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Posted Date: 5 June 2025

doi: 10.20944/preprints202411.1062.v7

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On the Method for Proving RH Using the Alcantara-Bode Equivalence

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Abstract: There is presented here a functional analysis - numerical solution for the Alcantara-Bode equivalent formulation of the Riemann Hypothesis (RH). The long-standing unsolved problem, posits that the non-trivial zeros of the Riemann Zeta function lie on the vertical line $\sigma = 1/2$. Alcantara-Bode equivalent (1993) obtained from Beurling equivalent of RH (1955) states that RH holds if and only if the null space of a specific integral operator T_ρ on $L^2(0,1)$ does not contain not null elements: $N_{T_\rho} = \{0\}$. The theory we introduced here is an update and extension of our previous work [1]. We provided methods for investigating the injectivity of linear bounded operators through their positivity properties on finite dimension subspaces. Here we separated the analysis of the operator restrictions from the approximations of the operator obtained by applying finite rank orthogonal projections. In both cases, the connection between the error estimations of an eligible zero and the positivity parameters dictates the operator injectivity. As a method, the Injectivity Criteria involving the adjoint operator introduced in [1] it is useful when no information we have to apply the finite rank approximation, like the compactness of the operator. The numerical evaluations using the finite rank operator approximations or operator restrictions on the same finite dimension subspaces, showed the injectivity of the integral operator from Alcantara-Bode equivalent. From Theorem 3 or Theorem 4 we obtained $N_{T_\rho} = \{0\}$, meaning that half from Alcantara-Bode equivalent of RH holds. Then, the other half should hold, i.e.: the Riemann Hypothesis is true.

Keywords: integral operators; Hilbert spaces; approximation methods; multi-level discretizations; Riemann Hypothesis

MSC: 31A10; 45P05; 47G10; 65R99; 11M26

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1. Introduction

Let H be a separable Hilbert space. A result obtained (Theorem 1 below) shows that the null space of a linear bounded operator strict positive on a dense set in H, does not contain non null elements. Consider F be a family of finite dimension including subspaces $S_n, n \geq 1$ such that their union S is a dense set in the separable Hilbert space H. Dense sets having such properties there exist, for example when $H := L^2(0,1)$ then S could be built in a multi-level fashion using indicator functions of the disjoint intervals of the domain partitions (see the paragraph 3). Such families could be obtained also from a basis in H, a subspace S_n being spanned by first n elements from the basis for example.
The positivity of a linear bounded operator T on S, $\langle Tv, v\rangle > 0 \ \ \forall v \in S$ not null, ensures that the null space of T contains from S only the element 0, i.e. $N_T \cap S = \{0\}$.
Now, a linear bounded operator T positive on a finite dimension subspace is in fact strictly positive on it: i.e. there exists $\alpha_n(T) > 0$ such that $\langle Tv, v\rangle \geq \alpha_n(T)\|v\|^2$ for every $v \in S_n$.
Suppose T positive on each subspace from the family F.

If there exists $\alpha > 0$ such that $\alpha_n \geq \alpha$ for any $n \geq 1$ then T is strict positive on the dense set S and, by Theorem 1 below $N_T = \{0\}$.

 doi:10.20944/preprints202411.1062.v7

If instead the sequence of the positivity parameters of T is converging to zero, $\alpha_n(T) \to 0$ with $n \to \infty$, we consider two directions for the investigation of its injectivity providing the theory and the methods needed for:

- involving the adjoint operator restrictions on the subspaces of the family, having as support Lemma 2 below;

- considering a sequence of positive operator approximations on subspaces, if there exists one such that the sequence is convergent in norm to the operator

$\|T - T_n\| \to 0$ with $n \to \infty$ and, whose corresponding sequence of positivity parameters is inferior bounded: there exists $\alpha > 0$ such that

$\langle T_n v, v\rangle \geq \alpha_n \|v\|^2$ for any $n \geq 1$ where $\alpha_n := \alpha_n(T_n) \geq \alpha$. The Theorem 2 and Lemma 1 below are dealing with this method.

While the criteria involving the adjoint operator (introduced in [1]) could be applied to any positive, linear bounded operator, for involving the operator approximations we have to find the means for obtaining a proper approximation schema in terms of the convergence of the approximations to the original operator and, such operator approximations should be positive on the subspaces in F having an inferior bound for the sequence of the positivity parameters.

However, note that the positivity of the operator could be solved by replacing it by its associated Hermitian (T^*T) that has the same null space with T and it is non negative definite on H.
Let observe that there is a connection between the two kind of positivity parameters on each subspace: if h is the length of the intervals in a partition of (0,1), $nh = 1$, then for the Hilbert-Schmidt integral operator on $L^2(0,1)$ of our interest, we obtained $\alpha_n(T) = n^{-1}\alpha_n(T_n)$, $n \geq 1$ with $\alpha_n(T_n)$ a constant mesh independent.

2. Two Theorems on Injectivity and Associated Methods

Let H be a separable Hilbert space and denote with $\mathcal{L}(H)$ the class of the linear bounded operators on H. If $T \in \mathcal{L}(H)$ is positive on a dense set $S \subset H$, i.e. $\langle Tv, v\rangle > 0$ $\forall v$ not null in S, then T has no zeros in the dense set. Otherwise, if there exists $w \in S$ such that $Tw = 0$ then $\langle Tw, w\rangle = 0$ contradicts its positivity.
Follows: its '*eligible*' zeros are all in the difference set $E := H \setminus S$, i.e. $N_T \subset E$.

Theorem 1. *If $T \in \mathcal{L}(H)$ is strict positive on a dense set of a separable Hilbert space then T is injective, equivalently $N_T = \{0\}$.*

Proof. Let's take in consideration only the set of eligible zeros that are on the unit sphere without restricting the generality, once for an element $w \in H, w \neq 0$ both w and $w/\|w\|$ are or are not in N_T.
The set $S \subset H$ is dense if its closure coincides with H. Then, if $w \in E := H \setminus S$, for every $\varepsilon > 0$ there exists $u_{\varepsilon,w} \in S$ such that $\|w - u_{\varepsilon,w}\| < \varepsilon$. Now, (1) results as follows. If $\|w\| \geq \|u_{\varepsilon,w}\|$:
$0 \leq \|w\| - \|u_{\varepsilon,w}\| = \|w - u_{\varepsilon,w} + u_{\varepsilon,w}\| - \|u_{\varepsilon,w}\| \leq \|w - u_{\varepsilon,w}\| + \|u_{\varepsilon,w}\| - \|u_{\varepsilon,w}\| < \varepsilon.$
If $\|u_{\varepsilon,w}\| \geq \|w\|$ instead, then:
$0 \leq \|u_{\varepsilon,w}\| - \|w\| = \|u_{\varepsilon,w} - w + w\| - \|w\| \leq \|w - u_{\varepsilon,w}\| < \varepsilon.$
So, given $w \in E$, for every $\varepsilon > 0$ there exists $u_{\varepsilon,w} \in S$ such that

$$| \|w\| - \|u_{\varepsilon,w}\| | < \varepsilon \tag{1}$$

Let w be an eligible element from the unit sphere, $\|w\| = 1$ and take $\varepsilon_n = 1/n$.
Then there exists at least one element $u_{\varepsilon_n,w} \in S$ such that $\|u_{\varepsilon_n,w} - w\| < \varepsilon_n$ holds. Follows from (1), | 1 - $\|u_{\varepsilon_n,w}\|$ | $< 1/n$ showing that, for any choices of a sequence approximating w, $u_{\varepsilon_n,w} \in S, n \geq 1$, it verifies $\|u_{\varepsilon_n,w}\| \to 1$.

If $T \in \mathcal{L}(H)$ is strict positive on S, then there exists $\alpha > 0$ such that $\forall u \in S$, $\langle Tu, u\rangle \geq \alpha\|u\|^2$.
Suppose that there exists $w \in E, \|w\| = 1$ a zero of T, i.e. $w \in N_T$ and consider a sequence of

 doi:10.20944/preprints202411.1062.v7

approximations of w, $u_{\varepsilon_n,w} \in S, n \geq 1$ that, as we showed, has its normed sequence converging in norm to 1. From the positivity of T on the dense set S, follows:

$$\alpha \|u_{\varepsilon_n,w}\|^2 \leq \langle Tu_{\varepsilon_n,w}, u_{\varepsilon_n,w} \rangle = \langle T(u_{\varepsilon_n,w} - w), u_{\varepsilon_n,w} \rangle < \varepsilon_n \|T\| \|u_{\varepsilon_n,w}\| \tag{2}$$

With c=$\|T\|/\alpha$, we obtain $\|u_{\varepsilon_n,w}\| \leq c/n$. Then, $\|u_{\varepsilon_n,w}\| \to 0$ with $n \to \infty$, in contradiction with its convergence $\|u_{\varepsilon_n,w}\| \to 1$ with $n \to \infty$.
Or, this happen for any choice of the sequence of approximations of w, verifying $\|w - u_{\varepsilon_n,w}\| < \varepsilon_n, n \geq 1$, when $Tw = 0$.
Thus $w \notin N_T$, valid for any $w \in E, \|w\| = 1$, proving the theorem because no zeros of T there are in S either. □

Suppose that the dense set S is the result of an union of finite dimension subspaces of a family F: $S = \cup_{n\geq 1} S_n, \overline{S} = H$. It is not mandatory but will ease our proofs considering that the subspaces are including: $S_n \subset S_{n+1}, n \geq 1$.

Observation 1. *Let $\beta_n(u) := \|u - u_n\|$ be the normed residuum of the eligible element $u \in E$ after its orthogonal projection on S_n. Then, $\beta_n(u) \to 0$ with $n \to \infty$.*

Proof. Given $\epsilon > 0$, from the density of the set S in H there exists $u_\epsilon \in S$ verifying $\|u - u_\epsilon\| < \epsilon$, as per the observations made in the proof of the Theorem 1. Let S_{n_ϵ} be the coarsest subspace, i.e. with the smallest dimension, from the family of subspaces containing u_ϵ. Because the best approximation of u in S_{n_ϵ} is its orthogonal projection, we obtain
$\beta_{n_\epsilon}(u) := \|u - P_{n_\epsilon} u\| \leq \|u - u_\epsilon\| < \epsilon$,
inequality valid for every $\epsilon > 0$, proving our assertion. □

Rewriting it, $\beta_n(u) := \|u - P_n u\| = \|(I - P_n)u\| \leq \|I - P_n\| \|u\| \to 0$ for $n \to \infty$ for any $u \in H$ with P_n the orthogonal projection onto S_n.

For $T \in \mathcal{L}(H)$, let $T_n, n \geq 1$ be a sequence of operator approximations on $S_n, n \geq 1$ having the property $\epsilon_n := \|T - T_n\| \to 0$ and, suppose that for every $n \geq 1$, the operator approximation T_n is positive on S_n and denote with $\alpha_n := \alpha_n(T_n)$ its positivity parameter.

Theorem 2. Let $T \in \mathcal{L}(H)$ be positive on the dense set S. If the sequence $\{T_n, n \geq 1\}$ of its approximations on the family F verifies*:*
i) $\epsilon_n := \|T - T_n\| \to 0$ *with* $n \to \infty$*;*
ii) $\langle T_n v, v \rangle \geq \alpha_n \|v\|^2, \forall v \in S_n, S_n \in F$*;*
iii) $\alpha_n \geq \alpha > 0, n \geq 1$*,*
then $N_T = \{0\}$.

Proof. Being positive on S, the operator has no zeros in the dense set.
For $u \in E := H \setminus S, \|u\| = 1$ denoting the not null orthogonal projection over S_n by $u_n := P_n u, n \geq n_0 := n_0(u)$, we have on any subspace S_n, $\|u\|^2 = \|u_n\|^2 + \beta_n^2(u)$ where $\beta_n(u) = \|u - P_n u\| := \beta_n$ is its (normed) residuum. We have: $\|u_n\| \uparrow 1$ and $\beta_n(u) \to 0$.
If there exists $u \in N_T \cap E, \|u\| = 1$ then for it:
$\alpha_n \|u_n\|^2 \leq \langle T_n u_n, u_n \rangle \leq \|T_n u_n\| \|u_n\|$
$= (\|T_n u_n - Tu_n + Tu_n - Tu\|)\|u_n\|$
$\leq (\|T - T_n\| \|u_n\| + \|T\| \|u - u_n\|)\|u_n\|$
$= (\epsilon_n \|u_n\| + \|T\| \beta_n)\|u_n\| \leq (\epsilon_n + \|T\| \beta_n)$
evaluation obtained because $\|u_n\| < 1$. Then from Observation 1 and iii) we have:
$\alpha \leq (\epsilon_n + \|T\| \beta_n) \to 0$.
The inequality is violated from a $n_1 \geq n_0$, involving $u \notin N_T$, valid for any supposed zero of T in E. Once T has no zeros in the dense set, $N_T = \{0\}$. □

We will deal now, with the special case of the approximations of the Hilbert-Schmidt integral operators that, being compact operators could be approximated in a proper manner on finite dimension subspaces so, the condition i) is satisfied ([3]).

Let $T := T_\varphi$ be a Hilbert-Schmidt integral operator. A technique for obtaining the approximations for an integral operator is used in [5]. Thus, the condition i) in the Theorem 2 is fulfilled when $T_n, n \geq 1$ are finite rank approximations on the subspaces of the family F obtained by orthogonal projection integral operators $T_n := P_n^r(T)$. Then, for every $u \in H$ not null:

$\|Tu - T_n u\| = \|(I - P_n^r)Tu\| \leq \|I - P_n^r\|\|Tu\| \to 0$

Lemma 1 ((Criteria for finite rank approximations))**.** If the finite rank approximations of a positive linear Hilbert-Schmidt integral operator T_φ on a dense set S are positive on the family of approximation subspaces F and the sequence of the positivity parameters is inferior bounded,
then T_φ is strict positive on the dense set so, it is injective.

Proof. The requests i), ii) in the Theorem 2 hold by the previous observations. From the convergence to zero of the sequence $\epsilon_n, n \geq 1$ there exists ϵ_0 a 'compactness' parameter verifying $\epsilon_0 := max_n\{\epsilon_n; \epsilon_n < \alpha\}$ corresponding to a subspace $S_{n_0}, n_0 < \infty$. The parameter ϵ_0 in independent from any $v \in S$ and, due to the including property, for any $n < n_0$ we have $S_n \subset S_{n_0}$. We could consider S_{n_0} be S_1 or, your choice, we could consider v as being inside of S_{n_0}. Then:

$\alpha_n \geq \alpha > \epsilon_0 \geq \epsilon_n$ for $n \geq 1$, resulting $(\alpha_n - \epsilon_n) > (\alpha - \epsilon_0) > 0\ \forall n \geq 1$.

For an arbitrary $v \in S$ there exists a coarser subspace (i.e. with a smaller dimension) $S_n, n \geq n_1 := n_1(v)$, for which $v \in S_n$. For it, we have:

$\langle Tv, v\rangle = \langle T_n v, v\rangle - \langle (T_n - T)v, v\rangle > 0$. Since T_n is positive on S_n,

$\langle Tv, v\rangle \geq \alpha_n\|v\|^2 - \langle (T_n - T)v, v\rangle$.

Now, T and T_n are positive on S_n. Then the inner product in the right side of the equality is real valued and, $|\langle (T_n - T)v, v\rangle| \leq \epsilon_n\|v\|^2$.

So, if $\langle (T_n - T)v, v\rangle \geq 0$, then $\langle (T_n - T)v, v\rangle \leq \epsilon_n\|v\|^2$. Because $\epsilon_n < \alpha_n$, follows:

$\langle Tv, v\rangle \geq (\alpha_n - \epsilon_n)\|v\|^2 \geq (\alpha - \epsilon_0)\|v\|^2$.

Now, if $\langle (T_n - T)v, v\rangle < 0$, $\langle Tv, v\rangle \geq \alpha_n\|v\|^2 \geq \alpha\|v\|^2 \geq (\alpha - \epsilon_0)\|v\|^2$.

Thus, taking $\alpha(T) = (\alpha - \epsilon_0)$,

$\langle Tv, v\rangle \geq \alpha(T)\|v\|^2$ for any $v \in S$ meaning that T is strict positive on the dense set S and from Theorem 1, $N_T = \{0\}$. □

Corollary 1. *If $Q \in \mathcal{L}(H)$ is a Hermitian compact operator verifying on a dense set the properties i) and ii) from Theorem 2, then it is injective.*

Proof. Being Hermitian, the operator verifies $\langle Qv, v\rangle \geq 0$, for every $v \in H$. Being compact it admits on a dense family of finite dimension subspaces a sequence of approximations. Then, for any $v \in S$,

$\langle Qv, v\rangle = \langle Q_n v, v\rangle - \langle (Q_n - Q)v, v\rangle \geq 0$ obtaining following the steps from the proof of Lemma 1 that in the hypotheses i) and ii) holds:

$\langle Qv, v\rangle \geq (\alpha - \epsilon_0)\|v\|^2$ meaning that Q is strict positive on the dense set. Thus, $N_Q = \{0\}$ due to the Theorem 1. Let observe that if $Q = T^*T$ then $N_T = N_Q = \{0\}$ result obtained without requesting the positivity of Q or T on the dense set. □

The following lemma is dealing with the cases in which a proper sequence of operator approximations could not be defined (see the Injectivity Criteria in [1]).

Lemma 2 ((Criteria for operator restrictions.))**.** Let $T \in \mathcal{L}(H)$ positive on the subspaces $S_n, n \geq 1$ whose union S is a dense set S, verifying: $\langle Tv, v\rangle \geq \alpha_n\|v\|^2$ for every $v \in S_n$, where $\alpha_n \to 0$ with $n \to \infty$. Consider now the parameters*:*

$\mu_n := \alpha_n(T)/\omega_n$ where ω_n verifies $\|T^*v\| \leq \omega_n\|v\|, \forall v \in S_n, n \geq 1$.

If there exists $C > 0$ *such that* $\mu_n \geq C$ *for every* $n \geq 1$, then $N_T = \{0\}$.

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Proof. Suppose that there exists $u \in (H \setminus S) \cap N_T$, $\|u\| = 1$ and let u_n its orthogonal projection on $S_n, n \geq 1$. Then, from the (strict) positivity of T on each of the subspaces $S_n, n \geq 1$ (see (2)):
$\alpha_n(T)\|u_n\|^2 \leq \langle Tu_n, u_n\rangle = \langle T(u_n - u), u_n\rangle = \langle (u_n - u), T^*u_n\rangle \leq \beta_n\omega_n\|u_n\|$
Then, from
$C \leq \mu_n \leq \beta_n/\sqrt{1-\beta_n^2} \to 0$ where $\beta_n := \beta_n(u) = \|u - u_n\|$, we obtain a contradiction. Thus, $u \notin N_T$ affirmation valid for any $u \in H \setminus S$. Follows: $N_T = \{0\}$. □

3. Dense sets in $L^2(0,1)$.

Let $H := L^2(0,1)$. The semi-open intervals of equal lengths $h = 2^{-m}, m \in N$, nh = 1, $\Delta_{h,k} = ((k-1)/2^m, k/2^m], k = 1, n-1$ together with the open $\Delta_{h,n}$ are defining for $m \geq 1$ a partition of (0,1), k=1,n, $n = 2^m, nh = 1$. Consider the interval indicator functions having the supports these intervals (k=1,n), nh=1:

$$I_{h,k}(t) = 1 \text{ for } t \in \Delta_{h,k} \text{ and } 0 \text{ otherwise} \tag{3}$$

The family F of finite dimensional subspaces S_h that are the linear spans of interval indicator functions of the h-partitions defined by (3) with disjoint supports, $S_h = span\{I_{h,k}; k = 1, n, nh = 1\}$, built on a multi-level structure, are including $S_h \subset S_{h/2}$ by halving the mesh h. In fact, the property is obtained from (3) observing that $S_h \ni I_{h,i} = I_{h/2,2i-1} + I_{h/2,2i} \in S_{h/2}, i = 1, n$.
The set $S = \cup_{n\geq 1} S_h, nh = 1$ is dense in H well known in literature.

Citing [5], (pg 986), the integral operator $P_h^r, n \geq 1$ having the kernel function:

$$r_h(y,x) = h^{-1} \sum_{k=1,n} I_{h,k}(y) I_{h,k}(x) \tag{4}$$

is a finite rank integral operator orthogonal projection having the spectrum {0, 1} with the eigenvalue 1 of the multiplicity n (nh=1) corresponding to the orthogonal eigenfunctions $I_{h,k}, k = 1, n$. We will show it, by proving that $\forall u \in H$, $P_h^r u \in S_h$ and, as a consequence, obviously $(P_h^r)^2 = P_h^r$ for $n \geq 2, nh = 1$.
For any $u \in H$,
$(P_h^r u)(y) = \int_{x\in(0,1)} \left(h^{-1} \sum_{k=1.n} I_{h,k}(y) I_{h,k}(x)\right) u(x)dx$
$= h^{-1} \sum_{k=1,n} c_k I_{h,k}(y)$, where $c_k := \int_{\Delta_{h,k}} u(x)dx$,
that is the standard orthogonal projection of u onto S_h.
Now, for f = $I_{h,j} \in S_h, j = 1, n, nh = 1$:
$P_h^r(f) = h^{-1} \sum_{k=1,n} c_k I_{h,k}$, where $c_k = \int_{\Delta_{h,k}} I_{h,j}(x)dx$, with $c_k = h$ for k=j and 0 for $k \neq j$. Then,
$P_h^r(I_{h,j}) = I_{h,j}$ and so, $P_h^r(v_h) = v_h$ for every $v_h \in S_h$ and, $(P_h^r)^2 u = P_h^r u$ for any $u \in H$.
Because P_h^r is an orthogonal projection onto S_h and due to the including property of the finite dimension subspaces whose union is dense, follows:
$\|I - P_h^r\| \to 0$ for $n \to \infty, nh = 1$. So, i) in Theorem 2 holds.

Remark 1. *The matrix representation of T_ρ on S_h is a sparse diagonal matrix: its elements outside the diagonal are zero valued.*

Proof. The inner product on the subspace S_h between $u \notin S_h$ and $v_h \in S_h$ is a result between the orthogonal projection of u and v_h, like an inner product between two step functions: $\langle u, v_h\rangle := \langle P_h^r u, v_h\rangle$. If $P_h^r u := u_h = \sum_{k=1,n} a_k I_{h,k}$ and $v_h = \sum_{j=1,n} c_j I_{h,j}$, due to the disjoint supports of the indicator interval functions, $\langle I_{h,k}, I_{h,j}\rangle = 0$ for $k \neq j$ and, their inner product is $\langle u_h, v_h\rangle = \sum_{k=1,n} a_k \overline{c_k} \langle I_{h,k}, I_{h,k}\rangle$.
Let T_ρ be a Hilbert-Schmidt integral operator on H. Now,
$T_\rho I_{h,k} = \int_0^1 \rho(y,x) I_{h,k}(x)dx = \int_{\Delta_{h,k}} \rho(y,x) I_{h,k}(x)dx$. Follows:
$\langle T_\rho I_{h,k}, I_{h,j}\rangle = \int_0^1 \left[\int_{\Delta_{h,k}} \rho(y,x) I_{h,k}(x)dx\right] I_{h,j}(y)dy$
$= \int_{\Delta_{h,k}} \int_{\Delta_{h,j}} I_{h,j}(y)\rho(y,x) I_{h,k}(x)dxdy = 0$ for $k \neq j$ because $I_{h,k}$ and $I_{h,j}$ have disjoint supports for $k \neq j$. Then, the matrix representation of T_ρ on S_h, $M_h(T_\rho)$ is a sparse diagonal matrix having the diagonal entries

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$d^h_{kk} := \int_{\Delta_{h,k}} \int_{\Delta_{h,k}} I_{h,k}(y)\rho(y,x)I_{h,k}(x)dxdy,\ \ k = 1, n, nh = 1$ and, with $v_h = \sum_{k=1,n} c_k I_{h,k}$,
$\langle T_\rho v_h, v_h\rangle = \sum_{k=1,n} c_k \overline{c_k} d^h_{kk}$. □

The integral operator approximation of T_ρ on S_h is a finite rank operator approximation, T_{ρ_h}, having the kernel function ([5])

$$\rho_h(y,x) = h^{-1} \sum_{k=1,n} I_{h,k}(y)\rho(y,x)I_{h,k}(x) := h^{-1} \sum_{k=1,n} \rho^k_h(y,x) \tag{5}$$

where the pieces $\rho^k_h, k = 1, n$ of the kernel function ρ_h in the sum have disjoint supports in $L^2(0,1)^2$ namely $\Delta_{h,k} \times \Delta_{h,k}, k = 1, n, nh = 1$.

Remark 2. *The matrix representation of T_{ρ_h} is a sparse diagonal matrix and, $M^r_h(T_\rho) = h^{-1} M_h(T_\rho)$. Evaluating the previous relationship for $v = I_{h,i}$, we obtain*
$(T_{\rho_h} I_{h,i})(y) = h^{-1}\left[\int_{\Delta_{h,i}} \rho(y,x)I_{h,i}(x)dx\right] I_{h,i}(y)$. *Then,*
$\langle T_{\rho_h} I_{h.i}, I_{h,j}\rangle = 0$ *for $i \neq j$ and the matrix representation of the finite rank operator $P^r_h(T_\rho) := T_{\rho_h}$, is:*
$M^r_h(T_\rho) = h^{-1} diag\left[d^h_{kk}\right]_{k=1,n}$, *a sparse diagonal because $d^h_{ij} = 0$ for $i \neq j$ and with the diagonal entries given by:*

$$d^h_{kk} = \int_{\Delta_{h,k}} \int_{\Delta_{h,k}} I_{h,k}(y)\rho(y,x)I_{h,k}(x)dxdy := \int_{\Delta_{h,k}} \int_{\Delta_{h,k}} \rho(y,x)dxdy, k = 1, n \tag{6}$$

Follows: $M^r_h(T_\rho) = h^{-1} M_h(T_\rho)$, i.e. both matrices are or are not simultaneous positive. □

Pointing out:
$\langle T_{\rho_h} v_h, v_h\rangle = h^{-1}\langle T_\rho v_h, v_h\rangle = h^{-1} \sum_{k=1,n} c_k \overline{c_k} d^h_{kk}$, *for any* $v_h = \sum_{k=1,n} c_k I_{h,k} \in S_h, v_h \neq 0$.

Remark 3. *If $d^h_{kk} > 0$, $\forall k = 1, n, nh = 1$, because $\|v_h\|^2 = h\sum_{k=1,n} c_k \overline{c_k}$ we obtain:*
$\langle T_{\rho_h} v_h, v_h\rangle \geq \alpha_h(T_{\rho_h})\|v_h\|^2$ *where*

$$\alpha_h(T_{\rho_h}) = h^{-2} min_{(k=1,n)} d^h_{kk} \tag{7}$$

is the positivity parameter of the finite rank operator approximation T_{ρ_h}.
From $\langle T_\rho v_h, v_h\rangle = \sum_{k=1,n} c_k \overline{c_k} d^h_{kk} = h\langle T_{\rho_h} v_h, v_h\rangle$ results that T_ρ is positive on S_h if and only if T_{ρ_h} is positive on S_h. Then, if on every subspace $S_h \in F$, $d^h_{kk} > 0$ $k = 1, n, nh = 1$, the following relationship holds

$$\alpha_h(T_\rho) = h^{-1} min_{(k=1,n)}(d^h_{kk}) := h\alpha_h(T_{\rho_h}), nh = 1 \tag{8}$$

Remark 4. *Thus, the positivity of the linear bounded integral operator T_ρ on the dense set S is determined by the diagonal entries in its matrix representations.*

4. Proof of the Alcantara-Bode Equivalent

We are now in position to prove RH showing that the integral operator
$(T_\rho u)(y) = \int_0^1 \rho(y,x)u(x)dx,\ u \in L^2(0,1)$, where $\rho(y,x) = \{y/x\}$ is the fractional function of the ratio y/x, has its null space $N_{T_\rho} = \{0\}$.
Alcantara-Bode ([2], pg. 151) in his theorem of the equivalent formulation of RH obtained from Beurling equivalent formulation ([4]), states:

The Riemann Hypothesis holds if and only if $N_{T_\rho} = \{0\}$.

Its kernel function $\rho \in L^2(0,1)^2$ defined by the fractional part of the ratio (y/x) is continue almost everywhere, the discontinuities in $(0,1)^2$ consisting in a set of numerable one dimensional lines of the form $y = kx, k \in N$, being of Lebesgue measure zero. The integral operator is Hilbert-Schmidt ([2]) and so compact, allowing us to consider its approximations on finite dimension subspaces ([3]).

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The entries in the diagonal matrix representation $M_h(T_{\rho_h})$ of the finite rank integral operator T_{ρ_h} are given by:

$d^h_{kk} = \int_{\Delta_{h,k}} \int_{\Delta_{h,k}} \rho(y,x)dxdy$, and valued (see also [1]) as follows:

$$d^h_{11} = h^2(3-2\gamma)/4; \quad d^h_{kk} = \frac{h^2}{2}(-1 + \frac{2k-1}{k-1} ln(\frac{k}{k-1})^{k-1}) \tag{9}$$

for $k \geq 2$, where γ is the Euler-Mascheroni constant ($\simeq 0.5772156...$). The formula (9) has been computed using for the fractional part the suggestion found in [4]: for $0 < a < b < 2a$, $\{b/a\} = (b/a) - 1$. Then,

$\int_{\Delta_{h,k}} \int_{\Delta_{h,k}} \rho(y,x)dxdy = \int_{\Delta_{h,k}} \left[\int_{\Delta_{h,k}} (y/x)dx - \int^y_{(k-1)h} dx \right] dy.$

The sequence from (9)

$f(k) := h^{-2} d^h_{kk} = (-1 + \frac{2k-1}{k-1} ln(\frac{k}{k-1})^{k-1})/2$ is monotone decreasing for $k \geq 2$ and converges to 0.5 for k $\to \infty$. For $k \geq 2$, we have: $d^h_{kk} > 0.5h^2 > d^h_{11}$. Then:

$$\alpha_h(T_{\rho_h}) = h^{-2} d^h_{11} = (3-2\gamma)/4 > 0, \; n \geq 2, nh = 1. \tag{10}$$

showing that the finite rank approximations of the integral operator have the sequence of the positivity parameters inferior bounded.

Theorem 3 (Finite Rank Approximations:)**.** The Alcantara-Bode equivalent of RH holds.

Proof. From (10) results that the sequence of the positivity parameters of the finite rank operator approximations T_{ρ_h} on the dense family F is inferior bounded, $\alpha_h(T_{\rho_h})$ being a constant mesh independent. Then, $N_{T_\rho} = \{0\}$ is obtained from Lemma 1 (or Theorem 2). □

We use now the method covered by Lemma 2 (see also [1], Injectivity Criteria). The integral operator T_ρ is (strict) positive on $S_h \in F, n \geq 2, nh = 1$ with the parameter valued from (8) and (10),

$\alpha_h(T_\rho) = h\alpha_h(T_{\rho_h}) = h(3-2\gamma)/4 \to 0$ with $n \to \infty$.

In order to apply Lemma 2, we should invoke the adjoint operator of T_ρ whose kernel function is $\rho^*(y,x) = \overline{\rho(x,y)} = \rho(x,y)$. For $v_h = \sum_{k=1,n} c_k I^k_h \in S_h$

$T^*_\rho v_h = \sum_{k=1,n} c_k \int_{\Delta_{h,k}} \rho(x,y) I^k_h(y)dy = \sum_{k=1,n} c_k \int_{\Delta_{h,k}} \rho_{h,k}(x,y)dy,$

where $\rho_{h,k} = I_{h,k}(x)\rho(x,y)I_{h,k}(y)$. Follows:

$\|T^*_\rho v_h\|^2 = \langle \sum_{k=1,n} c_k \int_{\Delta_{h,k}} \rho_{h,k}(x,y)dy, \sum_{j=1,n} c_j \int_{\Delta_{h,j}} \rho_{h,j}(x,y)dy \rangle$

$= \sum_{k=1,n} c_k \overline{c_k} (\int_{\Delta_{h,k}} \left[\int_{\Delta_{h,k}} \rho(x,y) I_{h,k}(y)dy \right]^2 I_{h,k}(x)dx).$

Because $\rho(x,y)$ is valued in [0,1), $\rho(y,x) < 1$ for every $x, y \in (0,1)$, we obtaining:

$\|T^*_\rho v_h\|^2 \leq \sum_{k=1,n} c_k \overline{c_k} h^3 = h^2 \|v_h\|^2$ and, $\|T^*_\rho v_h\| \leq h\|v_h\|$.

Taking $\omega_h(T^*_\rho) = h$, the injectivity parameter of T on S_h is given by:

$$\mu_h = (3-2\gamma)/4, \quad \text{a mesh independent constant } \forall n, nh = 1 \tag{11}$$

Theorem 4 ((*Injectivity Criteria*):)**.** The Alcantara-Bode equivalent of RH holds.

Proof. Because μ_h is a constant (see (11)) for any $h, nh = 1$, applying Lemma 2 we obtain $N_{T_\rho} = \{0\}$. □

Proposition 1. The Riemann Hypothesis is true.

Proof. From Theorem 3 or Theorem 4 we obtained $N_{T_\rho} = \{0\}$, meaning that half from Alcantara-Bode equivalent of RH holds. Then, the other half should hold, i.e.: the Riemann Hypothesis is true. □

Observations.

We considered the subspaces S_h spanned by indicator of semi-open intervals functions of a partition of the domain and so, the subspaces are including ($S_h \subset S_{h/2}$) providing the monotony of the positivity parameters. If we take instead the indicator open-intervals functions for generating the subspace S_h^o as well of the indicator closed-intervals functions generating the subspace S_h^c, $nh = 1, n \geq 1$ then both sets S^o and S^c are dense like S, easy to show ([11]).

The dense sets S and S^c have been used in [5] and respectively [6] for obtaining optimal evaluations of the decay rate of convergence to zero of the eigenvalues of Hermitian integral operators having the kernel like Mercer kernels ([9]).

Conflicts of Interest: No Competing Interests.

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