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Keywords: Prime numbers; Prime Number Theorem; binary Goldbach conjecture; Lagrange-Lemoine-Levy conjecture; Bezout-Goldbach conjecture; gaps between consecutive primes; Weak Chen or Goldbach(-) conjecture

Article

Proof of the Binary Goldbach Conjecture

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Abstract: In this article the proof of the binary Goldbach conjecture is established (Any integer greater than one is the mean arithmetic of two positive primes) . To this end the weak Chen conjecture is proved (Any even integer greater than One is the difference of two positive primes) and a " located " algorithm is developed for the construction of two recurrent sequences of primes (U_{2n}) and (V_{2n}), ((U_{2n}) dependent of (V_{2n})) such that for each integer $n \geq 2$ their sum is equal to $2n$. To form this a third sequence of primes (W_{2n}) is defined for any integer $n \geq 3$ by W_{2n} = Sup ($p \in \mathcal{P}$: $p \leq 2n$ - 3) , $\mathcal{P}$ being the infinite set of positive primes. The Goldbach conjecture has been proved for all even integers $2n$ between 4 and 4.10^{18}.. In the table of terms of Goldbach sequences given in Appendix 12 values of the order of $2n = 10^{1000}$ are reached. An analogous proof by recurrence « finite ascent and descent method » is developed and a majorization of U_{2n} by 0.7 $\ln^{2.2}(2n)$ is justified.. In addition, the Lagrange-Lemoine-Levy conjecture and its generalization called '' Bezout-Goldbach '' conjecture are proven by the same type of algorithm.

Keywords : Prime Number Theorem; Binary Goldbach Conjecture; Weak Chen Conjecture; Lagrange-Lemoine-Levy Conjecture; Bezout-Goldbach Conjecture; Gaps between consecutive Primes.

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1. Overview

Number theory " the queen of mathematics " studies the structures and properties defined on integers and primes (Euclid [13], Hadamard [15], Hardy and Wright [16], Landau [22], Tchebychev [35]). Numerous problems have been raised and conjectures made, the statements of which are often simple but very difficult to prove. These main components include

• **Elementary arithmetic .** Determination and properties of primes, operations on integers (basic operations, congruence, gcd, lcm,).

Decomposition of integers into products or sums of primes

(fundamental theorem of arithmetic, decomposition of large numbers, cryptography and Goldbach's conjecture).

• **Analytical number theory .** Distribution of primes (Prime Number Theorem,

Hadamard [15], De la Vallée-Poussin [36], Littlewood [25] and Erdos [12], the Riemann hypothesis,.....).

Gaps between consecutive primes (Bombieri,Davenport [3], Cramer [8], Baker,Harmann,Iwaniec, Pintz [4],[5],[20], Granville [14], Maynard [27], Tao [34], Shanks [30], Tchebychev [35] and Zhang [39]).

• Algebraic, probabilistic, combinatorial and algorithmic number theories . Modular arithmetic, diophantine approximations, equations, arithmetic functions and algebraic, diophantine and number geometry.

2. Definitions Notations and Background

The integers $n, k, p, q, r,$……..... used in this article are always positive.(2.1)

The symbol '' / '' means " in relation to". (2.2)

Let $\mathcal{P}$ be the infinite set of positive primes p_k (called simply primes) (2.3)

($p_1 = 2$; $p_2 = 3$; $p_3 = 5$; $p_4 = 7$; $p_5 = 11$; $p_6 = 13$; ....)

For any integer $K \geq 1 \mathcal{P}_K = \{ p \in \mathcal{P} : p \leq 2K \}$ (2.4)

The writing of large numbers (see appendix 12) is simplified using the following constants

M = 10^9 ; R = 4.10^8 ; G = 10^{100} ; S = 10^{500} ; T = 10^{1000} (2.5)

$\ln(x)$ denotes the neperian logarithm of the real $x > 0$

Let (W_{2n}) be the sequence of primes defined by

$\forall$ n $\in \mathbb{N} + 3$ W_{2n} = Sup(p $\in \mathcal{P}$: p $\leq$ 2n - 3) (2.6)

Any sequence denoted by $(G_{2n}) = (U_{2n}; V_{2n})$ verifying (2.6.1) is called a **Goldbach sequence.**

$\forall$ n $\in \mathbb{N} + 2$ U_{2n} ,$V_{2n} \in \mathcal{P}$ and$U_{2n} + V_{2n}$ = 2n(2.7)

U_{2n} and V_{2n} are also known as " **Goldbach decomponents** ".

Iwaniec,Pintz [20] have shown that for a sufficiently large integer n there is always a prime between $n - n^{23/42}$ and n . Baker,Harman [4],[5] concluded that there is a prime in the interval $[\, n \,; n + o\,(n^{0.525})]$. Thus this results provides an increase of the gap between two consecutive primes p_k and p_{k+1} of the form

$\forall\ \varepsilon > 0\ \exists\ k_\varepsilon \in \mathbb{N}^*$ / $\forall$ k $\in \mathbb{N}$ k $\geq$ k_ε $p_{k+1} - p_k < \varepsilon . p_k^{0.525}$(2.8)

The results obtained on the Cramer-Granville-Maier-Nicely conjecture [1],[3],[8],[14],[26],[28] imply the following majorization.

For any real $c > 2$ and for any integer $k \geq 500$

$p_{k+1} - p_k \leq 0.7 \ \ln^c(p_k)$ **(with probability one)** (2.9)

3. Introduction

Chen [6], Hardy,Littlewood [17], Hegfollt,Platt [18], Ramaré,Saouter [29], Tao [34],

Tchebychev [35] and Vinogradov [37] have taken important steps and obtained promising results on the Goldbach conjecture (Any integer $n \geq 2$ is the mean arithmetic of two primes).

Indeed, Helfgott,Platt [18] proved the weak Goldbach conjecture in 2013.

Silva,Herzog,Pardi [32] held the record for calculating the terms of Goldbach sequences after determining pairs of primes $(U_{2n}\ ;\ V_{2n})$ verifying

$\forall\ n \in \mathbb{N}$ / $4 \leq 2n \leq 4.10^{18}$ $U_{2n} + V_{2n} = 2n$ (3.1)

In previous research work there is no explicit construction of recurrent Goldbach sequences. In this article two sequences of primes are developed using a simple, efficient and « located » algorithm to compute for any integer $n \geq 3$ by successive iterations any term U_{2n} and V_{2n} .

Using Maxima scientific software on a personal computer Silva's record is broken and the values $2n = 10^{500}$ and even $2n = 10^{1000}$ are reached. The binary Goldbach conjecture can be established on the same principle by recurrence by using the weak Chen or Goldbach(-) conjecture (any even integer greater than three is the difference of two primes) demonstrated in Teorem 4.

● Remark.

1. **Chen conjecture :** For any integer K ≥ 1 there are infinitely many pairs of primes with a difference equal to 2K.

2. **Polignac conjecture :** Same as Chen, but with consecutive pairs of primes.

3. What we know :

April 2013, Yitang Zhang [39] demonstrates that the smallest even integer $2K$ verifying the conjecture is greater than 70 million.

In 2014, James Maynard [27] then Terence Tao [34] lowered this limit to 246.

We validate weak Chen or Goldbach(-) conjecture by verifying directly in the prime number tables that all even gaps from 2 to 246 are possible between primes.

In addition, the Lagrange-Lemoine-Lévy conjectures [9],[19],[21],[26],[28],[33],[38] and its generalization called '' Bezout-Goldbach « conjecture' are validated.

Using case disjunction reasoning we construct two recurrent sequences of primes (V_{2n}) and (U_{2n})

according to the sequence (W_{2n}) by the following process

For any integer $n \geq 2$

$U_4 = 2$ and $V_4 = 2$ (3.2)

Let $n \in \mathbb{N} + 3$

● Either

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(2n - W_{2n}) is a prime
then V_{2n} and U_{2n} are defined directly in terms of W_{2n} .
• Either
($2n$ - W_{2n}) is a composite number
then V_{2n} and U_{2n} are determined from the previous terms of the sequence (G_{2n}).

4. Theorem (Weak Chen or Goldbach(-) conjecture)

$\forall$ K $\in \mathbb{N}^*$ $\exists$ p , q $\in \mathcal{P}$ /
p - q = 2K3 ≤ q ≤ 2K and 3 + 2K ≤ p ≤ 4K if K ≥ 2 (4.1)
Practical method on some examples:
First of all (5 - 3 = 2), then we begin the process at (7 - 3 = 4), we will select the smallest primes for which the difference is precisely 6 (11 - 5 = 6) then 8 (11 - 3 = 8)
then 10 (13 - 3 = 10),......... then $2K$, then $2(K + 1)$ (demonstration established by strong recurrence, by the asurd and return).
All pairs of Goldbach(-) decomponents obtained by this method for K between 2 and 123 are listed in the table in Appendix 13.
Proof . The proof is established by strong recurrence on K . Let $\mathcal{P}_{Chen}$ (K) be the following property
« $\forall$ K $\in \mathbb{N}^*$ $\exists$ p, q $\in \mathcal{P}$ / p - q = 2K3 ≤ q ≤ 2K and 2K + 3 ≤ p ≤ 4K «(4.2)
► $\mathcal{P}_{Chen}$ (2) is true : 7 - 3 = 4 ; $q = 3 \leq 4$ and $p = 7 \leq 4 \times 2 = 8$
► Let's show
$\forall M \in \mathbb{N} / M \leq K$ then $\mathcal{P}_{Chen}$ $(M) \Longrightarrow \mathcal{P}_C$ $(K + 1)$
We reason through the absurd
$\forall p , q \in \mathcal{P}_K$ $/ p \geq q$; $\forall h, m \in \mathbb{N} / p + 2h , q + 2m \in \mathcal{P}$
we assume that
$p + 2h - q - 2m \neq 2(K + 1)$
Therefore
$p - q \neq 2(K + 1 - h + m)$.
You can always choose ($h \geq m$ and $h - m \leq K + 1$).
However the strong recurrence hypothesis asserts that
$\forall M \in \mathbb{N} / M \leq K \ \exists \ p, q \in \mathcal{P} \ / p - q = 2M$(4.3)
By choosing $M = K + 1 - h + m$
this contradicts (4.3).
So
$\exists$ $h,m \in \mathbb{N} / p + 2h - q - 2m = 2(K + 1)$ (p, p+$2h$, q, q+$2m$ $\in \mathcal{P}$: $h \geq m$ and $h - m \leq K + 1$) (4.4)
Thus validating the heredity of property $\mathcal{P}_{Chen}$ (K).
The property $\mathcal{P}_{Chen}$ (K) is therefore true. As a result Goldbach(-)'s conjecture is validated.

5. Corollary

Let (R_{2K}) and (Q_{2K}) two sequences of primes determined by
R_{2K} = Inf (p $\in \mathcal{P}$: p - 2K $\in \mathcal{P}$) and Q_{2K} = Inf (p $\in \mathcal{P}$: 2K + p $\in \mathcal{P}$) = R_{2K} - 2K (5.1)
They are defined for any integer K $\in \mathbb{N}^*$ and satisfy
(5.1) lim R_{2K} = +∞
(5.2) $\forall K \in \mathbb{N}^*$ R_{2K}, Q_{2K} $\in \mathcal{P}$and R_{2K} - Q_{2K} = 2K
(5.3) *For any integer K* / 2 ≤ K ≤ 163 ≤ Q_{2K} ≤ 2K and 2K + 3 ≤ R_{2K} ≤ 4K
(5.4) For any integer K ≥ 16
$3 \leq Q_{2K} \leq 2(2K)^{0.525}$ and $2K + 3 \leq R_{2K} \leq 2K + (2K)^{0.525}$

Proof. According to the previous theorem, the sequences (R_{2K}) and (Q_{2K}) are defined by strong recurrence and finite descent.
(5.1) $R_{2K} \geq 2K \Rightarrow \lim R_{2K} = +\infty$
(5.2) By construction, these sequences thus verify $R_{2K} - Q_{2K} = 2K$

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(5.3) The term-to-term property can be verified directly by examining the sequence proposed above.

(5.4) This property is verified up to $2K = 246$ by calculations on the previous list.

We prove this result by recurrence

First of all we order the Goldbach(-) decomponents at a fixed prime Q,

so as to obtain the estimate (5.4) more easily.

We examine the following sequences of primes ($PQ(K)$).

$P3(K) = 2K + 3$

$(P3(K)\,;\,2K) \longrightarrow$ (5;2);(7;4);(11;8);(13;10);(17;14);(19;16);(23;20);(29;26);(29;28);..........

$P5(K) = 2K + 5$

$(P5(K)\,;\,2K) \longrightarrow$ (7;2);(11;6);(13;8);(17;12);(19;14);(23;18);(29;24);(31;26);(37;32)........

$P7(K) = 2K + 7$

$(P7(K)\,;\,2K) \longrightarrow$ (11;4);(13;6);(17;10);(19;12);(23;16);(29;22);(31;24);(37;30)...........

$P11(K) = 2K + 11$

$(P11(K)\,;\,2K) \longrightarrow$ (13;2);(17;6);(19;8);(23;12);(29;18);(31;20);(37;26);(41;30);(43;34)......

$(P13(K)\,;\,2K) \longrightarrow$ (17;4);(19;6);(23;10);(29;16);(31;18);(37;24);(41;28);(43;30);(47;34)......

$PQ(K) = 2K + Q$ ($K \in \mathbb{N}^*$: $PQ(K)$ and Q are primes)

(see the table in Appendix 14)

For any integer K satisfying $2(2K)^{0.525} > Q$ the property holds for $PQ(K)$.

Therefore it is generally validated for all $K > 15$, since we obtain all possible cases of Chen's weak conjecture starting with $P3(K)$, then $P5(K)$, then $P7(K)$ ...... for $2(2K)^{0.525} \leq Q$.

(can be proved by strong recurrence using the same method as inTheorem 4 by "finite descent»).

Let $c_p = \frac{40}{21}$ and $\Pr(K)$ be the following property

« For any integer $M < (0.5Q_K)^{c_p}$, there exists at least a prime $Q < Q_K$ such that $2M + Q$ is a prime «

▶ Pr(15) is true (see Appendix 14).

▶ Let's show : $\Pr(K) \Rightarrow \Pr(K+1)$

$Q_{K+1} \leq Q_K + Q_K^{0.525}$

It is assumed that$M$ /

$P_{K+1} - Q_{K+1} \neq 2M$ $M < (0.5Q_{K+1})^{c_p}$ $P_{K+1} = p + 2h$ and $Q_{K+1} = q + 2s$

then

$p - q \neq 2(M + s - h)$

which is impossible according to the hypothesis of strong recurrence since

$2(M + s - h)$ is less than $(0.5Q_K)^{c_p}$ and that all primes p, q satisfy the recurrence hypothesis.

We deduce that $\Pr(K) \Rightarrow \Pr(K+1)$

Thus the property (5.4) is true.

6. Principle of Proof

To determine pairs of primes that verify Goldbach's conjecture three sequences of primes (W_{2n}), (V_{2n}), (U_{2n}) are defined and they verify the following properties

$\lim V_{2n} = +\infty$. (6.1)

(6.2) $\forall\, n \in \mathbb{N} + 2V_{2n}$ is defined as a function of $W_{2n} = \mathrm{Sup}(p \in \mathcal{P} : p \leq 2n - 3)$

(6.3) (W_{2n}) is an increasing sequence of primes that contains all primes except $p_1 = 2$

(6.4) $\lim W_{2n} = +\infty$

(6.5) (U_{2n}) is a complementary sequence of negligible primes with respect to $2n$

(6.6) For any integer $n \geq 3$

• If $(2n - \boldsymbol{W_{2n}})$ is a prime

then V_{2n} and U_{2n} are defined by

(6.7) $V_{2n} = W_{2n}$ and $U_{2n} = 2n - W_{2n}$

• Otherwise, if $(2n - \boldsymbol{W_{2n}})$ is a composite number

we search for two previous terms of the sequence (G_{2n}), $U_{2(n-k)}$ and $V_{2(n-k)}$ satisfying the following conditions

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(6.8) $U_{2(n-k)}$, $V_{2(n-k)}$ and $U_{2(n-k)}$ + 2k are primes $U_{2(n-k)} + V_{2(n-k)}$ = 2(n - k)
which is always possible (see Theorem 4)
So by setting
(6.9) $V_{2n} = V_{2(n-k)}$ and $U_{2n} = U_{2(n-k)}$ + 2k
two new primes V_{2n} and U_{2n} satisfying (4.10) are generated .
(6.10) $U_{2n} + V_{2n}$ = 2n
This process is then repeated incrementing n by one unit ($n \to n + 1$).

7. Theorem

There exists a recurrent sequence (G_{2n}) = (U_{2n} ; V_{2n}) of primes satisfying the following conditions.

For any integer n ≥ 2
U_{2n} , V_{2n} ϵ $\mathcal{P}$ and $U_{2n} + V_{2n}$ = 2n(7.1)
(Any integer n ≥ 2 is the mean arithmetic of two primes)
An algorithm can be used to explicitly compute any term U_{2n} and V_{2n} . (7.2)
Proof .
□ FIRST METHOD :
For any integer $n \geq 3$
● If (2n - $\boldsymbol{W_{2n}}$) is a prime
then V_{2n} and U_{2n} are defined by
(7.3) $V_{2n} = W_{2n}$ and U_{2n}= 2n - W_{2n}
● Otherwise, if (2n - $\boldsymbol{W_{2n}}$) is a composite number
we use the previous terms of the sequence (G_{2n}).
For any integer q such that $1 \leq q \leq n - 3$ we have
$3 \leq U_{2(n-q)} \leq$ n .
Then, there exists an integer k / $1 \leq k \leq n - 3$ following the Bertrand principle and Theorem 4
since all primes smaller than 2k are represented by $U_{2(n-j)}$, (if there were no such primes, we would have a contradiction with the Theorem 4 , even if it means transforming the indexing of the sequence (U_{2n}) . In fact, in an equivalent way we can copy the proof of Teorem 4 by performing a similar strong recurrence " finite descent return and absurd " directly on the set { }$U_{2(r\,-\,j)}$).
such that
(7.4) $R_{2n} = U_{2(n-k)}$ + 2k ∈ P
The smallest integer k / R_{2n} ∈ $\mathcal{P}$ is denoted by k_n .
So
(7.5) $U_{2n} = U_{2(n-k_n)} + 2k_n$ and$V_{2n} = V_{2(n-k_n)}$
(These two terms are primes)
In the previous steps two primes $U_{2(n-k_n)}$ and $V_{2(n-k_n)}$ whose sum is equal to 2(n - k_n) were determined.
(7.6) $U_{2(n-k_n)} + V_{2(n-k_n)}$ = 2(n - k_n)
By adding the term k_n to each member of the equality (5.6), it follows
(7.7) $U_{2(n-k_n)} + 2k_n + V_{2(n-k_n)}$ = 2(n - k_n) + 2k_n
(7.8) ⇔ [$U_{2(n-k_n)} + 2k_n$] + $V_{2(n-k_n)}$ = 2n
(7.9) ⇔ $U_{2n} + V_{2n}$ = 2n
Finally for any integer $n \geq 3$ this algorithm determines two sequences of primes (U_{2n}) and (V_{2n}) verifying Goldbach's conjecture.
□ SECOND METHOD :
The proof can be made using the following strong recurrence principle.
Let P(n) be the property defined for any integer $n \geq 2$ by
P(n) : " For any integer p satisfying $2 \leq p \leq n$ there exists two primes U_{2p} and V_{2p} such their sum is equal to 2p " .
(∀ p ∈ ℕ / 2 ≤ p ≤ n U_{2p} , V_{2p} ϵ $\mathcal{P}$ and $U_{2p} + V_{2p}$ = 2p)
Let's show by strong recurrence that P(n) is true for any integer $n \geq 2$
a) P(2) is true : it suffices to choose $U_4 = V_4$ = 2 .

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b) Let's show that the property P(n) is hereditary i.e $\forall\, k \in \mathbb{N}+2$P($n$) $\Rightarrow$ P(n+ 1)
Assume property P(n) is true,
• If (2(n + 1) - $\boldsymbol{W_{2(n+1)}}$) is a prime
then $V_{2(n+1)}$ and $U_{2(n+1)}$ are defined by
$V_{2(n+1)} = W_{2(n+1)}$ and$U_{2(n+1)}$= 2(n+1) - $W_{2(n+1)}$ (7.10)
• Otherwise, if (2(n+1) - $\boldsymbol{W_{2(n+1)}}$) is a composite number
there exists an integer k to obtain two terms $U_{2(n+1-k)}$ and $V_{2(n+1-k)}$ satisfying the following conditions
(7.11) $U_{2(n+1-k)}$, $V_{2(n+1-k)}$ and $U_{2(n+1-k)}$ + 2k are primes $U_{2(n+1-k)} + V_{2(n+1-k)}$ = 2(n +1- k)
(which is always possible : see FIRST METHOD and Theorem 4).
Thus by setting
(7.12) $V_{2(n+1)} = V_{2(n+1-k)}$ and $U_{2(n+1)} = U_{2(n+1-k)}$+ 2k
Two new primes $V_{2(n+1)}$ and $U_{2(n+1)}$ satisfying ($U_{2(n+1)} + V_{2(n+1)}$ = 2(n + 1)) are generated.
It follows that P(n + 1) is true. Then the property P(n) is hereditary (P(n) => P(n + 1)).
Therefore for any integer $n \geq 2$ the property P(n) is true.
it follows
$\forall\; n \in \mathbb{N}+2$there are two primes U_{2n} and V_{2n} and such their sum is $2n$: ($U_{2n} + V_{2n} = 2n$)

8. Lemma

The sequence (U_{2n}) verifies the following majorization
For any integer n ≥ 65
(8.1) $U_{2n} \leq (2n)^{0.55}$
Proof . According to the programm 11.2 and appendix 12 the majorization (8.1) is verified
For any integer n such that $65 \leq n \leq 2000$. For any integer $n > 2000$ the proof is established by recurrence. For this purpose let P1(n) be the following property
(8.2) P1(n) : " There exists a strictly increasing sequence of positive numbers (C_n) such that $U_{2n} \leq C_n (2n)^{0.525}$ " .
P1(2000) is true according to program 11.2 and the table in appendix 12.
For any integer $n \geq 2000$ let's show that P1(n) is hereditary i.e P1(n) ⇒ P1(n + 1).
Assume that P1(n) is true : then
• If (2(n + 1) - $\boldsymbol{W_{2(n+1)}}$) is a prime
then $V_{2(n+1)}$ and $U_{2(n+1)}$ are defined by
(8.3) $V_{2(n+1)} = W_{2(n+1)}$ and $U_{2(n+1)} = 2(n+1) - W_{2(n+1)}$
According to the results in [4],[5],[20] there is a constant $K > 0$ such that
$(n + 1) - K\,.\,[2(n+1)]^{0.525} < W_{2(n+1)} < 2(n + 1)$
$\Rightarrow U_{2(n+1)} < K\;.[2(n+1)]^{0.525}$
$\Rightarrow U_{2(n+1)} \leq C_{n+1}.[2(n+1)]^{0.525}$
• Otherwise, if (2(n + 1) - $\boldsymbol{W_{2(n+1)}}$) is a composite number
(8.4)$\exists$ p $\in \mathbb{N}^*$ / $U_{2(n+1)} = U_{2(n+1-p)} + 2p$
According to [4],[5],[18] the smallest integer p defined in (6.4) verifies
(8.5) 2p < K.[$U_{2(n+1-p)}]^{0.525}$ and $U_{2(n+1-p)} < C_{n+1-p}\,.[2(n+1-p)]^{0.525}$
It follows
$U_{2(n+1)} < K\,.C_{n+1-p}^{0.525}\,.[2(n+1-p)]^{0.275625} + C_{n+1-p}\,.[2(n+1-p)]^{0.525}$
Then
(8.6) $U_{2(n+1)} < C_{n+1}.[2(n+1)]^{0.525}$
and by setting $C_n = (2n)^{0.025}$
It follows
(8.7) $U_{2(n+1)} < [2(n+1)]^{0.55}$
P1(n + 1) is true then P1(n) is hereditary.
So for any integer $n \geq 2000$ the property P1(n) is true.
(The inequality (6.7) is verified with the aid of the software Maple studying the functions of the type $f : x \to a\,.x^{0.275625} + b\,.\,x^{0.525}$ increased by $g : x \to x^{0.55}$ a and b being two strictly positive real parameters).

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• **Remark.** A more precise estimate can be obtained using the Cipolla or Axler frames [7],[2].

9. Theorem

For any integer n $\geq$ 3 it is easy to check

(9.1) (W_{2n}) is a positive increasing sequence of primes.

(9.2) { W_{2n} : n ∈ IN + 3 } ∪ { 2 } = $\mathcal{P}$

(9.3) lim W_{2n} = +oo

(9.4) (U_{2n}) and (V_{2n}) are sequences of primes and the set $\{(U_{2k}) : k \leq n\}$ contains all primes less than ln(n)

(9.5) n $\leq V_{2n} \leq W_{2n}$

(9.6) $3 \leq 2n - W_{2n} \leq U_{2n} \leq$ n

(9.7) lim V_{2n} = +oo

Proof .

(9.1) For any integer n ≥ 2 $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ .Therefore, $W_{2n} \leq W_{2(n+1)}$. So the sequence (W_{2n}) is increasing.

(9.2) Any prime except p_1 = 2 is odd, hence the result.

(9.3) lim W_{2n} = lim p_k = +oo

(9.4) By definition $V_{2n} = W_{2n}$ or there exits an integer $k \leq n$ - 2 such that $V_{2n} = V_{2(n-k)}$;

so the terms of the sequence $(V_{2n)})$ are primes.

(9.5) According to Lemma 6, for any integer $n \geq 65$

$U_{2n} < (2n)^{0.55}$

therefore

$U_{2n} < (2n)^{0.55} <$ n

and

V_{2n} = 2n - U_{2n} > 2n - n > n

For any integer n / 3 ≤ n ≤ 65 verification is carried out according to the computer program in paragraph 11.2 and the table in appendix 12.

we can also see that by construction $V_{2n} \geq U_{2n}$ because if we assume the opposite then V_{2n} is not the largest prime number verifying $\frac{1}{2}$ $(U_{2n} + V_{2n}) = n$.

So

$V_{2n} \geq$ n

(9.6) According to (9.5) n $\leq V_{2n} \Rightarrow U_{2n}$ = 2n - V_{2n} ≤ 2n - n ≤ n

therefore

$V_{2n} \leq W_{2n} \Rightarrow$ 2n - W_{2n} ≤ 2n - $V_{2n} = U_{2n}$

(9.7) By (9.5) for any integer $n \geq 2$: $n \leq V_{2n}$

so

lim V_{2n} = +oo .

10. Remarks

10.1 For any integer $k \geq 2$ there are infinitely many integers n such that $U_{2n} = p_k$.

10.2 V_{2n} ~ $2n$ for ($n \rightarrow$ +oo) .

10.3 For any sufficiently large integer n / $n \geq 5000$

$U_{2n} \ll V_{2n}$andlim $(\frac{U_{2n}}{V_{2n}})$ = 0.

10.4 The smallest integer n such that

$U_{2n} \neq 2n$ - W_{2n} is obtained for n = 49 and G_{98} = (79 ; 19).

(This type of terms increases in the Goldbach sequence (G_{2n}) as n increases in the sense of the Schnirelmann density and there are an infinite number of them; their proportion per interval can be computed using the results given in [29]).

10.5 If q is an odd integer greater than four we could generalize this algorithm with sequences (W'_{2n})

defined by

(10.5.1) ∀ n ∈ ℕ / n ≥ $\frac{(q+3)}{2}$ W'_{2n} = Sup(p ∈ $\mathcal{P}$: p ≤ 2n - q)

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Other Goldbach's sequences (G'_{2n}) independent of (G_{2n})are thus generated.

10.6 The sequence (G_{2n}) is ''extremal'' in the sense that for any integer $n \geq 2$ V_{2n} and U_{2n} are the largest and smallest possible primes such that $U_{2n} + V_{2n} = 2n$.

10.7 The Cramer-Granville-Maier-Nicely conjecture [8],[14],[19],[21],[23],[24],[26],[28],[33] is verified with probability one. It leads to the following majorization

For any integer $p \geq 500$

(10.7.1) $U_{2p} \leq 0.7\,[\ln(2p)]^{(2.2 - \frac{1}{p})}$**(with probability one)**

The proof is similar to that of lemma 8 and is validated by the studying functions of the type $f : x \rightarrow a\,.g(x) + b[\ln(g(x))]^{c}$(a,b > 0 ; c > 2) with

$g : x \rightarrow 0.7\,[\ln(x)]^{(c - \frac{1}{x})}$ and $h : x \rightarrow 0.7\,[\ln(x)]^{(2.2 - \frac{1}{x})}$ using Maple software.

• **Remark.** A better estimate can be obtained via [26],[28],[30] .

10.8 According to Bombieri [3] and using the same method as in the proof of Lemma 8, on average, we obtain the following estimate of U_{2n}

(10.8.1) $\forall\ \varepsilon > 0\ \ U_{2n} = \mathbf{O}\,(\ln^{1.3+\varepsilon}(2n))$ **(on average)**

11. Algorithm

11.1. Algorithm Written in Natural Language

Inputs :

Input four integer variables : *k, N, n, P*

Input : $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, ..., p_N the first N primes.

: n = 3

: *P* = *M, R, G, S* or *T* as indicated in paragraph **2**

Algorithm body :

Compute : W_{2n} = Sup(p $\in \mathcal{P}$: p $\leq$ 2n - 3)

If $\boldsymbol{T_{2n}}$= (2n - $\boldsymbol{W_{2n}}$) is a prime

Let :

(11.1.1)$U_{2n} = T_{2n}$ and $V_{2n} = W_{2n}$

otherwise

If $\boldsymbol{T_{2n}}$ is a composite number

Let : $k = 1$

B.1) While $U_{2(n-k)} + 2k$ is a composite number

assign to k the value $k + 1$ ($k \rightarrow k + 1$).

return to **B1)**

End while

Assign to k the value k_n ($k \rightarrow k_n$)

(11.1.2) Let :

$U_{2n} = U_{2(n-k_n)} + 2k_n$ and $V_{2n} = V_{2(n-k_n)}$

Assign to n the value $n + 1$ ($n \rightarrow n + 1$ and return to **A)**

End :

Outputs for integers less than $\mathbf{10^{4}}$:

Print (2n = ● ; 2n - 3 = ● ; W_{2n} = ● ; T_{2n} = ● ; V_{2n} = ● ; U_{2n} = ●)

Outputs for large integers :

Print (2n - P = ● ; 2n - 3 - P = ● ; W_{2n} - P = ● ; T_{2n} = ● ; $V_{2n} - P$ = ● ; U_{2n} = ●)

11.2. Program Written with Maxima Software for 2n = 10^{500}

```
n1 : 10**500 ; for n :5*10**499 + 10000 thru 5*10**499 + 10010 do
(a :2*n , c :a - 3 , test : 0 , b : prev_prime(a - 1) ,d :a - b ,
if primep(d)
then print(a - n1 , c - n1 , b - n1 , d , b - n1 , d)
else (
while test = 0 do
```

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(*e* :*a* - *c* , if (primep(*c*) and primep(*e*))
then (test:1 , print(a - n1 , b - n1 , d , c - n1 , e ," ** "))
else (test : 0 , *c* : *c* - 2))) ;

12. Appendix

Application of Algorithm 11 : Table of $\boldsymbol{U_{2n}}$ and $\boldsymbol{V_{2n}}$ terms of the Goldbach sequence ($\boldsymbol{G_{2n}}$) computed from program 11.2 ($2 \le 2n \le \mathbf{10^{1000}} + 4020$).

The ** sign in the table below indicates the results given by the algorithm 11 in case **B)** of return to the previous terms of the sequence (G_{2n}). **WATCH OUT !** For large integers n ($2n > 10^9$ for example), to simplify the display of large numbers the results are entered as follows

2n - P , (2n - 3) - P , $\boldsymbol{W_{2n}}$ - P , $\boldsymbol{T_{2n}}$, $\boldsymbol{V_{2n}}$- P and $\boldsymbol{U_{2n}}$

with

P* = *M, R, G, S, or ***T*** constants defined in (2.3)

2n 2n - 3	W_{2n}	T_{2n}=*2n* - W_{2n}	V_{2n}	U_{2n}
4 1	X	X	**2**	**2**
63	3	3	**3**	**3**
85	5	3	**5**	**3**
1 107	7	3	**7**	**3**
112 9	7	5	**7**	**5**
14 11	11	3	**11**	**3**
16 13	13	3	**13**	**3**
18 15	13	5	**13**	**5**
20 17	17	3	**17**	**3**
22 19	19	3	**19**	**3**
24 21	19	5	**19**	**5**
26 23	23	3	**23**	**3**
28 25	23	5	**23**	**5**
30 27	23	7	**23**	**7**
32 29	29	3	**29**	**3**
34 31	31	3	**31**	**3**
36 33	31	5	**31**	**5**
38 35	31	7	**31**	**7**
40 37	37	3	**37**	**3**
8077	73	7	**73**	**7**
8279	79	3	**79**	**3**
8481	79	5	**79**	**5**
8683	83	3	**83**	**3**
8885	83	5	**83**	**5**
9087	83	7	**83**	**7**
9289	89	3	**89**	**3**
9491	89	5	**89**	**5**
9693	89	7	**89**	**7**
****98 95**	**89**	**9**	**79**	**19**
10097	97	3	**97**	**3**

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120117	113	7	**113**	**7**
**122 119	113	9	**109**	**13**
124121	113	11	**113**	**11**
126123	113	13	**113**	**13**
**128 125	113	15	**109**	**19**
130127	127	3	**127**	**3**
132129	127	5	**127**	**5**
134131	131	**3**	**131**	**3**
136133	131	5	**131**	**5**
138135	131	7	**131**	**7**
140137	137	3	**137**	**3**
500 497	**491**	**9**	**487**	**13**
502499	499	3	**499**	**3**
504501	499	5	**499**	**5**
506503	503	3	**503**	**3**
508505	503	5	**503**	**5**
510507	503	7	**503**	**7**
1000 997	997	3	**997**	**3**
1002 999	997	5	**997**	**5**
1004 1001	997	7	**997**	**7**
1006 1003	**997**	**9**	**983**	**23**
1008 1005	997	11	**997**	**11**
1010 1007	997	13	**997**	**13**
1012 1009	1009	3	**1009**	**3**
1014 1011	1009	5	**1009**	**5**
1016 1013	1013	3	**1013**	**3**
1018 1015	1013	5	**1013**	**5**
10002 9999	9973	29	**9973**	**29**
10004 10001	9973	31	**9973**	**31**
10006 10003	**9973**	**33**	**9923**	**83**
10008 10005	**9973**	**35**	**9967**	**41**
10010 10007	10007	3	**10007**	**3**
10012 10009	10009	3	**10009**	**3**
10014 10011	10009	5	**10009**	**5**
10016 10013	10009	7	**10009**	**7**
10018 10015	**10009**	**9**	**10007**	**11**
1002010017	10009	11	**10009**	**11**

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$2n - M$ $(2n - 3) - M$	$W_{2n} - M$	$T_{2n} = 2n - W_{2n}$	$V_{2n} - M$	U_{2n}
+1000 +997	+993	7	+993	7
**+1002 +999	+993	9	+931	71
+1004+1001	+993	11	+993	11
+1006+1003	+993	13	+993	13
**+1008+1005	+993	15	+919	89
+1010+1007	+993	17	+993	17
+1012+1009	+993	19	+993	19
+1014+1011	+1011	3	+1011	3
+1016+1013	+1011	5	+1011	5
+1018+1015	+1011	7	+1011	7
**+1020+1017	+1011	9	+931	89
$2n - R$ $(2n - 3) - R$	$W_{2n} - R$	$T_{2n} = 2n - W_{2n}$	$V_{2n} - R$	U_{2n}
**+1000 +997	+979	21	+903	97
+1002 +999	+979	23	+979	23
**+1004 +1001	+979	25	+951	53
**+1006+1003	+979	27	+903	103
+1008+1005	+979	29	+979	29
+1010+1007	+979	31	+979	31
**+1012 +1009	+979	33	+951	61
**+1014 +1011	+979	35	+ 781	233
+1016 +1013	+979	37	+979	37
**+1018 +1015	+979	39	+951	67
+1020+1017	+1017	3	+1017	3
$2n - G$ $(2n - 3) - G$	$W_{2n} - G$	$T_{2n} = 2n - W_{2n}$	$V_{2n} - G$	U_{2n}
**+10000 +9997	+9631	369	+7443	2557
**+10002 +9999	+9631	371	+9259	743
+10004 +10001	+9631	373	+9631	373
**+10006 +10003	+9631	375	+8583	1423
**+10008 + 10005	+9631	377	+6637	3371
+10010 +10007	+9631	379	+9631	379
**+10012 +10009	+9631	381	+8583	1429
+10014 +10011	+9631	383	+9631	383
**+10016 +10013	+9631	385	+9259	757
**+10018 +10015	+9631	387	+4491	5527
+10020 +10017	+9631	389	+9631	389
$2n-S$ $(2n-3)-S$	$W_{2n} - S$	$T_{2n} = 2n - W_{2n}$	$V_{2n} - S$	U_{2n}
**+20000 +19997	+18031	1969	+17409	2591
**+20002 +19999	+18031	1971	+ 17409	2593

+20004	+20001	+18031	1973	+18031	1973
**+20006	+20003	+18031	1975	+16663	3343
**+20008	+20005	+18031	1977	+16941	3067
+20010	+20007	+18031	1979	+18031	1979
**+20012	+20009	+18031	1981	+5671	14341
**+20014	+20011	+18031	1983	+4101	15913
**+20016	+20013	+18031	1985	+3229	16787
+20018	+20015	+18031	1987	+18031	1987
**+20020	+20017	+18031	1989	+16941	3079
2n-T	*(2n-3)-T*	W_{2n}-*T*	$T_{2n} = 2n - W_{2n}$	$V_{2n} - T$	U_{2n}
**+40000	+39997	+29737	10263	+ 21567	18433
**+40002	+39999	+29737	10265	+ 22273	17729
+40004	+40001	+29737	10267	+29737	10267
**+40006	+40003	+29737	10269	+21567	18439
+40008	+40005	+29737	10271	+29737	10271
+40010	+ 40007	+29737	10273	+29737	10273
**+40012	+40009	+29737	10275	+10401	29611
**+40014	+40011	+29737	10277	-56003	96017
**+40016	+40013	+29737	10279	+27057	12959
**+40018	+40015	+29737	10281	+25947	14071
**+40020	+40017	+29737	10283	+24493	15527

13. Appendix

7-3=4	11-5=6	11-3=8	13-3=10	17-5=12	17-3=14	19-3=16	23-5=18
23-3=20	29-7=22	29-5=24	29-3=26	31-3=28	37-7=30	37-5=32	37-3=34
41-5=36	41-3=38	43-3=40	47-5=42	47-3=44	53-7=46	53-5=48	53-3=50
59-7=52	59-5=54	59-3=56	61-3=58	67-7=60	67-5=62	67-3=64	71-5=66
71-3=68	73-3=70	79-7=72	79-5=74	79-3=76	83-5=78	83-3=80	89-7=82
89-5=84	89-3=86	101-13=88	97-7=90	97-5=92	97-3=94	101-5=96	101-3=98
103-3=100	107-5=102	107-3=104	109-3=106	113-5=108	113-3=110	131-19=112	127-13=114
127-11=116	131-13=118	127-7=120	127-5=122	127-3=124	131-5=126	131-3=128	137-7=130
137-5=132	137-3=134	139-3=136	149-11=138	151-11=140	149-7=142	149-5=144	149-3=146
151-3=148	157-7=150	157-5=152	157-3=154	163-7=156	163-5=158	163-3=160	167-5=162
167-3=164	173-7=166	173-5=168	173-3=170	179-7=172	179-5=174	179-3=176	181-3=178
191-11=180	193-11=182	191-7=184	191-5=186	191-3=188	193-3=190	197-5=192	197-3=194
199-3=196	211-13=198	211-11=200	233-31=202	211-7=204	211-5=206	211-3=208	223-13=210
229-17=212	227-13=214	223-7=216	223-5=218	223-3=220	227-5=222	227-3=224	229-3=226

233-5=228	233-3=230	239-7=232	239-5=234	239-3=236	241-3=238	251-11=240	271-29=242
251-7=244	251-5=246						

14. Appendix

$(PQ(K)\ ;2K)$

$Q=3$	$Q=5$	$Q=7$	$Q=11$	$Q=13$	$Q=17$	$Q=19$	$Q=23$	$Q=29$	$Q=31$
5;2	7;2		13;2		19;2			31;2	
7;4		11;4		17;4		23;4			
	11;6	13;6	17;6	19;6	23;6		29;6		37;6
11;8	13;8		19;8				31;8	37;8	
13;10				23;10		29;10			41;10
	17;12	19;12	23;12		29;12	31;12		41;12	43;12
17;14	19;14				31;14		37;14	43;14	
19;16		23;16		29;16					47;16
	23;18		29;18	31;18		37;18	41;18	47;18	
23;20			31;20		37;20		43;20		
		29;22				41;22			53;22
	29;24	31;24		37;24	41;24	43;24	47;24	53;24	
29;26	31;26		37;26		43;26				
31;28				41;28		47;28			59;28
		37;30	41;30	43;30	47;30		53;30	59;30	61;30
	37;32		43;32					61;32	
37;34		41;34		47;34		53;34			
	41;36	43;36	47;36		53;36		59;36		67;36
41;38	43;38						61;38	67;38	
43;40		47;40		53;40		59;40			71;40
	47;42		53;42		59;42	61;42		71;42	73;42
47;44					61;44		67;44	73;44	
		53;46		59;46					
	53;48		59;48	61;48		67;48	71;48		79;48
53;50			61;50		67;50		73;50	79;50	
		59;52				71;52			83;52
	59;54	61;54		67;54	71;54	73;54		83;54	
59;56	61;56		67;56		73;56		79;56		
61;58				71;58					89;58
		67;60	71;60	73;60		79;60	83;60	89;60	

15. Perspectives and Generalizations

15.1 Other Goldbach sequences (G'_{2n}) and $(G"_{2n})$ independent of (G_{2n}) may be studied using the increasing sequences of primes (W'_{2n}), (see 10.5) and ($W"_{2n}$) defined by

For any integer $n \geq 3$

$W"_{2n} = \text{Sup}(p \in \mathcal{P} : p \leq f(n))$

f is a function defined on the interval $I = [3\ ; +\infty[$ and satisfying the following conditions

- f is strictly increasing on the interval I
- $f(3) = 3$ and $\lim_{x \to +\infty} f(x) = +\infty$
- $\forall x \in I\ f(x) \leq 2x - 3$

For example, one of the following functions defined on I can be selected.

□ $f : x \to a x + 3 - 3a (a \in \mathbb{R} : 0 < a \leq 2)$

□ $g : x \to [\ 4\sqrt{3x} - 9\ ]$ ([x] is the integer part of the real number x)

□ $h : x \to 6 \ln\left(\frac{x}{3}\right) + 3$

15.2 Using this method it would be interesting to study the Schnirelmann density [31] of primes 3 , 5 , 7, 11 ,........ ... in the sequence (U_{2n}) on variable intervals .

15.3 It is possible to exceed the values shown in the table of $2n = 10^{1000}$ by perfecting this algorithm starting from n , exploiting the fact that one of Goldbach's decomponents can be chosen equal to $12p + 1$,

(the set of Goldbach decomponents consists of primes of the form $6p$ +/- 1) using Cipolla-Axler-Dusart type functions [2],[7],[10],[11] to better identify the terms of (G_{2n}), using supercomputers and more efficients software as Maple.

15.4 Diophantine equations and conjectures of the same nature (Lagrange-Lemoine-Levy conjecture [9],[19],[21],[23],[24],[33]) can be processed using similar reasoning and algorithms.

1) To validate the Lagrange_Lemoine-Levy conjecture we study the following sequences of primes

(Wl_{2n}), (Vl_{2n}) and (Ul_{2n}) defined by

For any integer $n \geq 3$ Wl_{2n}= Sup(p $\in \mathcal{P}$: $p \leq n - 1$)

• If $Tl_{2n} = (2n + 1 - 2\ Wl_{2n})$ is a **prime**

then let

$Vl_{2n} = Wl_{2n}$ and $Ul_{2n} = Tl_{2n}$

• If Tl_{2n} is a **composite number**

then there exists an integer k / $1 \leq k \leq n - 3$ such hat

$Ul_{2(n-k)}$ + 2k is a <u>prime</u>

then let

$Vl_{2n} = Vl_{2(n-k)}$ and $Ul_{2n} = Ul_{2(n-k)} + 2k$

Using the same type of reasoning a generalization called «Bezout-Goldbach conjecture» of the following form can be validated

• Let K and Q be two odd integers prime to each other :

For any integer n / $2n \geq 3(K + Q)$ there exist two primes Ub_{2n} and Vb_{2n} verifying

K . Ub_{2n} + Q. Vb_{2n} = 2n

• Let K and Q be two integers of different parity prime to each other :

For any integer n such that $2n \geq 3(K + Q)$ there are two primes Ub_{2n} and Vb_{2n} verifying

K .Ub_{2n} + Q .Vb_{2n} = 2n + 1.

15.5 Remark

GOLDBACH(-) :

R_{2K} = Inf (p ∈ $\mathcal{P}$: p - 2K ∈ $\mathcal{P}$) and <u>Q_{2K}</u> = Inf (p ∈ $\mathcal{P}$: 2K + p ∈ $\mathcal{P}$) = R_{2K} - 2K

GOLDBACH(+) :

V_{2K} = Sup (p ∈ $\mathcal{P}$: 2K - p ∈ $\mathcal{P}$) and <u>U_{2K}</u> = Inf (p ∈ $\mathcal{P}$: 2K - p ∈ $\mathcal{P}$) = 2K - V_{2K}

(Is it possible to envisage a symmetry in the Goldbach triangle parametrized by arithmetic sequences between the representations of primes and even integers ?)

16. Conclusions

16.1 A recurrent and explicit Goldbach sequence $(G_{2n}) = (U_{2n} ;\ V_{2n})$ verifying

$\forall$ n ∈ $\mathbb{N} + 2 U_{2n}$ and V_{2n} are primes and $U_{2n} + V_{2n}$= 2n

has been developed using an simple and efficient "located" algorithm.

16.2 The record of Silva [29] is beaten on a personal computer and ten Goldbach decomponents U_{2n} and V_{2n} are obtained for values of the order $2n = 10^{1000}$ for a computation time of less than three hours.

16.3 For a given integer $n \geq 49$ the evaluation of the terms U_{2n} and V_{2n} does not require the computing of all previous terms U_{2k} and V_{2k} / $1 \leq k < n - 1$. We just need to know the primes p_l and V_{2r} such that

(16.3.1) $p_l \leq 7.\ln^{1.3}(2n)$ and $2n - 7.\ln^{1.3}(2n) \leq V_{2r} \leq 2n$ <u>**(on average)**</u>

This property allows quick computing of U_{2n} and V_{2n} .

16.4 Therefore the Lagrange-Lemoine-Levy and the binary Goldbach(- & +) conjectures,

« Any even integer greater than three is the sum and difference of two primes » are true.

In fact, these two conjectures are intertwined.

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