Article

Not peer-reviewed version

Tri–Periodic Fibonacci Numbers and Tri–Periodic Leonardo Numbers

Bahadır Yılmaz and Yüksel Soykan *

Posted Date: 9 July 2025

doi: 10.20944/preprints202507.0739.v1

Keywords: Fibonacci numbers; Leonardo numbers; tri-periodic Fibonacci numbers; tri-periodic Leonardo numbers

Article

Tri–Periodic Fibonacci Numbers and Tri–Periodic Leonardo Numbers

Bahadır Yılmaz and Yüksel Soykan *

Department of Mathematics, Faculty of Science Zonguldak Bülent Ecevit University, 67100 Zonguldak, Turkey
* Correspondence: yuksel_soykan@hotmail.com

Abstract

In this study, we explore the properties of tri-periodic Fibonacci and tri-periodic Leonardo number sequences. Then we derive generating function of these sequences and give Binet's formula for the tri-periodic Fibonacci sequence. Furthermore, we present Cassani's identity associated with tri-periodic Fibonacci sequnce.

Keywords: Fibonacci numbers; Leonardo numbers; tri-periodic Fibonacci numbers; tri-periodic Leonardo numbers

PACS: 11B39; 11B83; 11B37; 11B50

---

1. Introduction

The number sequences have a crucial place in the literature. Because these sequences have wide-ranging applications, i.e, cryptology, computer science,art, architecture, finance and algorithm analysis to natural phenomena [1–10]. As a special cases of number sequences, named Fibonacci and Leonardo sequences have attracted the authors in the literature and have celebrated their recurrence relations. Also, they have connections to different mathematical problems.

Over the years, interest has grown in generalized forms of these sequences, particularly those governed by periodic recurrence relations. In this study, we focus on tri-periodic variants of the Fibonacci and Leonardo sequences, where the recurrence relation alternates cyclically every three terms. First, we remember some essential properties of Fibonacci and Leonardo numbers.

Fibonacci sequence is defined as reccursevily by

$$F_n = F_{n-1} + F_{n-2},\ F_0 = 0,\ F_1 = 1$$

respectively. The first few Fibonacci sequence numbers are given below

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...$$

Then the characteristic equation of these sequences is given by

$$x^2 - x - 1 = 0$$

that the roots are

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1+\sqrt{5}}{2}$$

Hence, the Binet's formula of Fibonacci sequence is

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and generating function of Fibonacci sequence is

$$\sum_{n=0}^{\infty} F_n z^n = \frac{z}{1-z-z^2}$$

There are many study in literature related to the Fibonacci sequence and so there are many properties found in the literature [10–16].

Now, we give the recurrance relation of Leonardo sequence as

$$L_n = L_{n-1} + L_{n-2} + 1, \ L_0 = 1, \ L_1 = 1$$

The first few Leonardo numbers are given below

$$1, 1, 3, 5, 9, 15, 25, 41, 67, 109, 177, 287, 465, 753, 1219, 1973, 3193, 5167, 8361, \ldots$$

The Leonardo numbers are related to the Fibonacci numbers given by

$$L_n = 2F_{n+1} - 1.$$

Hence, Binet's formula of Leonardo numbers can be given by

$$L_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} - 1.$$

Also there are many study in literature corresponding to the Leonardo sequence. So there are many relation between Leonardo numbers and Fibonacci numbers [17**?** –23].

The extensions of number sequences studied in literature are named bi-periodic number sequences, where the terms of bi-periodic sequences are generated based on odd and even indices. Unlike classical definition of number sequences, the structure of bi-periodic sequences makes them suitable for modeling systems with changing conditions for odd and even indices. Also many authors studied bi periodic sequences related to the special numbers and its properties, i.e , the generating function, Binet's formula, Catalan's and D'ocgane's identities. Now we present some examples from the literature raleted to the study of bi-periodic sequences. Next, we present some study that found in literature.

- Edson Yayenie whose first studied bi-periodic sequences based on Fibonacci numbers [24], defined bi-periodic Fibonacci sequences indicated by $\{q_n\}_{n\geq 0}$ which is defined as

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad \text{for } n \geq 2 \tag{1}$$

  with the initial conditions $q_0 = 0$ and $q_1 = 1$, where a and b are nonzero real numbers.
- Catarino and Spreafico [25] defined bi-periodic Leonardo sequences denoted by $\{GLe_n\}_{n\geq 0}$ which is defined as

$$GLe_n = \begin{cases} a\,GLe_{n-1} + GLe_{n-2} + a, & \text{if } n \text{ is even} \\ b\,GLe_{n-1} + GLe_{n-2} + b, & \text{if } n \text{ is odd} \end{cases} \quad \text{for } n \geq 2$$

  with the initial conditions $GLe_0 = 2a - 1$ and $GLe_1 = 2ab - 1$, where a and b are nonzero real numbers.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 July 2025 doi:10.20944/preprints202507.0739.v1

- Costa et al. [26] defined bi-periodic Eduard and Eduard-Lucas sequences denoted by $\left\{E_n^{(a,b)}\right\}_{n\geq 0}$ and $\left\{K_n^{(a,b)}\right\}_{n\geq 0}$, respecitively, which are defined as

$$E_n^{(a,b)} = \begin{cases} 6a\, E_{n-1}^{(a,b)} + E_{n-2}^{(a,b)}, & \text{if } n \text{ is even} \\ 6b\, E_{n-1}^{(a,b)} + E_{n-2}^{(a,b)}, & \text{if } n \text{ is odd} \end{cases} \quad \text{for } n \geq 2 \tag{2}$$

with the initial conditions $E_0^{(a,b)} = 0$ and $E_1^{(a,b)} = 1$, and

$$K_n^{(a,b)} = \begin{cases} 6a\, K_{n-1}^{(a,b)} + K_{n-2}^{(a,b)}, & \text{if } n \text{ is even} \\ 6b\, K_{n-1}^{(a,b)} + K_{n-2}^{(a,b)}, & \text{if } n \text{ is odd} \end{cases} \quad \text{for } n \geq 2 \tag{3}$$

with the initial conditions $K_0^{(a,b)} = 3$ and $K_1^{(a,b)} = 7$ where a and b are nonzero real numbers.

In this study we present the different expension of Fibonacci and Leonardo sequences named tri-periodic Fibonacci and tri-periodic Leonardo sequences where the terms of these sequences based on mod3 indices. Then we investigate generating function of tri-periodic Fibonacci and tri-periodic Leonardo sequences, Binet's formula of tri-periodic Fibonacci and tri-periodic Leonardo sequences and also Cassani's identity of tri-periodic Fibonacci numbers.

2. Tri-Fibonacci Numbers

In this section, we define generalization of Fibonacci numbers denoted $F_n^{(a,b,c)}$ called tri-periodic Fibonacci numbers and generalization of Leonardo numbers denoted $L_n^{(a,b,c)}$ called tri-periodic Leonardo numbers.

Definition 1. *For any real nonzero numbers a, b and c the tri-periodic Fibonacci sequence is defined as reccursevily by,*

$$F_n^{(a,b,c)} = \left[\begin{array}{ll} aF_{n-1}^{(a,b,c)} + F_{n-2}^{(a,b,c)} & n \equiv 0 \ (mod\ 3) \\ bF_{n-1}^{(a,b,c)} + F_{n-2}^{(a,b,c)} & n \equiv 1 \ (mod\ 3) \\ cF_{n-1}^{(a,b,c)} + F_{n-2}^{(a,b,c)} & n \equiv 2 \ (mod\ 3) \end{array} \right. \quad n \geq 3, \tag{4}$$

with the initial condition $F_0^{(a,b,c)} = 0$, $F_1^{(a,b,c)} = 1$,and $F_2^{(a,b,c)} = c$.

Definition 2. *For any real nonzero numbers a, b and c the tri-periodic Leonardo sequence is defined as recurrsively by,*

$$l_n^{(a,b,c)} = \left[\begin{array}{ll} al_{n-1}^{(a,b,c)} + l_{n-2}^{(a,b,c)} + a & n \equiv 0 \ (mod\ 3) \\ bl_{n-1}^{(a,b,c)} + l_{n-2}^{(a,b,c)} + b & n \equiv 1 \ (mod\ 3) \\ cl_{n-1}^{(a,b,c)} + l_{n-2}^{(a,b,c)} + c & n \equiv 2 \ (mod\ 3) \end{array} \right. \quad n \geq 3, \tag{5}$$

with the initial condition $l_0^{(a,b,c)} = 1$, $l_1^{(a,b,c)} = 1$,and $l_2^{(a,b,c)} = 2c + 1$.

Note that, if $a = b = 1$, (4) gives Fibonacci numbers and if $a = b = 1$, (4) gives k-Fibonacci numbers. Similarly, if $a = b = 1$, (5) gives Leonardo numbers and if $a = b = 1$, (5) gives k-Leonardo numbers. To write (4) and (5) in simple form, for all integer n, we define moduler Kronecker delta function that we denote $\delta_k(n)$ as

$$\delta_k(n) = \left[\begin{array}{ll} 1 & \text{if } n = k \ (\text{mod } 3) \\ 0 & \quad \text{otherwise} \end{array} \right.$$

where $k = 0, 1, 2$. Thus the recurrence relation (4) and (5) can be written as

$$F_n^{(a,b,c)} = a^{\delta_0(n)} b^{\delta_1(n)} c^{\delta_2(n)} F_{n-1}^{(a,b,c)} + F_{n-2}^{(a,b,c)}, \; n \geq 3 \tag{6}$$

$$l_n^{(a,b,c)} = a^{\delta_0(n)} b^{\delta_1(n)} c^{\delta_2(n)} l_{n-1}^{(a,b,c)} + l_{n-2}^{(a,b,c)} + a^{\delta_0(n)} b^{\delta_1(n)} c^{\delta_2(n)}, \; n \geq 3 \tag{7}$$

The (4) and (5) gives the following nonlinear quadratic equation for the tri-periodic Fibonacci sequence.

$$x^2 - abcx - abc = 0$$

with roots φ and ψ defined by

$$\varphi = \frac{abc + \sqrt{abc(abc+4)}}{2}, \; \psi = \frac{abc - \sqrt{abc(abc+4)}}{2}. \tag{8}$$

Note that, the following equalities are true,

$$\begin{aligned} \varphi + \psi &= abc, \; \varphi\psi = -abc \\ \frac{\psi^2}{abc} &= \psi + 1, \; \frac{\varphi^2}{abc} = \varphi + 1, \\ \varphi\psi &= -abc, \; \varphi + \psi = abc, \\ (\varphi+1)(\psi+1) &= 1\,, (\varphi+1)\psi = -\varphi, \; (\psi+1)\varphi = -\psi. \end{aligned}$$

In the Table 1, some terms of the tri-periodic Fibonacci sequence is given.

Table 1. A few values of the the tri-periodic Fibonacci sequence.

n	$F_n^{(a,b,c)}$
0	0
1	1
2	c
3	$ac + 1$
4	$b + c + abc$
5	$abc^2 + c^2 + ac + bc + 1$
6	$(ac+1)(a+b+c+abc) =$ $ba^2c^2 + a^2c + ac^2 + 2bac + a + c + b$

In the Table 2, some terms of the tri-periodic Leonardo sequence is given.

Table 2. A few values of the the tri-periodic Leonardo sequence.

n	$l_n^{(a,b,c)}$
0	1
1	1
2	$2c + 1$
3	$2ac + 2a + 1$
4	$2abc + 2ab + 2b + 2c + 1$
5	$2abc^2 + 2abc + 2a + 2c + 2ac + 2bc + 2c^2 + 1$
6	$2a^2bc^2 + 2a^2bc + 4abc + 2a + 2b + 2c + 2a^2c +$ $2ab + 2ac + 2ac^2 + 2a^2 + 1$

Theorem 1. *The tri-periodic Fibonacci sequence $F_n^{(a,b,c)}$ and tri-periodic Leonardo sequence are hold the following equality.*

(a) $F_{n+2}^{(a,b,c)} = (\omega - \delta_2(n)a - \delta_0(n)b - \delta_1(n)c)F_{n-1}^{(a,b,c)} + (a^{1-\delta_0(n)} b^{1-\delta_1(n)} c^{1-\delta_2(n)} + 1)F_{n-2}^{(a,b,c)}$.

doi:10.20944/preprints202507.0739.v1

(b) $l_{n+2}^{(a,b,c)} = (\omega - \delta_2(n)a - \delta_0(n)b - \delta_1(n)c)l_{n-1}^{(a,b,c)} + (a^{1-\delta_0(n)}b^{1-\delta_1(n)}c^{1-\delta_2(n)} + 1)l_{n-2}^{(a,b,c)} + \omega - \delta_2(n)a - \delta_0(n)b - \delta_1(n)c + a^{1-\delta_0(n)}b^{1-\delta_1(n)}c^{1-\delta_2(n)}.$

where $\omega = (abc + a + b + c)$.

Proof. (a) Let $n \equiv 0 \pmod 3$ then using (4), we obtain

$$\begin{aligned} F_n^{(a,b,c)} &= aF_{n-1}^{(a,b,c)} + F_{n-2}^{(a,b,c)}, \\ F_{n+1}^{(a,b,c)} &= bF_n^{(a,b,c)} + F_{n-1}^{(a,b,c)}, \\ F_{n+2}^{(a,b,c)} &= cF_{n+1}^{(a,b,c)} + F_n^{(a,b,c)}, \end{aligned}$$

and summing side by side and reorganizing these equalities we have

$$F_{n+2}^{(a,b,c)} = (c-1)F_{n+1}^{(a,b,c)} + bF_n^{(a,b,c)} + (a+1)F_{n-1}^{(a,b,c)} + F_{n-2}^{(a,b,c)}.$$

Hence, we obtain

$$\begin{aligned} F_{n+2}^{(a,b,c)} &= (c-1)(bF_n^{(a,b,c)} + F_{n-1}^{(a,b,c)}) + bF_n^{(a,b,c)} + (a+1)F_{n-1}^{(a,b,c)} + F_{n-2}^{(a,b,c)} \\ &= cb(aF_{n-1}^{(a,b,c)} + F_{n-2}^{(a,b,c)}) + (c+a)F_{n-1}^{(a,b,c)} + F_{n-2}^{(a,b,c)} \\ &= (abc + a + c)F_{n-1}^{(a,b,c)} + (bc+1)F_{n-2}^{(a,b,c)}. \end{aligned}$$

Let $n \equiv 1 \pmod 3$ then using (4), we obtain

$$\begin{aligned} F_n^{(a,b,c)} &= bF_{n-1}^{(a,b,c)} + F_{n-2}^{(a,b,c)}, \\ F_{n+1}^{(a,b,c)} &= cF_n^{(a,b,c)} + F_{n-1}^{(a,b,c)}, \\ F_{n+2}^{(a,b,c)} &= aF_{n+1}^{(a,b,c)} + F_n^{(a,b,c)}, \end{aligned}$$

and summing side by side and reorganizing these equalities we have

$$F_{n+2}^{(a,b,c)} = (a-1)F_{n+1}^{(a,b,c)} + cF_n^{(a,b,c)} + (b+1)F_{n-1}^{(a,b,c)} + F_{n-2}^{(a,b,c)}.$$

Hence, we obtain

$$\begin{aligned} F_{n+2}^{(a,b,c)} &= (a-1)(cF_n^{(a,b,c)} + F_{n-1}^{(a,b,c)}) + cF_n^{(a,b,c)} + (b+1)F_{n-1}^{(a,b,c)} + F_{n-2}^{(a,b,c)} \\ &= ac(bF_{n-1}^{(a,b,c)} + F_{n-2}^{(a,b,c)}) + (a+b)F_{n-1}^{(a,b,c)} + F_{n-2}^{(a,b,c)} \\ &= (abc + a + b)F_{n-1}^{(a,b,c)} + (ac+1)F_{n-2}^{(a,b,c)}. \end{aligned}$$

Similarly, Let $n \equiv 2 \pmod 3$ then using (4), we obtain

$$\begin{aligned} F_n^{(a,b,c)} &= cF_{n-1}^{(a,b,c)} + F_{n-2}^{(a,b,c)}, \\ F_{n+1}^{(a,b,c)} &= aF_n^{(a,b,c)} + F_{n-1}^{(a,b,c)}, \\ F_{n+2}^{(a,b,c)} &= bF_{n+1}^{(a,b,c)} + F_n^{(a,b,c)}, \end{aligned}$$

and summing side by side and reorganizing these equalities we have

$$F_{n+2}^{(a,b,c)} = (b-1)F_{n+1}^{(a,b,c)} + aF_n^{(a,b,c)} + (c+1)F_{n-1}^{(a,b,c)} + F_{n-2}^{(a,b,c)}.$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 July 2025 doi:10.20944/preprints202507.0739.v1

Hence, we obtain

$$\begin{aligned} F_{n+2}^{(a,b,c)} &= (b-1)(aF_n^{(a,b,c)} + F_{n-1}^{(a,b,c)}) + aF_n^{(a,b,c)} + (c+1)F_{n-1}^{(a,b,c)} + F_{n-2}^{(a,b,c)} \\ &= ab(aF_{n-1}^{(a,b,c)} + F_{n-2}^{(a,b,c)}) + (b+c)F_{n-1}^{(a,b,c)} + F_{n-2}^{(a,b,c)} \\ &= (abc+b+c)F_{n-1}^{(a,b,c)} + (ab+1)F_{n-2}^{(a,b,c)}. \end{aligned}$$

(b) Let $n \equiv 0 \pmod 3$ then using (4), we obtain

$$\begin{aligned} l_n^{(a,b,c)} &= al_{n-1}^{(a,b,c)} + l_{n-2}^{(a,b,c)} + a, \\ l_{n+1}^{(a,b,c)} &= bl_n^{(a,b,c)} + l_{n-1}^{(a,b,c)} + b, \\ l_{n+2}^{(a,b,c)} &= cl_{n+1}^{(a,b,c)} + l_n^{(a,b,c)} + c, \end{aligned}$$

and summing side by side and reorganizing these equalities we have

$$l_{n+2}^{(a,b,c)} = (c-1)l_{n+1}^{(a,b,c)} + bl_n^{(a,b,c)} + (a+1)l_{n-1}^{(a,b,c)} + l_{n-2}^{(a,b,c)} + a + b + c.$$

Hence, we obtain

$$\begin{aligned} l_{n+2}^{(a,b,c)} &= (c-1)(bl_n^{(a,b,c)} + l_{n-1}^{(a,b,c)} + b) + bl_n^{(a,b,c)} + (a+1)l_{n-1}^{(a,b,c)} + l_{n-2}^{(a,b,c)} + a + b + c \\ &= bc(al_{n-1}^{(a,b,c)} + l_{n-2}^{(a,b,c)} + a) + (a+c)l_{n-1}^{(a,b,c)} + l_{n-2}^{(a,b,c)} + bc + a + c \\ &= (abc+a+c)l_{n-1}^{(a,b,c)} + (bc+1)l_{n-2}^{(a,b,c)} + abc + bc + a + c. \end{aligned}$$

Let $n \equiv 1 \pmod 3$ then using (4), we obtain

$$\begin{aligned} l_n^{(a,b,c)} &= bl_{n-1}^{(a,b,c)} + l_{n-2}^{(a,b,c)} + b, \\ l_{n+1}^{(a,b,c)} &= cl_n^{(a,b,c)} + l_{n-1}^{(a,b,c)} + c, \\ l_{n+2}^{(a,b,c)} &= al_{n+1}^{(a,b,c)} + l_n^{(a,b,c)} + a, \end{aligned}$$

and summing side by side and reorganizing these equalities we have

$$l_{n+2}^{(a,b,c)} = (a-1)l_{n+1}^{(a,b,c)} + cl_n^{(a,b,c)} + (b+1)l_{n-1}^{(a,b,c)} + l_{n-2}^{(a,b,c)} + a + b + c.$$

Hence, we obtain

$$\begin{aligned} l_{n+2}^{(a,b,c)} &= (a-1)(cl_n^{(a,b,c)} + l_{n-1}^{(a,b,c)} + c) + cl_n^{(a,b,c)} + (b+1)l_{n-1}^{(a,b,c)} + l_{n-2}^{(a,b,c)} + a + b + c \\ &= ac(bl_{n-1}^{(a,b,c)} + l_{n-2}^{(a,b,c)} + b) + (a+b)l_{n-1}^{(a,b,c)} + l_{n-2}^{(a,b,c)} + ac + a + b \\ &= (abc+a+b)l_{n-1}^{(a,b,c)} + (ac+1)l_{n-2}^{(a,b,c)} + abc + ac + a + b. \end{aligned}$$

Similarly, Let $n \equiv 2 \pmod 3$ then using (4), we obtain

$$\begin{aligned} l_n^{(a,b,c)} &= cl_{n-1}^{(a,b,c)} + l_{n-2}^{(a,b,c)} + c, \\ l_{n+1}^{(a,b,c)} &= al_n^{(a,b,c)} + l_{n-1}^{(a,b,c)} + a, \\ l_{n+2}^{(a,b,c)} &= bl_{n+1}^{(a,b,c)} + l_n^{(a,b,c)} + b, \end{aligned}$$

and summing side by side and reorganizing these equalities we have

$$l_{n+2}^{(a,b,c)} = (b-1)l_{n+1}^{(a,b,c)} + al_n^{(a,b,c)} + (c+1)l_{n-1}^{(a,b,c)} + l_{n-2}^{(a,b,c)} + a + b + c.$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 July 2025 doi:10.20944/preprints202507.0739.v1

Hence, we obtain

$$\begin{aligned} l_{n+2}^{(a,b,c)} &= (b-1)(al_n^{(a,b,c)} + l_{n-1}^{(a,b,c)} + a) + al_n^{(a,b,c)} + (c+1)l_{n-1}^{(a,b,c)} + l_{n-2}^{(a,b,c)} + a + b + c \\ &= ab(cl_{n-1}^{(a,b,c)} + l_{n-2}^{(a,b,c)} + c) + (b+c)l_{n-1}^{(a,b,c)} + l_{n-2}^{(a,b,c)} + ab + b + c \\ &= (abc + b + c)l_{n-1}^{(a,b,c)} + (ab+1)l_{n-2}^{(a,b,c)} + abc + ab + b + c. \square \end{aligned}$$

Theorem 2. *The tri-periodic Fibonacci sequence $F_n^{(a,b,c)}$ and tri-periodic Leonardo sequence hold the following equality.*

$$F_n^{(a,b,c)} = \omega F_{n-3}^{(a,b,c)} + F_{n-6}^{(a,b,c)}, \tag{9}$$

$$l_n^{(a,b,c)} = \omega l_{n-3}^{(a,b,c)} + l_{n-6}^{(a,b,c)} + \omega, \tag{10}$$

where $\omega = (abc + a + b + c)$.

Proof. (a) For the case $n \equiv 0 \pmod 3$, using (4), we have

$$\begin{aligned} F_{3n}^{(a,b,c)} &= aF_{3n-1}^{(a,b,c)} + F_{3n-2}^{(a,b,c)} \\ &= acF_{3n-2}^{(a,b,c)} + aF_{3n-3}^{(a,b,c)} + F_{3n-2}^{(a,b,c)} \\ &= acbF_{3n-3}^{(a,b,c)} + acF_{3n-4}^{(a,b,c)} + aF_{3n-3}^{(a,b,c)} + F_{3n-2}^{(a,b,c)} \\ &= acbF_{3n-3}^{(a,b,c)} + cF_{3n-3}^{(a,b,c)} - cF_{3n-5}^{(a,b,c)} + aF_{3n-3}^{(a,b,c)} + F_{3n-2}^{(a,b,c)} \\ &= (abc + a + b + c)F_{3n-3}^{(a,b,c)} - (F_{3n-4}^{(a,b,c)} - F_{3n-6}^{(a,b,c)}) + F_{3n-4}^{(a,b,c)} \\ &= (abc + a + b + c)F_{3n-3}^{(a,b,c)} + F_{3n-6}^{(a,b,c)}. \end{aligned}$$

For the case $n \equiv 1 \pmod 3$, using (4), we have

$$\begin{aligned} F_{3n+1}^{(a,b,c)} &= bF_{3n}^{(a,b,c)} + F_{3n-1}^{(a,b,c)} \\ &= abF_{3n-1}^{(a,b,c)} + bF_{3n-2}^{(a,b,c)} + F_{3n-1}^{(a,b,c)} \\ &= abcF_{3n-2}^{(a,b,c)} + abF_{3n-3}^{(a,b,c)} + bF_{3n-2}^{(a,b,c)} + F_{3n-1}^{(a,b,c)} \\ &= abcF_{3n-2}^{(a,b,c)} + aF_{3n-2}^{(a,b,c)} - aF_{3n-4}^{(a,b,c)} + bF_{3n-2}^{(a,b,c)} + F_{3n-1}^{(a,b,c)} \\ &= (abc + a + b + c)F_{3n-2}^{(a,b,c)} - (F_{3n-3}^{(a,b,c)} - F_{3n-5}^{(a,b,c)}) + F_{3n-3}^{(a,b,c)} \\ &= (abc + a + b + c)F_{3n-2}^{(a,b,c)} + F_{3n-5}^{(a,b,c)}. \end{aligned}$$

For the case $n \equiv 2 \pmod 3$, using (4), we have

$$\begin{aligned} F_{3n+2}^{(a,b,c)} &= cF_{3n+1}^{(a,b,c)} + F_{3n}^{(a,b,c)} \\ &= cbF_{3n}^{(a,b,c)} + cF_{3n-1}^{(a,b,c)} + F_{3n}^{(a,b,c)} \\ &= abcF_{3n-1}^{(a,b,c)} + cbF_{3n-2}^{(a,b,c)} + cF_{3n-1}^{(a,b,c)} + F_{3n}^{(a,b,c)} \\ &= abcF_{3n-1}^{(a,b,c)} + bF_{3n-1}^{(a,b,c)} - bF_{3n-3}^{(a,b,c)} + cF_{3n-1}^{(a,b,c)} + F_{3n}^{(a,b,c)} \\ &= (abc + a + b + c)F_{3n-1}^{(a,b,c)} - (F_{3n-2}^{(a,b,c)} - F_{3n-5}^{(a,b,c)}) + F_{3n-2}^{(a,b,c)} \\ &= (abc + a + b + c)F_{3n-1}^{(a,b,c)} + F_{3n-5}^{(a,b,c)}. \square \end{aligned}$$

 doi:10.20944/preprints202507.0739.v1

(b) For the case $n \equiv 0 \pmod 3$, using (10), we have

$$\begin{aligned}
l_{3n}^{(a,b,c)} &= al_{3n-1}^{(a,b,c)} + l_{3n-2}^{(a,b,c)} + a \\
&= acl_{3n-2}^{(a,b,c)} + al_{3n-3}^{(a,b,c)} + l_{3n-2}^{(a,b,c)} + ac + a \\
&= acbl_{3n-3}^{(a,b,c)} + acl_{3n-4}^{(a,b,c)} + al_{3n-3}^{(a,b,c)} + l_{3n-2}^{(a,b,c)} + abc + ac + a \\
&= acbl_{3n-3}^{(a,b,c)} + cl_{3n-3}^{(a,b,c)} - cl_{3n-5}^{(a,b,c)} + al_{3n-3}^{(a,b,c)} + l_{3n-2}^{(a,b,c)} + abc + a \\
&= (abc + a + b + c)l_{3n-3}^{(a,b,c)} - (l_{3n-4}^{(a,b,c)} - l_{3n-6}^{(a,b,c)} - c) + l_{3n-4}^{(a,b,c)} + abc + a + b \\
&= (abc + a + b + c)l_{3n-3}^{(a,b,c)} + l_{3n-6}^{(a,b,c)} + abc + a + b + c.
\end{aligned}$$

For the case $n \equiv 1 \pmod 3$, using (10), we have

$$\begin{aligned}
l_{3n+1}^{(a,b,c)} &= bl_{3n}^{(a,b,c)} + l_{3n-1}^{(a,b,c)} + b \\
&= abl_{3n-1}^{(a,b,c)} + bl_{3n-2}^{(a,b,c)} + l_{3n-1}^{(a,b,c)} + ab + b \\
&= abcl_{3n-2}^{(a,b,c)} + abl_{3n-3}^{(a,b,c)} + bl_{3n-2}^{(a,b,c)} + l_{3n-1}^{(a,b,c)} + abc + ab + b \\
&= abcl_{3n-2}^{(a,b,c)} + al_{3n-2}^{(a,b,c)} - al_{3n-4}^{(a,b,c)} + bl_{3n-2}^{(a,b,c)} + l_{3n-1}^{(a,b,c)} + abc + b \\
&= (abc + a + b + c)l_{3n-2}^{(a,b,c)} - al_{3n-4}^{(a,b,c)} + l_{3n-3}^{(a,b,c)} + abc + b + c \\
&= (abc + a + b + c)l_{3n-2}^{(a,b,c)} + l_{3n-5}^{(a,b,c)} + abc + a + b + c.
\end{aligned}$$

For the case $n \equiv 2 \pmod 3$, using (10), we have

$$\begin{aligned}
l_{3n+2}^{(a,b,c)} &= cl_{3n+1}^{(a,b,c)} + l_{3n}^{(a,b,c)} + c \\
&= cbl_{3n}^{(a,b,c)} + cl_{3n-1}^{(a,b,c)} + l_{3n}^{(a,b,c)} + bc + c \\
&= abcl_{3n-1}^{(a,b,c)} + cbl_{3n-2}^{(a,b,c)} + cl_{3n-1}^{(a,b,c)} + l_{3n}^{(a,b,c)} + abc + bc + c \\
&= abcl_{3n-1}^{(a,b,c)} + bl_{3n-1}^{(a,b,c)} - bl_{3n-3}^{(a,b,c)} + cl_{3n-1}^{(a,b,c)} + l_{3n}^{(a,b,c)} + abc + c \\
&= (abc + a + b + c)l_{3n-1}^{(a,b,c)} - (l_{3n-2}^{(a,b,c)} - l_{3n-5}^{(a,b,c)} - b) + l_{3n-2}^{(a,b,c)} + abc + a + c \\
&= (abc + a + b + c)l_{3n-1}^{(a,b,c)} + l_{3n-5}^{(a,b,c)} + abc + a + b + c. \square
\end{aligned}$$

The recurrence (9) and (10) hold the following characteristic equation:

$$x^6 - \omega x^3 - 1 = \left(x^3 - \alpha\right)\left(x^3 - \beta\right) = 0 \tag{11}$$

where $\alpha = \dfrac{\omega + \sqrt{\omega^2 + 4}}{2}$ and $\beta = \dfrac{\omega - \sqrt{\omega^2 + 4}}{2}$.

Theorem 3. *Suppose that $F(z) = \sum_{n=0}^{\infty} F_n^{(a,b,c)} z^n$ is the ordinary generating function of the tri-periodic Fibonacci sequence. Then $F(z)$ is given by*

$$F(z) = \frac{F_0^{(a,b,c)} + F_1^{(a,b,c)}z + F_2^{(a,b,c)}z^2 + (F_3^{(a,b,c)} - \omega F_0^{(a,b,c)})z^3 + (F_4^{(a,b,c)} - \omega F_1^{(a,b,c)})z^4 + (F_5^{(a,b,c)} - \omega F_2^{(a,b,c)})z^5}{1 - \omega z^3 - z^6}.$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 July 2025 doi:10.20944/preprints202507.0739.v1

Proof. Consider the generating function as

$$\begin{aligned} F(z) &= \sum_{n=0}^{\infty} F_n^{(a,b,c)} z^n \\ &= \sum_{n=0}^{\infty} F_{3n}^{(a,b,c)} z^{3n} + \sum_{n=0}^{\infty} F_{3n+1}^{(a,b,c)} z^{3n+1} + \sum_{n=0}^{\infty} F_{3n+2}^{(a,b,c)} z^{3n+2} \\ &= G(z) + H(z) + T(z). \end{aligned} \tag{12}$$

Hence, if we multiply $G(z) = \sum_{n=0}^{\infty} F_{3n}^{(a,b,c)} z^{3n}$ with $(1 - \omega z^3 - z^6)$ and using (9), we obtain

$$\begin{aligned} \left(1 - \omega z^3 - z^6\right) G(z) &= \left(1 - \omega z^3 - z^6\right) \sum_{n=0}^{\infty} F_{3n}^{(a,b,c)} z^{3n} \\ &= \sum_{n=0}^{\infty} F_{3n}^{(a,b,c)} z^{3n} - \omega \sum_{n=1}^{\infty} F_{3n-3}^{(a,b,c)} z^{3n} - \sum_{n=2}^{\infty} F_{3n-6}^{(a,b,c)} z^{3n} \\ &= F_0^{(a,b,c)} + (F_3^{(a,b,c)} - \omega F_0^{(a,b,c)}) z^3 \\ &\quad + \sum_{n=2}^{\infty} \underbrace{(F_{3n}^{(a,b,c)} - \omega F_{3n-3}^{(a,b,c)} - F_{3n-6}^{(a,b,c)})}_{\text{equal to zero}} z^{3n}. \end{aligned}$$

That means

$$G(z) = \frac{F_0^{(a,b,c)} + (F_3^{(a,b,c)} - \omega F_0^{(a,b,c)}) z^3}{(1 - \omega z^3 - z^6)}$$

Next, if we multiply $H(z) = \sum_{n=0}^{\infty} F_{3n+1}^{(a,b,c)} z^{3n+1}$ with $(1 - \omega z^3 - z^6)$ and using (9), we obtain

$$\begin{aligned} \left(1 - \omega z^3 - z^6\right) H(z) &= \left(1 - \omega z^3 - z^6\right) \sum_{n=0}^{\infty} F_{3n+1}^{(a,b,c)} z^{3n+1} \\ &= \sum_{n=0}^{\infty} F_{3n+1}^{(a,b,c)} z^{3n+1} - \omega \sum_{n=1}^{\infty} F_{3n-2}^{(a,b,c)} z^{3n+1} - \sum_{n=2}^{\infty} F_{3n-5}^{(a,b,c)} z^{3n+1} \\ &= F_1^{(a,b,c)} z + (F_4^{(a,b,c)} - \omega F_1^{(a,b,c)}) z^4 \\ &\quad + \sum_{n=2}^{\infty} \underbrace{(F_{3n+1}^{(a,b,c)} - \omega F_{3n-2}^{(a,b,c)} - F_{3n-5}^{(a,b,c)})}_{\text{equal to zero}} z^{3n+1}. \end{aligned}$$

That means

$$H(z) = \frac{F_1^{(a,b,c)} z + (F_4^{(a,b,c)} - \omega F_1^{(a,b,c)}) z^4}{(1 - \omega z^3 - z^6)}$$

Eventually, if we multiply $T(z) = \sum_{n=0}^{\infty} F_{3n+2}^{(a,b,c)} z^{3n+2}$ with $(1 - \omega z^3 - z^6)$ and using (9), we obtain

$$\begin{aligned} \left(1 - \omega z^3 - z^6\right) T(z) &= \left(1 - \omega z^3 - z^6\right) \sum_{n=0}^{\infty} F_{3n+2}^{(a,b,c)} z^{3n+2} \\ &= \sum_{n=0}^{\infty} F_{3n+2}^{(a,b,c)} z^{3n+2} - \omega \sum_{n=1}^{\infty} F_{3n-1}^{(a,b,c)} z^{3n+2} - \sum_{n=2}^{\infty} F_{3n-4}^{(a,b,c)} z^{3n+2} \\ &= F_2^{(a,b,c)} z^2 + (F_5^{(a,b,c)} - \omega F_2^{(a,b,c)}) z^5 \\ &\quad + \sum_{n=2}^{\infty} \underbrace{(F_{3n+2}^{(a,b,c)} - \omega F_{3n-1}^{(a,b,c)} - F_{3n-4}^{(a,b,c)})}_{\text{equal to zero}} z^{3n+2}. \end{aligned}$$

That means

$$T(z)=\frac{F_2^{(a,b,c)}z^2+(F_5^{(a,b,c)}-\omega F_2^{(a,b,c)})z^5}{(1-\omega z^3-z^6)}$$

So, if we substitute the expressions $G(z)$, $H(z)$ and $T(z)$ into (12) we get the result.□

Theorem 4. *Suppose that* $l(z)=\sum_{n=0}^{\infty} l_n^{(a,b,c)}z^n$ *is the ordinary generating function of the tri-periodic Leonardo sequence. Then* $l(z)$ *is given by*

$$\begin{aligned} l(z) &= \frac{1}{(1-z)(1-\omega z^3-z^6)}(l_0^{(a,b,c)}(1-z)+l_1^{(a,b,c)}z(1-z)+l_2^{(a,b,c)}z^2(1-z) \\ &\quad +(l_3^{(a,b,c)}-\omega l_0^{(a,b,c)})z^3(1-z)+(l_4^{(a,b,c)}-\omega l_1^{(a,b,c)})z^4(1-z) \\ &\quad +(l_5^{(a,b,c)}-\omega l_2^{(a,b,c)})z^5(1-z)+\omega z^6).\square \end{aligned}$$

Proof. Consider the generating function as

$$\begin{aligned} l(z) &= \sum_{n=0}^{\infty} l_n^{(a,b,c)}z^n \\ &= \sum_{n=0}^{\infty} l_{3n}^{(a,b,c)}z^{3n}+\sum_{n=0}^{\infty} l_{3n+1}^{(a,b,c)}z^{3n+1}+\sum_{n=0}^{\infty} l_{3n+2}^{(a,b,c)}z^{3n+2} \\ &= K(z)+M(z)+N(z). \end{aligned} \tag{13}$$

Hence, if we multiply $K(z)=\sum_{n=0}^{\infty} l_{3n}^{(a,b,c)}z^{3n}$ with $\left(1-\omega z^3-z^6\right)$ and using (9), we obtain

$$\begin{aligned} \left(1-\omega z^3-z^6\right)K(z) &= \left(1-\omega z^3-z^6\right)\sum_{n=0}^{\infty} l_{3n}^{(a,b,c)}z^{3n} \\ &= \sum_{n=0}^{\infty} l_{3n}^{(a,b,c)}z^{3n}-\omega\sum_{n=1}^{\infty} l_{3n-3}^{(a,b,c)}z^{3n}-\sum_{n=2}^{\infty} l_{3n-6}^{(a,b,c)}z^{3n} \\ &= l_0^{(a,b,c)}+(l_3^{(a,b,c)}-\omega l_0^{(a,b,c)})z^3+\sum_{n=2}^{\infty}\underbrace{(F_{3n}^{(a,b,c)}-\omega F_{3n-3}^{(a,b,c)}-F_{3n-6}^{(a,b,c)})}_{\text{equal to } \omega}z^{3n} \\ &= l_0^{(a,b,c)}+(l_3^{(a,b,c)}-\omega l_0^{(a,b,c)})z^3+\omega\left(\frac{1}{1-z^3}-1-z^3\right) \\ &= \frac{l_0^{(a,b,c)}(1-z^3)+(l_3^{(a,b,c)}-\omega l_0^{(a,b,c)})z^3(1-z^3)+\omega z^6}{1-z^3}. \end{aligned}$$

That means

$$K(z)=\frac{l_0^{(a,b,c)}(1-z^3)+(l_3^{(a,b,c)}-\omega l_0^{(a,b,c)})z^3(1-z^3)+\omega z^6}{(1-\omega z^3-z^6)(1-z^3)}$$

Next, if we multiply $M(z)=\sum\limits_{n=0}^{\infty}F_{3n+1}^{(a,b,c)}z^{3n+1}$ with $\left(1-\omega z^3-z^6\right)$ and using (9), we obtain

$$\begin{aligned}\left(1-\omega z^3-z^6\right)M(z) &= \left(1-\omega z^3-z^6\right)\sum_{n=0}^{\infty}l_{3n+1}^{(a,b,c)}z^{3n+1}\\ &= \sum_{n=0}^{\infty}l_{3n+1}^{(a,b,c)}z^{3n+1}-\omega\sum_{n=1}^{\infty}l_{3n-2}^{(a,b,c)}z^{3n+1}-\sum_{n=2}^{\infty}l_{3n-5}^{(a,b,c)}z^{3n+1}\\ &= l_1^{(a,b,c)}z+(l_4^{(a,b,c)}-\omega l_1^{(a,b,c)})z^4+\sum_{n=2}^{\infty}\underbrace{(l_{3n+1}^{(a,b,c)}-\omega l_{3n-2}^{(a,b,c)}-l_{3n-5}^{(a,b,c)})}_{\text{equal to }\omega}z^{3n+1}\\ &= l_1^{(a,b,c)}z+(l_4^{(a,b,c)}-\omega l_1^{(a,b,c)})z^4+\omega z\left(\frac{1}{1-z^3}-1-z^3\right)\\ &= \frac{l_1^{(a,b,c)}z(1-z^3)+(l_4^{(a,b,c)}-\omega l_1^{(a,b,c)})z^4(1-z^3)+\omega z^7}{1-z^3}.\end{aligned}$$

That means

$$M(z)=\frac{l_1^{(a,b,c)}z(1-z^3)+(l_4^{(a,b,c)}-\omega l_1^{(a,b,c)})z^4(1-z^3)+\omega z^7}{(1-z^3)(1-\omega z^3-z^6)}$$

Eventually, if we multiply $N(z)=\sum\limits_{n=0}^{\infty}F_{3n+2}^{(a,b,c)}z^{3n+2}$ with $\left(1-\omega z^3-z^6\right)$ and using (9), we obtain

$$\begin{aligned}\left(1-\omega z^3-z^6\right)N(z) &= \left(1-\omega z^3-z^6\right)\sum_{n=0}^{\infty}l_{3n+2}^{(a,b,c)}z^{3n+2}\\ &= \sum_{n=0}^{\infty}l_{3n+2}^{(a,b,c)}z^{3n+2}-\omega\sum_{n=1}^{\infty}l_{3n-1}^{(a,b,c)}z^{3n+2}-\sum_{n=2}^{\infty}l_{3n-4}^{(a,b,c)}z^{3n+2}\\ &= l_2^{(a,b,c)}z^2+(l_5^{(a,b,c)}-\omega l_2^{(a,b,c)})z^5+\sum_{n=2}^{\infty}\underbrace{(l_{3n+2}^{(a,b,c)}-\omega l_{3n-1}^{(a,b,c)}-l_{3n-4}^{(a,b,c)})}_{\text{equal to }\omega}z^{3n+2}\\ &= l_2^{(a,b,c)}z^2+(l_5^{(a,b,c)}-\omega l_2^{(a,b,c)})z^5+\omega z^2\left(\frac{1}{1-z^3}-1-z^3\right)\\ &= \frac{l_2^{(a,b,c)}z^2(1-z^3)+(l_5^{(a,b,c)}-\omega l_2^{(a,b,c)})z^5(1-z^3)+\omega z^8}{1-z^3}.\end{aligned}$$

That means

$$T(z)=\frac{l_2^{(a,b,c)}z^2(1-z^3)+(l_5^{(a,b,c)}-\omega l_2^{(a,b,c)})z^5(1-z^3)+\omega z^8}{(1-z^3)(1-\omega z^3-z^6)}$$

So, if we substitute the expressions $K(z)$, $M(z)$ and $N(z)$ into (12), we obtain

$$\begin{aligned}l(z) &= \frac{1}{(1-z)(1-\omega z^3-z^6)}(l_0^{(a,b,c)}(1-z)+l_1^{(a,b,c)}z(1-z)+l_2^{(a,b,c)}z^2(1-z)\\ &\quad+(l_3^{(a,b,c)}-\omega l_0^{(a,b,c)})z^3(1-z)+(l_4^{(a,b,c)}-\omega l_1^{(a,b,c)})z^4(1-z)\\ &\quad+(l_5^{(a,b,c)}-\omega l_2^{(a,b,c)})z^5(1-z)+\omega z^6).\square\end{aligned}$$

Note that the following equality is true by using Maclaurin series expansion

$$\frac{1}{1-dz^3} = \sum_{n=0}^{\infty}d^n z^{3n},\ \left|dz^3\le 1\right|, \tag{14}$$

$$\frac{z}{1-dz^3} = \sum_{n=0}^{\infty}d^n z^{3n+1},\ \left|dz^3\le 1\right|, \tag{15}$$

$$\frac{z^2}{1-dz^3} = \sum_{n=0}^{\infty}d^n z^{3n+2},\ \left|dz^3\le 1\right|. \tag{16}$$

Theorem 5. *The Binet's formula of the tri-periodic Fibonacci sequence is given by*

$$F_n^{(a,b,c)} = \widetilde{F}_0\delta_0(n) + \widetilde{F}_1\delta_1(n) + \widetilde{F}_2\delta_2(n). \tag{17}$$

where

$$\begin{aligned}
\widetilde{F}_0 &= \frac{(1+ac)(\alpha^{\frac{n}{3}} - \beta^{\frac{n}{3}})}{\alpha-\beta}, \\
\widetilde{F}_1 &= \frac{(\alpha^{\frac{n-1}{3}+1} - \beta^{\frac{n-1}{3}+1}) - a(\alpha^{\frac{n-1}{3}} - \beta^{\frac{n-1}{3}})}{\alpha-\beta}, \\
\widetilde{F}_2 &= \frac{c(\alpha^{\frac{n-2}{3}+1} - \beta^{\frac{n-2}{3}+1}) + (\alpha^{\frac{n-2}{3}} - \beta^{\frac{n-2}{3}})}{\alpha-\beta}.
\end{aligned}$$

Proof. Using Theorem 3, (14)-(16) and (11), we obtain

$$\begin{aligned}
F(z) &= \frac{F_0^{(a,b,c)} + F_1^{(a,b,c)}z + F_2^{(a,b,c)}z^2 + (F_3^{(a,b,c)} - \omega F_0^{(a,b,c)})z^3 + (F_4^{(a,b,c)} - \omega F_1^{(a,b,c)})z^4 + (F_5^{(a,b,c)} - \omega F_2^{(a,b,c)})z^5}{1-\omega z^3 - z^6} \\
&= \frac{r_1z^2 + r_2z + r_3}{(1-\alpha z^3)} + \frac{r_4z^2 + r_5z + r_6}{(1-\beta z^3)} \\
&= \sum_{n=0}^{\infty} r_1\alpha^n z^{3n+2} + \sum_{n=0}^{\infty} r_2\alpha^n z^{3n+1} + \sum_{n=0}^{\infty} r_3\alpha^n z^{3n} + \sum_{n=0}^{\infty} r_4\beta^n z^{3n+2} + \sum_{n=0}^{\infty} r_5\beta^n z^{3n+1} + \sum_{n=0}^{\infty} r_6\beta^n z^{3n}.
\end{aligned} \tag{18}$$

Hence, we obtain
$F_0^{(a,b,c)} + F_1^{(a,b,c)}z + F_2^{(a,b,c)}z^2 + (F_3^{(a,b,c)} - \omega F_0^{(a,b,c)})z^3 + (F_4^{(a,b,c)} - \omega F_1^{(a,b,c)})z^4 + (F_5^{(a,b,c)} - \omega F_3^{(a,b,c)})z^5 = (r_3 + r_6) + z(r_2 + r_5) + z^2(r_1 + r_4) - z^3(\beta r_3 + \alpha r_6) - z^4(\beta r_2 + \alpha r_5) - z^5(\beta r_1 + \alpha r_4)$.
From this, we have the following equalities:

$$\begin{aligned}
r_3 + r_6 &= F_0^{(a,b,c)} \\
r_2 + r_5 &= F_1^{(a,b,c)} \\
r_1 + r_4 &= F_2^{(a,b,c)} \\
-\beta r_3 - \alpha r_6 &= F_3^{(a,b,c)} - \omega F_0^{(a,b,c)} \\
-\beta r_2 - \alpha r_5 &= F_4^{(a,b,c)} - \omega F_1^{(a,b,c)} \\
-\beta r_1 - \alpha r_4 &= F_5^{(a,b,c)} - \omega F_2^{(a,b,c)}
\end{aligned} \tag{19}$$

Hence, we have

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\beta & 0 & 0 & -\alpha \\ 0 & -\beta & 0 & 0 & -\alpha & 0 \\ -\beta & 0 & 0 & -\alpha & 0 & 0 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{pmatrix} = \begin{pmatrix} F_0^{(a,b,c)} \\ F_1^{(a,b,c)} \\ F_2^{(a,b,c)} \\ F_3^{(a,b,c)} - \omega F_0^{(a,b,c)} \\ F_4^{(a,b,c)} - \omega F_1^{(a,b,c)} \\ F_5^{(a,b,c)} - \omega F_3^{(a,b,c)} \end{pmatrix}.$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 July 2025 doi:10.20944/preprints202507.0739.v1

and

$$\begin{aligned} r_1 &= \frac{1}{\alpha-\beta}\left(F_2^{(a,b,c)}\alpha + F_5^{(a,b,c)} - \omega F_2^{(a,b,c)}\right), \\ r_2 &= \frac{1}{\alpha-\beta}\left(F_1^{(a,b,c)}\alpha + F_4^{(a,b,c)} - \omega F_1^{(a,b,c)}\right), \\ r_3 &= \frac{1}{\alpha-\beta}\left(F_0^{(a,b,c)}\alpha + F_3^{(a,b,c)} - \omega F_0^{(a,b,c)}\right), \\ r_4 &= -\frac{1}{\alpha-\beta}\left(F_2^{(a,b,c)}\beta + F_5^{(a,b,c)} - \omega F_2^{(a,b,c)}\right), \\ r_5 &= -\frac{1}{\alpha-\beta}\left(F_1^{(a,b,c)}\beta + F_4^{(a,b,c)} - \omega F_1^{(a,b,c)}\right), \\ r_6 &= -\frac{1}{\alpha-\beta}\left(F_0^{(a,b,c)}\beta + F_3^{(a,b,c)} - \omega F_0^{(a,b,c)}\right). \end{aligned}$$

Thus, (18) can be written as

$$\begin{aligned} F(z) &= \sum_{n=0}^{\infty}\frac{1}{\alpha-\beta}\left(F_0^{(a,b,c)}\alpha + F_3^{(a,b,c)} - \omega F_0^{(a,b,c)}\right)\alpha^n z^{3n} - \sum_{n=0}^{\infty}\frac{1}{\alpha-\beta}\left(F_0^{(a,b,c)}\beta + F_3^{(a,b,c)} - \omega F_0^{(a,b,c)}\right)\beta^n z^{3n} \\ &+ \sum_{n=0}^{\infty}\frac{1}{\alpha-\beta}\left(F_1^{(a,b,c)}\alpha + F_4^{(a,b,c)} - \omega F_1^{(a,b,c)}\right)\alpha^n z^{3n+1} - \sum_{n=0}^{\infty}\frac{1}{\alpha-\beta}\left(F_1^{(a,b,c)}\beta + F_4^{(a,b,c)} - \omega F_1^{(a,b,c)}\right)\beta^n z^{3n+1} \\ &+ \sum_{n=0}^{\infty}\frac{1}{\alpha-\beta}\left(F_2^{(a,b,c)}\alpha + F_5^{(a,b,c)} - \omega F_2^{(a,b,c)}\right)\alpha^n z^{3n+2} - \sum_{n=0}^{\infty}\frac{1}{\alpha-\beta}\left(F_2^{(a,b,c)}\beta + F_5^{(a,b,c)} - \omega F_2^{(a,b,c)}\right)\beta^n z^{3n+2}. \end{aligned}$$

So, using (9), we have

$$\begin{aligned} F(z) &= \sum_{n=0}^{\infty}\frac{1}{\alpha-\beta}\left(F_0^{(a,b,c)}(\alpha^{n+1}-\beta^{n+1}) + (F_3^{(a,b,c)} - \omega F_0^{(a,b,c)})(\alpha^n-\beta^n)\right)z^{3n} \\ &+ \sum_{n=0}^{\infty}\frac{1}{\alpha-\beta}\left(F_1^{(a,b,c)}(\alpha^{n+1}-\beta^{n+1}) + (F_4^{(a,b,c)} - \omega F_1^{(a,b,c)})(\alpha^n-\beta^n)\right)z^{3n+1} \\ &+ \sum_{n=0}^{\infty}\frac{1}{\alpha-\beta}\left(F_2^{(a,b,c)}(\alpha^{n+1}-\beta^{n+1}) + (F_5^{(a,b,c)} - \omega F_2^{(a,b,c)})(\alpha^n-\beta^n)\right)z^{3n+2}. \end{aligned}$$

This implies that

$$F_n^{(a,b,c)} = \begin{cases} \dfrac{F_0^{(a,b,c)}(\alpha^{k+1}-\beta^{k+1}) + (F_3^{(a,b,c)} - \omega F_0^{(a,b,c)})(\alpha^k-\beta^k)}{\alpha-\beta} & n = 3k \\ \dfrac{F_1^{(a,b,c)}(\alpha^{k+1}-\beta^{k+1}) + (F_4^{(a,b,c)} - \omega F_1^{(a,b,c)})(\alpha^k-\beta^k)}{\alpha-\beta} & n \equiv 3k+1 \ , k \in \mathbb{Z} \\ \dfrac{F_2^{(a,b,c)}(\alpha^{k+1}-\beta^{k+1}) + (F_5^{(a,b,c)} - \omega F_2^{(a,b,c)})(\alpha^k-\beta^k)}{\alpha-\beta} & n \equiv 3k+2 \end{cases}$$

Using (9), the above equality can be written as

$$\begin{aligned} F_n^{(a,b,c)} &= \begin{cases} \dfrac{F_0^{(a,b,c)}(\alpha^{\frac{n}{3}+1}-\beta^{\frac{n}{3}+1}) + F_{-3}^{(a,b,c)}(\alpha^{\frac{n}{3}}-\beta^{\frac{n}{3}})}{\alpha-\beta} & n \equiv 0 \pmod 3 \\ \dfrac{F_1^{(a,b,c)}(\alpha^{\frac{n-1}{3}+1}-\beta^{\frac{n-1}{3}+1}) + F_{-2}^{(a,b,c)}(\alpha^{\frac{n-1}{3}}-\beta^{\frac{n-1}{3}})}{\alpha-\beta} & n \equiv 1 \pmod 3 \quad n \geq 3, \\ \dfrac{F_2^{(a,b,c)}(\alpha^{\frac{n-2}{3}+1}-\beta^{\frac{n-2}{3}+1}) + F_{-1}^{(a,b,c)}(\alpha^{\frac{n-2}{3}}-\beta^{\frac{n-2}{3}})}{\alpha-\beta} & n \equiv 2 \pmod 3 \end{cases} \\ &= \widetilde{F}_0\delta_0(n) + \widetilde{F}_1\delta_1(n) + \widetilde{F}_2\delta_2(n). \ \square \end{aligned}$$

Theorem 6 (Cassani's identity). *For $n \geq 0$ we have following equalities.*

(a) *For $n \equiv 0$ (mod 3)*

$$F_{n-1}^{(a,b,c)}F_{n+1}^{(a,b,c)} - \left(F_n^{(a,b,c)}\right)^2 = \frac{1}{\omega^2+4}(A_0\sqrt[3]{\alpha^{2n}} + B_0\sqrt[3]{\beta^{2n}} + C_0(-1)^n)$$

where

$$\begin{aligned} A_0 &= \frac{(1+\alpha F_2^{(a,b,c)})(\alpha + F_{-2}^{(a,b,c)})}{\alpha} - \left(F_3^{(a,b,c)}\right)^2 \\ B_0 &= \frac{(1+\beta F_2^{(a,b,c)})(\beta + F_{-2}^{(a,b,c)})}{\beta} - \left(F_3^{(a,b,c)}\right)^2 \\ C_0 &= \omega^2 + 2\omega - 2\left(F_3^{(a,b,c)}\right)^2. \end{aligned}$$

(b) *For $n \equiv 1$ (mod 3)*

$$F_{n-1}^{(a,b,c)}F_{n+1}^{(a,b,c)} - \left(F_n^{(a,b,c)}\right)^2 = \frac{1}{\omega^2+4}(A_1\sqrt[3]{\alpha^{2n-2}} + B_1\sqrt[3]{\beta^{2n-2}} + C_1(-1)^n)$$

where

$$\begin{aligned} A_1 &= F_3^{(a,b,c)}(1+F_2^{(a,b,c)}\alpha) - (\alpha - F_2^{(a,b,c)})^2 \\ B_1 &= F_3^{(a,b,c)}(1+F_2^{(a,b,c)}\beta) - (\beta - F_2^{(a,b,c)})^2 \\ C_1 &= .F_3^{(a,b,c)}(-F_2^{(a,b,c)}\omega - 2) + 2F_{-2}^{(a,b,c)} - 2\left(F_{-2}^{(a,b,c)}\right)^2. \end{aligned}$$

(c) *For $n \equiv 2$ (mod 3)*

$$F_{n-1}^{(a,b,c)}F_{n+1}^{(a,b,c)} - \left(F_n^{(a,b,c)}\right)^2 = \frac{1}{\omega^2+4}(A_2\sqrt[3]{\alpha^{2n-4}} + B_2\sqrt[3]{\beta^{2n-4}} + C_2(-1)^n$$

where

$$\begin{aligned} A_2 &= F_3^{(a,b,c)}(\alpha^2 + F_{-2}^{(a,b,c)}\alpha) - (F_2^{(a,b,c)} + \alpha)^2 \\ B_1 &= F_3^{(a,b,c)}(\beta^2 + F_{-2}^{(a,b,c)}\beta) - (F_2^{(a,b,c)} + \beta)^2 \\ C_1 &= F_3^{(a,b,c)}(2 + F_{-2}^{(a,b,c)}\omega) - 2F_2^{(a,b,c)}\left(F_2^{(a,b,c)} + \omega\right) - 2. \end{aligned}$$

Proof. (a) For the case, $n = 3k$, $k \in \mathbb{Z}$ and using Theorem (5)

$$F_{n-1}^{(a,b,c)}F_{n+1}^{(a,b,c)} - \left(F_n^{(a,b,c)}\right)^2 = F_{3k-1}^{(a,b,c)}F_{3k+1}^{(a,b,c)} - \left(F_{3k}^{(a,b,c)}\right)^2.$$

Hence, we have

$$\begin{aligned} F_{3k-1}^{(a,b,c)}F_{3k+1}^{(a,b,c)} &= \frac{c(\alpha^k - \beta^k) + (\alpha^{k-1} - \beta^{k-1})}{\alpha - \beta}\,\frac{(\alpha^{k+1} - \beta^{k+1}) - a(\alpha^k - \beta^k)}{\alpha - \beta} \\ &= \frac{1}{(\alpha-\beta)^2}(\alpha^{2k-1}(1+c\alpha)(\alpha - a) + \beta^{2k-1}(1+c\beta)(\alpha - \beta) \\ &\quad +(-1)^k(\omega^2 + 2\omega). \end{aligned}$$

and

$$\begin{aligned}\left(F_{3k}^{(a,b,c)}\right)^2 &= \left(\frac{(1+ac)(\alpha^k-\beta^k)}{\alpha-\beta}\right)^2 \\ &= (1+ac)^2\alpha^{2k}+(1+ac)\beta^{2k}-2(-1)^k(1+ac)^2.\end{aligned}$$

Therefore, we obtain

$$F_{n-1}^{(a,b,c)}F_{n+1}^{(a,b,c)}-\left(F_n^{(a,b,c)}\right)^2=\frac{1}{\omega^2+4}(A_0\sqrt[3]{\alpha^{2n}}+B_0\sqrt[3]{\beta^{2n}}+C_0)$$

(b)For the case, $n=3k+1$, $k\in\mathbb{Z}$ and using Theorem (5)

$$F_{n-1}^{(a,b,c)}F_{n+1}^{(a,b,c)}-\left(F_n^{(a,b,c)}\right)^2=F_{3k}^{(a,b,c)}F_{3k+2}^{(a,b,c)}-\left(F_{3k+1}^{(a,b,c)}\right)^2.$$

Hence, we have

$$\begin{aligned}F_{3k}^{(a,b,c)}F_{3k+2}^{(a,b,c)} &= \frac{(1+ac)(\alpha^k-\beta^k)}{\alpha-\beta}\,\frac{c(\alpha^{k+1}-\beta^{k+1})+(\alpha^k-\beta^k)}{\alpha-\beta} \\ &= \frac{1}{(\alpha-\beta)^2}(\alpha^{2k}(1+ac)(1+c\alpha)+\beta^{2k}(1+ac)(1+c\beta) \\ &\quad +(-1)^k((1+ac)(-c\omega-2)))\end{aligned}$$

and

$$\begin{aligned}\left(F_{3k+1}^{(a,b,c)}\right)^2 &= \left(\frac{(\alpha^{k+1}-\beta^{k+1})-a(\alpha^k-\beta^k)}{\alpha-\beta}\right)^2 \\ &= \frac{1}{(\alpha-\beta)^2}(\alpha^{2k}(\alpha-a)^2+\beta^{2k}(\beta-a)^2-(-1)^k(2+2a\omega+2a^2))\end{aligned}$$

Therefore, we obtain

$$F_{3k}^{(a,b,c)}F_{3k+2}^{(a,b,c)}-\left(F_{3k+1}^{(a,b,c)}\right)^2=\frac{1}{\omega^2+4}(A_1\sqrt[3]{\alpha^{2n-2}}+B_1\sqrt[3]{\beta^{2n-2}}+C_1(-1)^n)$$

(c)For the case, $n=3k+2$, $k\in\mathbb{Z}$ and using Theorem (5)

$$F_{n-1}^{(a,b,c)}F_{n+1}^{(a,b,c)}-\left(F_n^{(a,b,c)}\right)^2=F_{3k+1}^{(a,b,c)}F_{3k+3}^{(a,b,c)}-\left(F_{3k+2}^{(a,b,c)}\right)^2.$$

Hence, we have

$$\begin{aligned}F_{3k+1}^{(a,b,c)}F_{3k+3}^{(a,b,c)} &= \frac{(\alpha^{k+1}-\beta^{k+1})-a(\alpha^k-\beta^k)}{\alpha-\beta}\,\frac{(1+ac)(\alpha^{k+1}-\beta^{k+1})}{\alpha-\beta} \\ &= \frac{1}{(\alpha-\beta)^2}(\alpha^{2k}((1+ac)(\alpha^2-a\alpha)+\beta^{2k}((1+ac)(\beta^2-a\beta) \\ &\quad +(-1)^k((1+ac)(2-a\omega))\end{aligned}$$

and

$$\begin{aligned}\left(F_{3k+2}^{(a,b,c)}\right)^2 &= \frac{1}{(\alpha-\beta)^2}(\alpha^{2k}(c+\alpha)^2+\beta^{2k}(c+\beta)^2 \\ &\quad +(-1)^k(2c^2-2c\omega-2))\end{aligned}$$

Hence, we have

$$F_{3k+1}^{(a,b,c)} F_{3k+3}^{(a,b,c)} - \left(F_{3k+2}^{(a,b,c)}\right)^2 = \frac{1}{\omega^2+4}\left(A_3\sqrt[3]{\alpha^{2n-4}} + B_3\sqrt[3]{\beta^{2n-4}} + C_3(-1)^n\right.$$

References

1. Liu, Z., (2023), Analysis of the application of Fibonacci sequence in different fields. College of Engineering, The Pennsylvania State University. Journal of Undergraduate Research in Engineering, 14(1), 45–58.
2. Devlin, K. (2011), The man of numbers: Fibonacci's arithmetic revolution. Walker & Company. Mathematics History Series, 3, 1–275.
3. Koshy, T. (2001), Fibonacci and Lucas numbers with applications. Wiley-Interscience. Mathematics in Engineering, 6, 1–676.
4. Stakhov, A. (2009), The mathematics of harmony: From Euclid to contemporary mathematics and computer science. World Scientific. Applied Mathematical Sciences, 22, 1–570.
5. Bergum, G. E., (1996), Applications of Fibonacci numbers: Volume 6. Springer. Proceedings of the Sixth International Conference, 6, 1–432.
6. Grigas, A. (2013). The Fibonacci sequence: Its history, significance, and manifestations in nature.
7. Chastkofsky, L. (2001). The subtractive Euclidean algorithm and Fibonacci numbers. The Fibonacci Quarterly, 39(4), 320–323.
8. Iliopoulos, C. S., Moore, D., & Smyth, W. F. (1997). A characterization of the squares in a Fibonacci string. Theoretical Computer Science, 172(1–2), 281–291.
9. Khattak, B. H. A., Shafi, I., Rashid, C. H., Safran, M., Alfarhood, S., & Ashraf, I. (2024). Profitability trend prediction in crypto financial markets using Fibonacci technical indicator and hybrid CNN model. Journal of Big Data, 11(1), 58. https://doi.org/10.1186/s40537-024-00858-9.
10. Négadi, T. (2015). A mathematical model for the genetic code(s) based on Fibonacci numbers and their q-analogues. arXiv preprint arXiv:1510.01278. https://arxiv.org/abs/1510.01278.
11. Hoggatt, V. E. Jr. (1969). Fibonacci and Lucas numbers. The Fibonacci Quarterly, 7(5), 482–500.
12. Horadam, A. F. (1965), Basic properties of a certain generalized sequence. The Fibonacci Quarterly, 3(3), 161–176.
13. Miles, E. P. Jr. (1960). Generalized Fibonacci numbers and associated matrices. The American Mathematical Monthly, 67(8), 745–752.
14. Lucas, É. (1877). Théorie des Fonctions Numériques Simplement Périodiques. American Journal of Mathematics, 1(4), 289–321.
15. Panwar, Y. K., Singh, B., & Gupta, V. K. (2014), Generalized Fibonacci Sequences and Its Properties. Palestine Journal of Mathematics, 3(2), 283–289.
16. Gosai, I. (2019), Fibonacci Sequence and Its Applications. International Journal of Research and Analytical Reviews, 6(2), 843–847.
17. Catarino, P., (2021), On Leonardo numbers, Acta Math. Univ. Comenianae, Vol. LXXXIX, 1, pp. 75–86.
18. Catarino, P., & Borges, A., (2020), On Leonardo numbers. Acta Mathematica Universitatis Comenianae, 89(2), 135–141.
19. Shannon, A. G., (2019), A note on generalized Leonardo numbers. Notes on Number Theory and Discrete Mathematics, 25(3), 97–101.
20. J. Bravo, C. A. Gomez, J. L. Herrera, (2019) On The Intersection of k-Fibonacci and Pell Numbers, Bulletin ofthe Korean Mathematical Society, 56(2), 535–547.
21. Vieira, M. Mangueria, F. Alves, P. Catarino, (2020), A forma matricial dos numeros de Leonardo, UniversidadeFederal de Santa Maria, 42, 1–12.
22. T. Yaying, B. Hazarika, et al., (2022) On New Banach Sequence Spaces Involving Leonardo Numbers and theAssociated Mapping Ideal, Hindawi Journal of Function Spaces, doi.org/10.1155/2022/8269000.
23. Gökbaş, H., k-Leonardo numbers, (2024), Palestine Journal of Mathematics, Vol 13(4), 1427–1435.
24. Edson, M., Yayenie, O. M., A New Generalization of Fibonacci Sequence & Extended Binet's Formula, Integers, 9(6):639-654, 2009.

25. Catarino, P. M. M., Spreafico, E. V. P., A Note on Bi-Periodic Leonardo Sequence, Armenian Journal of Mathematics, 16, 1–17, 2024.
26. A. Costa, E., V. P. Spreafico, E., M. M. C. Catarino, P., On The Bi-periodic Eduard and Eduard-Lucas Numbers, V International Conference on Mathematics and its Applications in Science and Engineering 2024, DOI: 10.13140/RG.2.2.13659.17446.