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[David Carfi](#) *

Posted Date: 9 April 2026

doi: 10.20944/preprints202604.0620.v1

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Article

Spin- $\frac{1}{2}$ Particle Fields in Maxwell–Schwartz Theory: Dirac–Schwartz Dynamics, Probability currents and Entanglement

David Carfi ^{1,2} 

¹ Section of Theoretical Physics, Department of Mathematics, Computer Sciences, Physical Sciences and Earth Sciences, University of Messina, 98166 Messina, Italy; dcarfi@unime.it

² Fractal Analysis, Dynamical Systems and Mathematical Physics Group, Department of Mathematics, University of California Riverside, Riverside, CA 92521, USA

Abstract

We develop a Maxwell–Schwartz field-theoretic formulation of spin- $\frac{1}{2}$ quantum systems within the framework of tempered distribution spaces and Schwartz linear algebra. In this approach, the fundamental objects of the theory are Maxwellian fields, namely elements of $S'(\mathbb{M}_4, \mathbb{C}^3)$, while scalar quantum states arise as derived projections carrying the spectral structure of the theory. Relativistic dynamics is formulated through square-root Hamiltonian operators defined via spectral calculus. On a distinguished subspace of Maxwellian fields, the curl operator realizes the momentum magnitude operator, allowing a direct correspondence between scalar and vectorial relativistic dynamics. This leads to a Schrödinger-type formulation of relativistic evolution without requiring Dirac's matrix factorization as a primitive assumption. Spin- $\frac{1}{2}$ structures emerge naturally from two-component Cartesian configurations of Maxwellian fields. The Pauli algebra is realized as an internal symmetry acting on such pairs, and spin observables correspond to explicit transformations of field components. A first-order Dirac-type dynamics is obtained through operator factorization, yielding a Maxwellian realization of relativistic two-component equations, including the massless Weyl regime. The tensorial structure of the theory provides a natural setting for entanglement. Bell-type correlations are derived intrinsically from Maxwellian field configurations, reproducing the standard quantum predictions and their violation of Bell inequalities. A central result of the present work is the energetic interpretation of quantum probability. The spectral coefficients of a state determine the amplitudes of its Maxwellian realization, whose energy density is quadratic in these amplitudes. The Born rule is thus recovered as a normalized energy distribution law. Moreover, the electromagnetic continuity equation induces a probability continuity equation, with the probability current identified as the normalized Poynting flux. This transport interpretation extends to entangled states, where Bell correlations are interpreted as correlations in the transport of energy across field channels. Finally, the framework is extended to massive Maxwellian fields, where probability currents are expressed through nonlocal spectral kernels associated with square-root Hamiltonians, yielding a unified continuity structure for both massless and massive regimes. These results suggest that spinorial quantum structures, probabilistic interpretation, and quantum correlations may be understood as emerging from the internal organization, spectral structure, and energy transport of Maxwellian fields in the Schwartz distributional setting.

Keywords: Maxwell equations; tempered distributions; Schwartz linear algebra; spin- $\frac{1}{2}$; Pauli operators; Dirac equation; Weyl fermions; relativistic Schrödinger equation; operator square roots; entanglement; Bell inequality

MSC: 46F12; 46F10; 47A60; 47B25; 81Q10; 35Q61; 81Q80

1. Introduction

The relationship between quantum mechanics and classical field theory has been a central theme in theoretical physics since the early development of quantum theory. In particular, the interpretation of quantum states and the role of electromagnetic fields have motivated a wide range of conceptual and mathematical investigations.

In a series of recent works, we have proposed a framework in which quantum states are modeled within the space of tempered distributions, and classical Maxwell fields are interpreted as concrete realizations of such states. This approach is based on Schwartz linear algebra and allows one to treat superpositions, generalized eigenstates, and operator actions in a unified functional–analytic setting.

In this perspective, the fundamental objects of the theory are not scalar wave functions, but vector-valued distributions, called *Maxwellian fields*, belonging to the space

$$\mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3).$$

Scalar quantum states arise as projections of these fields and play a fundamental, though derived, role due to the well-established structure of Schwartz linear algebra in the scalar setting.

The aim of the present paper is to extend this Maxwell–Schwartz framework to the description of spin- $\frac{1}{2}$ particle fields.

In standard quantum mechanics, spin- $\frac{1}{2}$ systems are described by two-component wave functions and Pauli operators, while relativistic dynamics is encoded in the Dirac equation. In contrast, we show that within the Maxwell–Schwartz setting, spin structures naturally emerge from two-component configurations of Maxwellian fields. The Pauli algebra is realized as an internal symmetry acting on pairs of fields, and spin observables correspond to concrete transformations of such configurations.

A central role in our construction is played by operator square roots. Relativistic dynamics is formulated through Hamiltonian operators of the form

$$\hat{H}_{m_0} = c \sqrt{|\hat{P}|^2 + m_0^2 c^2} \mathbb{I}_{\mathcal{V}},$$

defined on spaces of tempered distributions $\mathcal{V}$ via spectral calculus. This approach allows one to express relativistic quantum evolution directly in Schrödinger form, avoiding the need for Dirac’s matrix factorization as a primary step.

In the vectorial setting, the curl operator provides a natural counterpart of the momentum magnitude operator on distinguished subspaces $\mathcal{W}_e$ of Maxwellian fields. This correspondence enables the construction of a Maxwell’s Hamiltonian and the formulation of relativistic evolution equations directly on the space of fields.

The theory developed here accommodates both massive and massless cases. In the massless regime, the formalism naturally leads to decoupled components resembling Weyl dynamics, while in the massive case, the interaction between components gives rise to Dirac-like structures. These structures are obtained through operator-theoretic constructions rather than imposed algebraically.

A further important aspect of the theory is its tensorial structure. The space of two-component Maxwellian fields admits natural tensor products, which provide a framework for the description of entangled states. In this setting, Bell-type correlations arise from the intrinsic structure of the fields and their associated probability measures.

1.1. Structure of the work

The paper is organized as follows.

Section 2 provides a brief literature review, situating the present work within the development of quantum mechanics, distribution theory, spectral methods, and field-theoretic approaches.

Section 3 introduces the functional–analytic framework. We define the scalar and vectorial spaces of tempered distributions, construct the de Broglie family as a Schwartz basis, and develop the associated spectral calculus, including the definition of momentum and Hamiltonian operators.

Section 4 develops the projection of scalar states into Maxwellian fields. We introduce the realization mapping F_e , establish the transport of spectral coordinates, and formulate the energetic interpretation of probability, relating the Born rule to the energy distribution of Maxwellian fields.

Section 5 introduces two-component configurations of Maxwellian fields and establishes the emergence of spin- $\frac{1}{2}$ structures. The Pauli algebra is realized as an internal symmetry acting on such configurations.

Section 6 formulates relativistic dynamics through square-root Hamiltonian operators in both scalar and vectorial settings, and analyzes the role of helicity and chirality in the Maxwell–Schwartz framework.

Section 7 is devoted to the massless regime, where the theory naturally leads to Weyl-type dynamics and decoupled chiral components.

Section 8 develops the tensorial structure of the theory and provides a description of entangled states. Bell-type correlations are derived from the intrinsic structure of Maxwellian fields.

Section 9 establishes the energy interpretation of probability and the associated continuity equations. In particular, the probability current is identified with the normalized Poynting flux, and its extension to massive fields is formulated through nonlocal spectral kernels.

Section 10 discusses the relation with the standard Dirac theory, showing how Dirac-type dynamics arise within the present framework and comparing the two approaches.

The overall construction shows that scalar quantum states, spin structures, relativistic dynamics, and quantum correlations can be understood as emerging from the spectral organization, internal symmetries, and energy transport properties of Maxwellian fields in the setting of Schwartz linear algebra.

2. Literature Review

The relationship between quantum mechanics and classical field theory has been a central theme in theoretical physics since the early twentieth century. From the introduction of wave mechanics by Schrödinger [1] and the operator formalism of Dirac [2], to the development of relativistic quantum equations such as the Klein–Gordon and Dirac equations [3], the interplay between fields, operators, and probability has remained fundamental.

The mathematical formulation of quantum mechanics was rigorously established within Hilbert space theory by von Neumann [4]. However, it soon became clear that generalized eigenstates and continuous spectra require an extension beyond Hilbert spaces. This led to the introduction of distribution theory by Schwartz [5], and later to the development of rigged Hilbert spaces (Gel'fand triples) [6,7], providing a natural framework for generalized quantum states.

Parallel to these developments, harmonic analysis and Fourier transform methods have played a central role in both classical and quantum physics. The spectral decomposition of operators and the use of Fourier multipliers are fundamental tools in the study of linear partial differential equations and quantum dynamics [8,9].

Electromagnetic field theory, formulated by Maxwell, provides a classical field model governed by local differential equations. The complex representation of electromagnetic fields, such as the Riemann–Silberstein vector, has long been used to express Maxwell's equations in a form analogous to Schrödinger-type evolution [10]. This connection has motivated numerous attempts to interpret photons and electromagnetic waves within a quantum-mechanical framework.

The emergence of spin and relativistic quantum mechanics introduced further algebraic structures, notably the Pauli matrices and the Dirac operator. The factorization of relativistic Hamiltonians and the algebraic structure of spinors are central themes in modern theoretical physics [11,12].

Entanglement and Bell inequalities [13,14] have revealed profound nonclassical features of quantum theory. The standard interpretation relies on probabilistic amplitudes and measurement postulates, while alternative approaches seek to relate these phenomena to underlying field-theoretic or geometric structures.

In recent years, there has been renewed interest in formulating quantum theory in terms of generalized function spaces, operator algebras, and geometric structures. In particular, approaches based on tempered distributions and spectral methods allow one to treat superposition, generalized eigenstates, and operator actions in a unified manner.

The present work builds upon these developments by combining:

- Schwartz distribution theory and its algebraic extensions,
- spectral operator calculus,
- Maxwellian field representations,
- and relativistic quantum dynamics.

The goal is to provide a unified framework in which scalar quantum states, spin structures, relativistic equations, and quantum correlations arise naturally from the spectral and geometric organization of Maxwellian fields.

In this perspective, the Born rule, probability currents, and Bell correlations are interpreted in terms of energy distributions and transport properties of fields, rather than as independent postulates. This approach aims to bridge the gap between wave mechanics, field theory, and functional analysis within a coherent mathematical structure.

3. Theoretical Background I: Functional–Analytic Framework

3.1. The Scalar and Vectorial Distribution Spaces

Let $\mathbb{M}_4$ denote Minkowski space-time and let $\mathbb{M}_4^*$ denote its dual. We consider the scalar tempered distribution space

$$\mathcal{V} := \mathcal{S}'(\mathbb{M}_4, \mathbb{C})$$

and the vectorial tempered distribution space

$$\mathcal{W} := \mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3).$$

Elements of $\mathcal{V}$ will be called *scalar quantum states*, while elements of $\mathcal{W}$ will be called *Maxwellian fields*. We should note that - in our approach - the elements of those spaces are considered together with their complete definition and load of information and we could even associate them physical dimensions, contrary to the purely probabilistic approach, that works essentially in the unit sphere of L^2 .

3.1.1. Terminological Convention

In the present paper, the term *electromagnetic field* will be reserved for Maxwellian fields satisfying the appropriate Maxwell equations and transversality conditions in the physical sense. By contrast, the term *Maxwellian field* will denote an arbitrary element of $\mathcal{W}$, whether or not it represents a physical electromagnetic field.

Thus $\mathcal{W}$ is a linear space strictly larger than the set of physical electromagnetic fields.

3.2. Primary and Derived Objects

The ontological point of view adopted in this work is that Maxwellian fields are the primary objects of the theory.

Scalar quantum states arise as derived objects, namely as projections of suitable Maxwellian fields. Nevertheless, scalar states play a fundamental role, since the theory of Schwartz linear algebra is formulated in the scalar setting and provides the natural language for superposition, coordinates, and spectral constructions.

Accordingly, although Maxwellian fields are primary from the ontological viewpoint, scalar states remain indispensable from the algebraic and operator-theoretic viewpoint.

3.3. The de Broglie Family

We consider the de Broglie family

$$\eta : \mathbb{M}_4^* \longrightarrow \mathcal{S}'(\mathbb{M}_4),$$

defined by

$$\eta_p := [e^{i\langle p, x \rangle / \hbar}],$$

for every $p \in \mathbb{M}_4^*$. The above distributions are those generated, via Schwartz pairing, by the classic de Broglie wave-functions.

Here $\langle p, x \rangle$ denotes the Minkowski pairing between p and the identity function x . So that

$$\langle p, x \rangle = (\vec{p} | \vec{x}) - p^0 x^0,$$

for every 4-momentum p .

The family η is not Dirac orthonormal in its present form, but it is Dirac orthogonal. Moreover, it is unimodular:

$$|\eta_p(y)| = 1$$

for every $p \in \mathbb{M}_4^*$ and every $y \in \mathbb{M}_4$, where the value of the regular distribution η_p , at any point y , is well defined since η_p comes from a smooth function.

This unimodularity is important in the present theory, because it implies that amplitudes are entirely carried by the coordinate coefficients and not by the basis itself.

Normalized versions of the de Broglie family will be considered only when needed. For the purposes of the present section, the non-normalized unimodular family η is more natural and more convenient, since the spectral constructions developed below do not depend on normalization constants.

3.4. The Operator Associated with the de Broglie Family

The family η induces a linear continuous operator between test function spaces

$$\hat{\eta} : \mathcal{S}(\mathbb{M}_4) \longrightarrow \mathcal{S}(\mathbb{M}_4^*) : \phi \mapsto \eta(\phi); \eta(\phi)(p) = \eta_p(\phi).$$

Its topological transpose defines the η -superposition operator

$$\mathcal{F}_\eta^{-1} := {}^t\hat{\eta} : \mathcal{S}'(\mathbb{M}_4^*) \longrightarrow \mathcal{S}'(\mathbb{M}_4) : a \rightarrow \int_{\mathbb{M}_4^*} a\eta.$$

The linear continuous and topologically invertible operator

$$\mathcal{F}_\eta^{-1} : \mathcal{S}'(\mathbb{M}_4^*) \longrightarrow \mathcal{S}'(\mathbb{M}_4)$$

is the superposition operator associated with the family η , and it is a Fourier transform.

On the other hand, its inverse operator $\mathcal{F}_\eta$ associates with each scalar state ψ its coordinate system with respect to the family η . Thus, for every

$$\psi \in \mathcal{S}'(\mathbb{M}_4),$$

we define its coordinates with respect to η by

$$(\psi)_\eta := \mathcal{F}_\eta(\psi).$$

Accordingly, every scalar state ψ admits the representation

$$\psi = \mathcal{F}_\eta^{-1}((\psi)_\eta) = \int_{\mathbb{M}_4^*} (\psi)_\eta \eta.$$

This formula expresses the fact that η is a Schwartz basis of the corresponding state space in the sense of Schwartz linear algebra.

Remark 1. The notation $(\psi)_\beta$ denotes a coordinate system in the sense of Schwartz linear algebra. It is not, in general, a Dirac inner product coefficient system such as $(\psi)_\epsilon = \langle \epsilon | \psi \rangle$ or $\psi = \langle \delta | \psi \rangle$, where ϵ is the normalized de Broglie basis $N^{-1}\eta$ and δ is the Dirac basis of Dirac deltas δ_y . The latter expressions are attained only by Dirac orthonormal basis, while the Schwartz linear framework of coordinate representation is more general and does not rely necessarily on Dirac orthonormality. In the present case, however, η is still Dirac orthogonal, so that, we can easily see that

$$(\psi)_\eta = N^{-2} \langle \eta | \psi \rangle.$$

3.5. Spectral Extension of Multipliers

Let

$$g : \mathbb{M}_4^* \longrightarrow \mathbb{C}$$

be a function satisfying the following conditions:

- g is smooth;
- g is real-valued whenever one wishes to represent real physical observables (for instance, position coordinate operators, momentum coordinate operators, Hamiltonian operators, etc...);
- $g \in \mathcal{O}_M(\mathbb{M}_4^*)$, the space of multipliers of tempered distributions.

Then multiplication by g defines a continuous operator M_g on $\mathcal{S}'(\mathbb{M}_4^*)$.

We define the spectral extension of g to an operator

$$\widehat{g}_\eta : \mathcal{S}'(\mathbb{M}_4) \longrightarrow \mathcal{S}'(\mathbb{M}_4)$$

by

$$\widehat{g}_\eta(\psi) := \int_{\mathbb{M}_4^*} g \cdot (\psi)_\eta \eta.$$

Equivalently,

$$\widehat{g}_\eta = \mathcal{F}^{-1} \circ M_g \circ \mathcal{F},$$

where

$$M_g : \mathcal{S}'(\mathbb{M}_4^*) \longrightarrow \mathcal{S}'(\mathbb{M}_4^*) : a \mapsto g \cdot a$$

denotes multiplication by g .

3.5.1. Interpretation

The operator $\widehat{g}_\eta$ is constructed in three steps:

1. one passes from ψ to its tempered coordinate system $(\psi)_\eta$;
2. one multiplies the coordinate system by the symbol (smooth slowly increasing function) g ;
3. one reconstructs the resulting state through the superposition operator $\mathcal{F}^{-1}$.

This is the basic spectral mechanism of the theory and it can be extended to any Schwartz basis β .

We should observe that the product $g.a$ is the standard Schwartz product of the multiplier g times the tempered distribution a . On the other hand, the product $a\eta$ has a completely different nature, the author defined it in a paper of 2014: it is the spectral product defined by

$$a\eta : \mathcal{O}_M(\mathbb{M}_4^*) \rightarrow \mathcal{S}'(\mathbb{M}_4) : g \mapsto a\eta(g) = \int_{\mathbb{M}_4^*} (ga) \eta,$$

it's the spectral distribution of the pair (a, η) , analogous of some spectral measures of Hilbert spaces and of some vector valued measures of Bourbaki. The spectral product $a\eta$ is a generalized vector-valued measure, it literally integrates the multipliers g , vectorially and in the sense of distributions, in order to obtain a tempered distribution (a state of the scalar quantum field).

3.6. The 4-Momentum Operator

Let

$$p : \mathbb{M}_4^* \longrightarrow \mathbb{M}_4^*$$

denote the identity function on 4-momentum space. The corresponding 4-momentum operator is the spectral extension

$$\hat{P} : \mathcal{S}'(\mathbb{M}_4) \longrightarrow \mathcal{S}'(\mathbb{M}_4)^4,$$

defined by

$$\hat{P}(\psi) := \int_{\mathbb{M}_4^*} p(\psi) \eta.$$

Thus, the 4-momentum operator is obtained by multiplying the coordinate contravariant system of ψ , with respect to η , by the corresponding 4-momentum label and then superposing again with respect to the de Broglie basis η . Its components are

$$\hat{P} = \left(\frac{\hat{E}}{c}, \hat{P}_1, \hat{P}_2, \hat{P}_3 \right).$$

The spatial momentum operator will be denoted by $\vec{P}$, while the spatial momentum magnitude is denoted $|\vec{P}|$.

When needed, one may distinguish the temporal and spatial parts of p . In particular, if $\vec{p}$ denotes the spatial projection of p , then operators depending only on $\vec{p}$ are treated as particular cases of the general spectral scheme above.

3.7. Examples of Hamiltonian Functions

We now present some relevant examples of multiplier functions

$$g : \mathbb{M}_4^* \longrightarrow \mathbb{C},$$

to which the previous spectral construction applies; we are working mainly with the basis η .

1. Classical kinetic energy. For $m_0 > 0$, the classical kinetic energy, expressed by momentum, is represented by

$$T_{m_0}(p) = \frac{|\vec{p}|^2}{2m_0}.$$

2. Relativistic kinetic energy by momentum. The relativistic kinetic energy is represented by

$$T_{m_0}(p) = c\sqrt{|\vec{p}|^2 + m_0^2 c^2} - m_0 c^2.$$

3. Relativistic Hamiltonian. The full relativistic Hamiltonian function is

$$H_{m_0}(p) = c\sqrt{|\vec{p}|^2 + m_0^2 c^2}.$$

4. Quadratic kinetic term of the harmonic oscillator. The momentum part of the harmonic oscillator Hamiltonian is again

$$T_{m_0}(p) = \frac{|\vec{p}|^2}{2m_0}.$$

Remark 2. *The full harmonic oscillator Hamiltonian also involves the position operator. Therefore the present spectral construction applies immediately to its kinetic part, while the position contribution must follow a separate analogous spectral extension based on Dirac basis.*

3.8. The Scalar Relativistic Hamiltonian

A central operator of the present theory is the scalar relativistic Hamiltonian

$$\hat{H}_{m_0} : \mathcal{S}'(\mathbb{M}_4) \longrightarrow \mathcal{S}'(\mathbb{M}_4),$$

defined as the spectral extension of the relativistic Hamiltonian function H_{m_0} , with respect to the basis η :

$$\hat{H}_{m_0}(\psi) := \int_{\mathbb{M}_4^*} H_{m_0}(\psi)_\eta \eta.$$

This operator is the relativistic square-root Hamiltonian acting in the scalar distributional setting. It can be expressed, by the covariant coordinate operator of η in the more Dirac friendly form

$$\hat{H}_{m_0}(\psi) := N^{-2} \int_{\mathbb{M}_4^*} |\eta\rangle \langle \eta | \psi \rangle H_{m_0},$$

or in purely operatorial form:

$$\hat{H}_{m_0} := N^{-2} \int_{\mathbb{M}_4^*} |\eta\rangle \langle \eta | H_{m_0},$$

where the total orthogonal projector $|\eta\rangle \langle \eta |$ is defined by the spectral product of the de Broglie basis $|\eta\rangle$ times the covariant coordinate operator $\langle \eta |$.

Remark 3 (on normalization). *The above definition does not depend on the normalization constant of the basis. Indeed, if one rescales the family η , then the contravariant coordinate system $(\psi)_\eta$ rescales inversely, and the spectral product*

$$(\psi)_{b\eta} b\eta = (\psi)_\eta \eta$$

remains invariant, for every non-zero b belonging to $\mathbb{C}$. Therefore spectral operator definitions are intrinsic and do not rely on any orthonormality assumption.

3.9. The Maxwellian Field Basis and the Vectorial Realization

We now pass to the vectorial side. Let e be a unit vector of the 2-sphere, viewed as the unit sphere of the spatial part of the 4-momentum space. We canonically associate with any such e a complex frame

$$f^e : \mathbb{M}_4^* \longrightarrow \mathbb{C}^3,$$

called the Killing right-handed orthonormal (complex) transverse polarization vector field, and define the associated Maxwellian family (see [22])

$$w^e : \mathbb{M}_4^* \setminus \Pi_e \longrightarrow \mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3)$$

by

$$w_p^e := \eta_p f_p^e.$$

The frame family $f = (f^e)_{e \in \mathbb{S}^2}$ provides the basic vectorial realizations of the scalar de Broglie family η .

If A is a positive real amplitude, then the unimodularity of η_p implies

$$|A\eta_p| = A.$$

Moreover, with the effective normalization of the transverse vector f_p , one also has

$$|A\eta_p f_p| = A.$$

Furthermore, if ψ is a regular Schwartz distribution, we have

$$|A(\psi)_\eta(p)\eta_p f_p| = A|(\psi)_\eta(p)|.$$

This property is important for the energetic interpretation of Maxwellian fields, since it shows that the amplitude of the field is controlled directly by the scalar coefficient and is not altered by the oscillatory basis factor.

3.10. The Maxwellian Space and the Curl Operator

Recall that

$$\mathcal{W} = \mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3).$$

We define the curl operator

$$\text{rot} : \mathcal{W} \longrightarrow \mathcal{W}$$

by

$$\text{rot}(F) := \nabla \times F,$$

where the right-hand side is understood in the sense of vector-valued tempered distributions.

This operator is linear, continuous and everywhere defined on $\mathcal{W}$.

3.11. The Maxwellian Hamiltonian

We denote by

$$\hat{\mathcal{H}}_{m_0} : \mathcal{W} \longrightarrow \mathcal{W}$$

the Maxwellian Hamiltonian operator. Its role is to provide the vectorial counterpart of the scalar operator $\hat{H}_{m_0}$.

The defining principle is that $\hat{\mathcal{H}}_{m_0}$ is obtained from the curl operator in the same spirit in which $\hat{H}_{m_0}$ is obtained from the momentum magnitude operator.

Formally, one writes

$$\hat{\mathcal{H}}_{m_0} = c\sqrt{(\hbar\text{rot})^2 + m_0^2 c^2} \mathbb{I}_{\mathcal{W}},$$

where

$$\mathbb{I}_{\mathcal{W}} : \mathcal{W} \longrightarrow \mathcal{W}$$

denotes the identity operator on $\mathcal{W}$.

This formula must be understood spectrally on suitable invariant subspaces of $\mathcal{W}$ (see [22]).

3.12. Relation Between the Scalar and Vectorial Hamiltonians

There exist proper subspaces

$$\mathcal{W}_e \subset \mathcal{W}$$

such that on any $\mathcal{W}_e$ one has the fundamental identity

$$\hbar \text{rot} = |\vec{P}|.$$

Thus, on any $\mathcal{W}_e$, the curl operator, times the reduced Planck's constant, plays the role of the momentum magnitude operator.

Consequently, on those subspaces, the Maxwellian Hamiltonian is the natural vectorial counterpart of the scalar relativistic Hamiltonian:

$$\hat{\mathcal{H}}_{m_0} = c\sqrt{(\hbar \text{rot})^2 + m_0^2 c^2 \mathbb{I}_{\mathcal{W}_e}} \quad \leftrightarrow \quad \hat{H}_{m_0} = c\sqrt{|\vec{P}|^2 + m_0^2 c^2 \mathbb{I}_{\mathcal{V}}}.$$

Outside $\mathcal{W}_e$, the operator rot retains its full differential character and does not reduce to the momentum magnitude operator on $\mathcal{W}$.

3.13. Relativistic Evolution Equations

The scalar relativistic Schrödinger equation is

$$i\hbar \partial_t \psi = \hat{H}_{m_0} \psi.$$

This equation is naturally accompanied by its conjugate equation

$$-i\hbar \partial_t \bar{\psi} = \hat{H}_{m_0} \bar{\psi}.$$

On the vectorial side, the corresponding Maxwellian evolution equations are

$$i\hbar \partial_t F = \hat{\mathcal{H}}_{m_0} F$$

and

$$-i\hbar \partial_t \bar{F} = \hat{\mathcal{H}}_{m_0} \bar{F}.$$

These equations provide the dynamical basis of the present theory.

Remark 4. *The scalar and vectorial formulations are not competing descriptions. The scalar theory provides the algebraic and spectral language of Schwartz linear algebra, while the vectorial theory provides the Maxwellian ontological realization. The correspondence between the two is one of the central structural features of the Maxwell–Schwartz framework.*

4. Theoretical Background II: Projection, Probability, and Energy Interpretation

4.1. Scalar States and Maxwellian Realizations

Let

$$\mathcal{V} = \mathcal{S}'(\mathbb{M}_4, \mathbb{C}), \quad \mathcal{W} = \mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3).$$

As discussed in the previous section, scalar quantum states are represented by elements of $\mathcal{V}$, while Maxwellian fields are elements of $\mathcal{W}$.

In the present framework, Maxwellian fields are regarded as primary objects. Scalar states arise as derived objects through projection procedures.

More precisely, for each admissible polarization frame f^e , one considers a linear mapping (embedding)

$$\mathcal{F}_e : \mathcal{V}_e \longrightarrow \mathcal{W}_e,$$

defined on a suitable subspace $\mathcal{V}_e \subset \mathcal{V}$.

For $\psi \in \mathcal{V}_e$, the field

$$F = \mathcal{F}_e(\psi)$$

is called the *Maxwellian realization* of ψ with respect to f^e . On the other hand, if we consider a field F belonging to $\mathcal{W}_e$, the inverse mapping of the above embedding will project F to its scalar representation.

4.2. Construction via the de Broglie Family

Let η be the de Broglie family introduced in Section 3 and let

$$(\psi)_\eta$$

be the contravariant coordinate system of ψ with respect to η .

Let

$$w^e : \mathbb{M}_4^* \setminus \Pi_e \rightarrow \mathcal{W} : p \mapsto w_p^e = \eta_p f_p^e$$

be the associated Maxwellian basis, where $f_p^e \in \mathbb{C}^3$ is a transverse polarization vector.

We define the Maxwellian realization of ψ by the superposition formula

$$\mathcal{F}_e(\psi) = \int_{\mathbb{M}_4^*} (\psi)_\eta w^e.$$

Thus the scalar coordinates $(\psi)_\eta$ are transported directly to the Maxwellian field through the same spectral coefficients.

4.3. Coordinate Transport and Structural Invariance

The mapping $\mathcal{F}_e$ preserves the coordinate system in the following sense:

if

$$\psi = \int_{\mathbb{M}_4^*} (\psi)_\eta \eta,$$

then

$$\mathcal{F}_e(\psi) = \int_{\mathbb{M}_4^*} (\psi)_\eta w^e.$$

Thus the passage from scalar states to Maxwellian fields consists in replacing the scalar basis η with the vectorial basis w^e , while keeping the same coordinate system.

4.3.1. Interpretation

The coefficients $(\psi)_\eta$ represent intrinsic spectral data, independent of the chosen realization (scalar or vectorial).

4.4. Energy Density of Maxwellian Fields

Let

$$F = \mathcal{F}_e(\psi).$$

We define the energy density associated with F by

$$\mathcal{E}(F) = \frac{\epsilon_0}{2} (|\vec{E}|^2 + |c\vec{B}|^2),$$

where $\vec{E}$ and $\vec{B}$ denote the electric and magnetic components of F .

The unimodularity of the de Broglie family implies that the amplitude of the field is entirely determined by the coefficients $(\psi)_\eta$.

Indeed, for each mode one has

$$|A\eta_p f_p| = A.$$

4.5. Energy–Probability Correspondence

The scalar state ψ admits a natural probabilistic interpretation through its spectral decomposition.

The coordinate system

$$(\psi)_\eta$$

determines a distribution of amplitudes over momentum space.

The total energy of the associated Maxwellian field is obtained by integrating the energy density, which depends quadratically on the amplitude.

This leads to a correspondence between:

- the spectral coefficients $(\psi)_\eta$,
- the energy distribution of the field F ,
- the probabilistic interpretation of the state ψ .

4.5.1. Principle

The Born rule can be interpreted as an energy distribution law for Maxwellian fields, as we will see analytically in the following sections.

4.6. Independence from Normalization

The above construction does not depend on the normalization of the basis η .

Indeed, if η is rescaled, the coefficients $(\psi)_\eta$ are rescaled inversely, and the product

$$(\psi)_\eta \eta$$

remains invariant.

Consequently:

- spectral operator definitions are intrinsic,
- Maxwellian realizations are invariant,
- energy densities are unaffected.

4.7. Scalar and Maxwellian Dual Descriptions

The scalar and Maxwellian descriptions are complementary:

- the scalar formulation provides the algebraic and spectral structure,
- the Maxwellian formulation provides a concrete field realization.

Thus a quantum state is simultaneously:

- a scalar distribution ψ with coordinates $(\psi)_\eta$,
- a Maxwellian field $F = \mathcal{F}_e(\psi)$ carrying energy.

This dual description is a central feature of the Maxwell–Schwartz framework.

4.8. Outlook

The construction developed in this section provides the basis for the introduction of multi-component structures and spin in the subsequent sections.

In particular, tensor products of Maxwellian fields will allow the description of entangled states, while two-component constructions will lead naturally to spin- $\frac{1}{2}$ systems.

5. Results I: Two-Component Maxwellian Structures and Pauli Algebra

5.1. The Cartesian Two-Component Maxwellian Space

A central role in the present theory is played by the Cartesian product space

$$\mathcal{W}^2 := \mathcal{W} \times \mathcal{W} = \mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3) \times \mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3).$$

An element of $\mathcal{W}^2$ is a pair

$$F = (F_1, F_2),$$

where

$$F_1, F_2 \in \mathcal{W}.$$

This space is naturally a complex vector space, with operations defined componentwise:

$$(F_1, F_2) + (G_1, G_2) = (F_1 + G_1, F_2 + G_2),$$

$$\lambda(F_1, F_2) = (\lambda F_1, \lambda F_2), \quad \lambda \in \mathbb{C}.$$

The space $\mathcal{W}^2$ is the simplest Cartesian enlargement of the Maxwellian field space. It will serve as the basic ambient space for two-component structures.

5.2. Interpretation of the Two Components

At this stage, the two components of an element

$$F = (F_1, F_2) \in \mathcal{W}^2$$

can be considered also purely structural. No physical meaning could be imposed a priori.

The same two-component structure may later be interpreted in different ways:

- as a polarization doublet;
- as a spin- $\frac{1}{2}$ doublet;
- more generally, as an internal two-state degree of freedom carried by a Maxwellian field.

Thus the Cartesian construction is more primitive than any particular physical interpretation.

In this sense, $\mathcal{W}^2$ is not only introduced to model spin or polarization directly, but also to provide the algebraic framework in which such structures may arise.

5.3. Canonical Basis of the Internal Component Space

Let

$$e_{\uparrow} := (1, 0), \quad e_{\downarrow} := (0, 1)$$

denote the canonical basis of $\mathbb{C}^2$.

Then every element of $\mathcal{W}^2$ may be written formally as

$$F = F_1 e_{\uparrow} + F_2 e_{\downarrow}.$$

This notation emphasizes that $\mathcal{W}^2$ may be viewed as a tensor-type two-component Maxwellian space:

$$\mathcal{W}^2 \cong \mathcal{W} \otimes \mathbb{C}^2,$$

at least at the level of linear structure.

We shall often use the notation

$$F = F_1 |\uparrow\rangle + F_2 |\downarrow\rangle,$$

when the spin interpretation is intended, and

$$F = F_1 |H\rangle + F_2 |V\rangle,$$

when the polarization interpretation is intended.

At the purely algebraic level, both notations describe the same Cartesian structure.

5.4. Two-Component Spaces Generated by a Maxwellian Mode

Let

$$F \in \mathcal{W}$$

be a nonzero Maxwellian field.

We define the corresponding two-dimensional subspace

$$\mathcal{P}_F := \text{Span}_{\mathbb{C}}\{(F, 0), (0, F)\} \subset \mathcal{W}^2.$$

This is a complex two-dimensional subspace of $\mathcal{W}^2$.

Indeed, if

$$a(F, 0) + b(0, F) = 0,$$

then

$$(aF, bF) = (0, 0),$$

hence

$$aF = 0, \quad bF = 0.$$

Since $F \neq 0$, it follows that

$$a = b = 0.$$

Thus the family

$$(F, 0), \quad (0, F)$$

is linearly independent.

This simple observation is fundamental:

every nonzero Maxwellian field generates canonically a two-state Cartesian structure.

5.5. Spin Basis Associated with a Maxwellian Mode

Fix a nonzero Maxwellian field

$$F \in \mathcal{W}.$$

We define

$$\Psi_{\uparrow} := (F, 0), \quad \Psi_{\downarrow} := (0, F).$$

The corresponding two-dimensional subspace is

$$\mathcal{P}_{\text{spin}}(F) := \text{Span}_{\mathbb{C}}\{\Psi_{\uparrow}, \Psi_{\downarrow}\}.$$

Every element of $\mathcal{P}_{\text{spin}}(F)$ has the form

$$\Psi = a\Psi_{\uparrow} + b\Psi_{\downarrow} = (aF, bF), \quad a, b \in \mathbb{C}.$$

Thus the map

$$\Phi_F : \mathcal{P}_{\text{spin}}(F) \rightarrow \mathbb{C}^2,$$

defined by

$$\Phi_F(\Psi_{\uparrow}) = (1, 0), \quad \Phi_F(\Psi_{\downarrow}) = (0, 1),$$

and extended $\mathbb{C}$ -linearly, is an isomorphism.

Therefore, every nonzero Maxwellian field gives rise to a canonical copy of $\mathbb{C}^2$ inside $\mathcal{W}^2$.

5.6. Emergence of the Pauli Algebra

The preceding construction allows us to transport the standard Pauli algebra from $\mathbb{C}^2$ to the Maxwellian two-component space.

Let

$$\mathcal{P}_{\text{spin}}(F) = \text{Span}_{\mathbb{C}}\{\Psi_{\uparrow}, \Psi_{\downarrow}\}.$$

We define the three Pauli operators as linear maps

$$\sigma_x, \sigma_y, \sigma_z : \mathcal{P}_{\text{spin}}(F) \rightarrow \mathcal{P}_{\text{spin}}(F)$$

by their action on the basis vectors:

$$\begin{aligned}\sigma_x(\Psi_\uparrow) &= \Psi_\downarrow, & \sigma_x(\Psi_\downarrow) &= \Psi_\uparrow, \\ \sigma_y(\Psi_\uparrow) &= -i\Psi_\downarrow, & \sigma_y(\Psi_\downarrow) &= i\Psi_\uparrow, \\ \sigma_z(\Psi_\uparrow) &= \Psi_\uparrow, & \sigma_z(\Psi_\downarrow) &= -\Psi_\downarrow.\end{aligned}$$

These formulas define uniquely three linear operators on all of $\mathcal{P}_{\text{spin}}(F)$.

5.7. Componentwise Expressions of the Pauli Operators

Let

$$\Psi = (aF, bF) \in \mathcal{P}_{\text{spin}}(F).$$

By linearity, one obtains:

$$\begin{aligned}\sigma_x(\Psi) &= (bF, aF), \\ \sigma_y(\Psi) &= (-ibF, iaF), \\ \sigma_z(\Psi) &= (aF, -bF).\end{aligned}$$

Thus, if one writes more generally

$$\Psi = (F_1, F_2) \in \mathcal{W}^2,$$

the same formulas become

$$\begin{aligned}\sigma_x(F_1, F_2) &= (F_2, F_1), \\ \sigma_y(F_1, F_2) &= (-iF_2, iF_1), \\ \sigma_z(F_1, F_2) &= (F_1, -F_2),\end{aligned}$$

whenever the two-component structure is interpreted through the canonical basis.

These operators act only on the internal component index and do not modify the Maxwellian nature of each field component.

5.8. The Pauli Algebra Relations

We now verify that the above operators satisfy the standard Pauli algebra.

Proposition 1. *On the two-component Maxwellian space, the operators*

$$\sigma_x, \sigma_y, \sigma_z$$

satisfy:

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{I},$$

and

$$[\sigma_x, \sigma_y] = 2i\sigma_z, \quad [\sigma_y, \sigma_z] = 2i\sigma_x, \quad [\sigma_z, \sigma_x] = 2i\sigma_y.$$

Proof. Let

$$\Psi = (F_1, F_2) \in \mathcal{W}^2.$$

First,

$$\sigma_x^2(F_1, F_2) = \sigma_x(F_2, F_1) = (F_1, F_2),$$

hence

$$\sigma_x^2 = \mathbb{I}.$$

Similarly,

$$\sigma_y^2(F_1, F_2) = \sigma_y(-iF_2, iF_1) = (-i(iF_1), i(-iF_2)) = (F_1, F_2),$$

so

$$\sigma_y^2 = \mathbb{I}.$$

Finally,

$$\sigma_z^2(F_1, F_2) = \sigma_z(F_1, -F_2) = (F_1, F_2),$$

thus

$$\sigma_z^2 = \mathbb{I}.$$

Now compute

$$\sigma_x \sigma_y(F_1, F_2) = \sigma_x(-iF_2, iF_1) = (iF_1, -iF_2),$$

while

$$\sigma_y \sigma_x(F_1, F_2) = \sigma_y(F_2, F_1) = (-iF_1, iF_2).$$

Therefore

$$[\sigma_x, \sigma_y](F_1, F_2) = (2iF_1, -2iF_2) = 2i \sigma_z(F_1, F_2).$$

Thus

$$[\sigma_x, \sigma_y] = 2i\sigma_z.$$

The other two relations are obtained analogously:

$$[\sigma_y, \sigma_z] = 2i\sigma_x, \quad [\sigma_z, \sigma_x] = 2i\sigma_y.$$

□

5.9. Spin Observables Associated with Directions in Space

Let

$$\vec{n} = (n_1, n_2, n_3) \in \mathbb{R}^3$$

be a unit vector.

We define the corresponding spin observable by

$$\sigma_{\vec{n}} := n_1 \sigma_x + n_2 \sigma_y + n_3 \sigma_z.$$

This is a linear operator on the two-component Maxwellian space.

For

$$\Psi = (F_1, F_2),$$

one finds

$$\sigma_{\vec{n}}(F_1, F_2) = (n_3 F_1 + (n_1 - in_2)F_2, (n_1 + in_2)F_1 - n_3 F_2).$$

Thus a spatial direction determines a specific linear mixing of the two Maxwellian components.

This is the Maxwell–Schwartz realization of spin measurements.

5.10. Polarization Interpretation and Spin Interpretation

The same two-component Cartesian structure supports two distinct physical readings.

Polarization interpretation. If the two components are interpreted as horizontal and vertical polarization channels, then the two-dimensional space corresponds directly to the physical transverse plane,

and rotations of analyzers correspond directly to rotations of the component basis.

Spin interpretation. If the two components are interpreted as spin-up and spin-down states, then the same space carries the usual Pauli algebra and supports spin observables associated with spatial directions.

At the algebraic level, the same Cartesian space serves both purposes. The difference lies in the physical interpretation of the component basis and in the relation between space rotations and state-space transformations.

5.11. Formal Result

We recap here the mathematical content of this section.

Theorem 1 (Two-component Maxwellian structures and induced Pauli symmetry). *Let*

$$W := \mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3)$$

be the Maxwellian field space, and let

$$W^2 := W \times W$$

be its Cartesian square.

Then the following assertions hold.

1. *The space W^2 is a complex vector space under componentwise operations, namely*

$$(F_1, F_2) + (G_1, G_2) = (F_1 + G_1, F_2 + G_2),$$

$$\lambda(F_1, F_2) = (\lambda F_1, \lambda F_2), \quad \lambda \in \mathbb{C}.$$

It may be viewed, at the level of linear structure, as a tensor-type two-component Maxwellian space,

$$W^2 \cong W \otimes \mathbb{C}^2.$$

2. *The two components of an element*

$$F = (F_1, F_2) \in W^2$$

are, at the purely algebraic level, structural components. Consequently, the same space W^2 may later be interpreted either as a polarization doublet, or as a spin- $\frac{1}{2}$ doublet, or more generally as an internal two-state degree of freedom carried by a Maxwellian field.

3. *Let*

$$e_{\uparrow} := (1, 0), \quad e_{\downarrow} := (0, 1)$$

be the canonical basis of $\mathbb{C}^2$. Then every element of W^2 may be written formally as

$$F = F_1 e_{\uparrow} + F_2 e_{\downarrow}.$$

Equivalently, one may use the notations

$$F = F_1 |\uparrow\rangle + F_2 |\downarrow\rangle$$

or

$$F = F_1 |H\rangle + F_2 |V\rangle,$$

according to whether the spin or polarization interpretation is intended.

4. *For every nonzero Maxwellian field*

$$F \in W,$$

the family

$$(F, 0), \quad (0, F)$$

is linearly independent in W^2 . Therefore the subspace

$$P_F := \text{Span}_{\mathbb{C}}\{(F, 0), (0, F)\} \subset W^2$$

is two-dimensional over $\mathbb{C}$.

5. If one defines

$$\Psi_{\uparrow} := (F, 0), \quad \Psi_{\downarrow} := (0, F),$$

then the subspace

$$P_{\text{spin}}(F) := \text{Span}_{\mathbb{C}}\{\Psi_{\uparrow}, \Psi_{\downarrow}\}$$

is canonically isomorphic to $\mathbb{C}^2$ through the map

$$\Phi_F : P_{\text{spin}}(F) \rightarrow \mathbb{C}^2,$$

$$\Phi_F(\Psi_{\uparrow}) = (1, 0), \quad \Phi_F(\Psi_{\downarrow}) = (0, 1),$$

extended $\mathbb{C}$ -linearly.

6. The Pauli operators

$$\sigma_x, \sigma_y, \sigma_z : P_{\text{spin}}(F) \rightarrow P_{\text{spin}}(F)$$

defined by

$$\begin{aligned} \sigma_x(\Psi_{\uparrow}) &= \Psi_{\downarrow}, & \sigma_x(\Psi_{\downarrow}) &= \Psi_{\uparrow}, \\ \sigma_y(\Psi_{\uparrow}) &= -i\Psi_{\downarrow}, & \sigma_y(\Psi_{\downarrow}) &= i\Psi_{\uparrow}, \\ \sigma_z(\Psi_{\uparrow}) &= \Psi_{\uparrow}, & \sigma_z(\Psi_{\downarrow}) &= -\Psi_{\downarrow}, \end{aligned}$$

are well defined linear operators.

7. For every

$$\Psi = (aF, bF) \in P_{\text{spin}}(F), \quad a, b \in \mathbb{C},$$

one has the componentwise formulas

$$\begin{aligned} \sigma_x(\Psi) &= (bF, aF), \\ \sigma_y(\Psi) &= (-ibF, iaF), \\ \sigma_z(\Psi) &= (aF, -bF). \end{aligned}$$

More generally, when written on the whole Cartesian space W^2 , the same formulas become

$$\begin{aligned} \sigma_x(F_1, F_2) &= (F_2, F_1), \\ \sigma_y(F_1, F_2) &= (-iF_2, iF_1), \\ \sigma_z(F_1, F_2) &= (F_1, -F_2). \end{aligned}$$

8. The operators $\sigma_x, \sigma_y, \sigma_z$ satisfy the Pauli algebra relations

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I,$$

and

$$[\sigma_x, \sigma_y] = 2i\sigma_z, \quad [\sigma_y, \sigma_z] = 2i\sigma_x, \quad [\sigma_z, \sigma_x] = 2i\sigma_y.$$

9. For every unit vector

$$\vec{n} = (n_1, n_2, n_3) \in \mathbb{R}^3,$$

the associated spin observable

$$\sigma_{\vec{n}} := n_1\sigma_x + n_2\sigma_y + n_3\sigma_z$$

is a well-defined linear operator on W^2 , and for every

$$\Psi = (F_1, F_2) \in W^2$$

one has

$$\sigma_{\vec{n}}(F_1, F_2) = (n_3F_1 + (n_1 - in_2)F_2, (n_1 + in_2)F_1 - n_3F_2).$$

10. Consequently, every nonzero Maxwellian field canonically generates a two-state Cartesian structure carrying an induced Pauli symmetry. The same algebraic structure supports both the polarization interpretation and the spin- $\frac{1}{2}$ interpretation, the distinction lying only in the physical meaning assigned to the two components.

5.12. Why the Cartesian Model Is Particularly Suited to Spin

The Cartesian construction is especially well adapted to spin- $\frac{1}{2}$ systems. Indeed, once the two-component structure

$$\mathcal{W}^2 = \mathcal{W} \times \mathcal{W}$$

is available, the entire Pauli formalism is realized immediately as an internal algebra acting on field pairs.

No additional geometric structure is required at this stage.

In particular:

- basis states are represented by pairs of Maxwellian fields;
- Pauli operators act componentwise;
- spin observables are internal field transformations;
- tensor products of such spaces will later provide the natural setting for spin entanglement.

5.13. Transition to Relativistic Dynamics

The algebraic construction developed in the present section is independent of the specific dynamics of the underlying field F .

In the next section, we shall combine this two-component structure with the square-root Hamiltonian formalism introduced in Section 3.

This will lead to two-component relativistic equations of Pauli–Weyl–Dirac type within the Maxwell–Schwartz framework.

6. Theoretical Framework: Two-Component Relativistic Dynamics

6.1. Scalar Relativistic Dynamics Revisited

Recall that the scalar relativistic Hamiltonian is defined by

$$\hat{H}_{m_0} : \mathcal{V} \longrightarrow \mathcal{V},$$

$$\hat{H}_{m_0}(\psi) = \int_{\mathbb{M}_4^*} H_{m_0}(\psi)_\eta \eta,$$

where

$$H_{m_0}(p) = c\sqrt{|\vec{p}|^2 + m_0^2 c^2}.$$

The corresponding evolution equation is

$$i\hbar\partial_t\psi = \hat{H}_{m_0}\psi.$$

This equation is always accompanied by its conjugate:

$$-i\hbar\partial_t\bar{\psi} = \hat{H}_{m_0}\bar{\psi}.$$

Thus the scalar theory naturally involves a *pair* of coupled equations.

6.2. Maxwellian Relativistic Dynamics

On the vectorial side, we consider the Maxwellian Hamiltonian

$$\hat{\mathcal{H}}_{m_0} : \mathcal{W}_e \longrightarrow \mathcal{W}_e,$$

defined by analogy with the scalar case.

On the invariant subspace $\mathcal{W}_e$, one has

$$\hbar \text{rot} = |\hat{\vec{P}}|,$$

and therefore

$$\hat{\mathcal{H}}_{m_0} = c\sqrt{(\hbar \text{rot})^2 + (m_0c)^2} \mathbb{I}_{\mathcal{W}}.$$

The corresponding evolution equation is

$$i\hbar\partial_t F = \hat{\mathcal{H}}_{m_0} F.$$

6.3. Two-Component Scalar Dynamics

We now pass to the Cartesian two-component scalar space

$$\mathcal{V}^2 := \mathcal{V} \times \mathcal{V}.$$

Let

$$\Psi = (\psi_1, \psi_2) \in \mathcal{V}^2.$$

The simplest relativistic dynamics is given componentwise:

$$i\hbar\partial_t \Psi = (\hat{H}_{m_0}\psi_1, \hat{H}_{m_0}\psi_2).$$

This may be written compactly as

$$i\hbar\partial_t \Psi = \hat{H}_{m_0} \mathbb{I}_{\mathbb{C}^2} \Psi.$$

6.4. Coupled Two-Component Dynamics

The essential new feature arises when the two components are coupled.

Let

$$\sigma_x, \sigma_y, \sigma_z$$

be the Pauli operators introduced in Section 3.

We consider operators of the form

$$\hat{\mathbb{H}} = \hat{H}_{m_0} \mathbb{I}_{\mathbb{C}^2} + \sum_{j=1}^3 A_j \sigma_j,$$

where the coefficients A_j are suitable operators acting on $\mathcal{V}$.

This leads to coupled evolution equations

$$i\hbar\partial_t \Psi = \hat{\mathbb{H}} \Psi.$$

6.5. First-Order Factorization of the Hamiltonian

A central idea of relativistic quantum mechanics is the factorization of the square-root Hamiltonian.

In the present framework, we seek operators $\mathcal{D}$ such that

$$\mathcal{D}^2 = \hat{H}_{m_0}^2 \mathbb{I}_{\mathbb{C}^2}.$$

Using the Pauli algebra, we consider operators of the form

$$\mathcal{D} = c \vec{\sigma} \cdot \hat{\vec{P}} + m_0 c^2 \sigma_z.$$

Formally, one verifies using

$$\sigma_i \sigma_j = \delta_{ij} \mathbb{I} + i \varepsilon_{ijk} \sigma_k$$

that

$$\mathcal{D}^2 = c^2 |\hat{\vec{P}}|^2 + m_0^2 c^4,$$

provided the operators commute appropriately.

6.6. Dirac-like Equation in Two-Component Form

We define the two-component relativistic equation

$$i\hbar \partial_t \Psi = \mathcal{D} \Psi.$$

This equation is a Dirac-like equation formulated in the two-component Maxwell–Schwartz framework.

It provides a first-order representation of relativistic dynamics, replacing the square-root Hamiltonian by a linear operator in $\hat{\vec{P}}$.

6.7. Maxwellian Realization of Two-Component Dynamics

We now transport the above construction to the Maxwellian setting.

Let

$$\Psi = (\psi_1, \psi_2) \in \mathcal{V}^2.$$

We define

$$\mathcal{F}_e(\Psi) = (\mathcal{F}_e(\psi_1), \mathcal{F}_e(\psi_2)) \in \mathcal{W}^2.$$

Thus the two-component scalar state is realized as a two-component Maxwellian field.

6.8. Maxwellian Dirac-Type Operator

On $\mathcal{W}_e^2$, we define

$$\mathcal{D}_{\mathcal{W}} = c \vec{\sigma} \cdot (\hbar \text{rot}) + m_0 c^2 \sigma_z.$$

This operator acts on pairs of Maxwellian fields.

On $\mathcal{W}_e$, the identification

$$\hbar \text{rot} = |\hat{\vec{P}}|$$

ensures that $\mathcal{D}_{\mathcal{W}}$ corresponds to the scalar operator $\mathcal{D}$.

6.9. Maxwellian Dirac Equation

We define the Maxwellian Dirac equation

$$i\hbar \partial_t F = \mathcal{D}_{\mathcal{W}} F,$$

where

$$F \in \mathcal{W}^2.$$

This equation describes a two-component Maxwellian field evolving under a first-order relativistic operator.

6.10. Massless Case and Weyl-Type Dynamics

If $m_0 = 0$, the operator reduces to

$$\mathcal{D}_{\mathcal{W}} = c \vec{\sigma} \cdot (\hbar \text{rot}).$$

The equation

$$i\hbar \partial_t F = c \vec{\sigma} \cdot (\hbar \text{rot}) F$$

is a Maxwell–Weyl-type equation.

In this case, the two components decouple into chiral modes corresponding to different helicities.

6.11. Interpretation

The construction above shows that:

- the Pauli algebra arises naturally from Cartesian structure;
- relativistic first-order equations arise from spectral factorization;
- Maxwellian fields provide a concrete realization of two-component dynamics.

Thus spin- $\frac{1}{2}$ dynamics emerges as an internal two-component structure carried by Maxwellian fields, rather than as an abstract spinor construction.

6.12. Outlook

The framework developed in this section prepares the ground for:

- tensor product constructions;
- entangled two-particle states;
- Bell-type correlations in the Maxwell–Schwartz setting.

7. Results II: Entanglement and Bell Correlations

7.1. Tensor Product Structure

Let

$$\mathcal{W} = \mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3).$$

We consider the Cartesian tensor product space

$$\mathcal{W}^2 \otimes \mathcal{W}^2.$$

An element of this space represents a two-particle Maxwellian system.

For

$$F = (F_1, F_2) \in \mathcal{W}^2, \quad G = (G_1, G_2) \in \mathcal{W}^2,$$

their tensor product is defined as

$$F \otimes G.$$

By linearity, a general element of the tensor product space is a superposition of such elementary tensors.

7.2. Separable and Entangled States

A two-particle state is said to be *separable* if it can be written as

$$\Phi = F \otimes G.$$

Otherwise, it is said to be *entangled*.

Thus entanglement corresponds to the impossibility of factorizing the state into independent Maxwellian components.

7.3. Canonical Entangled States

Let us fix a Maxwellian mode $F \in \mathcal{W}$ and consider the associated two-component basis

$$\Psi_{\uparrow} = (F, 0), \quad \Psi_{\downarrow} = (0, F).$$

We define the Bell-type state

$$\Phi = \frac{1}{\sqrt{2}}(\Psi_{\uparrow} \otimes \Psi_{\downarrow} - \Psi_{\downarrow} \otimes \Psi_{\uparrow}).$$

This state is antisymmetric and cannot be factorized. It is therefore entangled.

7.4. Observables for the Two Subsystems

Let $\vec{a}, \vec{b} \in \mathbb{R}^3$ be unit vectors.

We define the observables

$$\sigma_{\vec{a}}, \quad \sigma_{\vec{b}}$$

as in Section 3.

On the tensor product space, the corresponding operators are

$$\sigma_{\vec{a}} \otimes \mathbb{I}, \quad \mathbb{I} \otimes \sigma_{\vec{b}}.$$

The joint observable is

$$\sigma_{\vec{a}} \otimes \sigma_{\vec{b}}.$$

7.5. Correlation Function

Let Φ be a normalized two-particle state.

We define the correlation function by

$$E(\vec{a}, \vec{b}) := \langle \Phi, (\sigma_{\vec{a}} \otimes \sigma_{\vec{b}}) \Phi \rangle.$$

In the present framework, this quantity is interpreted through the corresponding Maxwellian field realization and its energy distribution.

7.6. Computation for the Bell State

For the antisymmetric Bell state defined above, a direct computation yields

$$E(\vec{a}, \vec{b}) = -\vec{a} \cdot \vec{b}.$$

Thus the correlation depends only on the relative angle between the two directions.

7.7. Geometric Interpretation

The result

$$E(\vec{a}, \vec{b}) = -\vec{a} \cdot \vec{b}$$

has a direct geometric meaning:

- maximal anticorrelation for $\vec{a} = \vec{b}$;
- zero correlation for orthogonal directions;
- sinusoidal variation for intermediate angles.

This behavior coincides exactly with the predictions of quantum mechanics.

7.8. Bell Inequality

Consider four unit vectors

$$\vec{a}, \vec{a}', \vec{b}, \vec{b}'.$$

Define the Bell parameter

$$S = E(\vec{a}, \vec{b}) + E(\vec{a}, \vec{b}') + E(\vec{a}', \vec{b}) - E(\vec{a}', \vec{b}').$$

Classical local theories impose the bound

$$|S| \leq 2.$$

7.9. Violation of Bell Inequality

Using the correlation formula

$$E(\vec{a}, \vec{b}) = -\vec{a} \cdot \vec{b},$$

one finds configurations for which

$$|S| = 2\sqrt{2}.$$

Thus the Maxwell–Schwartz framework reproduces the quantum violation of Bell inequalities.

7.10. Formal Result

Here we recap the mathematical content of the section.

Theorem 2 (Entanglement and Bell correlations in the Maxwell–Schwartz framework). *Let*

$$W := \mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3)$$

be the Maxwellian field space, and let

$$W^2 := W \times W$$

be its two-component Cartesian extension.

Then the following assertions hold.

1. *The tensor product space*

$$W^2 \otimes W^2$$

provides a natural state space for two-particle Maxwellian systems.

2. *A state*

$$\Phi \in W^2 \otimes W^2$$

is separable if it can be written in the form

$$\Phi = F \otimes G, \quad F, G \in W^2.$$

If no such factorization exists, then Φ is entangled.

3. *Let*

$$F \in W$$

be a nonzero Maxwellian mode, and define the associated two-component basis

$$\Psi_{\uparrow} := (F, 0), \quad \Psi_{\downarrow} := (0, F).$$

Then the antisymmetric Bell-type state

$$\Phi = \frac{1}{\sqrt{2}} (\Psi_{\uparrow} \otimes \Psi_{\downarrow} - \Psi_{\downarrow} \otimes \Psi_{\uparrow})$$

is entangled.

4. For every pair of unit vectors

$$\vec{a}, \vec{b} \in \mathbb{R}^3,$$

let

$$\sigma_{\vec{a}}, \sigma_{\vec{b}}$$

be the spin observables induced by the Pauli algebra on the two subsystems. Then the corresponding joint observable on $W^2 \otimes W^2$ is

$$\sigma_{\vec{a}} \otimes \sigma_{\vec{b}}.$$

5. If Φ is normalized, the associated correlation function is

$$E(\vec{a}, \vec{b}) := \langle \Phi, (\sigma_{\vec{a}} \otimes \sigma_{\vec{b}}) \Phi \rangle.$$

6. For the Bell-type state

$$\Phi = \frac{1}{\sqrt{2}} (\Psi_{\uparrow} \otimes \Psi_{\downarrow} - \Psi_{\downarrow} \otimes \Psi_{\uparrow}),$$

the correlation function is given by

$$E(\vec{a}, \vec{b}) = -\vec{a} \cdot \vec{b}.$$

7. Consequently, the correlation exhibits the standard geometric behavior:

- maximal anticorrelation for $\vec{a} = \vec{b}$;
- vanishing correlation for $\vec{a} \perp \vec{b}$;
- sinusoidal angular dependence for intermediate directions.

8. For four unit vectors

$$\vec{a}, \vec{a}', \vec{b}, \vec{b}' \in \mathbb{R}^3,$$

define the CHSH–Bell parameter

$$S = E(\vec{a}, \vec{b}) + E(\vec{a}, \vec{b}') + E(\vec{a}', \vec{b}) - E(\vec{a}', \vec{b}').$$

Then there exist configurations of directions for which

$$|S| = 2\sqrt{2}.$$

Hence the Maxwell–Schwartz framework reproduces the quantum violation of Bell inequalities.

9. Therefore, the tensorial Maxwellian structure naturally supports:

- entangled two-particle states;
- spin correlations determined by the Pauli observables;
- Bell-type correlations identical to the standard quantum predictions;
- CHSH violations incompatible with classical local hidden-variable bounds.

10. In particular, Bell correlations arise here as intrinsic properties of global Maxwellian field configurations, rather than as external statistical prescriptions.

7.11. Maxwellian Interpretation of Correlations

In the present theory, correlations arise from the structure of Maxwellian fields.

The key points are:

- the state Φ corresponds to a coupled Maxwellian field;
- measurements correspond to internal transformations of the field;
- correlations are computed from the resulting field amplitudes and their energy distribution.

Thus Bell correlations are not abstract statistical features, but arise from the geometry and superposition of Maxwellian fields.

8. Results III: Energy Interpretation of the Born Rule

Let Φ be a two-particle Maxwellian state.

The probabilities of measurement outcomes are determined by the energy distribution of the field components.

More precisely:

- amplitudes determine field strength;
- energy density is quadratic in amplitude;
- probabilities correspond to normalized energy contributions.

Thus the Born rule is interpreted as an energy distribution law.

8.1. Norm Preservation in the Schwartz Framework

The following result clarifies a fundamental aspect of the Maxwell–Schwartz construction: the preservation of the norm between scalar states and their Maxwellian realizations.

Theorem 3 (Norm preservation under Maxwellian realization). *Let ψ be a Schwartz function on $\mathbb{M}_4$, and let*

$$F_\psi = \int_{\mathbb{M}_4^*} (\psi)_\eta w^e$$

be its Maxwellian realization, where

$$w^e = \eta^e f^e$$

and the constant vector field $f^e \in \mathbb{C}^3$ has unit effective modulus.

Then

$$\|\psi\| = \|F_\psi\|.$$

Equivalently,

$$\|\psi\|^2 = \|F_\psi\|^2.$$

Proof. Since

$$\psi = \int_{\mathbb{M}_4^*} (\psi)_\eta \eta = \mathcal{F}_\eta^{-1}(\mathcal{F}_\eta(\psi)),$$

the Parseval relation for the operator $\mathcal{F}_\eta^{-1}$ yields

$$\|\psi\|^2 = C \|(\psi)_\eta\|^2,$$

for a positive constant C depending only on the transform convention.

On the other hand,

$$F_\psi = \int_{\mathbb{M}_4^*} (\psi)_\eta w^e = \int_{\mathbb{M}_4^*} (\psi)_\eta \eta^e f^e = \int_{\mathbb{M}_4^*} [(\psi)_\eta f^e] \eta^e.$$

Thus F_ψ is obtained by applying the same superposition operator to the vector-valued coefficient $(\psi)_\eta f^e$. Therefore the same Parseval relation gives

$$\|F_\psi\|^2 = C \|(\psi)_\eta f^e\|^2.$$

Since f^e has unit effective modulus, one has

$$\|(\psi)_\eta f^e\|^2 = \|(\psi)_\eta\|^2.$$

Hence

$$\|F_\psi\|^2 = C \|(\psi)_\eta\|^2 = \|\psi\|^2.$$

This proves the claim. $\square$

Remark 5 (Schwartz-linear nature of the result). *The above theorem is intrinsically a statement of Schwartz linear algebra.*

Indeed, the equality of norms does not rely on the use of a normalized de Broglie family, but only on the fact that both ψ and F_ψ are reconstructed from the same coordinate system $(\psi)_\eta$ through the same superposition operator.

In particular:

- *the result is independent of normalization constants;*
- *it is formulated at the level of $\mathcal{S}$ and $\mathcal{S}'$;*
- *it reflects the invariance of the product $(\psi)_\eta \eta$ under rescaling;*
- *the vectorial extension $(\psi)_\eta \mapsto (\psi)_\eta f_e$ preserves the norm because f_e has unit effective modulus.*

Thus the norm preservation property is a consequence of the underlying Schwartz-linear structure, rather than of any Hilbert space orthonormality.

Remark 6 (Relation with L^2). *The above result extends naturally to L^2 by completion.*

However, in the present framework, the fundamental structure is already fully captured at the level of $\mathcal{S}$, while L^2 appears as a secondary Hilbertian closure.

8.2. Energy Interpretation of the Born Rule

We now establish the precise relation between the spectral decomposition of a scalar state and the energy distribution of its Maxwellian realization.

Theorem 4 (Energy–Born correspondence). *Let $\psi \in \mathcal{S}(\mathbb{M}_4, \mathbb{C})$ and let*

$$F_\psi = \int_{\mathbb{M}_4^*} (\psi)_\eta w^e$$

be its Maxwellian realization.

Then:

1. *the spectral coefficients $(\psi)_\eta$ determine the amplitude of each Maxwellian mode;*
2. *the energy density of the field F_ψ is quadratic in these amplitudes;*
3. *the total energy of F_ψ satisfies*

$$\|F_\psi\|^2 = C \|(\psi)_\eta\|^2,$$

for a constant $C > 0$ independent of ψ ;

4. *after normalization, the quantity*

$$|(\psi)_\eta(p)|^2$$

represents the relative energy distribution over 4-momentum space.

Proof. The Maxwellian field is given by

$$F_\psi = \int_{\mathbb{M}_4^*} (\psi)_\eta w^e = \int_{\mathbb{M}_4^*} (\psi)_\eta \eta^e f^e.$$

Since the de Broglie family is unimodular, one has

$$|\eta^e| = 1.$$

Thus, for each mode, the amplitude of the field is proportional to

$$|(\psi)_\eta|.$$

The energy density of a Maxwellian field is quadratic in the field amplitude. Therefore each spectral component contributes proportionally to

$$|(\psi)_\eta|^2.$$

By the Parseval relation applied to the superposition operator, one obtains

$$\|F_\psi\|^2 = C \|(\psi)_\eta\|^2,$$

with the same constant C as in the scalar case.

This shows that the total energy is distributed according to the squared modulus of the spectral coefficients. $\square$

Corollary 1 (Born rule as energy distribution). *Let ψ be normalized.*

Then the function

$$p \longmapsto |(\psi)_\eta(p)|^2$$

defines the probability distribution of measurement outcomes.

Proof. Since $\|\psi\| = 1$, one has

$$\|(\psi)_\eta\|^2 = \frac{1}{C}.$$

Thus the normalized energy density is given by

$$\frac{|(\psi)_\eta|^2}{\|(\psi)_\eta\|^2},$$

which defines a probability distribution. $\square$

Remark 7 (Physical meaning). *In the present framework:*

- *the coefficients $(\psi)_\eta$ determine the strength of each Maxwellian mode;*
- *the field amplitude is directly proportional to $(\psi)_\eta$;*
- *energy density is quadratic in amplitude;*
- *probabilities arise as normalized energy contributions.*

Thus the Born rule is not postulated but emerges from the energetic structure of Maxwellian fields.

Remark 8 (Independence from normalization). *The above construction does not depend on the normalization of the de Broglie family.*

Indeed, the product

$$(\psi)_\eta \eta$$

is invariant under rescaling, and the energy density depends only on the modulus of $(\psi)_\eta$.
Thus the probabilistic interpretation is intrinsic and does not rely on orthonormal bases.

8.3. Energy Density and Poynting Vector

Let

$$F = \vec{E} + ic\vec{B}$$

be an electromagnetic field, with real electric and magnetic components $\vec{E}$ and $\vec{B}$.

We define its energy density by

$$u_F := \frac{\epsilon_0}{2} (|\vec{E}|^2 + |c\vec{B}|^2),$$

and its Poynting vector by

$$\mathbf{S}_F := \frac{1}{\mu_0} \vec{E} \times \vec{B}.$$

Theorem 5 (Born density as normalized energy density). *Let $\psi \in \mathcal{S}(\mathbb{M}_4, \mathbb{C})$ and let*

$$F_\psi = \mathcal{F}_e(\psi)$$

be its electromagnetic realization.

Assume that F_ψ has finite total energy at each fixed time t :

$$\int_{\mathbb{R}^3} u_{F_\psi}(x, t) dx < \infty.$$

Then the normalized density

$$\rho_\psi(x, t) := \frac{u_{F_\psi}(x, t)}{\int_{\mathbb{R}^3} u_{F_\psi}(y, t) dy}$$

defines a probability density on physical space.

Moreover, this probability density coincides with the Born density associated with ψ .

Proof. By the norm-preservation theorem, the scalar state ψ and its Maxwellian realization F_ψ have the same total norm. On the Maxwellian side, this norm is represented by the total electromagnetic energy, namely

$$\|F_\psi\|^2 = \int_{\mathbb{R}^3} u_{F_\psi}(x, t) dx.$$

Therefore, after normalization, the local energy density determines a probability density.

Since the construction of F_ψ preserves the spectral coordinates $(\psi)_\eta$, the resulting normalized energy distribution coincides with the Born density of the scalar state. $\square$

Remark 9 (Probability current and Poynting flux). *The Poynting vector $\mathbf{S}_{F_\psi}$ describes the transport of electromagnetic energy through space.*

Accordingly, the normalized quantity

$$\mathbf{J}_\psi(x, t) := \frac{\mathbf{S}_{F_\psi}(x, t)}{\int_{\mathbb{R}^3} u_{F_\psi}(y, t) dy}$$

may be interpreted as the probability current associated with the state ψ .

Thus:

- the Born density corresponds to normalized energy density;
- the probability current corresponds to normalized Poynting flux.

In this way, the usual probabilistic interpretation of quantum mechanics is realized in terms of classical electromagnetic energy transport.

8.4. Continuity Equation and Probability Transport

In the present subsection, we assume that

$$F = \vec{E} + ic\vec{B}$$

is an electromagnetic field of class

$$\mathcal{S}(\mathbb{M}_4, \mathbb{C}^3),$$

with real electric and magnetic components $\vec{E}$ and $\vec{B}$ satisfying the vacuum Maxwell equations.

These assumptions are perfectly natural for the theorem below and do not restrict the conceptual scope of the model. They provide the appropriate regularity and decay conditions under which all differential and integral operations are classical and rigorous.

8.5. Energy Density and Poynting Vector: Continuity Equation

We define the electromagnetic energy density by

$$u_F(x, t) := \frac{\varepsilon_0}{2} \left(|\vec{E}(x, t)|^2 + |c\vec{B}(x, t)|^2 \right),$$

and the Poynting vector by

$$\mathbf{S}_F(x, t) := \frac{1}{\mu_0} \vec{E}(x, t) \times \vec{B}(x, t).$$

Theorem 6 (Electromagnetic continuity equation). *Let*

$$F = \vec{E} + ic\vec{B} \in \mathcal{S}(\mathbb{M}_4, \mathbb{C}^3)$$

be an electromagnetic field satisfying the vacuum Maxwell equations. Then the energy density u_F and the Poynting vector $\mathbf{S}_F$ satisfy the continuity equation

$$\partial_t u_F + \nabla \cdot \mathbf{S}_F = 0.$$

Proof. Since F is an electromagnetic field, its real components E and B satisfy the vacuum Maxwell equations

$$\begin{aligned} \nabla \cdot \vec{E} &= 0, & \nabla \cdot \vec{B} &= 0, \\ \partial_t \vec{B} &= -\nabla \times \vec{E}, & \partial_t \vec{E} &= c^2 \nabla \times \vec{B}. \end{aligned}$$

We differentiate the energy density:

$$\partial_t u_F = \frac{\varepsilon_0}{2} \partial_t \left(|\vec{E}|^2 + |c\vec{B}|^2 \right) = \varepsilon_0 \vec{E} \cdot \partial_t \vec{E} + \varepsilon_0 c^2 \vec{B} \cdot \partial_t \vec{B}.$$

Substituting the Maxwell equations, we obtain

$$\partial_t u_F = \varepsilon_0 c^2 \vec{E} \cdot (\nabla \times \vec{B}) - \varepsilon_0 c^2 \vec{B} \cdot (\nabla \times \vec{E}).$$

Using the vector identity

$$\nabla \cdot (\vec{E} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{B}),$$

we deduce

$$\partial_t u_F = -\varepsilon_0 c^2 \nabla \cdot (\vec{E} \times \vec{B}).$$

Since

$$\varepsilon_0 c^2 = \frac{1}{\mu_0},$$

it follows that

$$\partial_t u_F = -\nabla \cdot \left(\frac{1}{\mu_0} \vec{E} \times \vec{B} \right) = -\nabla \cdot \mathbf{S}_F.$$

Therefore

$$\partial_t u_F + \nabla \cdot \mathbf{S}_F = 0,$$

as claimed. $\square$

Corollary 2 (Conservation of total energy). *Under the same assumptions, the total electromagnetic energy*

$$\mathcal{E}_F(t) := \int_{\mathbb{R}^3} u_F(x, t) dx$$

is independent of time.

Proof. Integrating the continuity equation over $\mathbb{R}^3$, one obtains

$$\frac{d}{dt} \mathcal{E}_F(t) = - \int_{\mathbb{R}^3} \nabla \cdot \mathbf{S}_F(x, t) dx.$$

Since $F \in \mathcal{S}(\mathbb{M}_4, \mathbb{C}^3)$, all components of $\mathbf{S}_F$ decrease rapidly at spatial infinity. Therefore the divergence integral vanishes by Gauss' theorem, and one gets

$$\frac{d}{dt} \mathcal{E}_F(t) = 0.$$

$\square$

Theorem 7 (Born density and probability current). *Let*

$$\psi \in \mathcal{S}(\mathbb{M}_4, \mathbb{C})$$

and let

$$F_\psi = \mathcal{F}_e(\psi)$$

be an electromagnetic realization of ψ of class

$$\mathcal{S}(\mathbb{M}_4, \mathbb{C}^3).$$

Assume that the total energy

$$\mathcal{E}_{F_\psi} = \int_{\mathbb{R}^3} u_{F_\psi}(x, t) dx$$

is finite and nonzero.

Define

$$\rho_\psi(x, t) := \frac{u_{F_\psi}(x, t)}{\mathcal{E}_{F_\psi}},$$

and

$$\mathbf{J}_\psi(x, t) := \frac{\mathbf{S}_{F_\psi}(x, t)}{\mathcal{E}_{F_\psi}}.$$

Then:

1. $\rho_\psi(\cdot, t)$ is a probability density on $\mathbb{R}^3$ for every fixed time t ;

2. the pair $(\rho_\psi, \mathbf{J}_\psi)$ satisfies the continuity equation

$$\partial_t \rho_\psi + \nabla \cdot \mathbf{J}_\psi = 0.$$

Proof. By definition,

$$\int_{\mathbb{R}^3} \rho_\psi(x, t) dx = \frac{1}{\mathcal{E}_{F_\psi}} \int_{\mathbb{R}^3} u_{F_\psi}(x, t) dx = 1.$$

Thus $\rho_\psi(\cdot, t)$ is a probability density.

Moreover, by the previous theorem,

$$\partial_t u_{F_\psi} + \nabla \cdot \mathbf{S}_{F_\psi} = 0.$$

Since $\mathcal{E}_{F_\psi}$ is constant in time, division by $\mathcal{E}_{F_\psi}$ yields

$$\partial_t \rho_\psi + \nabla \cdot \mathbf{J}_\psi = 0.$$

□

Remark 10 (Born rule as normalized electromagnetic transport). *The previous theorem shows that the probabilistic interpretation of the state ψ may be expressed entirely in electromagnetic terms:*

- the Born density is the normalized electromagnetic energy density;
- the probability current is the normalized Poynting vector;
- the continuity equation of quantum probability is the continuity equation of electromagnetic energy transport.

Thus the Born rule is realized not merely as a static energy-density law, but as a full dynamical transport law for electromagnetic energy.

Remark 11 (Role of the Schwartz assumption). *The assumption*

$$F_\psi \in \mathcal{S}(\mathbb{M}_4, \mathbb{C}^3)$$

is mathematically and physically appropriate in the present theorem.

It guarantees:

- classical differentiability of the field;
- rapid decay at spatial infinity;
- rigorous validity of the continuity equation;
- conservation of total energy without boundary contributions.

Other pre-Hilbert subspaces of $\mathcal{S}'$ may require different natural assumptions and corresponding variants of Parseval theory, but the present case already captures the essential logic of the Maxwell–Schwartz interpretation.

8.6. Probability Transport in the Bell Setting

We now extend the energy–probability correspondence to the two-particle Maxwellian setting.

Let

$$\Phi \in \mathcal{W}^2 \otimes \mathcal{W}^2$$

be a Bell-type Maxwellian state of class

$$\mathcal{S}(\mathbb{M}_4, \mathbb{C}^3)^2 \otimes \mathcal{S}(\mathbb{M}_4, \mathbb{C}^3)^2.$$

Assume that Φ is decomposed in the canonical two-component basis as

$$\Phi = \sum_{\alpha, \beta} \Phi_{\alpha\beta} |\alpha\rangle \otimes |\beta\rangle,$$

where each coefficient $\Phi_{\alpha\beta}$ is an electromagnetic field of class $\mathcal{S}(\mathbb{M}_4, \mathbb{C}^3)$.

For each channel (α, β) , let

$$u_{\alpha\beta} := \frac{\epsilon_0}{2} (|\vec{E}_{\alpha\beta}|^2 + |c\vec{B}_{\alpha\beta}|^2)$$

be the corresponding electromagnetic energy density, and let

$$\mathbf{S}_{\alpha\beta} := \frac{1}{\mu_0} \vec{E}_{\alpha\beta} \times \vec{B}_{\alpha\beta}$$

be the corresponding Poynting vector.

We define the total Bell energy density by

$$u_{\Phi} := \sum_{\alpha, \beta} u_{\alpha\beta},$$

and the total Bell energy flux by

$$\mathbf{S}_{\Phi} := \sum_{\alpha, \beta} \mathbf{S}_{\alpha\beta}.$$

Theorem 8 (Continuity equation for Bell-type Maxwellian fields). *Under the above assumptions, the Bell energy density and Bell energy flux satisfy the continuity equation*

$$\partial_t u_{\Phi} + \nabla \cdot \mathbf{S}_{\Phi} = 0.$$

Proof. Each electromagnetic channel $\Phi_{\alpha\beta}$ satisfies the classical electromagnetic continuity equation

$$\partial_t u_{\alpha\beta} + \nabla \cdot \mathbf{S}_{\alpha\beta} = 0.$$

Summing over all channel indices (α, β) gives

$$\partial_t \left(\sum_{\alpha, \beta} u_{\alpha\beta} \right) + \nabla \cdot \left(\sum_{\alpha, \beta} \mathbf{S}_{\alpha\beta} \right) = 0,$$

that is,

$$\partial_t u_{\Phi} + \nabla \cdot \mathbf{S}_{\Phi} = 0.$$

□

Let

$$\mathcal{E}_{\Phi}(t) := \int_{\mathbb{R}^3} u_{\Phi}(x, t) dx$$

denote the total Bell energy.

Since Φ is of Schwartz class, one has

$$\frac{d}{dt} \mathcal{E}_{\Phi}(t) = 0.$$

We may therefore define the normalized Bell density

$$\rho_{\Phi}(x, t) := \frac{u_{\Phi}(x, t)}{\mathcal{E}_{\Phi}},$$

and the normalized Bell current

$$\mathbf{J}_\Phi(x, t) := \frac{\mathbf{S}_\Phi(x, t)}{\mathcal{E}_\Phi}.$$

Corollary 3 (Bell probability transport). *The normalized Bell density and Bell current satisfy*

$$\partial_t \rho_\Phi + \nabla \cdot \mathbf{J}_\Phi = 0.$$

Proof. Since $\mathcal{E}_\Phi$ is constant in time, dividing the Bell continuity equation by $\mathcal{E}_\Phi$ yields the result. $\square$

Remark 12 (Bell probabilities as transported energy). *The Bell state is therefore governed by a genuine transport law.*

Its probabilities are not merely static projection weights, but normalized energy distributions of a global Maxwellian field, and the corresponding Bell current is the normalized Poynting transport of that same field.

Thus Bell correlations may be interpreted dynamically as correlations in the transport and redistribution of electromagnetic energy across the channels of a global entangled field configuration.

8.7. Structural Conclusion

The preceding analysis shows that:

- entanglement arises naturally in the tensor product of Maxwellian fields;
- Pauli observables act as internal transformations;
- Bell correlations follow from geometric properties of the fields;
- the Born rule is realized as an energy distribution principle.

This provides a concrete Maxwellian realization of quantum correlations.

9. Results IV: Probability Currents for Massive Maxwellian Fields

9.1. Massive Square-Root Evolution and Quadratic Conservation Law

Let

$$F \in \mathcal{S}(\mathbb{M}_4, \mathbb{C}^3)$$

be a Maxwellian field satisfying the massive evolution equation

$$i\hbar \partial_t F = \hat{\mathcal{H}}_{m_0} F,$$

where

$$\hat{\mathcal{H}}_{m_0} = c \sqrt{(\hbar \operatorname{rot})^2 + (m_0 c)^2} \mathbb{I}_{\mathcal{W}}.$$

Its conjugate equation is

$$-i\hbar \partial_t \bar{F} = \hat{\mathcal{H}}_{m_0} \bar{F}.$$

Proposition 2. *The quadratic density*

$$\rho_F := |F|^2 = F \cdot \bar{F}$$

satisfies the identity

$$\partial_t |F|^2 = \frac{i}{\hbar} \left(F \cdot \hat{\mathcal{H}}_{m_0} \bar{F} - (\hat{\mathcal{H}}_{m_0} F) \cdot \bar{F} \right).$$

Proof. Differentiate

$$|F|^2 = F \cdot \bar{F}$$

with respect to time:

$$\partial_t |F|^2 = (\partial_t F) \cdot \bar{F} + F \cdot \partial_t \bar{F}.$$

Using

$$\partial_t F = -\frac{i}{\hbar} \widehat{\mathcal{H}}_{m_0} F, \quad \partial_t \bar{F} = \frac{i}{\hbar} \widehat{\mathcal{H}}_{m_0} \bar{F},$$

we obtain

$$\partial_t |F|^2 = -\frac{i}{\hbar} (\widehat{\mathcal{H}}_{m_0} F) \cdot \bar{F} + \frac{i}{\hbar} F \cdot \widehat{\mathcal{H}}_{m_0} \bar{F},$$

which is the desired identity. $\square$

Corollary 4. Assume that $\widehat{\mathcal{H}}_{m_0}$ is symmetric on the chosen pre-Hilbert space of Maxwellian fields. Then

$$\frac{d}{dt} \int_{\mathbb{R}^3} |F(x, t)|^2 dx = 0.$$

9.2. Spectral Current for the Massive Square-Root Hamiltonian

Let $\psi \in \mathcal{S}(\mathbb{M}_4, \mathbb{C})$ satisfy

$$i\hbar \partial_t \psi = \widehat{H}_{m_0} \psi,$$

where

$$\widehat{H}_{m_0}(\psi) = \int_{\mathbb{M}_4^*} H_{m_0}(\psi)_\eta \eta, \quad H_{m_0}(p) = c\sqrt{|\vec{p}|^2 + m_0^2 c^2}.$$

Assume that $(\psi)_\eta$ is supported in a region of $\mathbb{M}_4^*$ where $|\vec{p}| > 0$.

Define the current

$$J_\psi(x) := \frac{1}{\hbar} \iint_{\mathbb{M}_4^* \times \mathbb{M}_4^*} \frac{H_{m_0}(p) - H_{m_0}(q)}{|\vec{p} - \vec{q}|^2} (\vec{p} - \vec{q}) (\psi)_\eta(p) \overline{(\psi)_\eta(q)} e^{\frac{i}{\hbar}(\langle p, x \rangle - \langle q, x \rangle)} dp dq.$$

Then, formally,

$$\partial_t |\psi(x, t)|^2 + \nabla \cdot J_\psi(x, t) = 0.$$

9.3. Continuity Equation for the Massive Square-Root Hamiltonian

In the present subsection, we work on a fixed spatial slice and assume that

$$\psi(\cdot, t) \in \mathcal{S}(\mathbb{R}^3, \mathbb{C})$$

for every t , and that ψ satisfies the relativistic square-root equation

$$i\hbar \partial_t \psi = H_{m_0}(D) \psi,$$

where

$$H_{m_0}(\vec{p}) = c\sqrt{|\vec{p}|^2 + m_0^2 c^2}.$$

Let

$$\psi(x, t) = \int_{\mathbb{R}^3} a(\vec{p}) e^{\frac{i}{\hbar}(\vec{p} \cdot x - H_{m_0}(\vec{p})t)} d\vec{p}, \quad a \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}).$$

Define

$$K_{m_0}(\vec{p}, \vec{q}) := \frac{H_{m_0}(\vec{p}) - H_{m_0}(\vec{q})}{|\vec{p} - \vec{q}|^2} (\vec{p} - \vec{q}), \quad \vec{p} \neq \vec{q},$$

and

$$J_\psi(x, t) := \iint_{\mathbb{R}^3 \times \mathbb{R}^3} K_{m_0}(\vec{p}, \vec{q}) a(\vec{p}) \overline{a(\vec{q})} e^{\frac{i}{\hbar}((\vec{p} - \vec{q}) \cdot x - (H_{m_0}(\vec{p}) - H_{m_0}(\vec{q}))t)} d\vec{p} d\vec{q}.$$

In more clean way,

$$\vec{J}_\psi(ct, \vec{x}) := \iint_{\mathbb{R}^3 \times \mathbb{R}^3} K_{m_0} \cdot (a \otimes \bar{a}) e^{\frac{i}{\hbar} S_{\vec{p}}^{m_0}(ct, \vec{x})} \otimes e^{-\frac{i}{\hbar} S_{\vec{q}}^{m_0}(ct, \vec{x})} d\vec{p} d\vec{q},$$

where $(\vec{p}, \vec{q})$ is a standard coordinate system on the bi-momentum space $\mathbb{R}^3 \times \mathbb{R}^3$.

Theorem 9. The current J_ψ is well defined for every $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$ and satisfies

$$\partial_t |\psi(x, t)|^2 + \nabla_x \cdot J_\psi(x, t) = 0.$$

Proof. We first prove that J_ψ is well defined.

By the mean value theorem,

$$H_{m_0}(\vec{p}) - H_{m_0}(\vec{q}) = \int_0^1 \nabla H_{m_0}(\vec{q} + s(\vec{p} - \vec{q})) \cdot (\vec{p} - \vec{q}) ds.$$

Hence

$$K_{m_0}(\vec{p}, \vec{q}) = \int_0^1 \frac{\nabla H_{m_0}(\vec{q} + s(\vec{p} - \vec{q})) \cdot (\vec{p} - \vec{q})}{|\vec{p} - \vec{q}|^2} (\vec{p} - \vec{q}) ds.$$

Therefore

$$|K_{m_0}(\vec{p}, \vec{q})| \leq \int_0^1 |\nabla H_{m_0}(\vec{q} + s(\vec{p} - \vec{q}))| ds.$$

Since

$$\nabla H_{m_0}(\vec{p}) = c \frac{\vec{p}}{\sqrt{|\vec{p}|^2 + m_0^2 c^2}},$$

one has

$$|\nabla H_{m_0}(\vec{p})| \leq c$$

for all $\vec{p} \in \mathbb{R}^3$. Thus

$$|K_{m_0}(\vec{p}, \vec{q})| \leq c.$$

Because $a \in \mathcal{S}(\mathbb{R}^3)$, the function

$$(\vec{p}, \vec{q}) \mapsto a(\vec{p}) \overline{a(\vec{q})}$$

is Schwartz on $\mathbb{R}^3 \times \mathbb{R}^3$, hence integrable. Therefore the double integral defining $J_\psi(x, t)$ is absolutely convergent.

We now compute the divergence. Since the integrand is absolutely integrable and differentiation with respect to x only introduces the factor

$$\frac{i}{\hbar}(\vec{p} - \vec{q}),$$

dominated convergence allows differentiation under the integral sign. Hence

$$\nabla_x \cdot J_\psi(x, t) = \frac{i}{\hbar} \iint K_{m_0}(\vec{p}, \vec{q}) \cdot (\vec{p} - \vec{q}) a(\vec{p}) \overline{a(\vec{q})} e^{\frac{i}{\hbar}((\vec{p}-\vec{q}) \cdot x - (H_{m_0}(\vec{p}) - H_{m_0}(\vec{q}))t)} d\vec{p} d\vec{q}.$$

By definition of K_{m_0} ,

$$K_{m_0}(\vec{p}, \vec{q}) \cdot (\vec{p} - \vec{q}) = H_{m_0}(\vec{p}) - H_{m_0}(\vec{q}).$$

Therefore

$$\nabla_x \cdot J_\psi(x, t) = \frac{i}{\hbar} \iint (H_{m_0}(\vec{p}) - H_{m_0}(\vec{q})) a(\vec{p}) \overline{a(\vec{q})} e^{\frac{i}{\hbar}((\vec{p}-\vec{q}) \cdot x - (H_{m_0}(\vec{p}) - H_{m_0}(\vec{q}))t)} d\vec{p} d\vec{q}.$$

On the other hand,

$$|\psi(x, t)|^2 = \iint a(\vec{p}) \overline{a(\vec{q})} e^{\frac{i}{\hbar}((\vec{p}-\vec{q}) \cdot x - (H_{m_0}(\vec{p}) - H_{m_0}(\vec{q}))t)} d\vec{p} d\vec{q},$$

and differentiation with respect to t gives

$$\partial_t |\psi(x, t)|^2 = -\frac{i}{\hbar} \iint (H_{m_0}(\vec{p}) - H_{m_0}(\vec{q})) a(\vec{p}) \overline{a(\vec{q})} e^{\frac{i}{\hbar}((\vec{p}-\vec{q}) \cdot \vec{x} - (H_{m_0}(\vec{p}) - H_{m_0}(\vec{q}))t)} d\vec{p} d\vec{q}.$$

Comparing the last two identities yields

$$\partial_t |\psi(x, t)|^2 + \nabla_x \cdot J_\psi(x, t) = 0.$$

□

9.4. Hamilton–Jacobi Spectral Representation of the Current

Let

$$H_{m_0}(\vec{p}) = c\sqrt{|\vec{p}|^2 + m_0^2 c^2}$$

and let

$$S_{\vec{p}}^{m_0}(ct, \vec{x}) := \vec{p} \cdot \vec{x} - H_{m_0}(\vec{p})t$$

be the corresponding Hamilton–Jacobi action.

Then the massive current density may be written in the form

$$\vec{J}_\psi(ct, \vec{x}) := \iint_{\mathbb{R}^3 \times \mathbb{R}^3} K_{m_0}(\vec{p}, \vec{q}) (a \otimes \bar{a})(\vec{p}, \vec{q}) e^{\frac{i}{\hbar} S_{\vec{p}}^{m_0}(ct, \vec{x})} \otimes e^{-\frac{i}{\hbar} S_{\vec{q}}^{m_0}(ct, \vec{x})} d\vec{p} d\vec{q},$$

where

$$K_{m_0}(\vec{p}, \vec{q}) := \frac{H_{m_0}(\vec{p}) - H_{m_0}(\vec{q})}{|\vec{p} - \vec{q}|^2} (\vec{p} - \vec{q}).$$

This formula exhibits the current as a double spectral superposition of elementary Hamilton–Jacobi waves.

In the massless case $m_0 = 0$, one has

$$H_0(\vec{p}) = c|\vec{p}|, \quad S_{\vec{p}}^0(ct, \vec{x}) = \vec{p} \cdot \vec{x} - c|\vec{p}|t,$$

and therefore

$$K_0(\vec{p}, \vec{q}) = \frac{c|\vec{p}| - c|\vec{q}|}{|\vec{p} - \vec{q}|^2} (\vec{p} - \vec{q}).$$

Hence the current reduces to

$$\vec{J}_\psi^0(ct, \vec{x}) := \iint_{\mathbb{R}^3 \times \mathbb{R}^3} K_0(\vec{p}, \vec{q}) (a \otimes \bar{a})(\vec{p}, \vec{q}) e^{\frac{i}{\hbar} S_{\vec{p}}^0(ct, \vec{x})} \otimes e^{-\frac{i}{\hbar} S_{\vec{q}}^0(ct, \vec{x})} d\vec{p} d\vec{q}.$$

Thus the proper electromagnetic scalar-wave current appears as the special case $m_0 = 0$ of the general massive transport formula.

Theorem 10 (Two representations of the proper electromagnetic current). *Let*

$$\psi(ct, \vec{x}) = \int_{\mathbb{R}^3} a(\vec{p}) e^{\frac{i}{\hbar} S_{\vec{p}}^0(ct, \vec{x})} d\vec{p},$$

where

$$S_{\vec{p}}^0(ct, \vec{x}) := \vec{p} \cdot \vec{x} - c|\vec{p}|t,$$

and let

$$F_\psi = \vec{E} + i c \vec{B}$$

be the corresponding proper electromagnetic Maxwellian realization.

Then the probability current of ψ admits the two equivalent expressions

$$\vec{J}_\psi(ct, \vec{x}) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} K_0(\vec{p}, \vec{q}) (a \otimes \bar{a})(\vec{p}, \vec{q}) e^{\frac{i}{\hbar} S_p^0(ct, \vec{x})} e^{-\frac{i}{\hbar} S_q^0(ct, \vec{x})} d\vec{p} d\vec{q}$$

and

$$\vec{J}_\psi(ct, \vec{x}) = \frac{\vec{S}_{F_\psi}(ct, \vec{x})}{\int_{\mathbb{R}^3} u_{F_\psi}(ct, \vec{y}) d\vec{y}},$$

where

$$K_0(\vec{p}, \vec{q}) := \frac{c|\vec{p}| - c|\vec{q}|}{|\vec{p} - \vec{q}|^2} (\vec{p} - \vec{q}),$$

$$\vec{S}_{F_\psi}(ct, \vec{x}) := \vec{E}(ct, \vec{x}) \times c\vec{B}(ct, \vec{x}),$$

and

$$u_{F_\psi}(ct, \vec{x}) = \frac{\epsilon_0}{2} (|\vec{E}(ct, \vec{x})|^2 + |c\vec{B}(ct, \vec{x})|^2).$$

In one line, we can write

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} K_0(\vec{p}, \vec{q}) (a \otimes \bar{a})(\vec{p}, \vec{q}) e^{\frac{i}{\hbar} S_p^0(ct, \vec{x})} e^{-\frac{i}{\hbar} S_q^0(ct, \vec{x})} d\vec{p} d\vec{q} = \frac{\vec{E}(ct, \vec{x}) \times c\vec{B}(ct, \vec{x})}{\int_{\mathbb{R}^3} \frac{\epsilon_0}{2} (|\vec{E}(ct, \vec{y})|^2 + |c\vec{B}(ct, \vec{y})|^2) d\vec{y}}.$$

Or alternatively,

$$\vec{S}_{F_\psi}(ct, \vec{x}) = \|F_\psi\|^2 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} K_0(\vec{p}, \vec{q}) (a \otimes \bar{a})(\vec{p}, \vec{q}) e^{\frac{i}{\hbar} S_p^0(ct, \vec{x})} e^{-\frac{i}{\hbar} S_q^0(ct, \vec{x})} d\vec{p} d\vec{q}.$$

Theorem 11 (Direct identification of the proper electromagnetic current). *Let*

$$\psi(ct, \vec{x}) = \int_{\mathbb{R}^3} a(\vec{p}) e^{\frac{i}{\hbar} S_p^0(ct, \vec{x})} d\vec{p},$$

where

$$S_p^0(ct, \vec{x}) := \vec{p} \cdot \vec{x} - c|\vec{p}|t,$$

and let

$$F_\psi = \vec{E} + i c \vec{B}$$

be the corresponding proper electromagnetic Maxwellian realization.

Then the probability current of ψ admits the two equivalent expressions

$$\vec{J}_\psi(ct, \vec{x}) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} K_0(\vec{p}, \vec{q}) (a \otimes \bar{a})(\vec{p}, \vec{q}) e^{\frac{i}{\hbar} S_p^0(ct, \vec{x})} e^{-\frac{i}{\hbar} S_q^0(ct, \vec{x})} d\vec{p} d\vec{q}$$

and

$$\vec{J}_\psi(ct, \vec{x}) = \frac{\vec{E}(ct, \vec{x}) \times c\vec{B}(ct, \vec{x})}{\int_{\mathbb{R}^3} u_{F_\psi}(ct, \vec{y}) d\vec{y}},$$

where

$$K_0(\vec{p}, \vec{q}) := \frac{c|\vec{p}| - c|\vec{q}|}{|\vec{p} - \vec{q}|^2} (\vec{p} - \vec{q}),$$

and

$$u_{F_\psi}(ct, \vec{x}) = \frac{\epsilon_0}{2} (|\vec{E}(ct, \vec{x})|^2 + |c\vec{B}(ct, \vec{x})|^2).$$

Proof. We proceed in two steps.

Step 1. Spectral representation of the massless current. The general massive spectral current is

$$\vec{J}_\psi^{m_0}(ct, \vec{x}) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} K_{m_0}(\vec{p}, \vec{q}) (a \otimes \bar{a})(\vec{p}, \vec{q}) e^{\frac{i}{\hbar} S_{\vec{p}}^{m_0}(ct, \vec{x})} e^{-\frac{i}{\hbar} S_{\vec{q}}^{m_0}(ct, \vec{x})} d\vec{p} d\vec{q},$$

with

$$K_{m_0}(\vec{p}, \vec{q}) = \frac{H_{m_0}(\vec{p}) - H_{m_0}(\vec{q})}{|\vec{p} - \vec{q}|^2} (\vec{p} - \vec{q}),$$

and

$$H_{m_0}(\vec{p}) = c\sqrt{|\vec{p}|^2 + m_0^2 c^2}.$$

If $m_0 = 0$, then

$$H_0(\vec{p}) = c|\vec{p}|,$$

hence

$$K_0(\vec{p}, \vec{q}) = \frac{c|\vec{p}| - c|\vec{q}|}{|\vec{p} - \vec{q}|^2} (\vec{p} - \vec{q}),$$

and

$$S_{\vec{p}}^0(ct, \vec{x}) = \vec{p} \cdot \vec{x} - c|\vec{p}|t.$$

Therefore the spectral current reduces to

$$\vec{J}_\psi(ct, \vec{x}) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} K_0(\vec{p}, \vec{q}) (a \otimes \bar{a})(\vec{p}, \vec{q}) e^{\frac{i}{\hbar} S_{\vec{p}}^0(ct, \vec{x})} e^{-\frac{i}{\hbar} S_{\vec{q}}^0(ct, \vec{x})} d\vec{p} d\vec{q}.$$

Step 2. Electromagnetic realization of the same current. Since F_ψ is assumed to be a proper electromagnetic realization of ψ , the previous electromagnetic continuity theorem applies. Hence the normalized probability current is

$$\vec{J}_\psi(ct, \vec{x}) = \frac{\vec{S}_{F_\psi}(ct, \vec{x})}{\int_{\mathbb{R}^3} u_{F_\psi}(ct, \vec{y}) d\vec{y}},$$

where

$$\vec{S}_{F_\psi}(ct, \vec{x}) = \vec{E}(ct, \vec{x}) \times c\vec{B}(ct, \vec{x}).$$

Thus the same probability current admits both the spectral representation and the electromagnetic representation. Therefore

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} K_0(\vec{p}, \vec{q}) (a \otimes \bar{a})(\vec{p}, \vec{q}) e^{\frac{i}{\hbar} S_{\vec{p}}^0(ct, \vec{x})} e^{-\frac{i}{\hbar} S_{\vec{q}}^0(ct, \vec{x})} d\vec{p} d\vec{q} = \frac{\vec{E}(ct, \vec{x}) \times c\vec{B}(ct, \vec{x})}{\int_{\mathbb{R}^3} u_{F_\psi}(ct, \vec{y}) d\vec{y}}.$$

This proves the claim. $\square$

Corollary 5 (Spectral representation of the Poynting vector). *Under the assumptions of the previous theorem, one has*

$$\vec{E}(ct, \vec{x}) \times c\vec{B}(ct, \vec{x}) = \left(\int_{\mathbb{R}^3} u_{F_\psi}(ct, \vec{y}) d\vec{y} \right) \iint_{\mathbb{R}^3 \times \mathbb{R}^3} K_0 \cdot a \otimes \bar{a} e^{\frac{i}{\hbar} S_{\vec{p}}^0(ct, \vec{x})} e^{-\frac{i}{\hbar} S_{\vec{q}}^0(ct, \vec{x})} d\vec{p} d\vec{q}.$$

Theorem 12 (Intrinsic Schwartz-linear representation of the proper electromagnetic current). *Let*

$$\psi(ct, \vec{x}) = \int_{\mathbb{R}^3} (\psi)_\beta(\vec{p}) e^{\frac{i}{\hbar} S_{\vec{p}}^0(ct, \vec{x})} d\vec{p},$$

where β is the spatial de Broglie basis and

$$S_{\vec{p}}^0(ct, \vec{x}) := \vec{p} \cdot \vec{x} - c|\vec{p}|t.$$

Let

$$F_{\psi} = \vec{E} + i c \vec{B}$$

be the corresponding proper electromagnetic Maxwellian realization.

Then the probability current of ψ admits the two equivalent expressions

$$\vec{J}_{\psi}(ct, \vec{x}) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} K_0(\vec{p}, \vec{q}) ((\psi)_{\beta} \otimes \overline{(\psi)_{\beta}})(\vec{p}, \vec{q}) e^{\frac{i}{\hbar} S_{\vec{p}}^0(ct, \vec{x})} e^{-\frac{i}{\hbar} S_{\vec{q}}^0(ct, \vec{x})} d\vec{p} d\vec{q}$$

and

$$\vec{J}_{\psi}(ct, \vec{x}) = \frac{\vec{E}(ct, \vec{x}) \times c \vec{B}(ct, \vec{x})}{\int_{\mathbb{R}^3} u_{F_{\psi}}(ct, \vec{y}) d\vec{y}},$$

where

$$K_0(\vec{p}, \vec{q}) := \frac{c|\vec{p}| - c|\vec{q}|}{|\vec{p} - \vec{q}|^2} (\vec{p} - \vec{q}),$$

and

$$u_{F_{\psi}}(ct, \vec{x}) = \frac{\epsilon_0}{2} (|\vec{E}(ct, \vec{x})|^2 + |c \vec{B}(ct, \vec{x})|^2).$$

Proof. We proceed in two steps.

Step 1. Spectral Schwartz-linear expression. The massless scalar wave is reconstructed from its coordinate system with respect to the spatial de Broglie basis β :

$$\psi(ct, \vec{x}) = \int_{\mathbb{R}^3} (\psi)_{\beta}(\vec{p}) e^{\frac{i}{\hbar} S_{\vec{p}}^0(ct, \vec{x})} d\vec{p}.$$

Accordingly,

$$|\psi(ct, \vec{x})|^2 = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} ((\psi)_{\beta} \otimes \overline{(\psi)_{\beta}})(\vec{p}, \vec{q}) e^{\frac{i}{\hbar} S_{\vec{p}}^0(ct, \vec{x})} e^{-\frac{i}{\hbar} S_{\vec{q}}^0(ct, \vec{x})} d\vec{p} d\vec{q}.$$

The corresponding massless spectral current is therefore

$$\vec{J}_{\psi}(ct, \vec{x}) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} K_0(\vec{p}, \vec{q}) ((\psi)_{\beta} \otimes \overline{(\psi)_{\beta}})(\vec{p}, \vec{q}) e^{\frac{i}{\hbar} S_{\vec{p}}^0(ct, \vec{x})} e^{-\frac{i}{\hbar} S_{\vec{q}}^0(ct, \vec{x})} d\vec{p} d\vec{q},$$

where

$$K_0(\vec{p}, \vec{q}) = \frac{c|\vec{p}| - c|\vec{q}|}{|\vec{p} - \vec{q}|^2} (\vec{p} - \vec{q}).$$

Step 2. Electromagnetic expression of the same current. Since F_{ψ} is assumed to be a proper electromagnetic realization of ψ , the electromagnetic continuity theorem applies. Hence the probability current is also given by the normalized Poynting flux:

$$\vec{J}_{\psi}(ct, \vec{x}) = \frac{\vec{S}_{F_{\psi}}(ct, \vec{x})}{\int_{\mathbb{R}^3} u_{F_{\psi}}(ct, \vec{y}) d\vec{y}},$$

where

$$\vec{S}_{F_{\psi}}(ct, \vec{x}) = \vec{E}(ct, \vec{x}) \times c \vec{B}(ct, \vec{x}).$$

Thus the same current admits both the intrinsic Schwartz-linear spectral representation and the electromagnetic local representation. Therefore

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} K_0(\vec{p}, \vec{q}) ((\psi)_\beta \otimes \overline{(\psi)_\beta})(\vec{p}, \vec{q}) e^{\frac{i}{\hbar} S_\beta^0(ct, \vec{x})} e^{-\frac{i}{\hbar} S_q^0(ct, \vec{x})} d\vec{p} d\vec{q} = \frac{\vec{E}(ct, \vec{x}) \times c\vec{B}(ct, \vec{x})}{\int_{\mathbb{R}^3} u_{F_\psi}(ct, \vec{y}) d\vec{y}}.$$

This proves the claim. $\square$

Corollary 6 (Intrinsic spectral representation of the Poynting vector). *Under the assumptions of the previous theorem, one has*

$$\vec{E}(ct, \vec{x}) \times c\vec{B}(ct, \vec{x}) = \left(\int_{\mathbb{R}^3} u_{F_\psi}(ct, \vec{y}) d\vec{y} \right) \iint_{\mathbb{R}^3 \times \mathbb{R}^3} K_0 \cdot (\psi)_\beta \otimes \overline{(\psi)_\beta} e^{\frac{i}{\hbar} S_\beta^0(ct, \vec{x})} e^{-\frac{i}{\hbar} S_q^0(ct, \vec{x})} d\vec{p} d\vec{q}.$$

Remark 13. *The previous theorem shows that the proper electromagnetic current admits a double nature. On the one hand, it is the local Poynting transport*

$$\vec{E} \times c\vec{B}.$$

On the other hand, it is the intrinsic Schwartz-linear transport generated by the bi-coordinate system

$$(\psi)_\beta \otimes \overline{(\psi)_\beta}$$

through the Hamilton–Jacobi spectral kernel K_0 .

Thus the local electromagnetic flux and the spectral superposition flux are two expressions of the same underlying current.

9.5. Continuity Equation for Massive Maxwellian Fields

We now transport the previous continuity result to the Maxwellian setting.

Let

$$\psi(\cdot, t) \in \mathcal{S}(\mathbb{R}^3, \mathbb{C})$$

be a Schwartz-class solution of the relativistic square-root equation

$$i\hbar \partial_t \psi = H_{m_0}(D)\psi,$$

where

$$H_{m_0}(\vec{p}) = c\sqrt{|\vec{p}|^2 + m_0^2 c^2}.$$

Let

$$F_\psi = \mathcal{F}_e(\psi) = \int_{\mathbb{R}^3} (\psi)_\beta w^e$$

be its Maxwellian realization, where β denotes the spatial de Broglie basis and $w^e = \beta^e f_e$ is the associated Maxwellian basis.

Assume that f_e has unit effective modulus.

Define the normalized Maxwellian density by

$$\rho_{F_\psi}(x, t) := \frac{|F_\psi(x, t)|^2}{\|F_\psi\|^2}.$$

Let

$$K_{m_0}(\vec{p}, \vec{q}) := \frac{H_{m_0}(\vec{p}) - H_{m_0}(\vec{q})}{|\vec{p} - \vec{q}|^2} (\vec{p} - \vec{q}), \quad \vec{p} \neq \vec{q},$$

and define the Maxwellian current by

$$J_{F_\psi}(x, t) := \frac{1}{\|F_\psi\|^2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} K_{m_0}(\vec{p}, \vec{q}) (\psi)_\beta(\vec{p}) \overline{(\psi)_\beta(\vec{q})} e^{\frac{i}{\hbar}((\vec{p}-\vec{q}) \cdot x - (H_{m_0}(\vec{p}) - H_{m_0}(\vec{q}))t)} d\vec{p} d\vec{q}.$$

Theorem 13 (Massive Maxwellian continuity equation). *Under the above assumptions, the current J_{F_ψ} is well defined for every $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$ and the pair*

$$(\rho_{F_\psi}, J_{F_\psi})$$

satisfies the continuity equation

$$\partial_t \rho_{F_\psi}(x, t) + \nabla_x \cdot J_{F_\psi}(x, t) = 0.$$

Proof. The scalar current constructed in the previous subsection is well defined and satisfies

$$\partial_t |\psi(x, t)|^2 + \nabla_x \cdot J_\psi(x, t) = 0,$$

where

$$J_\psi(x, t) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} K_{m_0}(\vec{p}, \vec{q}) (\psi)_\beta(\vec{p}) \overline{(\psi)_\beta(\vec{q})} e^{\frac{i}{\hbar}((\vec{p}-\vec{q}) \cdot x - (H_{m_0}(\vec{p}) - H_{m_0}(\vec{q}))t)} d\vec{p} d\vec{q}.$$

On the other hand, by the norm-preservation theorem for the Maxwellian realization, one has

$$\|F_\psi\| = \|\psi\|.$$

Moreover, since the field F_ψ is obtained from ψ by replacing the scalar basis β with the vectorial basis $w^e = \beta^e f_e$ while keeping the same coordinates $(\psi)_\beta$, and since f_e has unit effective modulus, one has

$$|F_\psi(x, t)|^2 = |\psi(x, t)|^2$$

at the level of the induced quadratic density associated with the common spectral coordinates.

Therefore

$$\rho_{F_\psi}(x, t) = \frac{|F_\psi(x, t)|^2}{\|F_\psi\|^2} = \frac{|\psi(x, t)|^2}{\|\psi\|^2}.$$

Likewise,

$$J_{F_\psi}(x, t) = \frac{J_\psi(x, t)}{\|\psi\|^2}.$$

Dividing the scalar continuity equation by the constant $\|\psi\|^2$ yields

$$\partial_t \rho_{F_\psi}(x, t) + \nabla_x \cdot J_{F_\psi}(x, t) = 0.$$

This proves the claim. $\square$

Remark 14 (Massive probability transport). *The previous theorem shows that the probabilistic interpretation of a massive Maxwellian field is governed by a genuine transport law.*

In the massless electromagnetic case, the corresponding current is given by the normalized Poynting vector. In the massive case, the current is no longer expressed by a local bilinear vector identity, but by a spectral integral kernel determined by the relativistic Hamiltonian symbol.

Thus the passage from massless to massive fields replaces local electromagnetic transport by nonlocal spectral transport, while preserving the continuity structure.

Remark 15 (Interpretation in the Maxwell–Schwartz framework). *The density ρ_{F_ψ} is the normalized quadratic density of the massive Maxwellian field, while J_{F_ψ} is its associated transport current.*

Therefore:

- the Born density is realized as a normalized massive field density;
- the probability current is realized as a normalized massive field flux;
- the continuity equation expresses conservation of total probability in the Maxwell–Schwartz setting.

This gives the massive analogue of the energy–Born correspondence established earlier for electromagnetic fields.

9.6. Unifying Remarks: Local and Nonlocal Transport

Remark 16 (Two regimes of transport). *The Maxwell–Schwartz framework reveals the existence of two distinct, yet structurally unified, regimes of probability transport.*

Massless (electromagnetic) regime. *In the case of genuine electromagnetic fields, the probability density is given by the normalized energy density*

$$\rho = \frac{u_F}{\mathcal{E}_F},$$

and the probability current is given by the normalized Poynting vector

$$J = \frac{\mathbf{S}_F}{\mathcal{E}_F}.$$

The continuity equation

$$\partial_t \rho + \nabla \cdot J = 0$$

is local and follows from classical vector identities.

Massive (Maxwellian) regime. *In the massive case, the density is still quadratic,*

$$\rho = \frac{|F_\psi|^2}{\|F_\psi\|^2},$$

but the current is no longer local. It is defined by a spectral integral kernel associated with the relativistic Hamiltonian.

Thus:

- *massless fields propagate energy locally;*
- *massive fields propagate probability through nonlocal spectral transport.*

Despite this difference, both regimes satisfy the same continuity equation, which expresses conservation of total probability.

Remark 17 (Common spectral origin). *Both local and nonlocal transport laws originate from the same spectral structure.*

In both cases:

- *the state is described by its spectral coordinates $(\psi)_\eta$;*
- *the dynamics is determined by a multiplier $H(p)$ on momentum space;*
- *the continuity equation arises from the antisymmetric part of the Hamiltonian action.*

The difference between the two regimes is therefore not structural, but functional:

- *in the electromagnetic case, the symbol is linear and yields local identities;*
- *in the massive case, the square-root symbol produces a nonlocal kernel.*

Remark 18 (Born rule as transport law). *In both regimes, the Born rule admits a dynamical interpretation.*

Indeed:

- *the probability density is a normalized quadratic field density;*
- *the probability current describes the transport of this density;*
- *the continuity equation ensures conservation of total probability.*

Thus the Born rule is not merely a static prescription, but part of a transport law governing the evolution of the field.

Remark 19 (From locality to spectral nonlocality). *The passage from massless to massive fields may be interpreted as a transition from geometric locality to spectral nonlocality.*

In the electromagnetic case, transport is governed by differential operators and local vector identities.

In the massive case, transport is governed by the spectral geometry of the Hamiltonian, and is expressed through integral kernels.

This suggests that nonlocality in quantum mechanics is not an anomaly, but a natural consequence of spectral dynamics.

Remark 20 (Bell correlations revisited). *The transport interpretation developed above extends naturally to the Bell setting.*

In this perspective:

- *entangled states correspond to global Maxwellian field configurations;*
- *Bell probabilities correspond to normalized energy distributions across channels;*
- *Bell correlations arise from the joint transport of these distributions.*

Thus Bell-type phenomena reflect the global structure of the field and its transport, rather than the action of local hidden variables.

One may therefore interpret Bell nonlocality as a manifestation of global Maxwellian transport.

Remark 21 (Conceptual synthesis). *The Maxwell–Schwartz framework unifies:*

- *spectral representations of quantum states;*
- *electromagnetic field theory;*
- *probabilistic interpretation via the Born rule;*
- *transport laws expressed by continuity equations.*

Within this unified picture:

- *quantum states become field configurations;*
- *probability becomes energy distribution;*
- *dynamics becomes transport of that distribution.*

This provides a coherent and physically transparent interpretation of both massless and massive quantum systems.

9.7. Generalized Poynting Vector for Massive Maxwellian Fields

In the massive case, the probability current is given by a nonlocal spectral expression. It is therefore natural to introduce a corresponding unnormalized flux, playing the role of a generalized Poynting vector.

Let

$$F \in \mathcal{S}(\mathbb{M}_4, \mathbb{C}^3)$$

be a Maxwellian field, and let J_F denote its normalized probability current.

We define the massive Poynting vector by

$$\mathbf{S}_{F,m_0} := \|F\|^2 J_F.$$

Theorem 14 (Massive energy continuity equation). *The quadratic density*

$$u_{F,m_0} := |F|^2$$

and the flux $\mathbf{S}_{F,m_0}$ satisfy the continuity equation

$$\partial_t u_{F,m_0} + \nabla \cdot \mathbf{S}_{F,m_0} = 0.$$

Proof. The normalized density $\rho_F = |F|^2 / \|F\|^2$ and current J_F satisfy

$$\partial_t \rho_F + \nabla \cdot J_F = 0.$$

Since $\|F\|^2$ is constant in time, multiplication by $\|F\|^2$ yields

$$\partial_t |F|^2 + \nabla \cdot (\|F\|^2 J_F) = 0,$$

which proves the claim. $\square$

Remark 22 (Nonlocal nature of the flux). *In contrast with the electromagnetic case, the vector $\mathbf{S}_{F,m_0}$ is not given by a local bilinear expression such as a cross product.*

Instead, it is defined through a spectral integral kernel associated with the relativistic Hamiltonian.

Thus, the massive Poynting vector describes a nonlocal transport of energy, reflecting the spectral nature of the underlying dynamics.

Remark 23 (Unified interpretation). *Both massless and massive regimes admit a flux-density formulation:*

- *in the electromagnetic case, the flux is the classical Poynting vector;*
- *in the massive case, the flux is a generalized spectral Poynting vector.*

In both cases, probability arises as a normalized energy distribution, and its evolution is governed by a continuity equation expressing conservation of the total field energy.

The classical Poynting vector appears as the local limit of a more general spectral flux associated with relativistic Hamiltonians.

10. Relation with the Dirac Equation

In this section we clarify the relationship between the Maxwell–Schwartz framework developed above and the classical Dirac equation.

Our aim is not to reformulate Dirac theory from scratch, but to show that its solution space admits a natural decomposition in terms of solutions of the relativistic Schrödinger equation and its conjugate, within the setting of tempered distributions.

10.1. Dirac Equation and Its Solution Space

Let

$$\Psi \in \mathcal{S}'(\mathbb{M}_4, \mathbb{C}^4)$$

be a four-component spinor field satisfying the free Dirac equation

$$i\hbar \partial_t \Psi = \hat{D} \Psi,$$

where $\hat{D}$ denotes the Dirac Hamiltonian operator.

The space of solutions of this equation will be denoted by

$$\mathcal{D} \subset \mathcal{S}'(\mathbb{M}_4, \mathbb{C}^4).$$

It is well known that this space is generated by plane-wave solutions (de Broglie-type solutions) associated with the relativistic dispersion relation.

10.2. Relativistic Schrödinger Equation and Conjugate Equation

In the Maxwell–Schwartz framework, the relativistic scalar dynamics is governed by the square-root Hamiltonian

$$\hat{H}_{m_0} = c \sqrt{\hat{p}^2 + m_0^2 c^2} \mathbb{I},$$

acting on scalar distributions.

We consider the two equations:

$$i\hbar \partial_t \psi = \hat{H}_{m_0} \psi, \quad -i\hbar \partial_t \phi = \hat{H}_{m_0} \phi,$$

where $\psi, \phi \in S'(\mathbb{M}_4, \mathbb{C})$.

The second equation is the conjugate evolution equation associated with the first.

Let

$$\mathcal{H}_+, \quad \mathcal{H}_-$$

denote the corresponding solution spaces.

10.3. Cartesian Decomposition of Dirac Solutions

We now describe the structure of $\mathcal{D}$ in terms of $\mathcal{H}_+$ and $\mathcal{H}_-$.

Theorem 15 (Cartesian decomposition of Dirac solution space). *The space $\mathcal{D}$ of solutions of the free Dirac equation admits a representation of the form*

$$\mathcal{D} \cong \mathcal{H}_+^2 \times \mathcal{H}_-^2,$$

where:

- the first two components correspond to solutions of the relativistic Schrödinger equation;
- the last two components correspond to solutions of the conjugate equation.

Sketch of proof. The Dirac equation admits plane-wave solutions of the form

$$\Psi_p(x) = U(p) e^{\frac{i}{\hbar} \langle p, x \rangle},$$

where $U(p) \in \mathbb{C}^4$ is a spinor amplitude and p lies on the mass shell.

These solutions split naturally into two classes corresponding to positive and negative energy branches.

In the Maxwell–Schwartz framework, scalar solutions are generated by de Broglie waves

$$\eta_p(x) = e^{\frac{i}{\hbar} \langle p, x \rangle},$$

and their conjugates.

Therefore:

- the first two components of Ψ may be expressed as superpositions of η_p , hence belong to $\mathcal{H}_+$;
- the last two components are superpositions of conjugate waves, hence belong to $\mathcal{H}_-$.

This establishes the decomposition. $\square$

10.4. Interpretation

The previous result shows that the Dirac solution space is not an irreducible object from the point of view of Schwartz linear algebra.

Rather, it is generated by combining:

- solutions of the relativistic Schrödinger equation;
- solutions of its conjugate equation;

- arranged in a Cartesian four-component structure.

Thus, the Dirac spinor may be interpreted as a composite object built from scalar spectral components.

10.5. Factorization and First-Order Structure

The classical Dirac equation arises from a factorization of the relativistic energy operator into first-order differential operators.

In the present framework, this factorization appears as a secondary structure, derived from the square-root Hamiltonian acting on scalar states.

In particular:

- the relativistic Schrödinger equation provides the fundamental dynamics;
- the Dirac equation emerges from a two-component factorization of this dynamics;
- the four-component structure corresponds to the pairing of a solution with its conjugate.

10.6. Comparison with the Maxwell–Schwartz Approach

From the viewpoint of the present theory:

- scalar states ψ are primary objects carrying spectral information;
- Maxwellian fields F_ψ encode their geometric and energetic content;
- spin structures arise from Cartesian doubling of Maxwellian fields;
- Dirac spinors correspond to a further algebraic organization of these components.

Thus, the Dirac formalism is recovered as a derived representation within a broader Maxwell–Schwartz framework.

10.7. Conclusion

The analysis above suggests that the Dirac equation may be viewed as a factorized and reorganized version of a more fundamental spectral dynamics.

In this perspective:

- the square-root Hamiltonian provides the primary relativistic evolution;
- spinorial structures arise from internal symmetries and Cartesian constructions;
- the four-component Dirac equation reflects the coupling between a solution and its conjugate.

This interpretation is fully compatible with the classical theory, while offering a different structural viewpoint grounded in Schwartz linear algebra and Maxwellian field realizations.

11. Discussion: Conceptual Synthesis and Outlook

11.1. Summary of the Construction

In this work we have developed a Maxwell–Schwartz framework for the description of quantum states, polarization, and entanglement.

The construction is based on the following fundamental steps:

1. Scalar quantum states are modeled as tempered distributions

$$\psi \in \mathcal{S}'(\mathbb{M}_4, \mathbb{C}),$$

and described through their coordinate system $(\psi)_\eta$ with respect to the de Broglie family.

2. Spectral operators are defined intrinsically by superposition formulas of the form

$$\hat{H}(\psi) = \int_{\mathbb{M}_4^*} H(\psi)_\eta \, \eta,$$

independently of any normalization.

3. Maxwellian fields

$$F \in \mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3)$$

- are obtained by transporting the same coordinate system to a vectorial family.
- 4. The curl operator provides the Maxwellian realization of the wave-vector magnitude, while $\hbar$ realizes the momentum magnitude operator on suitable subspaces.
- 5. The relativistic Hamiltonian is represented by a square-root operator both in the scalar and in the Maxwellian setting.
- 6. Two-component Cartesian structures give rise naturally to the Pauli algebra and to spin-type observables.
- 7. Tensor products of Maxwellian fields produce entangled states and yield Bell-type correlations.

11.2. From Abstract Quantum States to Maxwellian Fields

A central conceptual outcome of the present work is the following:

Principle. Quantum states, usually described as abstract elements of Hilbert spaces, admit a concrete realization as Maxwellian fields in space-time.

In this perspective:

- the scalar state ψ encodes spectral data;
- the Maxwellian field F provides a physical realization of this data;
- operators act spectrally through coordinate multiplication;
- observables correspond to transformations of the underlying field.

11.3. Energy Interpretation of Probability

Another key result is the reinterpretation of the Born rule.

In the present framework:

- amplitudes determine the strength of Maxwellian modes;
- energy density is quadratic in the amplitude;
- probabilities correspond to normalized energy contributions.

Conclusion. The Born rule is realized as an energy distribution law for Maxwellian fields.

11.4. Entanglement as Field Coupling

Entanglement appears naturally as a structural property of tensor products of Maxwellian fields.

In particular:

- entangled states correspond to non-factorizable field configurations;
- correlations arise from the joint structure of the field;
- Bell inequality violations follow from the geometric properties of these configurations.

Thus entanglement is not an abstract feature, but a concrete manifestation of Maxwellian field superposition.

11.5. Relation with Standard Quantum Mechanics

The present framework is fully compatible with the standard formalism of quantum mechanics, while providing a different conceptual interpretation.

In particular:

- Hilbert space structures arise as secondary constructions associated with normalized bases;
- the primary structure is given by Schwartz linear algebra and spectral superposition;
- operators are defined intrinsically through multipliers on momentum space;
- spin and polarization emerge from Cartesian and tensor constructions.

11.6. On Normalization and Intrinsic Structures

A notable feature of the theory is the independence from normalization constants.
All spectral constructions depend only on the product

$$(\psi)_\eta \eta,$$

which is invariant under rescaling of the basis.

This shows that:

- orthonormality is not a fundamental requirement;
- normalized bases may be introduced when convenient;
- the underlying structure is intrinsically Schwartz-linear.

11.7. Perspectives on Spin-1/2 Fields

The two-component construction developed in this paper provides a natural entry point for the study of spin- $\frac{1}{2}$ systems.

In particular:

- Pauli operators arise from internal Cartesian structure;
- Dirac-type equations appear as first-order factorizations of the relativistic Hamiltonian;
- Maxwellian fields provide a concrete realization of spin dynamics.

A systematic development of spin- $\frac{1}{2}$ Maxwellian fields will be addressed in subsequent work.

11.8. Future Directions

Several directions naturally emerge from the present study:

- extension to multi-particle systems and higher tensor powers;
- rigorous treatment of EPR-type experiments in the Maxwell–Schwartz setting;
- development of a full theory of spin- $\frac{1}{2}$ Maxwellian fields;
- analysis of interactions and potentials within the same framework;
- systematic study of Fourier families and Schwartz bases.

11.9. Final Conclusion

The Maxwell–Schwartz framework developed in this paper provides a unified and concrete realization of quantum states, observables, and correlations.

It shows that:

- quantum states can be represented as space-time fields;
- spectral theory can be formulated without reliance on Hilbert space orthonormality;
- probability emerges from energy distribution;
- entanglement is encoded in field geometry.

This perspective opens the way to a deeper understanding of quantum phenomena in terms of continuous field structures governed by Maxwell-type equations.

12. Conclusions

12.1. Maxwell–Schwartz Quantum Theory: Synthesis and Perspectives

In this work we have developed a Maxwell–Schwartz formulation of relativistic quantum systems, in which the fundamental objects are Maxwellian fields in the space $\mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3)$, and scalar quantum states arise as spectral projections of these fields.

The construction is based on three structural ingredients:

- the spectral decomposition of states in terms of de Broglie families;
- the Maxwellian realization mapping scalar states to vectorial fields;

- the functional calculus of relativistic Hamiltonians in the Schwartz setting.

Within this framework, several central aspects of quantum theory are recovered and reinterpreted in a unified manner.

12.1.1. Relativistic Dynamics

Relativistic evolution is governed by square-root Hamiltonians acting on scalar states. On appropriate Maxwellian subspaces, the curl operator realizes the momentum magnitude operator, establishing a direct correspondence between scalar and vectorial dynamics.

First-order Dirac-type equations are obtained through factorization of the square-root operator, rather than postulated as fundamental.

12.1.2. Spin and Internal Symmetry

Spin- $\frac{1}{2}$ structures arise naturally from Cartesian doubling of Maxwellian fields. The Pauli algebra is realized as an internal symmetry acting on two-component configurations, and spin observables correspond to explicit transformations of field components.

12.1.3. Entanglement and Bell Correlations

The tensorial structure of Maxwellian fields provides a natural setting for entanglement. Bell-type correlations are derived intrinsically from field configurations and reproduce the standard quantum predictions, including the violation of Bell inequalities.

In this framework, Bell correlations are interpreted as correlations in the structure and transport of a global field, rather than as consequences of measurement postulates.

12.1.4. Energy and Probability

A central result of the present work is the identification of quantum probability with energy distribution.

The spectral coefficients of a state determine the amplitudes of its Maxwellian realization, whose energy density is quadratic in these amplitudes. The Born rule is thus recovered as a normalized energy density:

$$\rho = \frac{u_F}{\int u_F}.$$

12.1.5. Probability Transport

The electromagnetic continuity equation induces a probability continuity equation, with probability current given by the normalized Poynting vector.

This establishes a transport interpretation of quantum probability:

$$\partial_t \rho + \nabla \cdot J = 0.$$

This interpretation extends naturally to entangled states, where Bell correlations are associated with the joint transport of energy across field channels.

12.1.6. Massive Fields and Spectral Transport

The framework extends to massive Maxwellian fields governed by square-root Hamiltonians. In this case, probability currents are expressed through nonlocal spectral kernels.

A generalized Poynting vector can be defined, yielding a continuity equation of the form

$$\partial_t |F|^2 + \nabla \cdot \mathbf{S}_{F,m_0} = 0.$$

Thus, both massless and massive regimes admit a unified transport formulation.

12.1.7. Recapitulation

The main correspondences established in this work are summarized in the following table.

Quantum notion	Maxwellian realization	Structure
State ψ	Field F_ψ	Spectral realization
Spectral coefficients $(\psi)_\eta$	Mode amplitudes	Fourier–Schwartz
Norm $\ \psi\ ^2$	Total energy $\int u_F$	Parseval identity
Born density $ \psi ^2$	Energy density u_F	Quadratic structure
Probability current	Poynting vector $\mathbf{S}_F$	Local transport
Continuity equation	Energy conservation	Transport law
Spin- $\frac{1}{2}$	Two-component fields	Internal symmetry
Dirac equation	Factorized dynamics	Derived structure
Entanglement	Tensor Maxwellian fields	Global structure
Bell correlations	Energy correlations	Transport correlations
Massive dynamics	Spectral current	Nonlocal transport

12.1.8. Conceptual Perspective

The results of this work suggest a unified interpretation of quantum theory in which:

- quantum states are realized as field configurations;
- probability arises as normalized energy distribution;
- dynamics is expressed as transport of this distribution;
- nonlocality reflects spectral, rather than geometric, structure.

In this perspective, the distinction between classical and quantum descriptions is not based on the presence or absence of fields, but on the nature of their spectral organization and transport properties.

12.1.9. Outlook

Several directions remain open for further investigation:

- extension to interacting fields and external potentials;
- analysis of measurement processes and collapse within the field framework;
- geometric interpretation of spectral transport;
- extension to higher-spin systems and multi-particle configurations.

These developments may further clarify the role of Maxwellian structures in the foundations of quantum theory and their relation to classical field models.

Author Contributions: “Conceptualization, D.C.; methodology, D.C.; validation, D.C.; formal analysis, D.C.; writing—original draft preparation, D.C.; writing—review and editing, D.C.; visualization, D.C.; supervision, D.C.; project administration, D.C. All authors have read and agreed to the published version of the manuscript.”

Funding: This research received no external funding

Informed Consent Statement: Not applicable

Conflicts of Interest: The authors declare no conflicts of interest.

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