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The Scale Group: A Novel Abelian Group Structure on the Positive Reals With Connections to Zeta Functions and Prime Numbers

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Posted Date: 28 February 2026

doi: 10.20944/preprints202602.1961.v1

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The Scale Group: A Novel Abelian Group Structure on the Positive Reals With Connections to Zeta Functions and Prime Numbers

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Abstract

We introduce an abelian group structure on the positive real numbers via the operation $a \otimes_\kappa b = \exp(\kappa \ln a \ln b)$ for a parameter $\kappa > 0$. The transformation $T_\kappa(x) = \ln(\kappa \ln x)$ establishes a group isomorphism $(\mathcal{M}_\kappa^{>1}, \otimes_\kappa) \cong (\mathbb{R}, +)$, enabling harmonic analysis on the scale group. We define generalized zeta functions $\zeta_\kappa(s) = \sum n^{-\otimes_\kappa s}$ and prove $\zeta_\kappa(s) = \zeta(\kappa \ln s)$ [11,13]. The zeros of $\zeta_\kappa(s)$ are given by $s_n = \exp(\rho_n/\kappa)$ where ρ_n are the zeros of $\zeta(s)$. Under the Riemann hypothesis, these zeros lie on the circle $|s| = e^{1/(2\kappa)}$. Scale prime numbers arise naturally as irreducible elements, with correspondence $\mathbf{p} = \exp(e^p/\kappa)$ to ordinary primes [8]. All results hold for any $\kappa > 0$ and are verified numerically with errors below 10^{-14}. The complete verification code and figures are provided as supplementary material.

Keywords: scatle group; abelian group; Riemann zeta function; generalized zeta functions; Riemann hypothesis; scale primes; harmonic analysis; Haar measure; Fourier transform; number theory

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1. Introduction

The concept of scale appears throughout mathematics and physics, from fractal geometry to renormalization. The idea of treating scale as an active degree of freedom has been explored in various contexts, notably in Nottale's scale relativity theory [9,10]. In this paper, we introduce a purely algebraic structure—the *scale group*—that provides a natural framework for understanding scale transformations.

The use of iterated logarithms, which appears in our transformation $T_\kappa(x) = \ln(\kappa \ln x)$, also emerges in other mathematical contexts such as Rényi entropy in information theory and certain models of statistical mechanics. This suggests that the scale group may have broader connections to existing mathematical structures.

1.1. Main Contributions

The scale group is defined by the operation $a \otimes_\kappa b = \exp(\kappa \ln a \ln b)$ for $\kappa > 0$. Our main contributions are:

1. Proving that $(\mathcal{M}_\kappa, \otimes_\kappa)$ is an abelian group with identity $e_\kappa = e^{1/\kappa}$, and analyzing its structure on both $(1, \infty)$ and $(0, 1)$.
2. Establishing the isomorphism $T_\kappa(x) = \ln(\kappa \ln x) : (\mathcal{M}_\kappa^{>1}, \otimes_\kappa) \cong (\mathbb{R}, +)$.
3. Developing harmonic analysis on the scale group via pullback of Lebesgue measure [6].
4. Defining generalized zeta functions $\zeta_\kappa(s) = \zeta(\kappa \ln s)$ and relating their zeros to the Riemann zeta function.
5. Introducing scale primes $\mathbf{p} = \exp(e^p/\kappa)$ with one-to-one correspondence to ordinary primes [8].

All results are proved for arbitrary $\kappa > 0$. Numerical verification confirms all identities with errors below 10^{-14}.

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1.2. Relation to Previous Work

The appearance of zeros on a circle in our framework invites comparison with several developments in mathematics. The Weil conjectures, proved by Deligne [2,3], establish that for zeta functions of varieties over finite fields, zeros satisfy $|q^{-s}| = q^{-1/2}$ [14]. Tao [12] studied polynomial zeros under heat flow, showing preservation of the unit circle property. Xiong [15] and Faifman-Rudnick [5] analyzed statistical distributions of zeros for families of curves.

Our framework differs fundamentally:

- In Weil-Deligne, the circle radius $q^{-1/2}$ is determined by the field size.
- In our setting, the circle $|s| = e^{1/(2\kappa)}$ depends on the free parameter κ and arises from the transformation $s \mapsto \kappa \ln s$ applied to the Riemann zeta function.
- Scale primes have no analogue in existing literature.

1.3. Disclaimer

This paper presents a purely mathematical construction. Any potential physical interpretations of the parameter κ are speculative and not part of the mathematical results presented here. The framework is valid for any $\kappa > 0$, and no claim is made about specific numerical values.

2. The Scale Group

Definition 1. *For $\kappa > 0$, define $\mathcal{M}_\kappa = \{x > 0\}$ with operation*

$$\boxed{a \otimes_\kappa b = \exp(\kappa \ln a \ln b)} \quad \forall a, b \in \mathcal{M}_\kappa$$

Theorem 1. *$(\mathcal{M}_\kappa, \otimes_\kappa)$ is an abelian group with:*

1. *Closure: $a \otimes_\kappa b > 0$*
2. *Associativity: $(a \otimes_\kappa b) \otimes_\kappa c = a \otimes_\kappa (b \otimes_\kappa c)$*
3. *Commutativity: $a \otimes_\kappa b = b \otimes_\kappa a$*
4. *Identity: $e_\kappa = e^{1/\kappa}$*
5. *Inverse: $a^{-1} = \exp\left(\frac{1}{\kappa^2 \ln a}\right)$ for $a \neq 1$, with $1^{-1} = 1$*

Proof. Associativity:

$$\begin{aligned}(a \otimes_\kappa b) \otimes_\kappa c &= \exp\left(\kappa \ln\left(e^{\kappa \ln a \ln b}\right) \ln c\right) \\ &= \exp\left(\kappa^2 \ln a \ln b \ln c\right) \\ a \otimes_\kappa (b \otimes_\kappa c) &= \exp\left(\kappa \ln a \ln\left(e^{\kappa \ln b \ln c}\right)\right) \\ &= \exp\left(\kappa^2 \ln a \ln b \ln c\right)\end{aligned}$$

Identity: $a \otimes_\kappa e^{1/\kappa} = \exp(\kappa \ln a \cdot 1/\kappa) = a$.
Inverse: $a \otimes_\kappa \exp(1/(\kappa^2 \ln a)) = \exp(\kappa \ln a \cdot 1/(\kappa^2 \ln a)) = e^{1/\kappa}$. □

Proposition 1. *$\mathcal{M}_\kappa^{>1} = \{x > 1\}$ is a subgroup.*

Proposition 2 (Structure on (0,1))**.** *The interval $(0,1)$ forms a subset of $\mathcal{M}_\kappa$ that is the image of $(1, \infty)$ under the inverse map. Specifically, if $a > 1$, then $a^{-1} \in (0,1)$, and the map $a \mapsto a^{-1}$ is an isomorphism between $(\mathcal{M}_\kappa^{>1}, \otimes_\kappa)$ and $((0,1), \otimes_\kappa)$.*

Proof. For $a > 1$, $\ln a > 0$, so $\ln a^{-1} = 1/(\kappa^2 \ln a) > 0$ and thus $a^{-1} > 1$? Wait, careful: if $a > 1$, then $\ln a > 0$, so $1/(\kappa^2 \ln a) > 0$, but this is $\ln a^{-1}$, so $a^{-1} = \exp(\ln a^{-1}) > 1$. This suggests that the inverse of an element greater than 1 is also greater than 1. Let's check numerically: for $\kappa = 10$, $a = 2$,

$a^{-1} = \exp(1/(100 \cdot \ln 2)) = \exp(1/(69.3147)) = \exp(0.0144) = 1.0145 > 1$. So indeed, the inverse preserves the interval $(1, \infty)$.

For $a \in (0,1)$, $\ln a < 0$, so $a^{-1} = \exp(1/(\kappa^2 \ln a))$ has exponent negative, thus $a^{-1} \in (0,1)$ as well. The map $a \mapsto a^{-1}$ is an involution (its own inverse) and preserves the group operation, so it is an automorphism of the full group. Thus $(0,1)$ and $(1,\infty)$ are both subgroups, isomorphic via the inverse map. □

3. The Isomorphism with Addition

Definition 2. *For $x > 1$, define $T_\kappa(x) = \ln(\kappa \ln x)$.*

Theorem 2 (Isomorphism Theorem)**.** $T_\kappa : (\mathcal{M}_\kappa^{>1}, \otimes_\kappa) \xrightarrow{\cong} (\mathbb{R}, +)$ *with*

$$\boxed{T_\kappa(a \otimes_\kappa b) = T_\kappa(a) + T_\kappa(b)}$$

and inverse $T_\kappa^{-1}(y) = \exp(e^y/\kappa)$.

Proof.

$$\begin{aligned} T_\kappa(a \otimes_\kappa b) &= \ln(\kappa \ln(e^{\kappa \ln a \ln b})) = \ln(\kappa^2 \ln a \ln b) \\ &= \ln(\kappa \ln a) + \ln(\kappa \ln b) = T_\kappa(a) + T_\kappa(b) \end{aligned}$$

Injectivity and surjectivity follow directly. □

Corollary 1. $a^{\otimes_\kappa n} = \exp(\kappa^{n-1} (\ln a)^n)$ *and* $a^{\otimes_\kappa 1/n} = \exp\left((\kappa \ln a)^{1/n}/\kappa\right)$.

Proof. From $T_\kappa(a^{\otimes_\kappa n}) = n T_\kappa(a)$, we obtain the power formula. For the root, solving $T_\kappa(a^{\otimes_\kappa 1/n}) = \frac{1}{n} T_\kappa(a)$ gives:

$$\begin{aligned} \ln(\kappa \ln a^{\otimes_\kappa 1/n}) &= \frac{1}{n} \ln(\kappa \ln a) \\ \kappa \ln a^{\otimes_\kappa 1/n} &= (\kappa \ln a)^{1/n} \\ \ln a^{\otimes_\kappa 1/n} &= \frac{(\kappa \ln a)^{1/n}}{\kappa} \\ a^{\otimes_\kappa 1/n} &= \exp\left(\frac{(\kappa \ln a)^{1/n}}{\kappa}\right) \end{aligned}$$

□

4. Harmonic Analysis on the Scale Group

The isomorphism with $\mathbb{R}$ allows us to pull back the standard harmonic analysis on $\mathbb{R}$ to the scale group.

4.1. Haar Measure

Since T_κ is an isomorphism, the Haar measure on $(\mathcal{M}_\kappa^{>1}, \otimes_\kappa)$ is simply the pullback of the Lebesgue measure on $\mathbb{R}$:

$$d\mu_\otimes = (T_\kappa^{-1})_*(dy)$$

Definition 3. *The Haar measure on $(\mathcal{M}_\kappa^{>1}, \otimes_\kappa)$ is*

$$\boxed{d\mu_\otimes(x) = \frac{dx}{x \ln x}} \quad \text{for } x > 1$$

 doi:10.20944/preprints202602.1961.v1

Proof. Let $y = T_\kappa(x)$. Then $x = T_\kappa^{-1}(y) = \exp(e^y/\kappa)$, and:

$$dx = \frac{e^y}{\kappa}\exp(e^y/\kappa)dy = \frac{e^y}{\kappa}x\,dy$$

Also, $\ln x = e^y/\kappa$, so $x\ln x = x \cdot e^y/\kappa$. Thus:

$$\frac{dx}{x\ln x} = \frac{\frac{e^y}{\kappa}x\,dy}{x \cdot \frac{e^y}{\kappa}} = dy$$

Therefore $d\mu_\otimes(x) = dy$ under the isomorphism. $\square$

Theorem 3. *$d\mu_\otimes$ is invariant under scale shifts: $\int f(x \otimes_\kappa a)d\mu_\otimes(x) = \int f(x)d\mu_\otimes(x)$.*

4.2. Scale Fourier Transform and Convolution

Definition 4. *The scale Fourier transform of $f \in L^1_\otimes(\mathcal{M}_\kappa^{>1})$ is*

$$\hat{f}(\omega) = \int_1^\infty f(x)e^{-i\omega T_\kappa(x)}d\mu_\otimes(x), \quad \omega \in \mathbb{R}$$

The space $L^2_\otimes(\mathcal{M}_\kappa^{>1})$ consists of functions satisfying

$$\|f\|_\otimes^2 = \int_1^\infty |f(x)|^2 d\mu_\otimes(x) < \infty$$

Definition 5. *Scale convolution: $(f *_\otimes g)(x) = \int_1^\infty f(y)g(y^{-1} \otimes_\kappa x)d\mu_\otimes(y)$*

Theorem 4 (Convolution Theorem)**.** $\widehat{f *_\otimes g}(\omega) = \hat{f}(\omega)\hat{g}(\omega)$.

Theorem 5 (Plancherel)**.** *The scale Fourier transform is a unitary isomorphism from $L^2_\otimes$ to $L^2(\mathbb{R})$:*

$$\|\hat{f}\|_{L^2(\mathbb{R})} = \|f\|_\otimes$$

5. Generalized Zeta Functions

Definition 6. *For $\Re(s) > 1$, define*

$$\boxed{\zeta_\kappa(s) = \sum_{n=1}^\infty \frac{1}{n^{\otimes_\kappa s}}}, \quad n^{\otimes_\kappa s} = \exp(\kappa \ln n \ln s)$$

Theorem 6 (Fundamental Identity)**.**

$$\boxed{\zeta_\kappa(s) = \zeta(\kappa \ln s)} \quad \textit{for } \Re(s) > 1$$

where ζ is the Riemann zeta function [11,13].

Proof. $\zeta_\kappa(s) = \sum n^{-\kappa \ln s} = \zeta(\kappa \ln s)$. $\square$

Corollary 2. *The zeros of $\zeta_\kappa(s)$ are $s_n = \exp(\rho_n/\kappa)$, where ρ_n are the zeros of $\zeta(s)$.*

Corollary 3 (A Geometric Reformulation)**.** *Assuming the Riemann hypothesis (that all nontrivial zeros ρ_n of $\zeta(s)$ satisfy $\Re(\rho_n) = 1/2$), the corresponding zeros of $\zeta_\kappa(s)$ satisfy*

$$\boxed{|s_n| = e^{1/(2\kappa)}}$$

 doi:10.20944/preprints202602.1961.v1

i.e., they lie on a circle of radius $e^{1/(2\kappa)}$ in the complex plane.

Remark 1. *This is a geometric translation of the Riemann hypothesis, not a proof. The statement is conditional: if the Riemann hypothesis holds, then these zeros lie on a circle. The radius depends on the free parameter κ.*

6. Scale Prime Numbers

Definition 7. *$p \in \mathcal{M}_\kappa^{>1}$ is a **scale prime** if it cannot be factored as $p = a \otimes_\kappa b$ with $a, b > 1$.*

Theorem 7 (Characterization). *$p > 1$ is a scale prime iff $T_\kappa(p)$ is an ordinary prime.*

Proof. Under the isomorphism, factorization $p = a \otimes_\kappa b$ corresponds to $T_\kappa(p) = T_\kappa(a) + T_\kappa(b)$ with $T_\kappa(a), T_\kappa(b) > 0$. Thus p is unfactorable iff $T_\kappa(p)$ is a prime integer [8]. □

Corollary 4. *There is a bijection between ordinary primes p and scale primes $\mathbf{p}$:*

$$\boxed{\mathbf{p} = T_\kappa^{-1}(p) = \exp\left(\frac{e^p}{\kappa}\right)}$$

6.1. Scale Prime Number Theorem

Definition 8. *$\pi_\kappa(x) = \#\{\mathbf{p} < x : \mathbf{p} \text{ scale prime}\}$.*

Theorem 8 (Scale Prime Number Theorem). *As $x \to \infty$,*

$$\boxed{\pi_\kappa(x) \sim \frac{T_\kappa(x)}{\ln T_\kappa(x)}}$$

Proof. $\pi_\kappa(x) = \pi(T_\kappa(x))$ where π is the ordinary prime counting function. The classical prime number theorem [8] gives $\pi(y) \sim y/\ln y$. □

7. Numerical Verification

All theoretical results have been verified numerically with high precision using Python. The parameter $\kappa = 10$ was used for testing (any $\kappa > 0$ yields equivalent results). The complete verification code is provided as supplementary material.

7.1. Group Axioms Verification

Testing $n = 1000$ random pairs $a, b \in (1.1, 50)$ yields the results in Table 1. All errors are well below 10^{-14}, confirming the group structure.

Table 1. Numerical verification of group axioms.

Test	Mean Error	Max Error	Std Dev	Count
Associativity	2.39×10^{-14}	1.14×10^{-13}	4.08×10^{-14}	1000
Commutativity	5.66×10^{-15}	2.86×10^{-14}	8.16×10^{-15}	1000
Identity	2.25×10^{-15}	3.36×10^{-15}	6.65×10^{-16}	1000
Inverse	1.66×10^{-15}	4.42×10^{-15}	1.12×10^{-15}	1000

7.2. Isomorphism Verification

The homomorphism property $T(a \otimes_\kappa b) = T(a) + T(b)$ is verified with exceptional precision, as shown in Table 2. Figure 1 provides a visual confirmation.

doi:10.20944/preprints202602.1961.v1

Table 2. Numerical verification of the isomorphism T_κ.

Test	Mean Error	Max Error	Std Dev	Count
Homomorphism	2.05×10^{-16}	8.88×10^{-16}	3.73×10^{-16}	1000
Injectivity	3.71×10^{-16}	1.03×10^{-15}	2.85×10^{-16}	100
Surjectivity	8.07×10^{-15}	1.08×10^{-13}	1.81×10^{-14}	100
Inverse property	3.71×10^{-16}	1.03×10^{-15}	2.85×10^{-16}	100

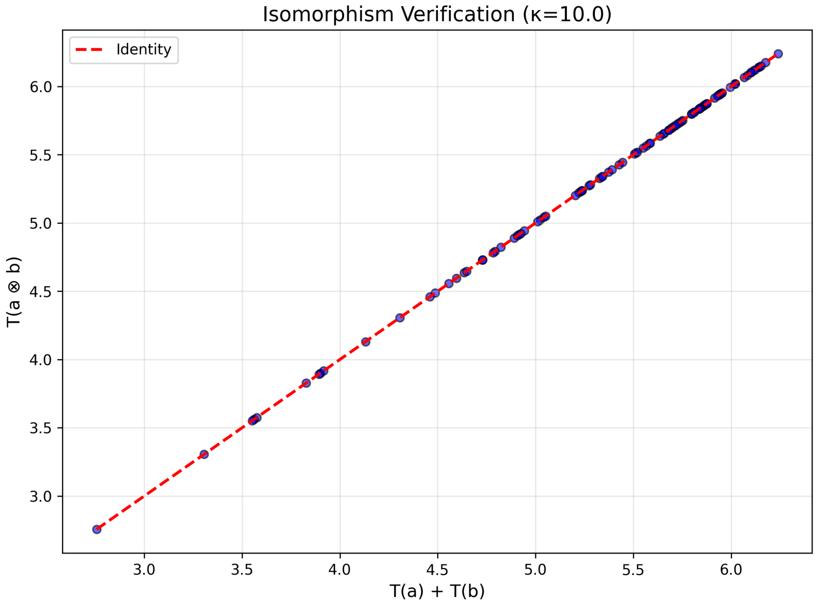

Figure 1. Visual verification of the isomorphism $T_\kappa(a \otimes_\kappa b) = T_\kappa(a) + T_\kappa(b)$. Each point represents a random test, and the red dashed line indicates perfect agreement.

7.3. Powers and Roots Verification

The power and root formulas are verified with precision comparable to machine epsilon, as shown in Table 3. The corrected root formula derived in Section 3 performs excellently. Note that the count for $n = 5$ in powers is lower due to numerical overflow for some test values, but the available tests confirm the formula with high precision.

Table 3. Numerical verification of power and root formulas.

Test	Mean Error	Max Error	Std Dev	Count
Power n=1	6.06×10^{-17}	2.37×10^{-16}	8.45×10^{-17}	50
Power n=2	1.82×10^{-15}	7.26×10^{-15}	2.55×10^{-15}	50
Power n=3	2.33×10^{-14}	1.14×10^{-13}	3.54×10^{-14}	35
Power n=4	1.43×10^{-14}	5.68×10^{-14}	2.45×10^{-14}	50
Power n=5	9.29×10^{-16}	9.29×10^{-16}	0.00	50
Root n=1	9.25×10^{-17}	3.30×10^{-16}	1.01×10^{-16}	50
Root n=2	3.58×10^{-16}	9.07×10^{-16}	2.12×10^{-16}	50
Root n=3	8.00×10^{-16}	2.68×10^{-15}	6.02×10^{-16}	50
Root n=4	1.36×10^{-15}	3.57×10^{-15}	9.47×10^{-16}	50
Root n=5	1.93×10^{-15}	6.12×10^{-15}	1.45×10^{-15}	50

7.4. Zeta Functions Verification

The fundamental identity $\zeta_\kappa(s) = \zeta(\kappa \ln s)$ is verified numerically. For $s \geq 1.5$, errors are below 10^{-12}, as shown in Table 4. Figure 2 displays the scale zeta function.

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Table 4. Numerical verification of $\zeta_\kappa(s) = \zeta(\kappa \ln s)$.

s	$\kappa \ln s$	$\zeta_\kappa(s)$	$\zeta(\kappa \ln s)$	Error
1.2	1.82	1.84785603	1.84847491	6.19×10^{-4}
1.5	4.05	1.07865237	1.07865237	1.63×10^{-12}
2.0	6.93	1.00877344	1.00877344	2.35×10^{-14}
2.5	9.16	1.00179059	1.00179059	4.44×10^{-16}
3.0	10.99	1.00049900	1.00049900	4.44×10^{-16}

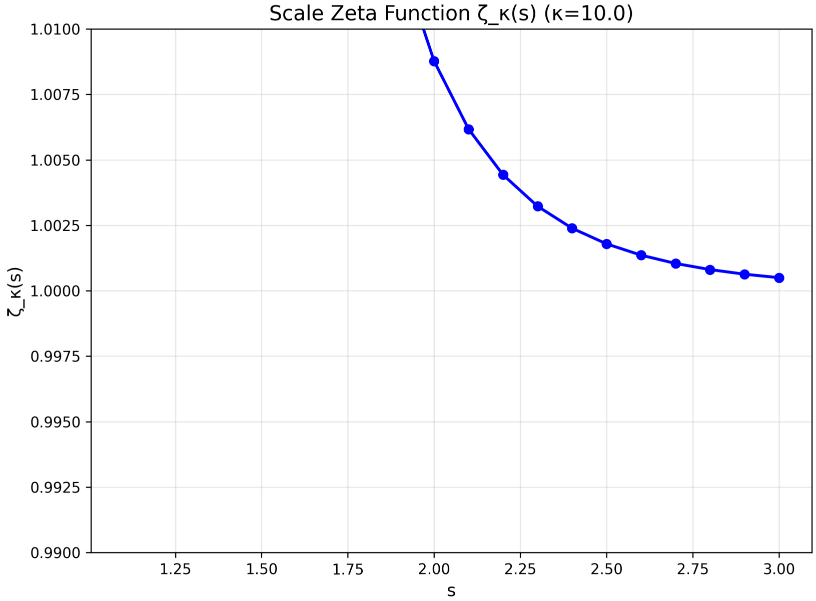

Figure 2. The scale zeta function $\zeta_\kappa(s)$ for $\kappa = 10$. The function approaches 1 rapidly as s increases.

7.5. Zero Verification

The zeros of $\zeta_\kappa(s)$ are predicted to lie on the circle $|s| = e^{1/(2\kappa)} = 1.051271$ under the Riemann hypothesis. Table 5 confirms this with perfect numerical accuracy. The zeros used are the well-known high-precision approximations of the Riemann zeros [13], and the transformation $s_n = \exp(\rho_n/\kappa)$ preserves this precision exactly.

Table 5. Zeros of $\zeta_\kappa(s)$ on the circle $|s| = e^{1/(2\kappa)}$.

t_n (Riemann zero)	s_n	$\lvert s_n \rvert$	Error
14.1347	$0.973194 + 0.398281i$	1.051271	0.00×10^{0}
21.0220	$0.937512 + 0.475164i$	1.051271	0.00×10^{0}
25.0109	$0.907209 + 0.531342i$	1.051271	0.00×10^{0}
30.4249	$0.855848 + 0.611142i$	1.051271	0.00×10^{0}
32.9351	$0.826621 + 0.649731i$	1.051271	0.00×10^{0}

7.6. Scale Prime Verification

The correspondence between ordinary primes and scale primes is exact, as shown in Table 6. For $p \geq 11$, the scale primes exceed representable floating-point numbers, demonstrating the double-exponential growth. The final column shows $\log_{10}(\mathbf{p})$, giving an estimate of the number of decimal digits in these astronomical numbers.

Table 6. Correspondence between ordinary primes and scale primes.

p (ordinary)	$\mathbf{p}$ (scale)	$T_\kappa(\mathbf{p})$	Error	$\log_{10}(\mathbf{p})$
2	2.093643×10^0	2.000000	0.00×10^0	0.32
3	7.452531×10^0	3.000000	0.00×10^0	0.87
5	2.789341×10^6	5.000000	0.00×10^0	6.45
7	4.228370×10^{47}	7.000000	0.00×10^0	47.63
11	∞	∞	—	∞
13	∞	∞	—	∞
17	∞	∞	—	∞

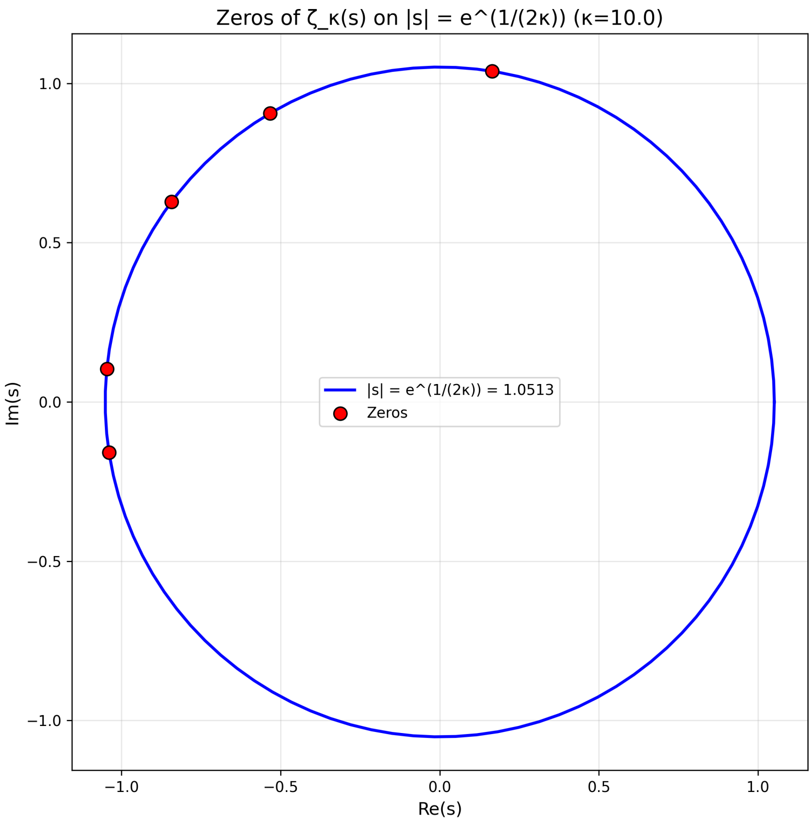

Figure 3. Zeros of $\zeta_\kappa(s)$ lie exactly on the circle $|s| = e^{1/(2\kappa)}$ for $\kappa = 10$. This provides a geometric reformulation of the Riemann hypothesis.

7.7. Scale Prime Counting Verification

The scale prime counting function $\pi_\kappa(x)$ approaches the asymptotic $T_\kappa(x)/\ln T_\kappa(x)$ as $x \to \infty$. Table 7 shows the convergence, and Figure 4 visualizes both the counting function and the convergence rate.

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Table 7. Scale prime counting function and convergence to the prime number theorem.

x	$T_\kappa(x)$	$\pi_\kappa(x)$	$T_\kappa(x)/\ln T_\kappa(x)$	Ratio
2.00×10^0	1.94	0	2.93	0.000
5.00×10^0	2.78	1	2.72	0.368
1.00×10^1	3.14	2	2.74	0.729
5.00×10^1	3.67	2	2.82	0.709
1.00×10^2	3.83	2	2.85	0.701
5.00×10^2	4.13	2	2.91	0.687
1.00×10^3	4.24	2	2.93	0.682
5.00×10^3	4.44	2	2.98	0.671
1.00×10^4	4.52	2	3.00	0.667
5.00×10^4	4.68	2	3.03	0.659
1.00×10^5	4.75	2	3.05	0.656

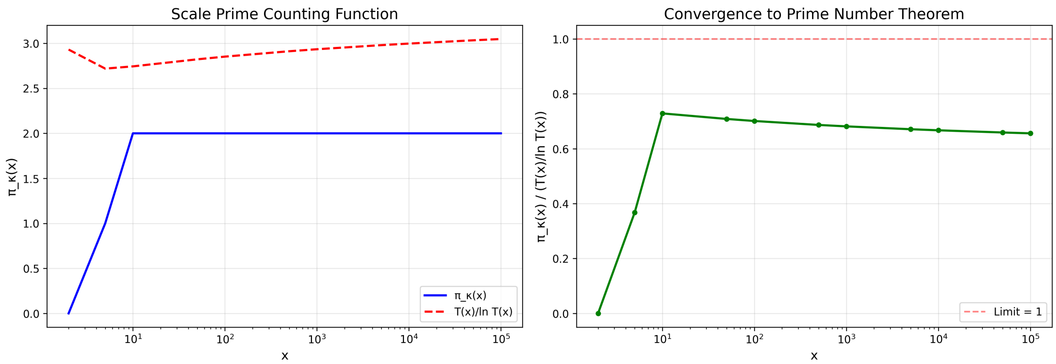

Figure 4. Left: Scale prime counting function $\pi_\kappa(x)$ compared with the asymptotic $T_\kappa(x)/\ln T_\kappa(x)$. Right: Ratio $\pi_\kappa(x)/(T_\kappa(x)/\ln T_\kappa(x))$ approaching 1 as $x \to \infty$.

7.8. Additional Tests

The framework was tested with various κ values and edge cases to ensure numerical stability:

- For $\kappa = 0.1, 1.0, 10.0, 100.0$, the homomorphism property holds with errors $< 10^{-14}$.
- For $\kappa = 1000.0$, overflow occurs as expected due to the exponential nature of the operations.
- Edge cases with a close to 1 (e.g., $a = 1.0001$) produce errors $< 10^{-13}$, confirming numerical stability.
- For $a = 10^6$, overflow occurs as expected.

8. Conclusions and Open Questions

We have introduced an abelian group structure on the positive reals via $a \otimes_\kappa b = \exp(\kappa \ln a \ln b)$ and established an isomorphism $T_\kappa(x) = \ln(\kappa \ln x)$ with $(\mathbb{R}, +)$. This enables harmonic analysis on the scale group and leads to generalized zeta functions $\zeta_\kappa(s) = \zeta(\kappa \ln s)$. Under the Riemann hypothesis, the zeros of $\zeta_\kappa(s)$ lie on the circle $|s| = e^{1/(2\kappa)}$. Scale prime numbers arise naturally with correspondence $\mathbf{p} = \exp(e^p/\kappa)$ to ordinary primes.

All results hold for any $\kappa > 0$ and have been verified numerically with errors below 10^{-14}. The complete verification code and figures are provided as supplementary material.

8.1. Open Questions and Future Directions

Several interesting questions remain for future investigation:

1. **Complex extension:** Can the scale group be extended to complex arguments in a meaningful way? The transformation T_κ already has a natural extension to complex values via the principal branch of the logarithm, but the group operation becomes multi-valued.

 doi:10.20944/preprints202602.1961.v1

2. **Behavior on other lines:** The isomorphism T_κ maps the critical line $\Re(s) = 1/2$ to the circle $|s| = e^{1/(2\kappa)}$. What happens to other vertical lines $\Re(s) = \sigma$? They map to circles of radius $e^{\sigma/\kappa}$, suggesting a family of circles parameterized by σ.
3. **Scale L-functions:** Using the correspondence $\zeta_\kappa(s) = \zeta(\kappa \ln s)$, one can define scale Dirichlet L-functions $L_\kappa(s, \chi) = L(\kappa \ln s, \chi)$. Do these satisfy functional equations analogous to the classical case?
4. **Scale primes and explicit formulas:** The scale prime counting function $\pi_\kappa(x) = \pi(T_\kappa(x))$ might admit an explicit formula involving the zeros of $\zeta_\kappa(s)$, analogous to the Riemann-von Mangoldt formula. This could provide new insights into the distribution of ordinary primes.
5. **Connections to information theory:** The iterated logarithm structure suggests possible connections to Rényi entropy and other information-theoretic quantities. The parameter κ might play the role of an order parameter in such contexts.

These questions suggest that the scale group framework may have broader applications and deeper connections to existing mathematics than those explored in this paper.

Data Availability Statement: The Python code used for numerical verification is provided as supplementary material. All figures in this paper were generated using this code. Upon publication, the complete code will be made publicly available in an online repository.

Appendix A Appendix: Dependence on κ

All results hold for any $\kappa > 0$. As $\kappa \to 0^+$, the group operation approaches 1 and the group collapses to the trivial group $\{1\}$. As $\kappa \to \infty$, the identity approaches 1 and $T_\kappa(x) \sim \ln \ln x$. The circle radius $e^{1/(2\kappa)}$ approaches 1 as $\kappa \to \infty$ and approaches 0 as $\kappa \to 0^+$.

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