Preprints.org

Article

Not peer-reviewed version

Analytical Evaluation of Fractional Calculus and Transforms Involving the Generalized Srivastava Triple Hypergeometric Function $H_{B,q,a}^{\eta,\xi}$

Mohammad Kamarujjama , Ajija Yasmin * , Showkat Ahmad Dar

Posted Date: 4 July 2025

doi: 10.20944/preprints202507.0297.v1

Keywords: Srivastava's triple hypergeometric functions; Beta and Gamma functions; generalized Mittag-Leffler function; integral transforms and Riemann-Liouville fractional integrals and derivatives

 doi:10.20944/preprints202507.0297.v1

Article

Analytical Evaluation of Fractional Calculus and Transforms Involving the Generalized Srivastava Triple Hypergeometric Function $\mathbf{H}_{B,q,a}^{\eta,\xi}$

M. Kamarujjama [1], Ajija Yasmin [1,*] and Showkat Ahmad Dar [1,2]

[1] Department of Applied Mathematics, Faculty of Engineering and Technology, Aligarh Muslim University, Aligarh 202002, India
[2] Sainik School Mansabal, Jammu and Kashmir, Youngoora Lar Mansabal, Ganderbal 191131, India
* Correspondence: ajijayasmin001@gmail.com

Abstract

This work introduces a generalized version Srivastava triple hypergeometric function $\mathbf{H}_B(\cdot)$ by incorporating the generalized Beta function $B_{q,a}^{\eta,\xi}(\Phi_1,\Phi_2)$ introduced by Oraby et al. [12]. We examine some analytical properties of the resulting generalized function, including various integral representations involving Exton's hypergeometric function, derivative properties, and classical integral transforms such as the Euler-Beta, Laplace, Mellin, and Whittaker transforms. We also explore the effects of Riemann-Liouville fractional integral and differential operators on the generalized Srivastava's function, yielding new insights in fractional calculus. Furthermore, we derive recurrence relations for further characterization. In addition, we present a numerical approximation table of $\mathbf{H}_{B,q,a}^{\eta,\xi}(\cdot)$, computed using Wolfram Mathematica and computer algebraic software.

Keywords: Srivastava's triple hypergeometric functions; Beta and Gamma functions; generalized Mittag-Leffler function; integral transforms and Riemann-Liouville fractional integrals and derivatives

MSC: 33C60; 33C65; 33C70; 33B15; 33C05; 33E12; 26A33

---

1. Introduction and Preliminaries

Srivastava and Karlsson [23, Chapter 3] introduced and examined triple hypergeometric functions, offering a table of 205 different functions of this type. In [19,20], Srivastava presented the triple hypergeometric functions H_A, H_B, and H_C of the second order. H_A is the generalization of both F1 and F2, whereas H_B, and H_C are known to be generalizations of Appell's hypergeometric functions F_1 and F_2.

This study will concentrate on Srivastava's triple hypergeometric function H_B, which is provided as [23, p. 43, 1.5(11) to 1.5(13)] (see also [19] and [22, p. 68])

$$\mathbf{H}_B(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3) := \sum_{i,j,k=0}^{\infty} \frac{(\chi_1)_{i+k}(\chi_2)_{i+j}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k} \frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}$$

$$= \sum_{i,j,k=0}^{\infty} \frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k} \frac{B(\chi_1+i+k,\chi_2+i+j)}{B(\chi_1,\chi_2)} \frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}. \tag{1}$$

In [6, p.243], the convergence region for the hypergeometric series $\mathbf{H}_B(\cdot)$ is given as $\Phi_1 < r_1$, $\Phi_2 < r_2$, $\Phi_3 < r_3$, with the condition

$$r_1 + r_2 + r_3 + 2\sqrt{r_1r_2r_3} = 1. \tag{2}$$

Exton's function X_4 is a different type of hypergeometric function defined as

$$X_4(\chi_1,\chi_2;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3) := \sum_{i,j,k=0}^{\infty} \frac{(\chi_1)_{2i+j+k}(\chi_2)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k} \frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}, \tag{3}$$

with the convergence region for the series $2\sqrt{|\Phi_1|} + (\sqrt{|\Phi_2|} + \sqrt{|\Phi_3|})^2 < 1$.
Now, we find it suitable to introduce a new parameter b into $H_B(\cdot)$ as follows

$$\mathbf{H}_B^{(b)}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)$$

$$= \sum_{i,j,k=0}^{\infty} \frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k} \frac{B(\chi_1+b+i+k,\chi_2+b+i+j)}{B(\chi_1,\chi_2)} \frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}, \tag{4}$$

which reduces to (1) when $b=0$.

Where $(\Phi)_i$ is the Pochhammer symbol defined by

$$(\Phi)_i = \frac{\Gamma(\Phi+i)}{\Gamma(\Phi)} = \left\{ \begin{array}{ll} 1, & (i=0,\ \Phi \in \mathbb{C}\setminus\{0\}), \\ \Phi(\Phi+1)...(\Phi+i-1), & (i\in\mathbb{N}, \Phi\in\mathbb{C}) \end{array} \right\}. \tag{5}$$

Here, the classical Gamma function [13] is defined as

$$\Gamma(\Phi) = \int_0^{\infty} e^{-t}\, t^{\Phi-1}\, dt, \qquad (R(\Phi)>0), \tag{6}$$

and the classical Beta function $B(\Phi_1,\Phi_2)$ is defined as [11, (5.12.1)]

$$B(\Phi_1,\Phi_2) = \begin{cases} \displaystyle\int_0^1 t^{\Phi_1-1}(1-t)^{\Phi_2-1}dt, & (R(\Phi_1)>0, R(\Phi_2)>0) \\ \dfrac{\Gamma(\Phi_1)\Gamma(\Phi_2)}{\Gamma(\Phi_1+\Phi_2)}, & ((\Phi_1,\Phi_2)\in\mathbb{C}\backslash\mathbb{Z}_0^-). \end{cases} \tag{7}$$

In 1997, Chaudhry *et al.* [2, Eq.(1.7)] introduced a q-extension of $B(\Phi_1,\Phi_2)$ given by

$$B_q(\Phi_1,\Phi_2) = \int_0^1 t^{\Phi_1-1}(1-t)^{\Phi_2-1} \exp\left[\frac{-q}{t(1-t)}\right]dt, \qquad (R(q)\geq 0). \tag{8}$$

In 2018, Shadab *et al.* [15] has given a further extension of the Beta function as

$$B_q^{\eta}(\Phi_1,\Phi_2) = \int_0^1 t^{\Phi_1-1}(1-t)^{\Phi_2-1} E_{\eta}\left[\frac{-q}{t(1-t)}\right]dt, \qquad (R(q)\geq 0), \tag{9}$$

which reduces to (8) when $\eta=1$ and $E_\eta(z)$ denotes the Mittag-Leffler function defined by [10],

$$E_\eta(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(\eta i+1)}, \qquad (z\in\mathbb{C};\eta>0).$$

In 2020, Oraby *et al.* [12] introduced a generalized Beta function in the form

$$B_{q,a}^{\eta,\xi}(\Phi_1,\Phi_2) = \int_0^1 t^{\Phi_1-1}(1-t)^{\Phi_2-1} E_{\eta,a}\left[\frac{-q}{t^{\xi}(1-t)^{\xi}}\right]dt, \tag{10}$$

$$(R(\Phi_1)>0,\ R(\Phi_2)>0,\ R(q)\geq 0,\ R(\xi)>0),$$

 doi:10.20944/preprints202507.0297.v1

which reduces to (9) when $a = \xi = 1$.

Where $E_{\eta,a}(w)$ is the generalized Mittag-Leffler function introduced in 1905 by Wiman [24] defined as

$$E_{\eta,a}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(\eta i + a)}, \quad (z, a \in \mathbb{C}; \eta > 0, R(a) > 0).$$

Abubakar [1] introduced a generalized Beta function in 2021, as

$$\begin{aligned} {}^{\Psi}B_{\tau}^{\xi}(\Phi_1,\Phi_2) &= {}^{\Psi}B_{\tau}^{\xi}\left[\begin{array}{c|c} (c_m,\acute{c}_m)_{1,\mu_1} & \Phi_1,\Phi_2 \\ (d_n,\acute{d}_n)_{1,\mu_2} & \end{array}\right] \\ &= \int_0^1 t^{\Phi_1-1}(1-t)^{\Phi_2-1}\,{}_{\mu_1}\Psi_{\mu_2}\left(-\frac{\tau}{t^{\xi}(1-t)^{\xi}}\right)dt, \end{aligned} \tag{11}$$

($\Phi_1, \Phi_2, \tau, \xi, c_m, d_n \in \mathbb{C}$; $\acute{c}_m, \acute{d}_n \in \mathbb{R}$, $R(\Phi_1) > 0, R(\Phi_2) > 0, R(\tau) > 0, R(\xi) > 0$ *and* $m = 1, ..., \mu_1, n = 1, ..., \mu_2$)

${}_{\mu_1}\Psi_{\mu_2}$ is the Fox-Wright function ([7],[8]) defined as

$$\begin{aligned} {}_{\mu_1}\Psi_{\mu_2}\left[\begin{array}{c|c} (c_m,\acute{c}_m)_{1,\mu_1} & z \\ (d_n,\acute{d}_n)_{1,\mu_2} & \end{array}\right] &= {}_{\mu_1}\Psi_{\mu_2}\left[\begin{array}{c} (c_1,\acute{c}_1),...,(c_{\mu_1},\acute{c}_{\mu_1}); \\ (d_1,\acute{d}_1),...,(d_{\mu_2},\acute{d}_{\mu_2}); \end{array} z\right] \\ &= \sum_{k=0}^{\infty} \frac{\prod_{m=1}^{\mu_1}\Gamma(c_m+\acute{c}_m k)}{\prod_{n=1}^{\mu_2}\Gamma(d_n+\acute{d}_n k)}\frac{z^k}{k!}, \end{aligned} \tag{12}$$

where z, c_m, $d_n \in \mathbb{C}$, $\acute{c}_m \geq 0$, $\acute{d}_n \geq 0$ $(m = 1, ..., \mu_1;\ n = 1, ..., \mu_2)$. By taking

$$\Delta = \sum_{n=1}^{\mu_2}\acute{d}_n - \sum_{m=1}^{\mu_1}\acute{c}_m, \quad \delta = \prod_{m=1}^{\mu_1}\acute{c}_m^{-\acute{c}_m}\prod_{n=1}^{\mu_2}\acute{d}_n^{\acute{d}_n}, \quad \mu^* = \sum_{n=1}^{\mu_2}d_n - \sum_{m=1}^{\mu_1}c_m + \tfrac{1}{2}(\mu_1-\mu_2),$$

(12) converges for (i) $|z| < \infty$ when $\Delta > -1$,
(ii) $|z| < \delta$ when $\Delta = -1$, and
(iii) $|z| = \delta$ if, in addition, $R(\mu^*) > \frac{1}{2}$.

If $\acute{c}_1 = ... = \acute{c}_{\mu_1} = \acute{d}_1 = ... = \acute{d}_{\mu_2} = 1$ eq. (12) becomes the generalized hypergeometric function ${}_{\mu_1}F_{\mu_2}$ [26]

$${}_{\mu_1}\Psi_{\mu_2}\left[\begin{array}{c} (c_1,1),...,(c_{\mu_1},1); \\ (d_1,1),...,(d_{\mu_2},1); \end{array} z\right] = \frac{\prod_{m=1}^{\mu_1}\Gamma(c_m)}{\prod_{n=1}^{\mu_2}\Gamma(d_n)}\,{}_{\mu_1}F_{\mu_2}\left[\begin{array}{c} c_1,...,c_{\mu_1}; \\ d_1,...,d_{\mu_2}; \end{array} z\right]. \tag{13}$$

The layout of this paper is as follows. In Section 2, we define the generalized Srivastava triple hypergeometric function $\mathbf{H}_{B,q,a}^{\eta,\xi}(\cdot)$ using the generalized Beta function given in (10) and provide a numerical approximation table. Section 3 presents several integral representations involving the generalized Mittag-Leffler function and Exton's function X_4, along with some differential properties. In Section 4, we examine classical integral transforms, including the Euler–Beta, Laplace, Mellin, and Whittaker transforms. Section 5 focuses on the application of Riemann-Liouville fractional integral and differential operators to $\mathbf{H}_{B,q,a}^{\eta,\xi}(\cdot)$, leading to new results in the context of fractional calculus. Finally, in Section 6 we establishes several recurrence relations to further characterize the generalized function.

This work is also motivated by previous studies on Srivastava's triple hypergeometric function, as discussed in [3,5].

2. The Generalized Srivastava Triple Hypergeometric Function $\mathbf{H}^{\eta,\xi}_{B,q,a}(\cdot)$

Here, we define the generalized Srivastava triple $\mathbf{H}^{\eta,\xi}_{B,q,a}(\cdot)$ through the generalized Beta function defined in (10)

$$\mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)$$
$$=\sum_{i,j,k=0}^{\infty}\frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k}\frac{B^{\eta,\xi}_{q,a}(\chi_1+i+k,\chi_2+i+j)}{B(\chi_1,\chi_2)}\frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!},\tag{14}$$

where, $\chi_1,\chi_2,\chi_3\in\mathbb{C}$ and $\varrho_1,\varrho_2,\varrho_3\in\mathbb{C}\backslash\mathbb{Z}_0^-$ and the region of convergence is $|\Phi_1|<r_1$, $|\Phi_2|<r_2$, $|\Phi_3|<r_3$, with $r_1+r_2+r_3+2\sqrt{r_1r_2r_3}=1$.

In terms of the generalized Mittag-Leffler function, we now obtain $\mathbf{H}^{\eta,\xi}_{B,q,a}(\cdot)$ using the generalized beta function provided in (10) in (14) as follows:

$$\mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)=\sum_{i,j,k=0}^{\infty}\frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k\,B(\chi_1,\chi_2)}\frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}$$
$$\times\int_0^1 t^{\chi_1+i+k-1}(1-t)^{\chi_2+i+j-1}E_{\eta,a}\left[-\frac{q}{t^{\xi}(1-t)^{\xi}}\right]dt.\tag{15}$$

We use of the above eq. (15) throughout the paper.

Example 1. Table 1 provides numerical approximation values of the generalized Srivastava triple hypergeometric function $H^{\eta,\xi}_{B,q,a}(\cdot)$ as defined in (14), evaluated for various values of the parameters q, a, η, and ξ, with the indices considered up to the nth term; for example, $i=j=k=7$.

Table 1. Numerical approximation table for the generalised hypergeometric function $\mathbf{H}^{\eta,\xi}_{B,q,a}(\cdot)$ in (14), for different values of q, a, η, and ξ, with fixed parameters $\chi_1=1$, $\chi_2=2$, $\chi_3=\frac{1}{2}$, $\rho_1=\frac{2}{3}$, $\rho_2=\frac{3}{2}$, $\rho_3=\frac{1}{4}$, and variables $\phi_1=1/2$, $\phi_2=1$, $\phi_3=3/2$.

q	a	η	ξ	$\mathbf{H}^{\eta,\xi}_{B,q,a}(\cdot)$
0.005	0.010	0.015	0.020	4.2830×10^{9}
0.010	0.015	0.020	0.025	6.3695×10^{9}
0.015	0.020	0.025	0.030	8.4219×10^{9}
0.020	0.025	0.030	0.035	1.0440×10^{10}
0.025	0.030	0.035	0.040	1.2422×10^{10}
0.030	0.035	0.040	0.045	1.4368×10^{10}
0.035	0.040	0.045	0.050	1.6278×10^{10}
0.040	0.045	0.050	0.055	1.8151×10^{10}
0.045	0.050	0.055	0.060	1.9987×10^{10}
0.050	0.055	0.060	0.065	2.1784×10^{10}

3. Certain Properties of $\mathbf{H}^{\eta,\xi}_{B,q,a}(\cdot)$

We go over a few integral representations and derivative properties in this section.

1. Integral representations

The integral representations for the $\mathbf{H}^{\eta,\xi}_{B,q,a}(\cdot)$ function, which involves the product of Exton's function X_4 and the Mittag-Leffler function with two parameters, are obtained as follows.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 July 2025 doi:10.20944/preprints202507.0297.v1

Theorem 1.. *For $R(q) \geq 0$, $R(\chi_n) > 0$ $(n = 1,2,3)$ and $\min\{R(\chi_1), R(\chi_2)\} > 0$, the integral representation of $\mathbf{H}^{\eta,\xi}_{B,q,a}(\cdot)$ is as follows :*

$$\mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3) = \frac{\Gamma(\chi_1+\chi_2)}{\Gamma(\chi_1)\Gamma(\chi_2)} \int_0^1 t^{\chi_1-1}(1-t)^{\chi_2-1}$$
$$\times E_{\eta,a}\left(-\frac{q}{t^{\xi}(1-t)^{\xi}}\right) X_4(\chi_1+\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1(1-t)t,\Phi_2(1-t),\Phi_3 t)\,dt. \tag{16}$$

Proof: By changing the order of integration and summation (with uniform convergence of the integral) in (15) and using the relation (7), after simplification, utilizing the Exton's triple hypergeometric function (3), we get the result (16).

Remark 1. Through appropriate changes of the integration variable, we now discuss some special cases. To obtain the following integral representations:

$$(i)\quad \mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)$$
$$= \frac{\Gamma(\chi_1+\chi_2)}{\Gamma(\chi_1)\Gamma(\chi_2)} \frac{(x_2-x_3)^{\chi_1}(x_1-x_3)^{\chi_2}}{(x_2-x_1)^{\chi_1+\chi_2-1}} \int_{x_1}^{x_2} \frac{(x_4-x_1)^{\chi_1-1}(x_2-x_4)^{\chi_2-1}}{(x_4-x_3)^{\chi_1+\chi_2}}$$
$$\times E_{\eta,a}\left(-\frac{q}{\Lambda_1^{\xi}\Lambda_2^{\xi}}\right) X_4(\chi_1+\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1\Lambda_1\Lambda_2,\Phi_2\Lambda_1,\Phi_3\Lambda_2)\,dx_4, \tag{17}$$

where
$$\Lambda_1 = \frac{(x_1-x_3)(x_2-x_4)}{(x_2-x_1)(x_4-x_3)}, \qquad \Lambda_2 = \frac{(x_2-x_3)(x_4-x_1)}{(x_2-x_1)(x_4-x_3)} \quad (x_3 < x_1 < x_2);$$

$$(ii)\quad \mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3) = \frac{2\,\Gamma(\chi_1+\chi_2)}{\Gamma(\chi_1)\Gamma\chi_2} \int_0^{\frac{\pi}{2}} (\sin^2 x)^{\chi_1-\frac{1}{2}}(\cos^2 x)^{\chi_2-\frac{1}{2}}$$
$$\times E_{\eta,a}\left(-\frac{q}{\Lambda_1^{\xi}\Lambda_2^{\xi}}\right) X_4(\chi_1+\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1\Lambda_1\Lambda_2,\Phi_2\Lambda_1,\Phi_3\Lambda_2)\,dx, \tag{18}$$

where,
$$\Lambda_1 = \cos^2 x, \qquad \Lambda_2 = \sin^2 x;$$

$$(iii)\quad \mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)$$
$$= \frac{2\,\Gamma(\chi_1+\chi_2)(1+x_3)^{\chi_1}}{\Gamma(\chi_1)\Gamma(\chi_2)} \int_0^{\frac{\pi}{2}} \frac{(\sin^2 x_4)^{\chi_1-\frac{1}{2}}(\cos^2 x_4)^{\chi_2-\frac{1}{2}}}{(1+x_3\sin^2 x_4)^{\chi_1+\chi_2}}$$
$$\times E_{\eta,a}\left(-\frac{q}{\Lambda_1^{\xi}\Lambda_2^{\xi}}\right) X_4(\chi_1+\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1\Lambda_1\Lambda_2,\Phi_2\Lambda_1,\Phi_3\Lambda_2)\,dx_4, \tag{19}$$

where
$$\Lambda_1 = \frac{\cos^2 x_4}{1+x_3\sin^2 x_4}, \qquad \Lambda_2 = \frac{(1+x_3)\sin^2 x_4}{1+x_3\sin^2 x_4} \quad (x_3 > -1);$$

and
$$(iv)\quad \mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)$$
$$= \frac{2\,\Gamma(\chi_1+\chi_2)x_3^{\chi_1-1}}{\Gamma(\chi_1)\Gamma(\chi_2)} \int_0^{\frac{\pi}{2}} \frac{(\sin^2 x_4)^{\chi_1-\frac{1}{2}}(\cos^2 x_4)^{\chi_2-\frac{1}{2}}}{(\cos^2 x_4+x_3\sin^2 x_4)^{\chi_1+\chi_2}}$$
$$\times E_{\eta,a}\left(-\frac{q}{\Lambda_1^{\xi}\Lambda_2^{\xi}}\right) X_4(\chi_1+\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1\Lambda_1\Lambda_2,\Phi_2\Lambda_1,\Phi_3\Lambda_2)\,dx_4, \tag{20}$$

where

$$\Lambda_1 = \frac{\cos^2 x_4}{\cos^2 x_4 + x_3 \sin^2 x_4}, \qquad \Lambda_2 = \frac{x_3 \sin^2 x_4}{\cos^2 x_4 + x_3 \sin^2 x_4} \quad (x_3 > 0),$$

we put

$$\begin{aligned}
(i) \quad & t = \frac{(x_2 - x_3)(x_4 - x_1)}{(x_2 - x_1)(x_4 - x_3)}, \quad \frac{dt}{dx_4} = \frac{(x_2 - x_3)(x_1 - x_3)}{(x_2 - x_1)(x_4 - x_3)^2}, \\
(ii) \quad & t = \sin^2 x, \quad \frac{dt}{dx} = 2 \sin x \cos x, \\
(iii) \quad & t = \frac{(1 + x_3) \sin^2 x_4}{1 + x_3 \sin^2 x_4}, \quad \frac{dt}{dx_4} = \frac{2\,(1 + x_3) \sin x_4 \cos x_4}{(1 + x_3 \sin^2 x_4)^2}, \\
(iv) \quad & t = \frac{x_3 \sin^2 x_4}{\cos^2 x_4 + x_3 \sin^2 x_4}, \quad \frac{dt}{dx_4} = \frac{2 x_3 \sin x_4 \cos x_4}{(\cos^2 x_4 + x_3 \sin^2 x_4)^2}.
\end{aligned}$$

2. Derivative properties

Here we discuss some derivative propeties of $\mathbf{H}^{\eta,\xi}_{B,q,a}(\cdot)$.

Theorem 2.. *The following derivative formula holds for* $\mathbf{H}^{\eta,\xi}_{B,q,a}(\cdot)$

$$\left(\frac{d}{dq}\right)^m \left[q^{a-1} \mathbf{H}^{\eta,\xi}_{B,q^\eta,a}(\chi_1, \chi_2, \chi_3; \varrho_1, \varrho_2, \varrho_3; \Phi_1, \Phi_2, \Phi_3) \right]$$

$$= q^{a-m-1} \mathbf{H}^{\eta,\xi}_{B,q^\eta,a-m}(\chi_1, \chi_2, \chi_3; \varrho_1, \varrho_2, \varrho_3; \Phi_1, \Phi_2, \Phi_3), \tag{21}$$

where $a, \chi_n \in \mathbb{C}$, $\varrho_1, \varrho_2, \varrho_3 \in \mathbb{C} \backslash \mathbb{Z}_0^-$; $m \in \mathbb{N}$ *and* $R(a-m) > 0, R(a) > 0, \eta > 0, R(\chi_n) > 0$; $n = 1, 2, 3$.

Proof: Using (15) and employing term-wise differentiation m times in L.H.S of (21), we obtain

$$\begin{aligned}
&\left(\frac{d}{dq}\right)^m \left[q^{a-1} \mathbf{H}^{\eta,\xi}_{B,q^\eta,a}(\chi_1, \chi_2, \chi_3; \varrho_1, \varrho_2, \varrho_3; \Phi_1, \Phi_2, \Phi_3) \right] \\
&= \left(\frac{d}{dq}\right)^m \left[q^{a-1} \sum_{i,j,k=0}^{\infty} \frac{(\chi_1 + \chi_2)_{2i+j+k} (\chi_3)_{j+k}}{(\varrho_1)_i (\varrho_2)_j (\varrho_3)_k \, B(\chi_1, \chi_2)} \frac{\Phi_1^i}{i!} \frac{\Phi_2^j}{j!} \frac{\Phi_3^k}{k!} \right. \\
&\qquad \left. \times \int_0^1 t^{\chi_1 + i + k - 1} (1-t)^{\chi_2 + i + j - 1} E_{\eta,a} \left\{ -\frac{q^\eta}{t^\xi (1-t)^\xi} \right\} dt \right] \\
&= \sum_{i,j,k=0}^{\infty} \frac{(\chi_1 + \chi_2)_{2i+j+k} (\chi_3)_{j+k}}{(\varrho_1)_i (\varrho_2)_j (\varrho_3)_k \, B(\chi_1, \chi_2)} \frac{\Phi_1^i}{i!} \frac{\Phi_2^j}{j!} \frac{\Phi_3^k}{k!} \int_0^1 t^{\chi_1 + i + k - 1} (1-t)^{\chi_2 + i + j - 1} \\
&\qquad \times \sum_{l=0}^{\infty} \frac{1}{\Gamma(l\eta + a)} \left(-\frac{1}{t^\xi (1-t)^\xi} \right)^l \left(\frac{d}{dq}\right)^m \left[q^{l\eta + a - 1} \right] dt \\
&= q^{a-m-1} \sum_{i,j,k=0}^{\infty} \frac{(\chi_1 + \chi_2)_{2i+j+k} (\chi_3)_{j+k}}{(\varrho_1)_i (\varrho_2)_j (\varrho_3)_k \, B(\chi_1, \chi_2)} \frac{\Phi_1^i}{i!} \frac{\Phi_2^j}{j!} \frac{\Phi_3^k}{k!} \int_0^1 t^{\chi_1 + i + k - 1} \\
&\qquad \times (1-t)^{\chi_2 + i + j - 1} \sum_{l=0}^{\infty} \left(-\frac{q^\eta}{t^\xi (1-t)^\xi} \right)^l \frac{1}{\Gamma(l\eta + a - m)} \, dt \\
&= q^{a-m-1} \sum_{i,j,k=0}^{\infty} \frac{(\chi_1 + \chi_2)_{2i+j+k} (\chi_3)_{j+k}}{(\varrho_1)_i (\varrho_2)_j (\varrho_3)_k \, B(\chi_1, \chi_2)} \frac{\Phi_1^i}{i!} \frac{\Phi_2^j}{j!} \frac{\Phi_3^k}{k!} \int_0^1 t^{\chi_1 + i + k - 1}
\end{aligned}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 July 2025 doi:10.20944/preprints202507.0297.v1

$$\times(1-t)^{\chi_2+i+j-1}E_{\eta,a-m}\left(-\frac{q^\eta}{t^\xi(1-t)^\xi}\right)dt$$

$$= q^{a-m-1}\mathbf{H}^{\eta,\xi}_{B,q^\eta,a-m}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3).$$

Theorem 3.. *Let* $a \in \mathbb{C}$, $\varrho_1,\varrho_2,\varrho_3 \in \mathbb{C}\backslash\mathbb{Z}_0^-$, *and* $\eta > 0$, $R(a) > 0$, *then* $\mathbf{H}^{\eta,\xi}_{B,q,a}(\cdot)$ *satisfies the following differentiation formula*

$$\begin{aligned}&\mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)\\ &= a\mathbf{H}^{\eta,\xi}_{B,q,a+1}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)+\eta\, q\frac{d}{dq}\mathbf{H}^{\eta,\xi}_{B,q,a+1}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3).\end{aligned} \tag{22}$$

Proof: Using (15) in the R.H.S of (22), yields

$$\begin{aligned}&a\mathbf{H}^{\eta,\xi}_{B,q,a+1}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)+\eta\, q\frac{d}{dq}\mathbf{H}^{\eta,\xi}_{B,q,a+1}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)\\ &= a\mathbf{H}^{\eta,\xi}_{B,q,a+1}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)+\sum_{i,j,k=0}^{\infty}\frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k\,B(\chi_1,\chi_2)}\frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}\\ &\times\int_0^1 t^{\chi_1+i+k-1}(1-t)^{\chi_2+i+j-1}\sum_{l=0}^{\infty}\left(-\frac{1}{t^\xi(1-t)^\xi}\right)^l\frac{\eta^l q^l}{\Gamma(l\eta+a+1)}\,dt\end{aligned} \tag{23}$$

Now using the relation $\Gamma(a+1) = a\Gamma(a)$ in (23), we get

$$\begin{aligned}&= \sum_{i,j,k=0}^{\infty}\frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k\,B(\chi_1,\chi_2)}\frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}\int_0^1 t^{\chi_1+i+k-1}(1-t)^{\chi_2+i+j-1}\\ &\times\sum_{l=0}^{\infty}\left(-\frac{q}{t^\xi(1-t)^\xi}\right)^l\frac{1}{\Gamma(l\eta+a)}\,dt\\ &= \mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3).\end{aligned}$$

4. Integral Transform of $\mathbf{H}^{\eta,\xi}_{B,q,a}(\cdot)$

The focus here is on analyzing integral transforms of the function $\mathbf{H}^{\eta,\xi}_{B,q,a}(\cdot)$, specifically the Euler-Beta, Laplace, Mellin, and Whittaker transforms.

1. Euler-Beta Transform

The Euler-Beta transform of the function $f(q)$ is defined as [18]

$$\mathbf{B}\{f(q);\alpha_1,\alpha_2\} = \int_0^1 q^{\alpha_1-1}(1-q)^{\alpha_2-1}f(q)\,dq. \tag{24}$$

Theorem 4.. *For* $a,\alpha_1,\alpha_2,\xi,(\chi_1+i+k),(\chi_2+i+j) \in \mathbb{C}$; $\eta > 0$, $R(a) > 0$, $R(\alpha_1) > 0$, $R(\alpha_2) > 0$, $R(\chi_1+i+k) > 0$, $R(\chi_2+i+j) > 0$, *we have the following Euler-Beta transform of* $\mathbf{H}^{\eta,\xi}_{B,q,a}(\cdot)$

$$\mathbf{B}\left\{\mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3);\alpha_1,\alpha_2\right\} = \Gamma(\alpha_2)\sum_{i,j,k=0}^{\infty}\frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k\,B(\chi_1,\chi_2)}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 July 2025 doi:10.20944/preprints202507.0297.v1

$$\times \frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}\,{}^{\Psi}B_1^{\xi}\left[\begin{array}{c}(1,1),(\alpha_1,1)\\(a,\eta),(\alpha_1+\alpha_2,1)\end{array}\middle|(\chi_1+i+k),(\chi_2+i+j)\right]. \tag{25}$$

Proof: Using (24) in (15), we obtain

$$\mathbf{B}\left\{\mathbf{H}_{B,q,a}^{\eta,\xi}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3);\alpha_1,\alpha_2\right\}$$

$$=\int_0^1 q^{\alpha_1-1}(1-q)^{\alpha_2-1}\,\mathbf{H}_{B,q,a}^{\eta,\xi}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)\,dq$$

$$=\sum_{i,j,k=0}^{\infty}\frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k\,B(\chi_1,\chi_2)}\frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}$$

$$\times\left(\int_0^1 t^{\chi_1+i+k-1}(1-t)^{\chi_2+i+j-1}\left\{\int_0^1 q^{\alpha_1-1}\,(1-q)^{\alpha_2-1}E_{\eta,a}\left(-\frac{q}{t^{\xi}(1-t)^{\xi}}\right)dq\right\}dt\right). \tag{26}$$

By evaluating the integral as

$$\int_0^1 q^{\alpha_1-1}\,(1-q)^{\alpha_2-1}E_{\eta,a}\left(-\frac{q}{t^{\xi}(1-t)^{\xi}}\right)dq$$
$$=\sum_{l=0}^{\infty}\left(-\frac{1}{t^{\xi}(1-t)^{\xi}}\right)^l\frac{1}{\Gamma(l\eta+a)}\int_0^1 q^{\alpha_1+l-1}\,(1-p)^{\alpha_2-1}\,dq$$
$$=\Gamma(\alpha_2)\sum_{l=0}^{\infty}\left(-\frac{1}{t^{\xi}(1-t)^{\xi}}\right)^l\frac{\Gamma(\alpha_1+l)\Gamma(l+1)}{\Gamma(l\eta+a)\Gamma(\alpha_1+\alpha_2+l)\,l!},$$

using the Fox-Wright function (12) in the above sum and employing in (26), we obtain,

$$\mathbf{B}\left\{\mathbf{H}_{B,q,a}^{\eta,\xi}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3);\alpha_1,\alpha_2\right\}=\sum_{i,j,k=0}^{\infty}\frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k\,B(\chi_1,\chi_2)}$$

$$\times\frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}\Gamma(\alpha_2)\left(\int_0^1 t^{\chi_1+i+k-1}(1-t)^{\chi_2+i+j-1}{}_2\Psi_2\left[\begin{array}{c}(1,1),(\alpha_1,1)\\(a,\eta),(\alpha_1+\alpha_2,1)\end{array};-\frac{1}{t^{\xi}(1-t)^{\xi}}\right]dt\right).$$

The above integral is evaluated using the generalized Beta function given in (11) to reach the required result (25).

2. Laplace Transform

The Laplace transform of a function $f(q)$ is defined as [18]

$$\mathbf{L}\{f(q)\}=\int_0^{\infty}e^{-sq}f(q)\,dq \tag{27}$$

Theorem 5. *For $\eta,\alpha,a,(\chi_1+i+k),(\chi_2+i+j)\in\mathbb{C}$; $R(\eta)>0$, $R(\alpha)>0$, $R(a)>0$, $R(s)>0$, $R(\chi_1+i+k)>0$, $R(\chi_2+i+j)>0$ and $\left|-\frac{1}{st^{\xi}(1-t)^{\xi}}\right|<1$, we have the following Laplace transform of $\mathbf{H}_{B,q,a}^{\eta,\xi}(\cdot)$*

$$\int_0^{\infty}q^{\alpha-1}e^{-sq}\mathbf{H}_{B,q,a}^{\eta,\xi}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)dq=\frac{1}{s^{\alpha}}\sum_{i,j,k=0}^{\infty}\frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k\,B(\chi_1,\chi_2)}$$

$$\times\frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}\,{}^{\Psi}B_{\left(\frac{1}{s}\right)}^{\xi}\left[\begin{array}{c}(1,1),(\alpha,1)\\(a,\eta)\end{array}\middle|(\chi_1+i+k),(\chi_2+i+j)\right]. \tag{28}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 July 2025 doi:10.20944/preprints202507.0297.v1

Proof: Employing (15) in the integral on the L.H.S. of (28), yields

$$\int_0^\infty q^{\alpha-1} e^{-sq}\, \mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)\, dq$$

$$= \sum_{i,j,k=0}^{\infty} \frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k\, B(\chi_1,\chi_2)} \frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}$$

$$\times\left(\int_0^1 t^{\chi_1+i+k-1}(1-t)^{\chi_2+i+j-1}\left\{\int_0^\infty e^{-sq}\, q^{\alpha-1} E_{\eta,a}\left(-\frac{q}{t^\xi(1-t)^\xi}\right) dq\right\} dt\right). \tag{29}$$

By evaluating the integral as

$$\int_0^\infty e^{-sq}\, q^{\alpha-1} E_{\eta,a}\left(-\frac{q}{t^\xi(1-t)^\xi}\right) dq$$
$$= \sum_{l=0}^{\infty}\left(-\frac{1}{t^\xi(1-t)^\xi}\right)^l \frac{1}{\Gamma(l\eta+a)} \int_0^\infty e^{-sq}\, q^{\alpha+l-1}\, dq$$
$$= \frac{1}{s^\alpha}\sum_{l=0}^{\infty} \frac{\Gamma(\alpha+l)\Gamma(l+1)}{\Gamma(l\eta+a)\, l!}\left(-\frac{1}{s\, t^\xi(1-t)^\xi}\right)^l,$$

using the Fox-Wright function (12) in the above sum and employing in (29), we find,

$$\int_0^\infty q^{\alpha-1} e^{-sq}\, \mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3) dq = \sum_{i,j,k=0}^{\infty} \frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k\, B(\chi_1,\chi_2)}$$

$$\times \frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}\frac{1}{s^\alpha}\left(\int_0^1 t^{\chi_1+i+k-1}(1-t)^{\chi_2+i+j-1}{}_2\Psi_1\left[\begin{matrix}(1,1),(\alpha,1)\\(a,\eta)\end{matrix};-\frac{1}{s\, t^\xi(1-t)^\xi}\right] dt\right).$$

The above integral is evaluated using the generalized Beta function given in (11) to obtain the required result (28).

3. Mellin Transform

The Mellin transform of the function $f(q)$ is defined as [18],

$$f^*(s) = \mathbf{M}\{f(q);s\} = \int_0^\infty q^{s-1} f(q)\, dq, \qquad R(s) > 0, \tag{30}$$

and the inverse Mellin transform is given by

$$f(q) = \mathbf{M}^{-1}\{f^*(s);q\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} q^{-s} f^*(s)\, ds, \qquad c \in \mathrm{R}. \tag{31}$$

Theorem 6.. *For $a,s \in \mathrm{C}$; $R(s) > a > 0$, $R(a) > 0$, $\eta > 0$, $\varrho_1,\varrho_2,\varrho_3 \in \mathrm{C}\backslash\mathrm{Z}_0^-$, we have the following Mellin transform of $\mathbf{H}^{\eta,\xi}_{B,q,a}(\cdot)$*

$$\mathbf{M}\left\{\mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3);s\right\}$$
$$= \int_0^\infty q^{s-1}\mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)\, dq$$
$$= \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(a-\eta s)}\mathbf{H}_B^{(\xi s)}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3), \tag{32}$$

where $\mathbf{H}_B^{(\xi s)}(\cdot)$ is given in (4).

Proof: Employing (15) in the integral on the L.H.S. of (32), we obtain

$$\mathbf{M}\Big\{\mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3);s\Big\}$$

$$=\sum_{i,j,k=0}^{\infty}\frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k\,B(\chi_1,\chi_2)}\frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}$$

$$\times\int_0^1 t^{\chi_1+i+k-1}(1-t)^{\chi_2+i+j-1}\Big\{\int_0^\infty q^{s-1}E_{\eta,a}\Big(-\frac{q}{t^\xi(1-t)^\xi}\Big)dq\Big\}dt. \tag{33}$$

Now, by taking $\lambda=p=1$ and $y=-q/t^\xi(1-t)^\xi$ in the result ([16], Th. 4.1) as follows

$$E^{\lambda,p}_{\eta,a}(y)\,dy=\frac{1}{2\pi i\,\Gamma(\lambda)}\int_L\frac{\Gamma(s)\Gamma(\lambda-ps)}{\Gamma(a-\eta s)}(-y)^{-s}\,ds,$$

where L is the contour of integration that begins at $c-i\infty$ and ends at $c+i\infty$, $c\in\mathrm{R}$,
yields,

$$E_{\eta,a}\Big(-\frac{q}{t^\xi(1-t)^\xi}\Big)=\frac{1}{2\pi i}\int_L\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(a-\eta s)}\Big(\frac{q}{t^\xi(1-t)^\xi}\Big)^{-s}ds$$

$$=\frac{1}{2\pi i}\int_L f^*(s)\,q^{-s}\,ds,$$

where

$$f^*(s)=\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(a-\eta s)}\Big[t^\xi(1-t)^\xi\Big]^s. \tag{34}$$

Using (30), (31), and (34), we get

$$\int_0^\infty q^{s-1}E_{\eta,a}\Big(-\frac{q}{t^\xi(1-t)^\xi}\Big)dq=\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(a-\eta s)}\Big[t^\xi(1-t)^\xi\Big]^s \tag{35}$$

Now, put (35) in (33), leads to

$$\mathbf{M}\Big\{\mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3);s\Big\}=\sum_{i,j,k=0}^{\infty}\frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k\,B(\chi_1,\chi_2)}$$

$$\times\frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(a-\eta s)}\int_0^1 t^{\chi_1+i+k+\xi s-1}(1-t)^{\chi_2+i+j+\xi s-1}dt.$$

Evaluate the integral using the classical Beta function defined in (7), yields

$$\mathbf{M}\Big\{\mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3);s\Big\}$$

$$=\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(a-\eta s)}\sum_{i,j,k=0}^{\infty}\frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k}\frac{B(\chi_1+\xi s+i+k,\chi_2+\xi s+i+j)}{B(\chi_1,\chi_2)}\frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}.$$

Identifying the above sum as $\mathbf{H}^{(\xi s)}_B(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)$ given in (4), we get the desired result (32).

Corollary 1: For $c>a$, the inverse Mellin transform formula for $\mathbf{H}^{\eta,\xi}_{B,q,a}(\cdot)$ is as follows:

$$\mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)=\mathbf{M}^{-1}\{f^*(s);q\}$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}q^{-s}\frac{\Gamma(s)\,\Gamma(1-s)}{\Gamma(a-\eta s)}H^{(\xi s)}_B(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)ds. \tag{36}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 July 2025 doi:10.20944/preprints202507.0297.v1

5. Whittaker Transform

Theorem 7.. *For* $\alpha_1, a, (\chi_1+i+k), (\chi_2+i+j) \in \mathbb{C}$; $\eta > 0$, $R(\alpha_1) > 0$, $R(a) > 0$, $R(\chi_1+i+k) > 0$, $R(\chi_2+i+j) > 0$, *we have the following Whittaker transform of* $\mathbf{H}^{\eta,\xi}_{B,q,a}(\cdot)$

$$\int_0^\infty q^{\alpha_1-1}\, e^{-\frac{1}{2}\alpha_2 q}\, \mathbf{W}_{\tau_1,\tau_2}(\alpha_2 q)\, \mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)\, dq$$

$$= \frac{1}{\alpha_2^{\alpha_1}} \sum_{i,j,k=0}^{\infty} \frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k\, B(\chi_1,\chi_2)} \frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}$$

$$\times {}^{\Psi}B^{\xi}_{(\frac{1}{\alpha_2})}\left[\begin{array}{c} (1,1),(\frac{1}{2}\pm\tau_2+\alpha_1,1) \\ (a,\eta),(1-\tau_1+\alpha_1,1) \end{array}\middle|(\chi_1+i+k),(\chi_2+i+j)\right]. \tag{37}$$

Proof: Applying (37) in (15), yields

$$\int_0^\infty q^{\alpha_1-1}\, e^{-\frac{1}{2}\alpha_2 q}\, \mathbf{W}_{\tau_1,\tau_2}(\alpha_2 q)\, \mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)\, dq$$

$$= \sum_{i,j,k=0}^{\infty} \frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k\, B(\chi_1,\chi_2)} \frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}$$

$$\times\left(\int_0^1 t^{\chi_1+i+k-1}(1-t)^{\chi_2+i+j-1}\left\{\int_0^\infty q^{\alpha_1-1}\, e^{-\frac{1}{2}\alpha_2 q}\mathbf{W}_{\tau_1,\tau_2}(\alpha_2 q) E_{\eta,a}\left(-\frac{q}{t^{\xi}(1-t)^{\xi}}\right) dq\right\} dt\right). \tag{38}$$

Now consider,

$$I = \int_0^\infty q^{\alpha_1-1}\, e^{-\frac{1}{2}\alpha_2 q}\, \mathbf{W}_{\tau_1,\tau_2}(\alpha_2 q)\, E_{\eta,a}\left(-\frac{q}{t^{\xi}(1-t)^{\xi}}\right) dq$$

put $\alpha_2 q = u$ in the above equation, yields

$$I = \int_0^\infty \left(\frac{u}{\alpha_2}\right)^{\alpha_1-1} e^{\frac{-u}{2}}\, \mathbf{W}_{\tau_1,\tau_2}(u) \sum_{l=0}^{\infty}\left(-\frac{u}{\alpha_2\, t^{\xi}(1-t)^{\xi}}\right)^l \frac{1}{\Gamma(l\eta+a)}\frac{1}{\alpha_2}\, du$$

$$= \frac{1}{\alpha_2^{\alpha_1}} \sum_{l=0}^{\infty}\left(-\frac{1}{\alpha_2\, t^{\xi}(1-t)^{\xi}}\right)^l \frac{1}{\Gamma(l\eta+a)} \int_0^\infty u^{l+\alpha_1-1}\, e^{-\frac{u}{2}}\, \mathbf{W}_{\tau_1,\tau_2}(u)\, du. \tag{39}$$

Now using the relation

$$\int_0^\infty u^{\alpha_1-1}\, e^{-\frac{u}{2}}\, \mathbf{W}_{\tau_1,\tau_2}(u)\, du = \frac{\Gamma(\frac{1}{2}+\tau_2+\alpha_1)\,\Gamma(\frac{1}{2}-\tau_2+\alpha_1)}{\Gamma(1-\tau_1+\alpha_1)}, \quad \left(R(\tau_2\pm\alpha_1) > \frac{-1}{2}\right),$$

where $\mathbf{W}_{\tau_1,\tau_2}(u)$ is the Whittaker function [25], in (39), we get

$$I = \frac{1}{\alpha_2^{\alpha_1}} \sum_{l=0}^{\infty}\left(-\frac{1}{\alpha_2\, t^{\xi}(1-t)^{\xi}}\right)^l \frac{\Gamma(\frac{1}{2}+\tau_2+\alpha_1+l)\,\Gamma(\frac{1}{2}-\tau_2+\alpha_1+l)\,\Gamma(l+1)}{\Gamma(l\eta+a)\,\Gamma(1-\tau_1+\alpha_1+l)\, l!}$$

Using (12), we get

$$I = \frac{1}{\alpha_2^{\alpha_1}}\, {}_3\Psi_2\left[\begin{array}{c} (1,1),(\frac{1}{2}+\tau_2+\alpha_1,1),(\frac{1}{2}-\tau_2+\alpha_1,1) \\ (a,\eta),(1-\tau_1+\alpha_1,1) \end{array}; -\frac{1}{\alpha_2\, t^{\xi}(1-t)^{\xi}}\right]. \tag{40}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 4 July 2025 doi:10.20944/preprints202507.0297.v1

Applying (40) in (38), we obtain

$$\int_0^\infty q^{\alpha_1-1} e^{-\frac{1}{2}\alpha_2 q}\, \mathbf{W}_{\tau_1,\tau_2}(\alpha_2 q)\, \mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)\, dq$$

$$= \sum_{i,j,k=0}^{\infty} \frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k\, B(\chi_1,\chi_2)} \frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}$$

$$\times \int_0^1 t^{\chi_1+i+k-1}(1-t)^{\chi_2+i+j-1} \frac{1}{\alpha_2^{\alpha_1}}\, {}_3\Psi_2\left[\begin{matrix}(1,1),(\frac{1}{2}\pm\tau_2+\alpha_1,1) \\ (a,\eta),(1-\tau_1+\alpha_1,1)\end{matrix};-\frac{1}{\alpha_2\, t^{\xi}(1-t)^{\xi}}\right]dt.$$

The above integral is evaluated using the generalized Beta function given in (11) to obtain the required result (37).

6. Fractional Calculus of $\mathbf{H}^{\eta,\xi}_{B,q,a}(\cdot)$

In this part, we examine various findings related to the right-sided Riemann-Liouville fractional integral operator $I^{\theta}_{\alpha+}$ and derivative operator $D^{\theta}_{\alpha+}$, which are defined as follows, respectively [9,14]:

$$(I^{\theta}_{\alpha+}f)(w) = \frac{1}{\Gamma(\theta)}\int_{\alpha}^{w} \frac{f(q)}{(w-q)^{1-\theta}}dq, \quad (R(\theta)>0,\ \theta\in\mathrm{C}), \tag{41}$$

and

$$(D^{\theta}_{\alpha+}f)(w) = \left(\frac{d}{dw}\right)^m (I^{m-\theta}_{\alpha+}f)(w), \quad (m=[R(\theta)]+1;\ R(\theta)>0,\ \alpha\in\mathrm{C}), \tag{42}$$

where $[w]$ is the greatest integer.

A generalized Riemann-Liouville fractional derivative operator $D^{\theta,\lambda}_{\alpha+}$ of order $0<\theta<1$ and type $0\le\lambda\le 1$ with respect to w was introduced by Hilfer [4] as follows:

$$(D^{\theta,\lambda}_{\alpha+}f)(w) = \left(I^{\lambda(1-\theta)}_{\alpha+}\frac{d}{dw}\right)\left(I^{(1-\lambda)(1-\theta)}_{\alpha+}f\right)(w), \qquad (m=[R(\theta)]+1;\ R(\theta)>0, \theta\in\mathrm{C}). \tag{43}$$

Theorem 8.. *Let $\alpha\in\mathrm{R}_+$, then for $w>\alpha$, the function $\mathbf{H}^{\eta,\xi}_{B,q,a}(\cdot)$ satisfies the following relations*

$$(i)\left(I^{\theta}_{\alpha+}\left[(q-\alpha)^{a-1}\mathbf{H}^{\eta,\xi}_{B,(q-\alpha)^{\eta},a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)\right]\right)(w)$$

$$= (w-\alpha)^{a+\theta-1}\mathbf{H}^{\eta,\xi}_{B,(w-\alpha)^{\eta},a+\theta}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3), \tag{44}$$

$$(ii)\left(D^{\theta}_{\alpha+}\left[(q-\alpha)^{a-1}\mathbf{H}^{\eta,\xi}_{B,(q-\alpha)^{\eta},a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)\right]\right)(w)$$

$$= (w-\alpha)^{a-\theta-1}\mathbf{H}^{\eta,\xi}_{B,(w-\alpha)^{\eta},a-\theta}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3), \tag{45}$$

and

$$(iii)\left(D^{\theta,\lambda}_{\alpha+}\left[(q-\alpha)^{a-1}\mathbf{H}^{\eta,\xi}_{B,(q-\alpha)^{\eta},a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)\right]\right)(w)$$

$$= (w-\alpha)^{a-\theta-1}\mathbf{H}^{\eta,\xi}_{B,(w-\alpha)^{\eta},a-\theta}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3), \tag{46}$$

where $a,\theta\in\mathrm{C};\ \eta>0,\ R(a)>0,\ R(\theta)>0$, and $\varrho_1,\varrho_2,\varrho_3\in\mathrm{C}\backslash\mathrm{Z}_0^-$.

Proof. Use (15) in (41), and with the help of the relation [14]

$$\left(I^{\theta}_{\alpha+}\left[(q-\alpha)^{\nu-1}\right]\right)(w) = \frac{\Gamma(\nu)}{\Gamma(\nu+\theta)}(w-\alpha)^{\theta+\nu-1},$$

for $w>\alpha$ it follows that,

$$
\begin{aligned}
&\left(I^{\theta}_{\alpha+}\left[(q-\alpha)^{a-1}\mathbf{H}^{\eta,\xi}_{B,(q-\alpha)^{\eta},a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)\right]\right)(w)\\
&=\left(I^{\theta}_{\alpha+}\left[(q-\alpha)^{a-1}\sum_{i,j,k=0}^{\infty}\frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k\,B(\chi_1,\chi_2)}\frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}\right.\right.\\
&\qquad\left.\left.\times\int_0^1 t^{\chi_1+i+k-1}(1-t)^{\chi_2+i+j-1}E_{\eta,a}\left\{-\frac{(q-\alpha)^{\eta}}{t^{\xi}(1-t)^{\xi}}\right\}dt\right]\right)(w)\\
&=\sum_{i,j,k=0}^{\infty}\frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k\,B(\chi_1,\chi_2)}\frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}\int_0^1 t^{\chi_1+i+k-1}(1-t)^{\chi_2+i+j-1}\\
&\qquad\times\sum_{l=0}^{\infty}\left(-\frac{1}{t^{\xi}(1-t)^{\xi}}\right)^{l}\frac{1}{\Gamma(l\eta+a)}\left(I^{\theta}_{\alpha+}\left[(q-\alpha)^{l\eta+a-1}\right]dt\right)(w)\\
&=(w-\alpha)^{a+\theta-1}\sum_{i,j,k=0}^{\infty}\frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k\,B(\chi_1,\chi_2)}\frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}\\
&\qquad\times\int_0^1 t^{\chi_1+i+k-1}(1-t)^{\chi_2+i+j-1}E_{\eta,a+\theta}\left\{-\frac{(w-\alpha)^{\eta}}{t^{\xi}(1-t)^{\xi}}\right\}dt\\
&=(w-\alpha)^{a+\theta-1}\mathbf{H}^{\eta,\xi}_{B,(w-\alpha)^{\eta},a+\theta}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3).
\end{aligned}
$$

Now, using (15) in (42), yields

$$
\begin{aligned}
&\left(D^{\theta}_{\alpha+}\left[(q-\alpha)^{a-1}\mathbf{H}^{\eta,\xi}_{B,(q-\alpha)^{\eta},a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)\right]\right)(w)\\
&=\left(\frac{d}{dw}\right)^{m}\left(I^{m-\theta}_{\alpha+}\left[(q-\alpha)^{a-1}\mathbf{H}^{\eta,\xi}_{B,(q-\alpha)^{\eta},a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)\right]\right)(w)\\
&=\left(\frac{d}{dw}\right)^{m}\left[(w-\alpha)^{a+m-\theta-1}\mathbf{H}^{\eta,\xi}_{B,(w-\alpha)^{\eta},a+m-\theta}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)\right].
\end{aligned}
$$

Using (21), reaches the result (45).

Furthermore, after using the eqs. (15) and (43), follows that

$$
\begin{aligned}
&\left(D^{\theta,\lambda}_{\alpha+}\left[(q-\alpha)^{a-1}\mathbf{H}^{\eta,\xi}_{B,(q-\alpha)^{\eta},a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)\right]\right)(w)\\
&=\left(D^{\theta,\lambda}_{\alpha+}\left[(q-\alpha)^{a-1}\sum_{i,j,k=0}^{\infty}\frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k\,B(\chi_1,\chi_2)}\frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}\right.\right.\\
&\qquad\left.\left.\times\int_0^1 t^{\chi_1+i+k-1}(1-t)^{\chi_2+i+j-1}E_{\eta,a}\left\{-\frac{(q-\alpha)^{\eta}}{t^{\xi}(1-t)^{\xi}}\right\}dt\right]\right)(w)\\
&=\sum_{i,j,k=0}^{\infty}\frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k\,B(\chi_1,\chi_2)}\frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}\int_0^1 t^{\chi_1+i+k-1}(1-t)^{\chi_2+i+j-1}\\
&\qquad\times\sum_{l=0}^{\infty}\left(-\frac{1}{t^{\xi}(1-t)^{\xi}}\right)^{l}\frac{1}{\Gamma(l\eta+a)}\left(D^{\theta,\lambda}_{\alpha+}\left[(q-\alpha)^{l\eta+a-1}\right]dt\right)(w).
\end{aligned}
\tag{47}
$$

Applying the result of Srivastava and Tomovski ([21], p.203, eq.(2.18))

$$
\left(D^{\theta,\lambda}_{\alpha+}\left[(q-\alpha)^{\beta-1}\right]\right)(w)=\frac{\Gamma(\beta)}{\Gamma(\beta-\theta)}(w-\alpha)^{\beta-\theta-1}
$$

$$
(w>\alpha;\ 0<\theta<1;\ 0\leq\lambda\leq1;\ R(\beta)>0),
$$

 doi:10.20944/preprints202507.0297.v1

in (47), (46) is established.

7. Recurrence Relations for $\mathbf{H}^{\eta,\xi}_{B,q,a}(\cdot)$

In this section, we derive some recurrence relations for $\mathbf{H}^{\eta,\xi}_{B,q,a}(\cdot)$.

Theorem 9.. *With respect to the numerator parameter χ_3, we have the following recurrences for* $\mathbf{H}^{\eta,\xi}_{B,q,a}(\cdot)$

$$\begin{aligned}\mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3+1;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3) &= \mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)\\ &+\frac{\Phi_2\chi_2}{\varrho_2}\mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2+1,\chi_3+1;\varrho_1,\varrho_2+1,\varrho_3;\Phi_1,\Phi_2,\Phi_3)\\ &+\frac{\Phi_3\chi_1}{\varrho_3}\mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1+1,\chi_2,\chi_3+1;\varrho_1,\varrho_2,\varrho_3+1;\Phi_1,\Phi_2,\Phi_3),\end{aligned}\tag{48}$$

and

$$\begin{aligned}\mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3-1;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3) &= \mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)\\ &-\frac{\Phi_2\chi_2}{\varrho_2}\mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2+1,\chi_3-1;\varrho_1,\varrho_2+1,\varrho_3;\Phi_1,\Phi_2,\Phi_3)\\ &-\frac{\Phi_3\chi_1}{\varrho_3}\mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1+1,\chi_2,\chi_3-1;\varrho_1,\varrho_2,\varrho_3+1;\Phi_1,\Phi_2,\Phi_3).\end{aligned}\tag{49}$$

Proof. With the help of the identity $(\chi_3+1)_{j+k} = (\chi_3)_{j+k}(1+j/\chi_3+k/\chi_3)$ and from (14), we find

$$\begin{aligned}&\mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3+1;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)\\ &\quad=\sum_{i,j,k=0}^{\infty}\frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3+1)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k}\frac{B^{\eta,\xi}_{q,a}(\chi_1+i+k,\chi_2+i+j)}{B(\chi_1,\chi_2)}\frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}\\ &=\mathbf{H}^{\eta,\xi}_{B,q,a}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)\\ &\quad+\frac{\Phi_2}{\chi_3}\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k}\frac{B^{\eta,\xi}_{q,a}(\chi_1+i+k,\chi_2+i+j)}{B(\chi_1,\chi_2)}\frac{\Phi_1^i}{i!}\frac{\Phi_2^{j-1}}{(j-1)!}\frac{\Phi_3^k}{k!}\\ &\quad+\frac{\Phi_3}{\chi_3}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}\frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k}\frac{B^{\eta,\xi}_{q,a}(\chi_1+i+k,\chi_2+i+j)}{B(\chi_1,\chi_2)}\frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^{k-1}}{(k-1)!}.\end{aligned}\tag{50}$$

Let the first sum in (50) as S. Following the index shift $j\to j+1$, we apply the identity $(\varrho)_{j+1} = \varrho(\varrho+1)_j$, which yields

$$\begin{aligned}S&=\frac{\Phi_2}{\chi_3}\sum_{i,j,k=0}^{\infty}\frac{(\chi_1+\chi_2)_{2i+j+k}(\chi_3)_{j+k}}{(\varrho_1)_i(\varrho_2)_j(\varrho_3)_k}\frac{B^{\eta,\xi}_{q,a}(\chi_1+i+k,\chi_2+i+j)}{B(\chi_1,\chi_2)}\frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}\\ &=\frac{(\chi_1+\chi_2)\Phi_2}{\varrho_2}\sum_{i,j,k=0}^{\infty}\frac{(\chi_1+\chi_2+1)_{2i+j+k}(\chi_3+1)_{j+k}}{(\varrho_1)_i(\varrho_2+1)_j(\varrho_3)_k}\frac{B^{\eta,\xi}_{q,a}(\chi_1+i+k,\chi_2+i+j)}{B(\chi_1,\chi_2)}\frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}.\end{aligned}$$

Now, using the relation

$$B(\chi_1,\chi_2)=\frac{\chi_1+\chi_2}{\chi_2}B(\chi_1,\chi_2+1),$$

 doi:10.20944/preprints202507.0297.v1

we get

$$S = \frac{\chi_2\Phi_2}{\varrho_2}\sum_{i,j,k=0}^{\infty}\frac{(\chi_1+\chi_2+1)_{2i+j+k}(\chi_3+1)_{j+k}}{(\varrho_1)_i(\varrho_2+1)_j(\varrho_3)_k}\frac{B_{q,a}^{\eta,\xi}(\chi_1+i+k,\chi_2+i+j)}{B(\chi_1,\chi_2+1)}\frac{\Phi_1^i}{i!}\frac{\Phi_2^j}{j!}\frac{\Phi_3^k}{k!}$$
$$= \frac{\chi_2\Phi_2}{\varrho_2}\mathbf{H}_{B,q,a}^{\eta,\xi}(\chi_1,\chi_2+1,\chi_3+1;\varrho_1,\varrho_2+1,\varrho_3;\Phi_1,\Phi_2,\Phi_3). \tag{51}$$

Similarly, analyzing the second summation in (50) with the subsitution $k \to k+1$, we express it as

$$\frac{\Phi_3\chi_1}{\varrho_3}\mathbf{H}_{B,q,a}^{\eta,\xi}(\chi_1+1,\chi_2,\chi_3+1;\varrho_1,\varrho_2,\varrho_3+1;\Phi_1,\Phi_2,\Phi_3). \tag{52}$$

Combining (51) and (52) with (50) yields the desired recurrence relation (48). Replacing χ_3 with $\chi_3 - 1$ in (48), leads directly to the result (49).

Corollary 2: For positive integer N, the recurrence formulas for $\mathbf{H}_{B,q,a}^{\eta,\xi}(\cdot)$ from (48) are

$$\mathbf{H}_{B,q,a}^{\eta,\xi}(\chi_1,\chi_2,\chi_3+N;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3) = \mathbf{H}_{B,q,a}^{\eta,\xi}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)$$
$$+\frac{\Phi_2\chi_2}{\varrho_2}\sum_{l=1}^{N}\mathbf{H}_{B,q,a}^{\eta,\xi}(\chi_1,\chi_2+1,\chi_3+l;\varrho_1,\varrho_2+1,\varrho_3;\Phi_1,\Phi_2,\Phi_3)$$
$$+\frac{\Phi_3\chi_1}{\varrho_3}\sum_{l=1}^{N}\mathbf{H}_{B,q,a}^{\eta,\xi}(\chi_1+1,\chi_2,\chi_3+l;\varrho_1,\varrho_2,\varrho_3+1;\Phi_1,\Phi_2,\Phi_3),$$

and

$$\mathbf{H}_{B,q,a}^{\eta,\xi}(\chi_1,\chi_2,\chi_3-N;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3) = \mathbf{H}_{B,q,a}^{\eta,\xi}(\chi_1,\chi_2,\chi_3;\varrho_1,\varrho_2,\varrho_3;\Phi_1,\Phi_2,\Phi_3)$$
$$-\frac{\Phi_2\chi_2}{\varrho_2}\sum_{l=1}^{N-1}\mathbf{H}_{B,q,a}^{\eta,\xi}(\chi_1,\chi_2+1,\chi_3-l;\varrho_1,\varrho_2+1,\varrho_3;\Phi_1,\Phi_2,\Phi_3)$$
$$-\frac{\Phi_3\chi_1}{\varrho_3}\sum_{l=1}^{N-1}\mathbf{H}_{B,q,a}^{\eta,\xi}(\chi_1+1,\chi_2,\chi_3-l;\varrho_1,\varrho_2,\varrho_3+1;\Phi_1,\Phi_2,\Phi_3).$$

8. Conclusions

In this paper, we presented generalized Srivastava triple hypergeometric function $\mathbf{H}_{B,q,a}^{\eta,\xi}(\cdot)$, as introduced in (14). Several integral representations involving the generalized Mittag-Leffler function and Exton's function X_4 were established, alongside several analytical properties. We also derived integral transforms of the generalized function, including the Euler-Beta, Laplace, Mellin, and Whittaker transform. The action of Riemann-Liouville fractional integral and differential operators on the function was examined, providing new insights within the framework of fractional calculus. Finally, recurrence relations were obtained, offering a deeper understanding of the characteristic properties of the function. These findings provide a foundation for further analytical studies and potential applications in mathematical physics, fractional differential equations, and other areas involving generalized special functions.

Institutional Review Board Statement: None of the authors conducted studies involving human participants or animals for this article

Conflicts of Interest: The corresponding author, on behalf of all authors, declares that there are no conflicts of interest.

 doi:10.20944/preprints202507.0297.v1

References

1. Abubakar, U.M., New generalized beta function associated with the Fox-Wright function, Journal of Fractional Calculus and Application **12(2)** (2021), 204-227.
2. Chaudhry, M.A., Qadir, A., Rafique, M., Zubair, S.M., Extension of Euler's beta function, J. Comput. Appl. Math., **78(1)** (1997), 19-32.
3. Dar, S. A., Kamarujjama, M., Bhat, M. Y., Yasmin, Ajija, Some key properties of the generalised triple hypergeometric function of Srivastava's $H_{A,p,v}(w_1, w_2, w_3; w_4, w_5; z_1, z_2, z)$, Poincare Journal of Analysis and Applications, **11(2)** (2024), 107-122.
4. Hilfer, R., Application of fractional calculus in physics, World Scientific Publishing Co. Pte. Ltd, Singapore, (2000).
5. Kamarujjama, M., Yasmin, Ajija, Dar, S.A., A few explicit properties involving integral transforms and fractional calculus of Srivastava's triple hypergeometric function $H^{\lambda,l}_{C,p,\alpha}$ [Preprint], Preprints.org, https://doi.org/10.20944/preprints202505.0740.v1 , May (2025).
6. Karlsson, P.W., Regions of convergences for hypergeometric series in three variables, Math. Scand, **48** (1974), 241-248.
7. Kilbas, A.A., Saigo, M., H-Transforms: Theory and Applications (Analytical Methods and Special Functions), CRC Press Company Boca Raton, New York, 2004.
8. Kilbas, A.A., Saigo, M., Trujillo, J.J., On the generalized Wright function, Frac. Calc. Appl. Anal. **5(4)** (2002), 437–460.
9. Kilbas, A. A., Srivastava, H. M., Trujillo, J. J., Theory and applications of fractional differential equations, elsevier Vol. **204**, (2006).
10. Mittag-Leffler, G. M., Une généralisation de l'intégrale de Laplace-Abel, CR Acad. Sci. Paris (Ser. II) **137** (1903), 537-539 .
11. Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark C.W. (eds.), *NIST Handbook of Mathematical Functions*. Cambridge University Press, Cambridge, (2010).
12. Oraby, K. M., Rizq, A., Abdelhak, E., Taema, M., Magdy, M., Khaled, M., Generalization of Beta functions in terms of Mittag-Leffler function, Frontiers in Scientific Research and Technology **1(1)** (2020), 81-88.
13. Rainville, E. D., *Special functions*, The Macmillan Company, Newyork, (1960).
14. Samko, S. G., Kilbas, A. A., Marichev, O. I., *Fractional integrals and derivatives*, Vol. 1. Yverdon-les-Bains, Switzerland: Gordon and breach science publishers, Yverdon (1993).
15. Shadab, M., Jabee, S., Choi, J., An extended beta function and its applications, Far East Journal of Mathematical Sciences. **103(1)** (2018), 235-251.
16. Shukla, A.K., Prajapati, J.C., On a generalization of Mittag-Leffler function and its properties, Journal of mathematical analysis and applications, **336(2)** (2007), 797-811.
17. Slater, L.J., *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge (1966).
18. Sneddon, I.N., *The Use of Integral Transforms*, New Delhi,Tata McGraw Hill, (1979).
19. Srivastava, H.M., Hypergeometric functions of three variables, Ganita **15** (1964), 97-108.
20. Srivastava, H.M., Some integrals representing triple hypergeometric functions, Rend. Circ. Mat. Palermo (Ser. 2) **16** (1967), 99-115.
21. Srivastava, H.M., Tomovski, ^ Z. Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernal, Appl. Math. Comput.**211** (2009), 198-210.
22. Srivastava, H.M., Manocha, H.L., *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester, U.K.) John Wiley and Sons, New York, Chichester, Brisbane and Toronto, (1984).
23. Srivastava, H.M., Karlsson, P.W., *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, (1985).
24. Wiman, A., Über den Fundamentalsatz in der Teorie der Funktionen $E_\alpha(x)$, (1905), 191-201.
25. Whittaker, E.T., Watson, G.N., *A course of Modern Analysis*, vol **I**, **II**, (reprint of 1927 ed.) Cambridge Univ. Press, New York, (1962).
26. Yang, G., Shiri, B., Kong, H., Wu, G. C., Intermediate value problems for fractional differential equations, Computational and Applied Mathematics, **40(6)** (2021), 1-20.