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Article

Two Undecidable Decision Problems on an Ordered Pair of Non-Negative Integers

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Abstract

For $n \in \mathbb{N}$, let $E_n = \{1 = x_k, x_i + x_j = x_k, x_i \cdot x_j = x_k : i, j, k \in \{0, \dots, n\}\}$. For $n \in \mathbb{N}$, $f(n)$ denotes the smallest $b \in \mathbb{N}$ such that if a system of equations $S \subseteq E_n$ has a solution in $\mathbb{N}^{n+1}$, then S has a solution in $\{0, \dots, b\}^{n+1}$. The author proved earlier that the function $f : \mathbb{N} \rightarrow \mathbb{N}$ is computable in the limit and eventually dominates every computable function $g : \mathbb{N} \rightarrow \mathbb{N}$. We present a short program in *MuPAD* which for $n \in \mathbb{N}$ prints the sequence $\{f_i(n)\}_{i=0}^\infty$ of non-negative integers converging to $f(n)$. Since f is not computable, no algorithm takes as input non-negative integers n and m and decides whether or not $\forall (x_0, \dots, x_n) \in \mathbb{N}^{n+1} \exists (y_0, \dots, y_n) \in \{0, \dots, m\}^{n+1} (\forall k \in \{0, \dots, n\} (1 = x_k \Rightarrow 1 = y_k)) \wedge (\forall i, j, k \in \{0, \dots, n\} (x_i + x_j = x_k \Rightarrow y_i + y_j = y_k)) \wedge (\forall i, j, k \in \{0, \dots, n\} (x_i \cdot x_j = x_k \Rightarrow y_i \cdot y_j = y_k))$. Similarly, no algorithm takes as input non-negative integers n and m and decides whether or not $\forall (x_0, \dots, x_n) \in \mathbb{N}^{n+1} \exists (y_0, \dots, y_n) \in \{0, \dots, m\}^{n+1} (\forall j, k \in \{0, \dots, n\} (x_j + 1 = x_k \Rightarrow y_j + 1 = y_k)) \wedge (\forall i, j, k \in \{0, \dots, n\} (x_i \cdot x_j = x_k \Rightarrow y_i \cdot y_j = y_k))$. For $n \in \mathbb{N}$, $\beta(n)$ denotes the smallest $b \in \mathbb{N}$ such that if a system of equations $S \subseteq E_n$ has a unique solution in $\mathbb{N}^{n+1}$, then this solution belongs to $\{0, \dots, b\}^{n+1}$. The author proved earlier that the function $\beta : \mathbb{N} \rightarrow \mathbb{N}$ is computable in the limit and eventually dominates every function $\delta : \mathbb{N} \rightarrow \mathbb{N}$ with a single-fold Diophantine representation. The computability of β is unknown. We present a short program in *MuPAD* which for $n \in \mathbb{N}$ prints the sequence $\{\beta_i(n)\}_{i=0}^\infty$ of non-negative integers converging to $\beta(n)$.

Key words and phrases: computable function; eventual domination; limit-computable function; single-fold Diophantine representation; undecidable decision problem

2020 Mathematics Subject Classification: 03D20; 11U05

1. The Collatz Problem Leads to a Short Computer Program That Computes in the Limit a Function $\gamma : \mathbb{N} \rightarrow \{0, 1\}$ of Unknown Computability

Definition 1. (cf. [10, pp. 233–235]). A computation in the limit of a function $f : \mathbb{N} \rightarrow \mathbb{N}$ is a semi-algorithm which takes as input a non-negative integer n and for every $m \in \mathbb{N}$ prints a non-negative integer $\xi(n, m)$ such that $\lim_{m \rightarrow \infty} \xi(n, m) = f(n)$.

By Definition 1, a function $f : \mathbb{N} \rightarrow \mathbb{N}$ is computable in the limit when there exists an infinite computation which takes as input a non-negative integer n and prints a non-negative integer on each iteration and prints $f(n)$ on each sufficiently high iteration.

It is known that there exists a limit-computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ which is not computable, see Theorem 1. Every known proof of this fact does not lead to the existence of a short computer program that computes f in the limit. So far, short computer programs can only compute in the limit functions from $\mathbb{N}$ to $\mathbb{N}$ whose computability is proven or unknown.

Lemma 1. For every $n \in \mathbb{N}$,

$$\frac{\text{sign}(n-1) \cdot (2n + (1 - (-1)^n) \cdot (5n + 2))}{4} = \begin{cases} 0, & \text{if } n = 1 \\ \frac{n}{2}, & \text{if } n \text{ is even} \\ 3n + 1, & \text{if } n \text{ is odd and } n \neq 1 \end{cases}$$

MuPAD is a part of the Symbolic Math Toolbox in MATLAB R2019b. By Lemma 1, the following program in MuPAD computes in the limit a function $\gamma : \mathbb{N} \rightarrow \{0, 1\}$.

```
input("Input a non-negative integer n",n):
while TRUE do
print(sign(n)):
n:=sign(n-1)*(2*n+(1-(-1)^n)*(5*n+2))/4:
end_while:
```

The computability of γ is unknown, see [1, p. 79]. The Collatz conjecture implies that $\gamma(n) = 0$ for every $n \in \mathbb{N}$.

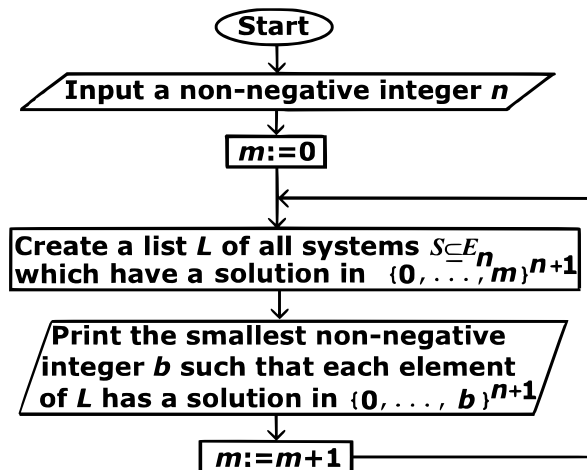
2. A Limit-Computable Function $f : \mathbb{N} \rightarrow \mathbb{N}$ Which Eventually Dominates Every Computable Function $g : \mathbb{N} \rightarrow \mathbb{N}$

For $n \in \mathbb{N}$, let

$$E_n = \{1 = x_k, x_i + x_j = x_k, x_i \cdot x_j = x_k : i, j, k \in \{0, \dots, n\}\}$$

Theorem 1. ([9, p. 118]). There exists a limit-computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ which eventually dominates every computable function $g : \mathbb{N} \rightarrow \mathbb{N}$.

We present an alternative proof of Theorem 1. For $n \in \mathbb{N}$, $f(n)$ denotes the smallest $b \in \mathbb{N}$ such that if a system of equations $S \subseteq E_n$ has a solution in $\mathbb{N}^{n+1}$, then S has a solution in $\{0, \dots, b\}^{n+1}$. The function $f : \mathbb{N} \rightarrow \mathbb{N}$ is computable in the limit and eventually dominates every computable function $g : \mathbb{N} \rightarrow \mathbb{N}$, see [12]. The term "dominated" in the title of [12] means "eventually dominated". Flowchart 1 shows a semi-algorithm which computes $f(n)$ in the limit, see [12].

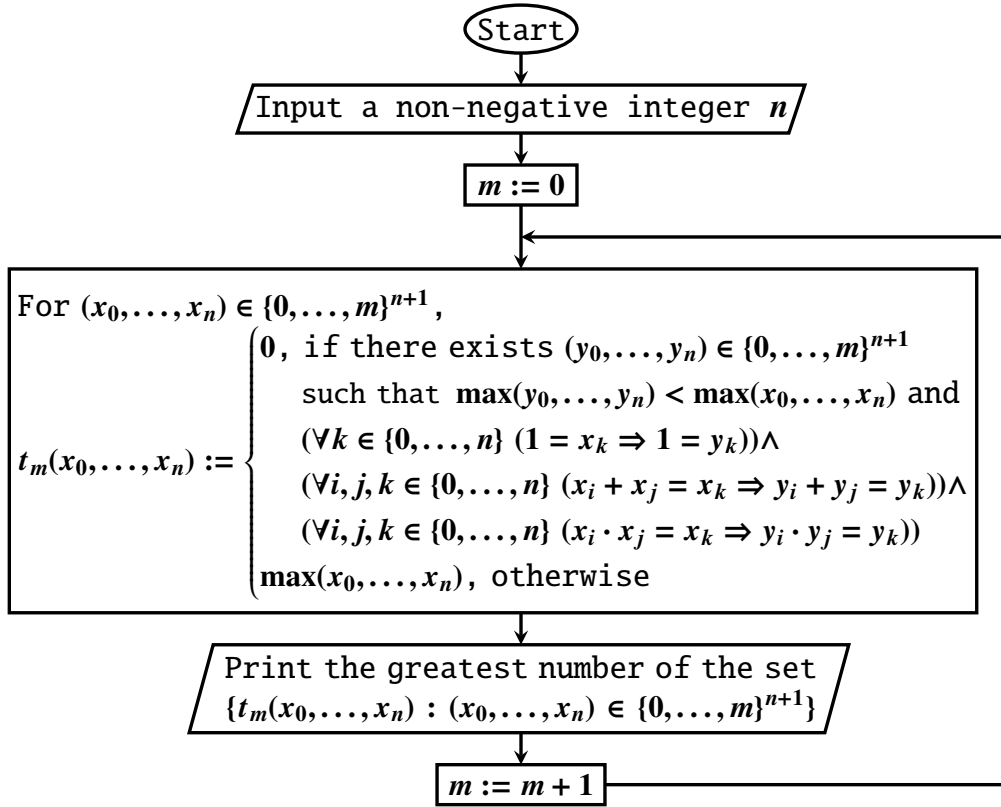


Flowchart 1

A semi-algorithm which computes $f(n)$ in the limit

3. The First Undecidable Decision Problem on an Ordered Pair of Non-Negative Integers

Flowchart 2 shows a simpler semi-algorithm which computes $f(n)$ in the limit.



Flowchart 2

A simpler semi-algorithm which computes $f(n)$ in the limit

Lemma 2. For every $n, m \in \mathbb{N}$, the number printed by Flowchart 2 does not exceed the number printed by Flowchart 1.

Proof. For every $(a_0, \dots, a_n) \in \{0, \dots, m\}^{n+1}$,

$$\begin{aligned}
 E_n &\supseteq \{1 = x_k : (k \in \{0, \dots, n\}) \wedge (1 = a_k)\} \cup \\
 &\{x_i + x_j = x_k : (i, j, k \in \{0, \dots, n\}) \wedge (a_i + a_j = a_k)\} \cup \\
 &\{x_i \cdot x_j = x_k : (i, j, k \in \{0, \dots, n\}) \wedge (a_i \cdot a_j = a_k)\}
 \end{aligned}$$

□

Lemma 3. For every $n, m \in \mathbb{N}$, the number printed by Flowchart 1 does not exceed the number printed by Flowchart 2.

Proof. Let $n, m \in \mathbb{N}$. For every system of equations $\mathcal{S} \subseteq E_n$, if $(a_0, \dots, a_n) \in \{0, \dots, m\}^{n+1}$ and $(a_0, \dots, a_n)$ solves $\mathcal{S}$, then $(a_0, \dots, a_n)$ solves the following system of equations:

$$\begin{aligned}
 &\{1 = x_k : (k \in \{0, \dots, n\}) \wedge (1 = a_k)\} \cup \\
 &\{x_i + x_j = x_k : (i, j, k \in \{0, \dots, n\}) \wedge (a_i + a_j = a_k)\} \cup \\
 &\{x_i \cdot x_j = x_k : (i, j, k \in \{0, \dots, n\}) \wedge (a_i \cdot a_j = a_k)\}
 \end{aligned}$$

□

Theorem 2. For every $n, m \in \mathbb{N}$, Flowcharts 1 and 2 print the same number.

Proof. It follows from Lemmas 2 and 3. $\square$

Definition 2. An approximation of a tuple $(x_0, \dots, x_n) \in \mathbb{N}^{n+1}$ is a tuple $(y_0, \dots, y_n) \in \mathbb{N}^{n+1}$ such that

$$\begin{aligned} & (\forall k \in \{0, \dots, n\} (1 = x_k \Rightarrow 1 = y_k)) \wedge \\ & (\forall i, j, k \in \{0, \dots, n\} (x_i + x_j = x_k \Rightarrow y_i + y_j = y_k)) \wedge \\ & (\forall i, j, k \in \{0, \dots, n\} (x_i \cdot x_j = x_k \Rightarrow y_i \cdot y_j = y_k)) \end{aligned}$$

Observation 1. For every $n \in \mathbb{N}$, there exists a set $\mathcal{A}(n) \subseteq \mathbb{N}^{n+1}$ such that

$$\text{card}(\mathcal{A}(n)) \leq 2^{\text{card}(E_n)} = 2^n + 1 + 2 \cdot (n+1)^3$$

and every tuple $(x_0, \dots, x_n) \in \mathbb{N}^{n+1}$ possesses an approximation in $\mathcal{A}(n)$.

Observation 2. For every $n \in \mathbb{N}$, $f(n)$ equals the smallest $b \in \mathbb{N}$ such that every tuple $(x_0, \dots, x_n) \in \mathbb{N}^{n+1}$ possesses an approximation in $\{0, \dots, b\}^{n+1}$.

Observation 3. For every $n, m \in \mathbb{N}$, Flowcharts 1 and 2 print the smallest $b \in \{0, \dots, m\}$ such that every tuple $(x_0, \dots, x_n) \in \{0, \dots, m\}^{n+1}$ possesses an approximation in $\{0, \dots, b\}^{n+1}$.

Theorem 3. No algorithm takes as input non-negative integers n and m and returns the logical value of the following sentence: every tuple $(x_0, \dots, x_n) \in \mathbb{N}^{n+1}$ possesses an approximation in $\{0, \dots, m\}^{n+1}$.

Proof. Since the function f is not computable, it follows from Observation 2. $\square$

4. A Short Program in MuPAD That Computes f in the Limit

The following program in MuPAD implements the semi-algorithm shown in Flowchart 2.

```
input("Input a non-negative integer n",n):
m:=0:
while TRUE do
X:=combinat::cartesianProduct([s $s=0..m] $t=0..n):
Y:=[max(op(X[u])) $u=1..(m+1)^(n+1)]:
for p from 1 to (m+1)^(n+1) do
for q from 1 to (m+1)^(n+1) do
v:=1:
for k from 1 to n+1 do
if 1=X[p][k] and 1<>X[q][k] then v:=0 end_if:
for i from 1 to n+1 do
for j from i to n+1 do
if X[p][i]+X[p][j]=X[p][k] and X[q][i]+X[q][j]<>X[q][k] then v:=0 end_if:
if X[p][i]*X[p][j]=X[p][k] and X[q][i]*X[q][j]<>X[q][k] then v:=0 end_if:
end_for:
end_for:
end_for:
if max(op(X[q]))<max(op(X[p])) and v=1 then Y[p]:=0 end_if:
end_for:
end_for:
print(max(op(Y))):
m:=m+1:
end_while:
```

5. The Second Undecidable Decision Problem on an Ordered Pair of Non-Negative Integers

For $n \in \mathbb{N}$, $h(n)$ denotes the smallest $b \in \mathbb{N}$ such that if a system of equations $\mathcal{S} \subseteq \{x_j + 1 = x_k, x_i \cdot x_j = x_k : i, j, k \in \{0, \dots, n\}\}$ has a solution in $\mathbb{N}^{n+1}$, then $\mathcal{S}$ has a solution in $\{0, \dots, b\}^{n+1}$. From [12] and Lemma 3 in [11], it follows that the function $h : \mathbb{N} \rightarrow \mathbb{N}$ is computable in the limit and eventually dominates every computable function $g : \mathbb{N} \rightarrow \mathbb{N}$. A bit shorter program in *MuPAD* computes h in the limit.

Theorem 4. *No algorithm takes as input non-negative integers n and m and returns the logical value of the following sentence:*

$$\begin{aligned} & \forall (x_0, \dots, x_n) \in \mathbb{N}^{n+1} \exists (y_0, \dots, y_n) \in \{0, \dots, m\}^{n+1} \\ & (\forall j, k \in \{0, \dots, n\} (x_j + 1 = x_k \Rightarrow y_j + 1 = y_k)) \wedge \\ & (\forall i, j, k \in \{0, \dots, n\} (x_i \cdot x_j = x_k \Rightarrow y_i \cdot y_j = y_k)) \end{aligned}$$

Proof. It holds because the function h is not computable. $\square$

6. A Limit-Computable Function $\beta : \mathbb{N} \rightarrow \mathbb{N}$ of Unknown Computability Which Eventually Dominates Every Function $\delta : \mathbb{N} \rightarrow \mathbb{N}$ with a Single-Fold Diophantine Representation

The Davis-Putnam-Robinson-Matiyasevich theorem states that every listable set $\mathcal{M} \subseteq \mathbb{N}^n$ ($n \in \mathbb{N} \setminus \{0\}$) has a Diophantine representation, that is

$$(a_1, \dots, a_n) \in \mathcal{M} \iff \exists x_1, \dots, x_m \in \mathbb{N} \ W(a_1, \dots, a_n, x_1, \dots, x_m) = 0 \quad (\text{R})$$

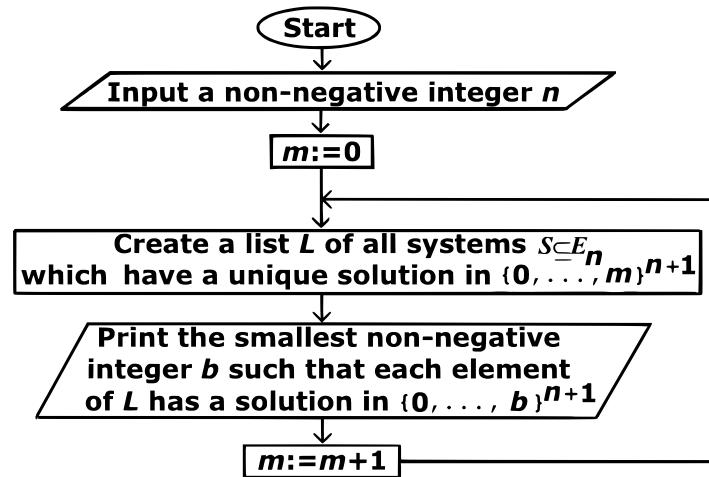
for some polynomial W with integer coefficients, see [6]. The representation (R) is said to be single-fold, if for any $a_1, \dots, a_n \in \mathbb{N}$ the equation $W(a_1, \dots, a_n, x_1, \dots, x_m) = 0$ has at most one solution $(x_1, \dots, x_m) \in \mathbb{N}^m$.

Hypothesis 1. ([2–4], [5, pp. 341–342], [7, p. 42], [8, p. 745]). *Every listable set $\mathcal{X} \subseteq \mathbb{N}^k$ ($k \in \mathbb{N} \setminus \{0\}$) has a single-fold Diophantine representation.*

For $n \in \mathbb{N}$, $\beta(n)$ denotes the smallest $b \in \mathbb{N}$ such that if a system of equations $\mathcal{S} \subseteq E_n$ has a unique solution in $\mathbb{N}^{n+1}$, then this solution belongs to $\{0, \dots, b\}^{n+1}$. The computability of β is unknown.

Theorem 5. *The function $\beta : \mathbb{N} \rightarrow \mathbb{N}$ is computable in the limit and eventually dominates every function $\delta : \mathbb{N} \rightarrow \mathbb{N}$ with a single-fold Diophantine representation.*

Proof. This is proved in [12]. Flowchart 3 shows a semi-algorithm which computes $\beta(n)$ in the limit, see [12].



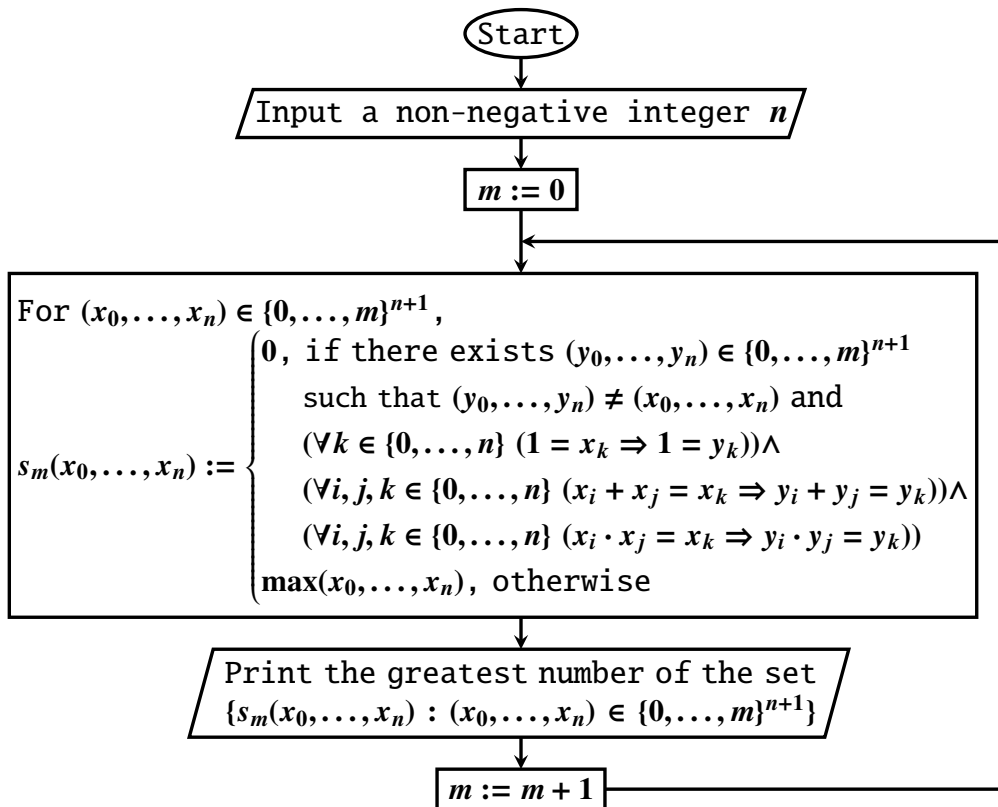
Flowchart 3

A semi-algorithm which computes $\beta(n)$ in the limit

□

7. A Short Program in MuPAD That Computes β in the Limit

Flowchart 4 shows a simpler semi-algorithm which computes $\beta(n)$ in the limit.



Flowchart 4

A simpler semi-algorithm which computes $\beta(n)$ in the limit

Lemma 4. For every $n, m \in \mathbb{N}$, the number printed by Flowchart 4 does not exceed the number printed by Flowchart 3.

Proof. For every $(a_0, \dots, a_n) \in \{0, \dots, m\}^{n+1}$,

$$\begin{aligned} E_n \supseteq & \{1 = x_k : (k \in \{0, \dots, n\}) \wedge (1 = a_k)\} \cup \\ & \{x_i + x_j = x_k : (i, j, k \in \{0, \dots, n\}) \wedge (a_i + a_j = a_k)\} \cup \\ & \{x_i \cdot x_j = x_k : (i, j, k \in \{0, \dots, n\}) \wedge (a_i \cdot a_j = a_k)\} \end{aligned}$$

□

Lemma 5. For every $n, m \in \mathbb{N}$, the number printed by Flowchart 3 does not exceed the number printed by Flowchart 4.

Proof. Let $n, m \in \mathbb{N}$. For every system of equations $\mathcal{S} \subseteq E_n$, if $(a_0, \dots, a_n) \in \{0, \dots, m\}^{n+1}$ is a unique solution of $\mathcal{S}$ in $\{0, \dots, m\}^{n+1}$, then $(a_0, \dots, a_n)$ solves the system $\hat{\mathcal{S}}$, where

$$\begin{aligned} \hat{\mathcal{S}} = & \{1 = x_k : (k \in \{0, \dots, n\}) \wedge (1 = a_k)\} \cup \\ & \{x_i + x_j = x_k : (i, j, k \in \{0, \dots, n\}) \wedge (a_i + a_j = a_k)\} \cup \\ & \{x_i \cdot x_j = x_k : (i, j, k \in \{0, \dots, n\}) \wedge (a_i \cdot a_j = a_k)\} \end{aligned}$$

By this and the inclusion $\hat{\mathcal{S}} \supseteq \mathcal{S}$, $\hat{\mathcal{S}}$ has exactly one solution in $\{0, \dots, m\}^{n+1}$, namely $(a_0, \dots, a_n)$. □

Theorem 6. For every $n, m \in \mathbb{N}$, Flowcharts 3 and 4 print the same number.

Proof. It follows from Lemmas 4 and 5. □

The following program in MuPAD implements the semi-algorithm shown in Flowchart 4.

```
input("Input a non-negative integer n",n):
m:=0:
while TRUE do
X:=combinat::cartesianProduct([s $s=0..m] $t=0..n):
Y:=[max(op(X[u])) $u=1..(m+1)^(n+1)]:
for p from 1 to (m+1)^(n+1) do
for q from 1 to (m+1)^(n+1) do
v:=1:
for k from 1 to n+1 do
if 1=X[p][k] and 1<>X[q][k] then v:=0 end_if:
for i from 1 to n+1 do
for j from i to n+1 do
if X[p][i]+X[p][j]=X[p][k] and X[q][i]+X[q][j]<>X[q][k] then v:=0 end_if:
if X[p][i]*X[p][j]=X[p][k] and X[q][i]*X[q][j]<>X[q][k] then v:=0 end_if:
end_for:
end_for:
end_for:
if q<>p and v=1 then Y[p]:=0 end_if:
end_for:
end_for:
print(max(op(Y))):
m:=m+1:
```


end_while:

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