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Article

Attack on the Riemann Hypothesis II

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Abstract

Let $\xi(z)$ be the Riemann xi function. In a previous paper we prove the boundedness of coefficients of the power series expansion of $\xi'(1/z)/\xi(1/z)$ and thus give a proof of the Riemann hypothesis. In this paper we generalize the method there to the study of the extended Riemann hypothesis for general number fields.

Keywords: Keiper-Li coefficients; extended Riemann hypothesis

MSC: 11M06; 11M26

1. Introduction

This paper is a continuation of [8]. Throughout we write $z = \sigma + it$ with $\sigma, t \in \mathbb{R}$.

1.1. Keiper and Li

Let $\zeta(z)$ be the Riemann zeta function and

$$\xi(z) = \frac{z(z-1)}{2} \pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \zeta(z)$$

the Riemann xi function where $\Gamma(z)$ is the gamma function. The product formula of $\xi(z)$ is

$$\xi(z) = \frac{1}{2} \prod_{\rho} \left(1 - \frac{z}{\rho}\right)$$

where ρ runs over the nontrivial complex zeros of the Riemann zeta function. The Riemann hypothesis asserts that all complex zeros of $\zeta(z)$ satisfy $\sigma = 1/2$.

In 1992 Keiper [5] studied the power series

$$\frac{\xi'(1/z)}{\xi(1/z)} = \sum_{n=0}^{\infty} \tau_n (1-z)^n, \tag{1.1}$$

$$\log(2\xi(1/z)) = \sum_{n=0}^{\infty} \lambda_n^K (1-z)^n, \tag{1.2}$$

and he observed the fact that the boundedness of $|\tau_n|$ in (1.1) implies the Riemann hypothesis. In a previous article [8] we find a proof of the boundedness of $|\tau_n|$ and thus give a proof of the Riemann hypothesis.

In 1997 Li [6] defined the closely related numbers

$$\lambda_n^L = \frac{1}{(n-1)!} \frac{d^n}{dz^n} \left[z^{n-1} \log \xi(z) \right] \Big|_{z=1} \tag{1.3}$$

and they are also the coefficients of the power series expansion of

$$\frac{\varphi'(z)}{\varphi(z)} = \sum_{n=0}^{\infty} \lambda_{n+1}^L z^n, \tag{1.4}$$

where

$$\varphi(z) = \zeta\left(\frac{1}{1-z}\right). \quad (1.5)$$

Li's λ_n^L has the expression

$$\lambda_n^L = \sum_{\rho} \left[1 - \left(1 - \frac{1}{\rho} \right)^n \right]. \quad (1.6)$$

The relation between Keiper's λ_n^K and Li's λ_n^L is

$$\lambda_n^L = n\lambda_n^K.$$

It follows from (1.1) and (1.2) that

$$\tau_n = (n+1)\lambda_{n+1}^K - 2n\lambda_n^K + (n-1)\lambda_{n-1}^K, \quad (1.7)$$

or equivalently

$$\tau_n = \lambda_{n+1}^L - 2\lambda_n^L + \lambda_{n-1}^L. \quad (1.8)$$

In the following we call both λ_n^K and λ_n^L the Keiper-Li coefficients. In 1997 Li [6] showed the following equivalence, which is now known as Li's criterion:

Riemann Hypothesis $\iff$ Keiper-Li coefficients are nonnegative.

The Keiper-Li coefficient was extensively studied, see for example [1–3] and the references therein. By (1.6) and (1.8) we have

$$\tau_n = - \sum_{\rho} \frac{1}{\rho^2} \left(1 - \frac{1}{\rho} \right)^{n-1}. \quad (1.9)$$

1.2. Extended Riemann Hypothesis

Let k be a number field of degree $\delta = r_1 + 2r_2$ where r_1 (resp. r_2) denotes the number of real places (resp. complex places) of k . Let $\zeta_k(z)$ be the Dedekind zeta function of k and d the discriminant of k . Let

$$\tilde{\zeta}_k(z) := \frac{1}{2} z(z-1) \left(\frac{|d|}{4^{r_2} \pi^{\delta}} \right)^{z/2} \Gamma\left(\frac{z}{2}\right)^{r_1} \Gamma(z)^{r_2} \zeta_k(z) \quad (1.10)$$

be the xi function for k . By analytic number theory $\tilde{\zeta}_k(z)$ is an entire function of order 1 and satisfies the functional equation

$$\tilde{\zeta}_k(z) = \tilde{\zeta}_k(1-z). \quad (1.11)$$

The extended Riemann hypothesis says that all zeros of $\zeta_k(z)$, equivalently $\tilde{\zeta}_k(z)$, in the strip $0 < \sigma < 1$ satisfy $\sigma = 1/2$.

The product formula of $\tilde{\zeta}_k(z)$ is [6, (3.1)]:

$$\tilde{\zeta}_k(z) = \prod_{\rho} \left(1 - \frac{z}{\rho} \right) \quad (1.12)$$

where ρ runs over the nontrivial complex zeros of the Dedekind zeta function $\zeta_k(z)$. From this one sees that $\tilde{\zeta}_k(0) = 1$.

Li [6] studied $\tilde{\zeta}_k(z)$ by using the arguments analogous to that of $\zeta(z)$. Namely he defined

$$\lambda_n(k)^L = \frac{1}{(n-1)!} \frac{d^n}{dz^n} \left[z^{n-1} \log \tilde{\zeta}_k(z) \right] \Big|_{z=1} \quad (1.13)$$

and showed that

$$\lambda_n(k)^L = \sum_{\rho} \left[1 - \left(1 - \frac{1}{\rho} \right)^n \right]. \quad (1.14)$$

Moreover he proved

Extended Riemann Hypothesis $\iff \lambda_n(k)^L > 0$ for every positive integer n .

In this paper we use the method in [8] and prove the following.

Theorem 1.1. *Let k be a number field of degree $\delta \leq 6$, then the extended Riemann Hypothesis holds for k .*

After suitable modification of the proof process we can remove the restriction $\delta \leq 6$.

Theorem 1.2. *Let k be a number field, then the extended Riemann Hypothesis holds for k .*

2. Proof of Theorem 1.1

As in Keiper [5] we define the coefficients $\tau_n(k)$ and $\lambda_n(k)^K$ by

$$\frac{\tilde{\zeta}'_k(1/z)}{\tilde{\zeta}_k(1/z)} = \sum_{n=0}^{\infty} \tau_n(k)(1-z)^n, \tag{2.1}$$

$$\log\left(\frac{\tilde{\zeta}_k(1/z)}{\tilde{\zeta}_k(0)}\right) = \log(\tilde{\zeta}_k(1/z)) = \sum_{n=0}^{\infty} \lambda_n(k)^K(1-z)^n. \tag{2.2}$$

Proceeding completely the same as in Keiper [5] one may deduce that

$$\lambda_n(k)^L = n\lambda_n(k)^K. \tag{2.3}$$

It follows from (2.1) and (2.2) that

$$\tau_n(k) = (n+1)\lambda_{n+1}(k)^K - 2n\lambda_n(k)^K + (n-1)\lambda_{n-1}(k)^K, \tag{2.4}$$

or equivalently

$$\tau_n(k) = \lambda_{n+1}(k)^L - 2\lambda_n(k)^L + \lambda_{n-1}(k)^L. \tag{2.5}$$

Notice that (1.7) and (1.8) are special case of (2.4) and (2.5) when $k = \mathbb{Q}$.

By (2.1) one sees that if $|\tau_n(k)|$ are bounded, then the extended Riemann hypothesis is true for k . By (1.14) and (2.5) we have

$$\tau_n(k) = -\sum_{\rho} \frac{1}{\rho^2} \left(1 - \frac{1}{\rho}\right)^{n-1}, \tag{2.6}$$

where ρ runs over the nontrivial complex zeros of the Dedekind zeta function $\zeta_k(z)$. Let $N_k(T)$ denote the number of zeros of $\zeta_k(z)$ in the region $0 < \sigma < 1, 0 < t < T$. Then it is well known [[Theorem 5.31] [4]] that as $T \rightarrow \infty$,

$$N_k(T) = \frac{\delta}{2\pi} T \log T + \frac{1}{2\pi} \log \frac{|d|}{(2\pi e)^{\delta}} T + o(T). \tag{2.7}$$

As a consequence of (2.7) we have the following estimate.

Lemma 2.1. *Let the nontrivial complex zeros $\beta + i\gamma$ of $\zeta_k(z)$ with $\gamma > 0$ be arranged in a sequence $\rho_n = \beta_n + i\gamma_n$ so that $\gamma_{n+1} \geq \gamma_n$, then as $n \rightarrow \infty$,*

$$|\rho_n| \sim \gamma_n \sim \frac{2\pi n}{\delta \log n}. \tag{2.8}$$

Proof. The proof is analogous as in [[p. 214] [7]]. By (2.7) we have

$$N_k(T) \sim \frac{\delta}{2\pi} T \log T$$

and thus

$$\frac{2\pi}{\delta} N_k(\gamma_n \pm 1) \sim (\gamma_n \pm 1) \log(\gamma_n \pm 1) \sim \gamma_n \log \gamma_n. \quad (2.9)$$

Also

$$N_k(\gamma_n - 1) \leq n \leq N_k(\gamma_n + 1) \quad (2.10)$$

and therefore

$$\frac{2\pi}{\delta} n \sim \gamma_n \log \gamma_n, \quad (2.11)$$

and then taking logarithms on both sides of (2.11) we have

$$\log n \sim \log \gamma_n. \quad (2.12)$$

Combining (2.11) and (2.12) gives

$$\gamma_n \sim \frac{2\pi n}{\delta \log n}. \quad (2.13)$$

That $|\rho_n| \sim \gamma_n$ is obvious. $\square$

We now prove Theorem 1.1.

Proof of Theorem 1.1. By (2.6) and $|z_1 - z_2| \leq |z_1| + |z_2|$ we have

$$|\tau_n(k)| \leq \sum_{\rho} \frac{1}{|\rho|^2} \left(1 + \frac{1}{|\rho|}\right)^{n-1}. \quad (2.14)$$

Together with (2.8) we have

$$\sum_{\rho} \frac{1}{|\rho|^2} \left(1 + \frac{1}{|\rho|}\right)^{n-1} \ll \sum_{n=1}^{\infty} \frac{\delta^2 \log^2 n}{(2\pi n)^2} \left(1 + \frac{\delta \log n}{2\pi n}\right)^{n-1}. \quad (2.15)$$

Since

$$(n-1) \log \left(1 + \frac{\delta \log n}{2\pi n}\right) \leq n \cdot \frac{\delta \log n}{2\pi n} = \frac{\delta \log n}{2\pi}, \quad (2.16)$$

and thus we have

$$\left(1 + \frac{\delta \log n}{2\pi n}\right)^{n-1} \leq n^{\delta/2\pi}. \quad (2.17)$$

When $\delta \leq 6$, one sees that

$$\sum_{\rho} \frac{1}{|\rho|^2} \left(1 + \frac{1}{|\rho|}\right)^{n-1} \ll \sum_{n=1}^{\infty} \frac{\log^2 n}{n^{2-\delta/2\pi}} = \zeta'' \left(2 - \frac{\delta}{2\pi}\right) = O(1). \quad (2.18)$$

which, together with (2.14), implies the boundedness of $|\tau_n(k)|$ and thus the radius of convergence of the series in (2.1) is 1. Consequently Theorem 1.1 holds. $\square$

3. Proof of Theorem 1.2

Recall from (2.1) and (2.2) that

$$\frac{\zeta'_k(1/z)}{\zeta_k(1/z)} = \sum_{n=0}^{\infty} \tau_n(k) (1-z)^n, \quad (3.1)$$

$$\log(\zeta_k(1/z)) = \sum_{n=0}^{\infty} \lambda_n(k) (1-z)^n. \quad (3.2)$$

Taking derivative for (3.2) we have

$$\frac{\xi'_k(1/z)}{\xi_k(1/z)} \cdot \left(-\frac{1}{z^2}\right) = -\sum_{n=0}^{\infty} n\lambda_n(k)^K(1-z)^{n-1}, \quad (3.3)$$

which is

$$\frac{\xi'_k(1/z)}{\xi_k(1/z)} = \sum_{n=0}^{\infty} n\lambda_n(k)^K(1-z)^{n-1}z^2. \quad (3.4)$$

since

$$z^2 = 1 - 2(1-z) + (1-z)^2, \quad (3.5)$$

and thus we have

$$\frac{\xi'_k(1/z)}{\xi_k(1/z)} = \sum_{n=0}^{\infty} n\lambda_n(k)^K(1-z)^{n-1} - \sum_{n=0}^{\infty} 2n\lambda_n(k)^K(1-z)^n + \sum_{n=0}^{\infty} n\lambda_n(k)^K(1-z)^{n+1}. \quad (3.6)$$

Comparing (3.1) and (3.6) gives that

$$\tau_n(k) = (n+1)\lambda_{n+1}(k)^K - 2n\lambda_n(k)^K + (n-1)\lambda_{n-1}(k)^K, \quad (3.7)$$

which is (2.4) we stated before. From this we are finally led to (2.6):

$$\tau_n(k) = -\sum_{\rho} \frac{1}{\rho^2} \left(1 - \frac{1}{\rho}\right)^{n-1}. \quad (3.8)$$

We then continue to take derivative for (3.1) and get the central difference of $\tau_n(k)$, and thus by (3.7) the higher order difference of $n\lambda_n(k)^K$ or equivalently $\lambda_n(k)^L$. We have

$$\frac{\xi''_k(1/z)\xi_k(1/z) - (\xi'_k(1/z))^2}{\xi_k(1/z)^2} \cdot \left(-\frac{1}{z^2}\right) = -\sum_{n=0}^{\infty} n\tau_n(k)(1-z)^{n-1}. \quad (3.9)$$

We write

$$\frac{\xi''_k(1/z)\xi_k(1/z) - (\xi'_k(1/z))^2}{\xi_k(1/z)^2} = \sum_{n=0}^{\infty} c_n(k)(1-z)^n. \quad (3.10)$$

As the observation of Keiper one sees that if $|c_n(k)|$ are bounded, then the extended Riemann hypothesis is true for k . As we did from (3.4) to (3.7) we have

$$c_n(k) = (n+1)\tau_{n+1}(k) - 2n\tau_n(k) + (n-1)\tau_{n-1}(k). \quad (3.11)$$

Inserting (3.8) into (3.11) we have

$$c_n(k) = -n \sum_{\rho} \frac{1}{\rho^4} \left(1 - \frac{1}{\rho}\right)^{n-2} + \sum_{\rho} \frac{1}{\rho^3} \left(1 - \frac{1}{\rho}\right)^{n-1} + \sum_{\rho} \frac{1}{\rho^3} \left(1 - \frac{1}{\rho}\right)^{n-2}. \quad (3.12)$$

Then analogously to the proof of Theorem 1.1 one may deduce

$$|c_n(k)| \ll \left| \sum_{n=1}^{\infty} \frac{\log^4 n}{n^{3-\delta/2\pi}} \right| + \left| \sum_{n=1}^{\infty} \frac{\log^3 n}{n^{3-\delta/2\pi}} \right|, \quad (3.13)$$

and thus when $\delta \leq 12$, $|c_n(k)|$ is bounded. This improves the degree bound $\delta \leq 6$ in Theorem 1.1. One may continue to take derivative for (3.10) along the same way and this will further improve the degree bound of number fields. However the computation will be complicated.

To prove Theorem 1.2 we directly generate new coefficients from (3.1). In the following for simplicity we write τ_n for $\tau_n(k)$.

Proof of Theorem 1.2. We multiply by $(1 - z)$ both sides of

$$\frac{\xi'_k(1/z)}{\xi_k(1/z)} = \sum_{n=0}^{\infty} \tau_n (1-z)^n, \quad (3.14)$$

and obtain

$$(1-z) \frac{\xi'_k(1/z)}{\xi_k(1/z)} = \sum_{n=0}^{\infty} \tau_n (1-z)^{n+1}. \quad (3.15)$$

The difference (3.14)-(3.15) is

$$z \frac{\xi'_k(1/z)}{\xi_k(1/z)} = \tau_0 + (\tau_1 - \tau_0)(1-z) + (\tau_2 - \tau_1)(1-z)^2 + \dots. \quad (3.16)$$

The general coefficients of (3.16) are the first order differences of τ_n and by (3.8) we have

$$|\Delta \tau_n| = |\tau_{n+1} - \tau_n| = \left| \sum_{\rho} \frac{1}{\rho^3} \left(1 - \frac{1}{\rho}\right)^{n-1} \right|. \quad (3.17)$$

Argued as before this implies the degree bound $\delta \leq 12$ for Theorem 1.2. We continue to multiply by $(1 - z)$ both sides of (3.16) and get

$$z(1-z) \frac{\xi'_k(1/z)}{\xi_k(1/z)} = \tau_0(1-z) + (\tau_1 - \tau_0)(1-z)^2 + (\tau_2 - \tau_1)(1-z)^3 + \dots. \quad (3.18)$$

The difference (3.16)-(3.18) is

$$z^2 \frac{\xi'_k(1/z)}{\xi_k(1/z)} = \tau_0 + (\tau_1 - 2\tau_0)(1-z) + (\tau_2 - 2\tau_1 + \tau_0)(1-z)^2 + \dots. \quad (3.19)$$

The general coefficients of (3.19) are the second order differences of τ_n and by (3.17) we have

$$|\Delta^2 \tau_n| = |\tau_{n+1} - 2\tau_n + \tau_{n-1}| = \left| \sum_{\rho} \frac{1}{\rho^4} \left(1 - \frac{1}{\rho}\right)^{n-2} \right|. \quad (3.20)$$

In general for an integer $\ell \geq 1$ in view of (3.14),

$$z^\ell \cdot \frac{\xi'_k(1/z)}{\xi_k(1/z)} = \sum_{n=0}^{\infty} \tau_n (1-z)^n z^\ell = \sum_{n=0}^{\infty} \tau_n (1-z)^n \sum_{m=0}^{\ell} (-1)^m \binom{\ell}{m} (1-z)^m. \quad (3.21)$$

The general coefficients of (3.21) are the ℓ -th order differences of τ_n and we have

$$|\Delta^\ell \tau_n| = \left| \sum_{\rho} \frac{1}{\rho^{\ell+2}} \left(1 - \frac{1}{\rho}\right)^{n-\ell} \right| \ll \left| \sum_{n=1}^{\infty} \frac{\log^{\ell+2} n}{n^{\ell+2-\delta/2\pi}} \right|. \quad (3.22)$$

When $\ell + 2 - \delta/2\pi > 1$, that is for every number field of fixed degree

$$\delta < 2\pi(\ell + 1), \quad (3.23)$$

there is as $n \rightarrow \infty$,

$$|\Delta^\ell \tau_n| = \left| \sum_{\rho} \frac{1}{\rho^{\ell+2}} \left(1 - \frac{1}{\rho}\right)^{n-\ell} \right| = O(1), \quad (3.24)$$

and thus the radius of convergence of (3.21) is 1 and hence the extended Riemann Hypothesis holds for k of degree δ . Since ℓ can be arbitrarily large the proof of Theorem 1.2 is complete. $\square$

It is possible to study more general L -functions in [[Chapter 5] [4]] using the method in this paper.

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