}{\psi(v)} < \infty$.

The set $S = S(\mathbb{I})$ is the set of Lebesgue measurable functions "means" on $(\mathbb{I})$ connected with the metric

$$d(y, x) = \inf_{\rho>0}[\epsilon + meas\{s : |y(s) - x(s)| \geq \rho\}]$$

is a complete space. Additionally, the convergence in measure on $\mathbb{I}$ is equivalent to the convergence regarding d (cf. [25]). The compactness in S is called "compactness in measure".

Lemma 3 ([19]). *Assume that $Y \subset L_\psi(\mathbb{I})$ is a bounded set, and $\exists$ a family $(\Omega_r)_{0\leq r\leq d} \subset \mathbb{I}$ s.t. meas $\Omega_r = r$ for every $r \in [0, d]$, and for every $y \in Y$,*

$$y(s_1) \geq y(s_2), \quad (s_1 \in \Omega_r,\ t_2 \notin \Omega_r).$$

Then, Y is compact in measure in $L_\psi(\mathbb{I})$.

Definition 1 ([26]). *The Hausdorff measure of noncompactness ($\mathcal{MNC}$) $\beta_H(X)$ for a bounded set $\emptyset \neq Y \subset L_\psi$ is known as*

$$\beta_H(Y) = \inf\{r > 0 : \exists\, X \subset L_\psi \text{ s.t. } Y \subset\ X\ +\ B_r\ \},$$

where $B_r = \{y \in L_\psi(\mathbb{I}) : \|y\|_\psi \leq r\}$ is the ball centered at the origin with radius r.

Denote a measure of equi-integrability c of $Y \in L_\psi(\mathbb{I})$ by :

$$c(Y) \ = \ \lim_{\epsilon\to 0} \ \sup_{measD\leq\epsilon} \ \sup_{y\in Y} \|y \cdot \chi_D\|_\psi,$$

where $\epsilon\ >\ 0$ and χ_A points to the characteristic function $A \subset \mathbb{I}$ (see [25] or [27]).

Lemma 4 ([19,27]). *Assume that $\emptyset \neq Y \subset L_\psi$ is a bounded set and compact in measure. Then, we have:*

$$\beta_H(Y) = c(Y).$$

Theorem 1 ([26]). *Assume that $\emptyset \neq \Omega \subset L_\psi$ is a convex, bounded, and closed set and $T : \Omega\ \to\ \Omega$ is continuous mapping and a contraction regarding to β_H, i.e.:*

$$\beta_H(T(Y))\ \leq\ k\beta_H(Y),\ \ 0 \leq k < 1$$

for any $\emptyset \neq Y \subset \Omega$. Then, the map T has at least one fixed point in Ω.

3. Generalized Fractional Operators

We provide and prove some properties and concepts of the generalized fractional (or g-fractional) integral operator In $L_\psi(\mathbb{I})$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 3 December 2024 doi:10.20944/preprints202412.0168.v1

Definition 2 ([2,5]). *The g-fractional, or generalized fractional, integral of a well-defined function y of order α concerning a different function g(s) is given by*

$$J_g^{\alpha} y(v) = \frac{1}{\Gamma(\alpha)} \int_0^v \frac{y(s)}{\Big(g(v) - g(s)\Big)^{1-\alpha}} g'(s) ds, \; \alpha > 0, \tag{2}$$

where g is a positive-increasing function on $(0, \infty]$ *and has a continuous derivative on* $(0, \infty)$.

Remark 1.

- *If* $g(v) = v$*, the operator* J_g^{α} *(2) becomes the Riemann–Liouville fractional operator, which has been analyzed in [2,3,22]:*

$$J_v^{\alpha} y(v) = \frac{1}{\Gamma(\alpha)} \int_0^v \frac{y(s)}{(v-s)^{1-\alpha}} ds.$$

- *If* $g(v) = \log(v)$*, the operator* J_g^{α} *(2) becomes the Hadamard fractional operator, which has been analyzed in [2,3,23]:*

$$J_{\log(v)}^{\alpha} y(v) = \frac{1}{\Gamma(\alpha)} \int_1^v \left(\log \frac{v}{s}\right)^{\alpha-1} \frac{y(s)}{s} \, ds.$$

- *If* $g(v) = v^{\beta}$, $\beta > 0$*, the operator* J_g^{α} *(2) becomes the Erdlyi–Kober's operator, which has been analyzed in [2,3,30]:*

$$J_{v^{\beta}}^{\alpha} y(v) = \frac{1}{\Gamma(\alpha)} \int_0^v \frac{\beta s^{\beta-1} y(s)}{(v^{\beta} - s^{\beta})^{1-\alpha}} \, ds.$$

- *If* $g(v) = v^2$*, the operator* J_g^{α} *(2) becomes the fractional integral operator of the form (Sneddon [28]):*

$$J_{v^2}^{\alpha} y(v) = \frac{2}{\Gamma(\alpha)} \int_0^v \frac{y(s)}{(v^2 - s^2)^{1-\alpha}} s ds.$$

Now, we examine the monotonicity of the operator J_g^{α} (2).

Lemma 5. *The operator* J_g^{α}, $\alpha > 0$ *with* $g(0) = 0$ *maps nonnegative and a.e. non-decreasing functions into functions having similar properties.*

Proof. Let $v_1, v_2 \in \mathbb{I}$, $v_1 \le v_2$, and y be a.e. nondecreasing-nonnegative function, then by putting $g(s) = g(v_1) \cdot u$, we have

$$\begin{aligned} J_g^{\alpha} y(v_1) &= \frac{1}{\Gamma(\alpha)} \int_0^{v_1} \frac{y(s)}{\Big(g(v_1) - g(s)\Big)^{1-\alpha}} g'(s) ds \\ &= \frac{g^{\alpha}(v_1)}{\Gamma(\alpha)} \int_0^1 \frac{y\Big(g^{-1}(g(v_1) \cdot u)\Big)}{(1-u)^{1-\alpha}} du. \end{aligned}$$

Since the function g is an increasing and positive function on $(0, \infty]$, then its inverse g^{-1} exists and has also the same properties (cf. [29]), then we get

$$J_g^{\alpha} y(v_1) \le \frac{g^{\alpha}(v_2)}{\Gamma(\alpha)} \int_0^1 \frac{y\Big(g^{-1}(g(v_2) \cdot u)\Big)}{(1-u)^{1-\alpha}} du = J_g^{\alpha} y(v_2).$$

Therefore, $0 \le J_g^{\alpha} y(v_1) \le J_g^{\alpha} y(v_2)$ for $v_1 \le v_2$. □

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 3 December 2024 doi:10.20944/preprints202412.0168.v1

Proposition 1 ([15]). *Let P be a (YF) Young function, then for $v \in \mathbb{R}^+$ and any $0 < \alpha < 1$, the set*

$$\mathbb{P}(s) = \left\{ \epsilon > 0 : \frac{1}{\|g'\|_P} \int_0^{g(v)\sigma^{\frac{1}{1-\alpha}}} P(u^{\alpha-1})\, du \leq \sigma^{\frac{1}{1-\alpha}} \right\} \neq \emptyset, \;\; \sigma = \frac{\epsilon}{\|g'\|_P},$$

is increasing and continuous functions with $\mathbb{P}(0) = 0$, where the function g is defined in Definition 2.

Lemma 6. *Let (P, Q) be a complementary pair of N-functions and ψ be an N-function, where P verifies $\int_0^{g(v)} P(u^{\alpha-1})\, du < \infty$, $\alpha \in (0,1)$. Then the operator $J_g^\alpha : L_Q(\mathbb{I}) \to L_\psi(\mathbb{I})$ is continuous, where*

$$k(v) = \frac{\sigma^{\frac{1}{\alpha-1}}}{\|g'\|_P} \int_0^{g(v)\sigma^{\frac{1}{1-\alpha}}} P(u^{\alpha-1})\, du \in E_\psi(\mathbb{I}), \;\; \epsilon > 0, \;\; \sigma = \frac{\epsilon}{\|g'\|_P}$$

for a.e. $v \in \mathbb{I}$.

Proof. Assume that

$$K(v,s) = \begin{cases} \frac{\left(g(v)-g(s)\right)^{\alpha-1} g'(s)}{\Gamma(\alpha)} & \text{if } s \in [0, v],\; v > 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $y \in L_Q(\mathbb{I})$ and by utilizing Hölder inequality, we get:

$$\begin{aligned} |J_g^\alpha y(v)| &= \left| \int_0^\infty K(v,s) y(s)\, ds \right| \\ &\leq 2\|K(v,\cdot)\|_P \|y\|_Q \\ &\leq \frac{2}{\Gamma(\alpha)} \inf_{\epsilon>0} \left\{ \int_{\mathbb{I}} P\left(\frac{\left(g(v)-g(s)\right)^{\alpha-1} g'(s)}{\epsilon} \right) ds \leq 1 \right\} \|y\|_Q \\ &= \frac{2}{\Gamma(\alpha)} \inf_{\epsilon>0} \left\{ \int_{\mathbb{I}} P\left(\frac{\left(g(v)-g(s)\right)^{\alpha-1} g'(s)\|g'\|_P}{\epsilon \cdot \|g'\|_P} \right) ds \leq 1 \right\} \|y\|_Q \\ &\leq \frac{2}{\Gamma(\alpha)} \inf_{\epsilon>0} \left\{ \int_{\mathbb{I}} P\left(\frac{\left(g(v)-g(s)\right)^{\alpha-1} \|g'\|_P}{\epsilon} \right) \frac{g'(s)}{\|g'\|_P}\, ds \leq 1 \right\} \|y\|_Q. \end{aligned}$$

Put $u = (g(v) - g(s))\sigma^{\frac{1}{1-\alpha}}$, where $\sigma = \frac{\epsilon}{\|g'\|_P}$, we have

$$\begin{aligned} \|J_g^\alpha y\|_\psi &\leq \frac{2}{\Gamma(\alpha)} \left\| \inf_{\epsilon>0} \left\{ \frac{1}{\|g'\|_P} \int_0^{g(v)\sigma^{\frac{1}{1-\alpha}}} P(u^{\alpha-1})\, du \leq \sigma^{\frac{1}{1-\alpha}} \right\} \right\|_\psi \|y\|_Q \\ &\leq \frac{2}{\Gamma(\alpha)} \|k\|_\psi \|y\|_Q, \end{aligned}$$

where $k \in E_\psi(\mathbb{I})$. Then, by recalling Proposition 1 and [17, Lemma 16.3], we have $J_g^\alpha : L_Q(\mathbb{I}) \to L_\psi(\mathbb{I})$ and is continuous. □

4. Main Results

Equation (1) can take the form:

$$y = B(y) = f + U(y),$$

where

$$U(y) = F_{h_1}(y) \cdot A(y), \ \ A(y)(v) = J_g^{\alpha} F_{h_2}(y)(v),$$

such that J_g^{α} is defined in Definition 2 and $F_{h_i}, i = 1, 2$ are known as the superposition operators.

Next, we shall discuss our existence theorem in L_{ψ}-spaces in the most interesting case, where the generating N-functions verifying the Δ_2-condition (cf. [1,20,21,30]). These allow us to utilize some general conditions for the studied functions.

Theorem 2. *Assume that ψ_1, ψ_2, and ψ are N-functions and (P, Q) is a complementary pair of N-functions, such that Q, ψ, ψ_1 satisfy the Δ_2 condition and $\int_0^{g(v)} P(u^{\alpha-1})\, du < \infty$, $\alpha \in (0,1)$ and that:*

(G1) $\exists\ k_1 > 0$ *s.t. for* $y_1 \in L_{\psi_1}(\mathbb{I})$ *and* $y_2 \in L_{\psi_2}(\mathbb{I})$ *we get* $\|y_1 y_2\|_{\psi} \leq k_1 \|y_1\|_{\psi_1} \|y_2\|_{\psi_2}$.
(C1) $f \in E_{\psi}(\mathbb{I})$ *is a.e. nondecreasing on* $\mathbb{I}$.
(C2) $h_i : \mathbb{I} \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$ *satisfy Carathéodory conditions and* $(s, y) \to h_i(s, y)$ *are nondecreasing.*
(C3) $\exists\ d_i \geq 0,\ i = 1, 2$ *and functions* $b_1 \in E_{\psi_1}(\mathbb{I})$, *and* $b_2 \in E_Q(\mathbb{I})$, *s.t.*

$$|h_1(s,y)| \leq b_1(s) + d_1 \psi_1^{-1}\Big(\psi(y)\Big), \ \ |h_2(s,y)| \leq b_2(s) + d_2 Q^{-1}\Big(\psi(y)\Big). \tag{3}$$

(C4) Assume that for a.e. $v \in \mathbb{I}$, $\exists\, \epsilon > 0$, *s.t.*

$$k(v) = \frac{\sigma^{\frac{1}{\alpha-1}}}{\|g'\|_P} \int_0^{\sigma^{\frac{1}{1-\alpha}} g(v)} P(u^{\alpha-1})\, du \in E_{\psi}(\mathbb{I}), \ \ \sigma = \frac{\epsilon}{\|g'\|_P}.$$

(C5) Assume that, $r > 0$ *on* $I_0 = [0, d_0] \subset \mathbb{I}$ *verifying*

$$\int_{I_0} \psi\left(|f(v)| + \frac{2k_1}{\Gamma(\alpha)} \|k\|_{\psi_2} \Big(\|b_1\|_{\psi_1} + d_1 r\Big)\Big(\|b_2\|_Q + d_2 r\Big)\right) dv \leq r$$

and

$$\left[\frac{2d_1 k_1}{\Gamma(\alpha)} \|k\|_{\psi_2} \Big(\|b_2\|_Q + d_2 \cdot r\Big)\right] < 1.$$

Then, there is a.e. nondecreasing solution $y \in E_{\psi}(I_0)$ *of (1) on* $I_0 \subset \mathbb{I}$.

Proof. I. Lemma 1 and assumptions (C2), (C3), imply that $F_{h_1} : E_{\psi}(\mathbb{I})) \to Ł_{\psi_1}(\mathbb{I})$, $F_{h_2} : E_{\psi}(\mathbb{I})) \to Ł_Q(\mathbb{I})$ and are continuous. Lemma 6) gives us that $A = J_g^{\alpha} F_{h_2} :\ E_{\psi}(\mathbb{I}) \to E_{\psi_2}(\mathbb{I})$ is continuous. Assumption (G1) implies that $U : E_{\psi}(\mathbb{I}) \to E_{\psi}(\mathbb{I})$, and by assumption (C1), $B : E_{\psi}(\mathbb{I}) \to E_{\psi}(\mathbb{I})$ is continuous.

II. Next, we should inspect and show that the operator B is bounded in $E_{\psi}(\mathbb{I})$.

Let Ω point to the closure of the set $\{y \in E_{\psi}(I_0) : \int_0^{d_0} \psi(|y(s)|)\, ds \leq r - 1\}$. Clearly, Ω is not a ball in $E_{\psi}(I_0)$, but $\Omega \subset B_r(E_{\psi}(I_0))$ (cf. [17] p. 222) and the set $\overline{\Omega}$ is a bounded, convex, and closed subset of $E_{\psi}(I_0)$.

By using Theorem 10.5 with constant k = 1 [17], then for arbitrary $y \in \Omega$ and $v \in I_0$, we get:

$$\left\|\psi_1^{-1}\Big(\psi(|y|)\Big)\right\|_{\psi_1} \leq \|y\|_{\psi} = 1 + \int_{I_0} \psi(y(s))ds \ \text{ and } \ \left\|Q^{-1}\Big(\psi(y)\Big)\right\|_Q \leq \|y\|_{\psi} = 1 + \int_{I_0} \psi(y(s))ds. \tag{4}$$

Therefore, by using Lemma 6 and our assumptions, we get

$$\begin{aligned}
|B(y)(v)| &\le |f(v)| + |U(y)(v)| \\
&\le |f(v)| + k_1 \|F_{h_1}(y)\|_{\varphi_1} \|A(y)\|_{\varphi_2} \\
&\le |f(v)| + k_1 \left\| b_1 + d_1 \varphi_1^{-1}\Big(\psi(|y|)\Big) \right\|_{\varphi_1} \cdot \|J_g^{\alpha} F_{h_2}(y)\|_{\varphi_2} \\
&\le |f(v)| + k_1 \left(\|b_1\|_{\varphi_1} + d_1 \left\| \varphi_1^{-1}\Big(\psi(|y|)\Big) \right\|_{\varphi_1} \right) \frac{2}{\Gamma(\alpha)} \|k\|_{\varphi_2} \left(\|b_2\|_Q + d_2 \left\| Q^{-1}\Big(\psi(|y|)\Big) \right\|_Q \right) \\
&\le |f(v)| + \frac{2k_1}{\Gamma(\alpha)} \|k\|_{\varphi_2} \left(\|b_1\|_{\varphi_1} + d_1 + d_1 \int_{I_0} \psi(y(s))ds \right) \left(\|b_2\|_Q + d_2 + d_2 \int_{I_0} \psi(y(s))ds \right) \\
&\le |f(v)| + \frac{2k_1}{\Gamma(\alpha)} \|k\|_{\varphi_2} \Big(\|b_1\|_{\varphi_1} + d_1 + d_1(r-1) \Big) \Big(\|b_2\|_Q + d_2 + d_2(r-1) \Big).
\end{aligned}$$

Recalling assumption (C5), we get

$$\int_{I_0} \psi(B(y)(v))\, dv \le \int_{I_0} \psi \left[|f(v)| + \frac{2k_1}{\Gamma(\alpha)} \|k\|_{\varphi_2} \Big(\|b_1\|_{\varphi_1} + d_1 r \Big) \Big(\|b_2\|_Q + d_2 r \Big) \right] dv \le r,$$

then $B(\Omega) \subset \Omega$, and $B(\overline{\Omega}) \subset \overline{B(\Omega)} \subset \overline{\Omega} = \Omega$. Then, the operator $B : \Omega \to \Omega$ is continuous on $\Omega \subset B_r(E_\psi(I_0))$.

III. Let $\Omega_r \subset \Omega$ contain all a.e. monotonic (nondecreasing) functions on I_0. The set $\emptyset \ne \Omega_r$ is bounded, closed, compact in measure, and convex in $L_\psi(I_0)$ (cf. [20]).

IV. The monotonicity of the functions is preserved via the operator B.

Take $y \in \Omega_r$, then y is a.e. nondecreasing on I_0 and, consequently, the operators $F_{h_i}, i = 1, 2$ are also a.e. nondecreasing on I_0. By Lemma 5), A is a.e. nondecreasing on I_0, then $U = F_{h_1} A$ is also a.e. nondecreasing on I_0. Using (C1), we get $B : \Omega_r \to \Omega_r$ is continuous.

V. Now, we show that B satisfies contraction condition w.r. to β_H.

Suppose there is a set $D \subset I_0$, with meas $D \le \varepsilon$, $\varepsilon > 0$. Therefore, for $y \in Y$ and $\emptyset \ne Y \subset Q_r$, we have:

$$\begin{aligned}
\|B(y) \cdot \chi_D\|_\psi &\le \|f \cdot \chi_D\|_\psi + \|F_{h_1}(y) A(y) \cdot \chi_D\|_\psi \\
&\le \|f \cdot \chi_D\|_\psi + k_1 \|F_{h_1}(y) \cdot \chi_D\|_{\varphi_1} \|A(y) \cdot \chi_D\|_{\varphi_2} \\
&\le \|f \cdot \chi_D\|_\psi + k_1 \left\| \left(b_1 + d_1 \varphi_1^{-1}\Big(\psi(|y|)\Big) \right) \cdot \chi_D \right\|_{\varphi_1} \cdot \left\| J_g^{\alpha} F_{h_2} \right\|_{\varphi_2} \\
&\le \|f \cdot \chi_D\|_\psi + k_1 \left(\|b_1 \cdot \chi_D\|_{\varphi_1} + d_1 \left\| \varphi_1^{-1}\Big(\psi(|y|)\Big) \cdot \chi_D \right\|_{\varphi_1} \right) \\
&\quad \times \frac{2}{\Gamma(\alpha)} \|k\|_{\varphi_2} \left(\|b_2\|_Q + d_2 \left\| Q^{-1}\Big(\psi(|y|)\Big) \right\|_Q \right) \\
&\le \|f \cdot \chi_D\|_\psi + \frac{2k_1}{\Gamma(\alpha)} \|k\|_{\varphi_2} \Big(\|b_1 \cdot \chi_D\|_{\varphi_1} + d_1 \|y \cdot \chi_D\|_\psi \Big) \Big(\|b_2\|_Q + d_2 \cdot r \Big).
\end{aligned}$$

Since $f \in E_\psi$, $b_1 \in E_{\varphi_1}$, then we have

$$\lim_{\varepsilon \to 0} \{ \sup_{meas\ D \le \varepsilon} [\sup_{y \in Y} \{ \|f \cdot \chi_D\|_\psi \}]\} = 0$$

and

$$\lim_{\varepsilon \to 0} \{ \sup_{meas\ D \le \varepsilon} [\sup_{y \in Y} \{ \|b_1 \cdot \chi_D\|_{\varphi_1} \}]\} = 0.$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 3 December 2024 doi:10.20944/preprints202412.0168.v1

By using the formula of of $c(Y)$, we get

$$c(B(Y)) \leq \left(\frac{2d_1k_1}{\Gamma(\alpha)}\|k\|_{\psi_2}\left(\|b_2\|_Q + d_2 \cdot r\right)\right)c(Y).$$

Based on the previously established properties, we may apply Lemma 4 to get

$$\beta_H(B(Y)) \leq \left(\frac{2d_1k_1}{\Gamma(\alpha)}\|k\|_{\psi_2}\left(\|b_2\|_Q + d_2 \cdot r\right)\right)\beta_H(Y).$$

The above inequality with $\left(\frac{2d_1k_1}{\Gamma(\alpha)}\|k\|_{\psi_2}\left[(]\|b_2\|_Q + d_2 \cdot r\right)\right] < 1$ allows us to apply Theorem 1. That ends the proof. □

4.1. Uniqueness of the Solution

Now, we may prove and discuss the uniqueness of the solutions of Eq. (1).

Theorem 3. *Assume that assumptions of Theorem 2 are verified, but replace the inequalities (3) with:*

(C6) $|h_i(v,0)| \leq b_i(v),\ b_1 \in E_{\psi_1}(\mathbb{I}),\ \ b_2 \in E_Q(\mathbb{I})$ *and*

$$|h_1(v,y) - h_1(v,z)| \leq d_1\psi_1^{-1}\Big(\psi(|y-z|)\Big),\ \ |h_2(v,y) - h_2(v,z)| \leq d_2Q^{-1}\Big(\psi(|y-z|)\Big),\ \ y,z \in \Omega_r,$$

where $d_i \geq 0$, *and* Ω_r *is as in Theorem 2 for* $i = 1,2$.

(C7) Assume that

$$\frac{2k_1\|k\|_{\psi_2}}{\Gamma(\alpha)}\left(d_2\Big(\|b_1\|_{\psi_1} + d_1 \cdot r\Big) + d_1\Big(\|b_2\|_{\psi_2} + d_2 \cdot r\Big)\right) < 1,$$

where r *is given in assumption (C5), then (1) has a unique solution* $y \in E_\psi$ *in* Ω_r.

Proof. Using assumption (C6), we get

$$\begin{aligned} \Big||h_1(v,y)| - |h_1(v,0)|\Big| &\leq |h_1(v,x) - h_1(v,0)| \leq d_1\psi_1^{-1}\Big(\psi(y)\Big) \\ \Rightarrow |h_1(v,y)| &\leq |h_1(v,0)| + d_1\psi_1^{-1}\Big(\psi(y)\Big) \leq b_1(v) + d_1\psi_1^{-1}\Big(\psi(y)\Big). \end{aligned}$$

Similarly, $|h_2(v,y)| \leq b_2(v) + d_2Q^{-1}\Big(\psi(y)\Big)$. Thus, Theorem 2 implies that, there exists a.e. nondecreasing solution $y \in E_\psi$ of (1) in Ω_r.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 3 December 2024 doi:10.20944/preprints202412.0168.v1

Next, Let $y,\ z \in \Omega_r$ be two distinct solutions of equation (1), then by using the inequalities (4) and assumption (C6), we obtain

$$\begin{aligned}
\|y-z\|_\psi &\leq \|F_{h_1}(y)A(y) - F_{h_1}(z)A(z)\|_\psi \\
&\leq \|F_{h_1}(y)A(y) - F_{h_1}(y)A(z)\|_\psi + \|F_{h_1}(y)A(z) - F_{h_1}(z)A(z)\|_\psi \\
&\leq k_1\|F_{h_1}(y)\|_{\psi_1}\|A(y)-A(z)\|_{\psi_2} + k_1\|F_{h_1}(y)-F_{h_1}(z)\|_{\psi_1}\|A(z)\|_{\psi_2} \\
&\leq k_1\left\|b_1 + d_1\psi_1^{-1}\Big(\psi(y)\Big)\right\|_{\psi_1}\left\|J_g^\alpha\Big(F_{h_2}(y)-F_{h_2}(z)\Big)\right\|_{\psi_2} + k_1\left\|d_1\psi_1^{-1}\Big(\psi(|y-z|)\Big)\right\|_{\psi_1}\left\|J_g^\alpha F_{h_2}(z)\right\|_{\psi_2} \\
&\leq k_1\Big(\|b_1\|_{\psi_1} + d_1\cdot r\Big)\frac{2\|k\|_{\psi_2}}{\Gamma(\alpha)}\left\|F_{h_2}(y)-F_{h_2}(z)\right\|_Q + d_1k_1\|y-z\|_\psi\frac{2\|k\|_{\psi_2}}{\Gamma(\alpha)}\left\|F_{h_2}(z)\right\|_Q \\
&\leq \frac{2k_1\|k\|_{\psi_2}}{\Gamma(\alpha)}\Big(\|b_1\|_{\psi_1} + d_1\cdot r\Big)\left\|d_2Q^{-1}\Big(\psi(|y-z|)\Big)\right\|_Q + \frac{2d_1k_1\|k\|_{\psi_2}}{\Gamma(\alpha)}\|y-z\|_\psi\left\|b_2 + d_2Q^{-1}\Big(\psi(|z|)\Big)\right\|_Q \\
&\leq \frac{2k_1d_2\|k\|_{\psi_2}}{\Gamma(\alpha)}\Big(\|b_1\|_{\psi_1} + d_1\cdot r\Big)\left\|y-z\right\|_\psi + \frac{2d_1k_1\|k\|_{\psi_2}}{\Gamma(\alpha)}\left\|y-z\right\|_\psi\Big(\|b_2\|_{\psi_2} + d_2\cdot r\Big) \\
&= \frac{2k_1\|k\|_{\psi_2}}{\Gamma(\alpha)}\bigg(d_2\Big(\|b_1\|_{\psi_1} + d_1\cdot r\Big) + d_1\Big(\|b_2\|_{\psi_2} + d_2\cdot r\Big)\bigg)\left\|y-z\right\|_\psi.
\end{aligned}$$

The above estimate with the assumption (C7) concludes the proof.

□

5. Conclusions

In the article, we prove and illustrate several novel features of the g-fractional type operator, encompassing boundedness, monotonicity, and continuity within Orlicz spaces L_ψ. The g-fractional operator combines and unifies many forms of fractional operators such as the Hadamard, Riemann-Liouville, and Erdlyi–Kober fractional operators into one form.

References

1. R.P. Agarwal, D. O'Regan, P. Wong, Solutions of a system of integral equations in Orlicz spaces. *J. Integral Equations Appl.* 2009, 21, 469–498.
2. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier Science B.V.: Amsterdam, The Netherlands, 2006; Volume 204.
3. Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*; John Wiley & Sons: New York, NY, USA, 1993.
4. Lakshmikantham, V.; Leela, S.; Vasundara Devi, J. *Theory of Fractional Dynamic Systems*; Cambridge Scientific Publishers: Cambridge, UK, 2009.
5. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives*; Gordon and Breach Science: Amsterdam, The Netherlands, 1993.
6. Srivastava, H.M.; Saxena, R.K. Operators of fractional integration and their applications. *Appl. Math. Comput.* **2001**, *118*, 1–52.
7. Ahmad, B.; Nieto, J.J. Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Leray-Schauder degree theory. *Topol. Methods Nonlinear Anal.* **2010**, *35*, 295–304.
8. Benchohra, M.; Henderson, J.; Ntouyas, S.K.; Ouahab, A. Existence results for fractional order functional differential equations with infinite delay. *J. Math. Anal. Appl.* **2008**, *338*, 1340–1350.
9. Zhang, S. Existence of positive solution for some class of nonlinear fractional differential equations. *J. Math. Anal. Appl.* **2003**, *278*, 136–148.
10. Hashem, H.H.; Zaki, M.S. Carthéodory theorem for quadratic integral equations of Erdélyi-Kober type. *J. Fract. Calc. Appl.* **2013**, *4*, 56–72.
11. Wang, J.; Dong, X.; Zhou, Y. Analysis of nonlinear integral equations with Erdélyi-Kober fractional operator. *Commun. Nonlinear Sci. Numer. Simul.* **2012**, *17*, 3129–3139.

12. Xin, J.; Zhu, c.; Wang J.; Chen, F. Nondecreasing solutions of fractional quadratic integral equations involving Erdélyi-Kober singular kernels. *Topol. Methods Nonlinear Anal.* **2014**, *44*, 73–88. https://doi.org/10.12775/TMNA.2014.036.
13. Hu, S.; Khavanin, M.; Zhuang, W. Integral equations arising in the kinetic theory of gases. *Appl. Anal.* **1989**, *34*, 261–266.
14. Kelly, C.T. Approximation of solutions of some quadratic integral equations in transport theory. *J. Integral Equ.* **1982**, *4*, 221–237.
15. H. Salem, M. Cichoń, W. Shammakh, Generalized fractional calculus in Banach spaces and applications to existence results for boundary value problems. *Bound. Value. Probl.* **2023**, *2023*, 57. https://doi.org/10.1186/s13661-023-01745-y
16. Chandrasekhar, S. *Radiative Transfer*; Dover Publications: New York, NY, USA, 1960.
17. Krasnosel'skii, M.A.; Rutitskii, Y. *Convex Functions and Orlicz Spaces*; Nordhoff, Gröningen, The Netherlands, 1961.
18. Majewski, W.A.; Labuschagne, L.E. On applications of Orlicz spaces to statistical physics. *Ann. Henri Poincaré* **2014**, *15*, 1197–1221. https://doi.org/10.1007/s00023-013-0267-3.
19. Cichoń, M.; Metwali, M. On solutions of quadratic integral equations in Orlicz spaces. *Mediterr. J. Math.* **2015**, *12*, 901–920.
20. Cichoń, M.; Metwali, M. Existence of monotonic L_ϕ-solutions for quadratic Volterra functional integral equations. *Electron. J. Qual. Theory Differ. Equ.* **2015**, *13*, 1–16.
21. Metwali, M. On perturbed quadratic integral equations and initial value problem with nonlocal conditions in Orlicz spaces. *Demonstratio Math.* **2020**, *53*, 86–94.
22. Metwali, M. On some properties of Riemann-Liouville fractional operator in Orlicz spaces and applications to quadratic integral equations. *Filomat* **2022**, *36*, 6009–6020.
23. Metwali, M. Solvability of quadratic Hadamard-type fractional integral equations in Orlicz spaces. *Rocky Mountain J. Math.* **2023**, *53*, 531–540. https://doi.org/10.1216/rmj.2023.53.531.
24. Maligranda, L. *Orlicz Spaces and Interpolation*; Departamento de Matemática, Universidade Estadual de Campinas: Campinas, Brazil, 1989.
25. Väth, M. *Volterra and Integral Equations of Vector Functions*; Marcel Dekker: New York, NY, USA; Basel, Switzerland, 2000.
26. Banaś, J.; Goebel, K. *Measures of Noncompactness in Banach Spaces*; Commentationes Mathematicae Universitatis Carolinae: Praha, Czech Republic, 1980; Volume 21, pp. 131–143.
27. Erzakova, N. Compactness in measure and measure of noncompactness. *Sib. Math. J.* **1997**, *38*, 926–928.
28. Sneddon, I.N. *Mixed Boundary Value Problems in Potential Theory*; North-Holand Publishing Co.: Amsterdam, The Netherlands, 1966.
29. K.G. Binmore, *Mathematical analysis : a straightforward approach*, Cambridge, England : Cambridge University Press, 1977.
30. Metwali, M.M.A.; Alsallami, S.A.M. On Erdlyi–Kober Fractional Operator and Quadratic Integral Equations in Orlicz Spaces. *Mathematics* **2023**, *11*, 3901. https://doi.org/10.3390/math11183901
31. E. Nakai, On generalized fractional integrals. *Taiwan. J. Math.* **2001**, *5*, 587–602.
32. R. O'Neil, Fractional integration in Orlicz spaces. *I. Trans. Am. Math. Soc.* **1965**, *115*, 300–328.