Short Note

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Non-Archimedean John, Non-Archimedean Dvoretzky-Milman, Non-Archimedean Type-Cotype and Non-Archimedean Kwapien Problems

K. Mahesh Krishna *

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Short Note

Non-Archimedean John, Non-Archimedean Dvoretzky-Milman, Non-Archimedean Type-Cotype and Non-Archimedean Kwapien Problems

K. Mahesh Krishna

School of Mathematics and Natural Sciences, Chanakya University Global Campus, NH-648, Haraluru Village, Devanahalli Taluk, Bengaluru North District, Karnataka State, 562 110, India; kmaheshak@gmail.com

Abstract

We ask for non-Archimedean version of following four: (1) John Theorem, (2) Dvoretzky-Milman Theorem, (3) Type-Cotype of Banach space, (4) Kwapien Theorem.

Keywords: John Theorem; Dvoretzky Theorem; Type-Cotype of Banach space; Kwapien Theorem

MSC: 47B10; 47L20; 12J25; 46S10

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1. Non-Archimedean John and Dvoretzky-Milman Problems

Let $\mathcal{X}$ and $\mathcal{Y}$ be two finite dimensional Banach spaces having same dimension. The **Banach-Mazur distance** between $\mathcal{X}$ and $\mathcal{Y}$ is defined as

$$d_{BM}(\mathcal{X},\mathcal{Y}) := \inf\{\|T\|\|T^{-1}\| : T : \mathcal{X} \to \mathcal{Y} \text{ is invertible linear operator}\}. \tag{1}$$

For $n \in \mathbb{N}$, let $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ be the standard Euclidean space. Famous John's theorem says following.

Theorem 1. [1,2] ***(John Theorem)*** *If $\mathcal{X}$ is any n-dimensional real Banach space, then*

$$d_{BM}(\mathcal{X}, (\mathbb{R}^n, \langle \cdot, \cdot \rangle)) \leq \sqrt{n}.$$

Following surprising result says that every finite dimensional Banach space has a subspace which is close (in the Banach-Mazur distance) to Euclidean space [3].

Theorem 2. [2–4] ***(Dvoretzky-Milman Theorem)*** *There is a universal number $C > 0$ satisfying the following property: If $\mathcal{X}$ is any n-dimensional real Banach space and $0 < \varepsilon < \frac{1}{3}$, then for every natural number*

$$k \leq C\frac{\varepsilon^2}{|\log \varepsilon|} \log n,$$

there exists a k-dimensional subspace $\mathcal{Y}$ of $\mathcal{X}$ such that

$$d_{BM}(\mathcal{Y}, (\mathbb{R}^k, \langle \cdot, \cdot \rangle)) < 1 + \varepsilon.$$

We refer [4–20] for more information on Dvoretzky-Milman theorem.

Let $\mathbb{K}$ be a field. A map $|\cdot| : \mathbb{K} \to [0, \infty)$ is said to be a non-Archimedean valuation if following conditions are satisfied.

(i) If $\lambda \in \mathbb{K}$ is such that $|\lambda| = 0$, then $\lambda = 0$.
(ii) $|\lambda\mu| = |\lambda||\mu|$ for all $\lambda, \mu \in \mathbb{K}$.
(iii) (Ultra-triangle inequality) $|\lambda + \mu| \leq \max\{|\lambda|, |\mu|\}$ for all $\lambda, \mu \in \mathbb{K}$.

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In this case, $\mathbb{K}$ is called as a non-Archimedean valued field [21]. If $\mathbb{K}$ is complete in the ultra-metric

$$d(\lambda,\mu) := |\lambda-\mu|, \quad \forall \lambda,\mu \in \mathbb{K},$$

then $\mathbb{K}$ is called as a complete non-Archimedean valued field. In the paper, we assume that $\mathbb{K}$ is complete. Let $\mathcal{X}$ be a vector space over a complete non-Archimedean valued field $\mathbb{K}$ with valuation $|\cdot|$. A map $\|\cdot\| : \mathcal{X} \to [0,\infty)$ is said to be a non-Archimedean norm if following conditions holds.

(i) If $x \in \mathcal{X}$ is such that $\|x\| = 0$, then $x = 0$.
(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{K}$, for all $x \in \mathcal{X}$.
(iii) (Ultra-norm inequality) $\|x+y\| \leq \max\{\|x\|,\|y\|\}$ for all $x,y \in \mathcal{X}$.

In this case, $\mathcal{X}$ is called as a non-Archimedean linear space (NALS) [22]. If $\mathcal{X}$ is complete in the ultra-metric

$$d(x,y) := \|x-y\|, \quad \forall x,y \in \mathcal{X},$$

then $\mathcal{X}$ is called as a non-Archimedean Banach space. Let $\mathcal{X}$ and $\mathcal{Y}$ be finite dimensional non-Archimedean Banach spaces having same dimension. The non-Archimedean Banach-Mazur distance between $\mathcal{X}$ and $\mathcal{Y}$ is defined as

$$d_{BM}(\mathcal{X},\mathcal{Y}) := \inf\{\|T\|\|T^{-1}\| : T : \mathcal{X} \to \mathcal{Y} \text{ is invertible linear operator}\}.$$

Given $n \in \mathbb{N}$ and a complete non-Archimedean valued field $\mathbb{K}$, the standard vector space $\mathbb{K}^n$ is a non-Archimedean Banach space equipped with norm

$$\|(\lambda_j)_{j=1}^n\| := \max_{1\leq j\leq n} |\lambda_j|, \quad \forall (\lambda_j)_{j=1}^n \in \mathbb{K}^n.$$

Problem 1. ***(Non-Archimedean John Problem)*** *Let $\mathcal{K}$ be the set of all complete non-Archimedean valued fields. What is the best function $\Psi : \mathcal{K} \times \mathbb{N} \to (0,\infty)$ satisfying the following property: If $\mathcal{X}$ is any n-dimensional non-Archimedean Banach space over $\mathbb{K}$, then*

$$d_{BM}(\mathcal{X},(\mathbb{K}^n,\langle\cdot,\cdot\rangle)) \leq \phi(\mathbb{K},n).$$

Problem 2. ***(Non-Archimedean Dvoretzky-Mliman Problem)*** *Let $\mathcal{K}$ be the set of all complete non-Archimedean valued fields. What is the best function $\Psi : \mathcal{K} \times (0,\infty) \times \mathbb{N} \to (0,\infty)$ satisfying the following property: If $\mathcal{X}$ is any n-dimensional non-Archimedean Banach space over a complete non-Archimedean valued field $\mathbb{K}$ and $\varepsilon > 0$, then for every natural number*

$$k \leq \Psi(\mathbb{K},\varepsilon,n),$$

there exists a k-dimensional complete subspace $\mathcal{Y}$ of $\mathcal{X}$ such that

$$d_{BM}(\mathcal{Y},(\mathbb{K}^k,\|\cdot\|)) < 1+\varepsilon.$$

2. Non-Archimedean Type-Cotype and Kwapien Problems

Let $\mathcal{H}$ be a Hilbert space and $n \in \mathbb{N}$. Then for $h_1,\dots,h_n \in \mathcal{H}$, we have the generalized parallelogram identity

$$\frac{1}{2^n}\sum_{\varepsilon_1,\dots,\varepsilon_n \in \{-1,1\}} \left\|\sum_{j=1}^n \varepsilon_j h_j\right\|^2 = \sum_{j=1}^n \|h_j\|^2. \tag{2}$$

On the other hand, celebrated Jordan-von Neumann result says that Equality (2) characterizes the Hilbert space [23]. Equality (2) motivated the definition of Type and Cotype for Banach spaces.

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Definition 1. *[2] Let $1 \leq p \leq 2$. A Banach space $\mathcal{X}$ is said to be of* ***(Rademacher) Type*** *p if there exists $T_p(\mathcal{X}) > 0$ such that*

$$\left(\frac{1}{2^n}\sum_{\varepsilon_1,\dots,\varepsilon_n\in\{-1,1\}}\left\|\sum_{j=1}^{n}\varepsilon_j x_j\right\|^p\right)^{\frac{1}{p}} \leq T_p(\mathcal{X})\left(\sum_{j=1}^{n}\|x_j\|^p\right)^{\frac{1}{p}}, \quad \forall x_1,\dots,x_n \in \mathcal{X}, \forall n \in \mathbb{N}.$$

Definition 2. *[2] Let $2 \leq q < \infty$. A Banach space $\mathcal{X}$ is said to be of* ***(Rademacher) Cotype*** *q if there exists $C_q(\mathcal{X}) > 0$ such that*

$$\left(\sum_{j=1}^{n}\|x_j\|^q\right)^{\frac{1}{q}} \leq C_q(\mathcal{X})\left(\frac{1}{2^n}\sum_{\varepsilon_1,\dots,\varepsilon_n\in\{-1,1\}}\left\|\sum_{j=1}^{n}\varepsilon_j x_j\right\|^q\right)^{\frac{1}{q}}, \quad \forall x_1,\dots,x_n \in \mathcal{X}, \forall n \in \mathbb{N}.$$

There is a vast literature on Type-Cotype of Banach spaces, see [2,24–32]. In 1972, Kwapien proved the following breakthrough result using Type-Cotype notions [33].

Theorem 3. *[33,34]* ***(Kwapien Theorem)*** *A Banach space $\mathcal{X}$ has Type 2 and Cotype 2 if and only if it is isomorphic to a Hilbert space.*

We ask for non-Archimedean version of previous Type-Cotype and Kwapien theorem as follows. Let $\mathbb{K}$ be a complete non-Archimedean valued field. Define

$$c_0(\mathbb{N},\mathbb{K}) := \{\{x_n\}_{n=1}^{\infty} : x_n \in \mathbb{K}, \forall n \in \mathbb{N}, \lim_{n\to\infty}|x_n| = 0\}.$$

Then $c_0(\mathbb{N},\mathbb{K})$ is a non-Archimedean Banach space w.r.t. the norm

$$\|\{x_n\}_{n=1}^{\infty}\| := \max_{n\in\mathbb{N}}|x_n|, \quad \forall \{x_n\}_{n=1}^{\infty} \in c_0(\mathbb{N},\mathbb{K}).$$

The space $c_0(\mathbb{N},\mathbb{K})$ is very important (among other non-Archimedean Banach spaces) and is known as p-adic Hilbert space, see [35–37].

Problem 3. ***(Non-Archimedean Type-Cotype and Non-Archimedean Kwapien Problems)*** *Whether there is a way to define non-Archimedean Type and non-Archimedean Cotype for non-Archimedean Banach spaces which will characterize the standard non-Archimedean Banach spaces $c_0(\mathbb{N},\mathbb{K})$? i.e., whether we have a non-Archimedean Kwapien theorem?*

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