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Article

Analytic Resonance and Model Harmony in Non-Linear Oscillators

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Abstract: Oscillons, fundamental to various physical domains, are examined in-depth in this paper. From Josephson junctions in superconducting circuits to classical and quantum field theories, oscillators reveal profound insights into diverse phenomena. The study primarily focuses on oscillons, distinctive field configurations with enduring localization, establishing intriguing connections with well-known models like the Sine-Gordon breather. Surprising parallels in the weak coupling limit illuminate the rich dynamics of oscillators. A semi-classical quantization method, discretizing the field into spatially homogeneous regions, exposes stability angles, providing insights into the quantum nature of oscillons. The application of this quantization framework to specific cases, including the Sine-Gordon breather, showcases its versatility. The results offer a comprehensive perspective on the quantization of oscillons, unraveling the intricate interplay between classical and quantum dynamics. In conclusion, this paper provides a profound exploration of quantum oscillators, unveiling novel connections and presenting a rigorous quantization framework. The gained insights contribute significantly to the broader understanding of oscillatory systems and their pivotal role in diverse physical phenomena.

Keywords: quantum oscillators; oscillons; Sine-Gordon breather; weak coupling limit; semi-classical quantization; stability angles; field configurations; mathematical physics

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1. Introduction

This paper extends work at Oscillons [6], a fundamental class of physical systems playing a crucial role in various areas of theoretical and experimental physics. From Josephson junctions in superconducting circuits to classical and quantum field theories, the study of oscillators provides insights into diverse phenomena. This paper delves into the intricate realm of oscillators, focusing on their dynamics and quantum characteristics. The investigation begins with a detailed exploration of classical and quantum properties of oscillators, emphasizing their ubiquity in different physical contexts. Oscillons, localized and long-lived field configurations, emerge as a central theme, showcasing their relevance in classical field theories. The paper further draws connections between oscillons and well-established models like the Sine-Gordon breather, unveiling unexpected correspondences in the weak coupling limit. A key aspect of the analysis involves the quantization of oscillons, introducing a semi-classical approach that discretizes the field into spatially homogeneous regions. By employing a separation of variables technique, the quantization is extended to individual regions, revealing stability angles and their implications for the quantum nature of oscillons. The mathematical sophistication of the approach ensures a comprehensive understanding of the quantum aspects of oscillatory systems. The general quantization theory is then applied to specific cases, such as the Sine-Gordon breather, showcasing the versatility and applicability of the proposed framework. The results provide a unified perspective on the quantization of oscillons, shedding light on the intricate interplay between classical and quantum dynamics. In conclusion, this paper offers a deep exploration of quantum oscillators, uncovering novel connections and providing a rigorous framework for their quantization. The insights gained from this analysis contribute to the broader understanding of oscillatory systems and their role in diverse physical phenomena.

2. Nonlinear Scalar Field Dynamics

In the context of a flat Minkowski background, the dynamical behavior of a scalar field is described by the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}\int dxdt \left[\dot{\phi}^2 - (\phi')^2 - m^2\left(\phi^2 - \frac{g}{2}\phi^4 + \frac{g^2}{3}\phi^6\right)\right]. \tag{1}$$

The corresponding Hamiltonian for this theory is given by:

$$H = \int dx\frac{1}{2}\left[\dot{\phi}^2 + (\phi')^2 + m^2\left(\phi^2 - \frac{g}{2}\phi^4 + \frac{g^2}{3}\phi^6\right)\right]. \tag{2}$$

The equation of motion derived from this Lagrangian is expressed as:

$$\ddot{\phi} - \phi'' + m^2\left(\phi - g\phi^3 + g^2\phi^5\right) = 0. \tag{3}$$

A particular solution to this equation, known as the oscillon solution, is given by:

$$\phi(t,x) = \epsilon\sqrt{\frac{8}{3g}}\cos\left(\frac{2\pi t}{\tau}\right)\operatorname{sech}(mx\epsilon), \tag{4}$$

where $\tau = \frac{2\pi}{m\sqrt{1-\epsilon^2}}$ represents the period of the oscillon solution.

This nonlinear scalar field dynamics exhibits intriguing periodic behavior, characterized by oscillons that arise as stable, localized structures with a well-defined temporal evolution. The emergence of such solutions highlights the rich dynamics inherent in this system. Further exploration and analysis of these solutions provide valuable insights into the complex behavior of nonlinear scalar fields in flat Minkowski spacetime.

3. Quantization Procedure for Periodic Fields

In the quantization of the periodic field, we adopt a procedure inspired by Dashen et al.'s work on the breather doublet solution of the Sine-Gordon model. The detailed derivation can be found in Dashen et al.'s paper and is succinctly summarized by Rajaraman in "Solitons and Instantons in Quantum Field Theory."

Let $\tilde{Q}(E) \equiv \mathrm{Tr}\left(\frac{1}{E-H}\right)$ and $G(T) \equiv \mathrm{Tr}\left[\exp\left(-\frac{iHT}{\hbar}\right)\right] = \int \mathcal{D}[\phi(\mathbf{x},t)]\exp\left\{\frac{i}{\hbar}S[\phi(\mathbf{x},t)]\right\}$ as defined by Rajaraman.

The Stationary Phase Approximation is employed to extract periodic classical solutions $\phi_{\mathrm{cl}}(x,t)$ from $G(T)$, following the approach outlined in Rajaraman's Chapter 6, "Functional Integrals and the WKB Method." This process yields quantized energy states analogous to the derivation of Bohr-Sommerfeld quantization conditions in Quantum Mechanics.

The quantization of a periodic quantum field involves determining stability angles ν_i, where i labels different solutions to Equation (5). These angles are obtained by solving the stability equation:

$$\left[-\frac{\partial^2}{\partial t^2} + \nabla^2 - \left(\frac{\partial^2 U}{\partial\phi^2}\right)\bigg|\phi\mathrm{cl}\right]\xi(x,t) = 0. \tag{5}$$

Since ϕ_{cl} is periodic, the solutions to Equation (5) will also be periodic. The stability angle is then given by:

$$\xi_i(x,t+\tau) = e^{i\nu_i}\xi_i(x,t). \tag{6}$$

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To handle potential energy divergences, counter terms are introduced to the field Lagrangian for energy (mass) spectrum renormalization. The energy is expressed as:

$$E = E_{\text{cl}}[\phi_{\text{cl}}] + E_{\text{ct}}[\phi_{\text{cl}}] + \sum_{i,p_i} \left(p_i + \frac{1}{2}\right) \hbar \frac{\partial \nu_i}{\partial \tau}, \tag{7}$$

where $p_i = 0, 1, 2, ..., \infty$, and the quantization-imposing conditions must satisfy:

$$W_{p_i}(E) = 2m\pi\hbar, \; W_{p_i}(E) \qquad = S_{\text{cl}}[\phi_{\text{cl}}] + S_{\text{ct}}[\phi_{\text{cl}}] + E\tau[\phi_{\text{cl}}] \qquad - \sum_{i,p_i=0}^{\infty} \left(p_i + \frac{1}{2}\right) \hbar\nu_i.$$

This quantization procedure, grounded in the principles of classical and quantum field theory, provides a systematic approach to handling periodic fields and extracting meaningful physical information from the system.

4. Classical Energy and Action Analysis

The classical energy of the oscillon field is given by:

$$\begin{aligned} E_{\text{cl}}[\phi_{\text{cl}}] &= \int dx \frac{1}{2} \left[\dot{\phi}^2 + (\phi')^2 + m^2 \left(\phi^2 - \frac{g}{2}\phi^4 + \frac{g^2}{3}\phi^6 \right) \right] \\ &= \frac{8m\epsilon}{1215g} \left[-360\epsilon^2 \cos^4(mt\sqrt{1-\epsilon^2}) + 512\epsilon^4 \cos^6(mt\sqrt{1-\epsilon^2}) \right. \\ &\quad \left. +135 \left(3 - \epsilon^2 + 2\epsilon^2 \cos(2mt\sqrt{1-\epsilon^2}) \right) \right] \\ &= \frac{8m}{3g}\epsilon - \frac{8m}{27g} \left(3 - 6\cos(2mt) + 8\cos^4(mt) \right) \epsilon^3 + O(\epsilon^5). \end{aligned} \tag{8}$$

Alternatively, expressing the energy in terms of the period, and expanding spatial and temporal terms, we get:

$$
\begin{aligned}
E_{\text{cl}}[\phi_{\text{cl}}] &= \int dx \frac{1}{2}\left[\dot{\phi}^2 + (\phi')^2 + m^2\left(\phi^2 - \frac{g}{2}\phi^4 + \frac{g^2}{3}\phi^6\right)\right] \\
&= \frac{8\epsilon}{3gm\tau^2}\left[m^2\tau^2\cos^2\left(\frac{2\pi t}{\tau}\right) + 4\pi^2\sin^2\left(\frac{2\pi t}{\tau}\right)\right] \\
&\quad - \frac{8m\epsilon^3}{27g}\cos^2\left(\frac{2\pi t}{\tau}\right)\left[1 + 4\cos\left(\frac{4\pi t}{\tau}\right)\right] \\
&\quad + \frac{4096m\epsilon^5}{1215g}\cos^6\left(\frac{2\pi t}{\tau}\right) \\
&= \frac{8m\epsilon}{3g}\left[\cos^2\left(\frac{2\pi t}{\tau}\right) + \left(1-\epsilon^2\right)\sin^2\left(\frac{2\pi t}{\tau}\right)\right] \\
&\quad - \frac{8m\epsilon^3}{27g}\cos^2\left(\frac{2\pi t}{\tau}\right)\left[1 + 4\cos\left(\frac{4\pi t}{\tau}\right)\right] \\
&\quad + \frac{4096m\epsilon^5}{1215g}\cos^6\left(\frac{2\pi t}{\tau}\right) \\
&= \frac{8m\epsilon}{3g}\left[1 - \epsilon^2\sin^2\left(\frac{2\pi t}{\tau}\right)\right] \\
&\quad - \frac{8m\epsilon^3}{27g}\cos^2\left(\frac{2\pi t}{\tau}\right)\left[1 + 4\cos\left(\frac{4\pi t}{\tau}\right)\right] \\
&\quad + \frac{4096m\epsilon^5}{1215g}\cos^6\left(\frac{2\pi t}{\tau}\right) \\
&= \frac{8m}{3g}\epsilon - \frac{8m\epsilon^3}{3g}\sin^2\left(\frac{2\pi t}{\tau}\right) \\
&\quad - \frac{8m}{27g}\cos^2\left(\frac{2\pi t}{\tau}\right)\left[1 + 4\cos\left(\frac{4\pi t}{\tau}\right)\right]\epsilon^3 + O(\epsilon^5) \\
&= \frac{8m}{3g}\epsilon - \frac{8m\epsilon^3}{27g}\left(6 - 2\cos\left(\frac{4\pi t}{\tau}\right) + \cos\left(\frac{8\pi t}{\tau}\right)\right).
\end{aligned} \tag{9}
$$

The time-dependence in the expression arises due to the approximate nature of our solution to the equation of motion.

5. Classical Action Analysis over One Period

The classical action over one period is given by:

$$
\begin{aligned}
S_{\text{cl}}[\phi_{\text{cl}}] &= \frac{1}{2}\int_0^\tau dt \int_{-\infty}^{\infty} dx \left[\dot{\phi}^2 - (\phi')^2 - m^2\left(\phi^2 - \frac{g}{2}\phi^4 + \frac{g^2}{3}\phi^6\right)\right] \\
&= \frac{4\epsilon}{3gm\tau}\left(4\pi^2 - m^2\tau^2\right) + \frac{4m\tau\epsilon^3}{9g} - \frac{256m\tau\epsilon^5}{243g} \\
&= -\frac{8m\tau\epsilon^3}{9g} - \frac{256m\tau\epsilon^5}{243g}.
\end{aligned} \tag{10}
$$

Verifying that $E_{\text{cl}} = -\frac{dS_{\text{cl}}}{d\tau}$, we obtain the part that is not time-dependent. The results are consistent up to the time dependence of Equation (11), leading to:

$$
E_{\text{cl}} = -\frac{dS_{\text{cl}}}{d\tau} = \frac{8m\epsilon}{3g} - \frac{16m\epsilon^3}{9g}. \tag{11}
$$

Note that in future analysis, we will treat τ and ϵ as independent, and therefore, no relation connecting them will be used. For instance, the relevant expression for classical action in our analysis will be the second line of Equation (12), where the identity $\tau = \frac{2\pi}{m\sqrt{1-\epsilon^2}}$ has not yet been employed.

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6. Stability Angles

The stability equation is obtained by taking $\phi = \phi_{\rm cl} + \xi$ and expanding the equation of motion to the first order. The stability angles can be found by solving this equation.

Assuming $\xi(t,x)$ is separable, we can write $\xi = \xi_T(t)\xi_X(x)$. Up to the third order in ϵ, keeping the period non-expanded, the stability equation is given by

$$\begin{aligned} &\left[-\partial_t^2 + \partial_x^2 - m^2\left(1 - 8\cos^2\left(\frac{2\pi t}{\tau}\right)\operatorname{sech}^2(mx\epsilon)\epsilon^2\right)\right]\xi(t,x) = 0 \\ &\left[-\partial_t^2 + \partial_x^2 - m^2\left(1 - 8\cos^2\left(\frac{2\pi t}{\tau}\right)\epsilon^2\right)\right]\xi_T\xi_X = 0. \end{aligned} \tag{12}$$

The x dependence only appears at the fourth order in ϵ. Therefore, we solve a wave equation with an oscillating spatially flat potential. After calculating the mass spectrum up to ϵ^3, higher-order contributions must be considered to include the x dependence in the stability equation.

This leads to two ordinary differential equations:

$$\frac{d^2}{dx^2}\xi_X + C^2\xi_X = 0, \tag{13}$$

$$\frac{d^2}{dt^2}\xi_T + m^2\left(1 + \frac{C^2}{m^2} - 8\epsilon^2\cos^2\left(\frac{2\pi t}{\tau}\right)\right)\xi_T = 0. \tag{14}$$

By following the theory of Mathieu functions, we derive the Mathieu equation for ξ_T:

$$\begin{aligned} &\ddot{\xi}_T + \left[\left(C^2 + m^2 - 4\epsilon^2 m^2\right) - 4\epsilon^2 m^2\cos\left(2\frac{2\pi t}{\tau}\right)\right]\xi_T = 0 \\ &\ddot{\xi}_T + \left[\left(C^2 + m^2 - 4\epsilon^2 m^2\right)\frac{\tau^2}{4\pi^2} - \frac{\tau^2}{\pi^2}\epsilon^2 m^2\cos\left(2\frac{2\pi t}{\tau}\right)\right]\xi_T = 0. \end{aligned} \tag{15}$$

Identifying a, q, and z as:

$$\begin{aligned} a &= \frac{(C^2 + m^2 - 4m^2\epsilon^2)\tau^2}{4\pi^2}, \\ q &= \frac{m^2\epsilon^2\tau^2}{2\pi^2}, \\ z &= \frac{2\pi t}{\tau}, \end{aligned} \tag{16}$$

we can write the general analytic solution in terms of Mathieu functions:

$$\xi_T(t) = A_1\mathrm{Ce}\left[a,q,z\right] + A_2\mathrm{Se}\left[a,q,z\right]. \tag{17}$$

The stability angles are obtained from the periodic solutions using Floquet's Theorem. The only stability angle is given by:

$$\cos\frac{\nu}{2} = \frac{\mathrm{Ce}[a,q,\pi]}{\mathrm{Ce}[a,q,0]}. \tag{18}$$

Thus, the formula for the Mathieu characteristic exponent is:

$$\cos(\mu\pi) = \frac{\mathrm{Ce}[a,q,\pi]}{\mathrm{Ce}[a,q,0]}. \tag{19}$$

For small q, we use the known expansion identity for Mathieu cosine function and find that:

$$\mu = \pm\left(\frac{\tau\sqrt{C^2+m^2}}{2\pi} - \frac{m^2\tau\epsilon^2}{\pi\sqrt{C^2+m^2}}\right). \tag{20}$$

Thus, the stability angles up to order ϵ^2 are given by:

$$\nu = \tau\sqrt{C^2+m^2} - \frac{2m^2\tau\epsilon^2}{\sqrt{C^2+m^2}}. \tag{21}$$

When imposing periodic boundary conditions, the stability angles involve an integral over all values of C, resulting in a quadratically and logarithmically divergent term.

7. Summation at Stability Angles

Let us insert the expression for ϵ in terms of τ into the equation for stability angles and action. When expanding in terms of ϵ, the insertion does not matter since τ is of order 1. Solving for ϵ from the identity for τ gives two solutions. However, since we have to square ϵ, it does not matter which one we take, as they are $\pm$ the same expression.

The stability angle is then

$$\begin{aligned}\nu &= \tau\sqrt{C^2+m^2} - \frac{2\left(m^2\tau^2-4\pi^2\right)}{\tau\sqrt{C^2+m^2}}\\ &= \frac{2\pi\sqrt{C^2+m^2}}{m} + \frac{\left(C^2-3m^2\right)\pi\epsilon^2}{m\sqrt{C^2+m^2}}\end{aligned} \tag{22}$$

Its derivative with respect to the period is

$$\frac{d\nu}{d\tau} = \sqrt{C^2+m^2} - \frac{2\left(m^2\tau^2+4\pi^2\right)}{\tau^2\sqrt{C^2+m^2}} \tag{23}$$

The sum over all stability angles is now an integral over all values of C. Since the integral will diverge, we can introduce a cut-off which we later take to infinity.

$$\begin{aligned}\frac{d}{d\tau}\sum\nu &= \int_{-\infty}^{\infty}\frac{d\nu}{d\tau}dC\\ &\to \int_{-\Lambda}^{\Lambda}\frac{d\nu}{d\tau}dC\end{aligned} \tag{24}$$

This gives:

$$\begin{aligned}\int_{-\Lambda}^{\Lambda}\frac{d\nu}{d\tau}dC &= \int_{-\Lambda}^{\Lambda}\left[\sqrt{C^2+m^2} - \frac{2\left(m^2\tau^2+4\pi^2\right)}{\tau^2\sqrt{C^2+m^2}}\right]dC\\ &= \Lambda\sqrt{\Lambda^2+m^2} - \frac{16\pi^2}{\tau^2}\ln\left(\frac{\Lambda+\sqrt{\Lambda^2+m^2}}{m}\right) - m^2\left[\operatorname{arcsinh}\frac{\Lambda}{m} + 2\ln\left(\frac{\Lambda+\sqrt{\Lambda^2+m^2}}{m}\right)\right]\end{aligned} \tag{25}$$

Now notice that

$$\operatorname{arcsinh}\frac{\Lambda}{m} = \ln\left(\frac{\Lambda+\sqrt{\Lambda^2+m^2}}{m}\right). \tag{26}$$

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Hence

$$\int_{-\Lambda}^{\Lambda} \frac{d\nu}{d\tau} dC = \Lambda\sqrt{\Lambda^2+m^2} - \frac{16\pi^2+3m^2\tau^2}{\tau^2} \ln\left(\frac{\Lambda+\sqrt{\Lambda^2+m^2}}{m}\right),$$
$$\frac{1}{2}\frac{d}{d\tau}\sum \nu = \lim_{\Lambda\to\infty}\left[\frac{1}{2}\Lambda\sqrt{\Lambda^2+m^2} - \left(\frac{8\pi^2}{\tau^2}+\frac{3m^2}{2}\right)\ln\left(\frac{\Lambda+\sqrt{\Lambda^2+m^2}}{m}\right)\right].$$
$$\frac{1}{2}\sum \nu = \lim_{\Lambda\to\infty}\left[\frac{1}{2}\Lambda\sqrt{\Lambda^2+m^2}\tau + \left(\frac{8\pi^2}{\tau}-\frac{3m^2\tau}{2}\right)\ln\left(\frac{\Lambda+\sqrt{\Lambda^2+m^2}}{m}\right)\right].$$

Integrating without any $\epsilon - \tau$ substitutions, with periodic BC:

$$\begin{aligned}
\frac{1}{2}\sum_n \nu &= \sum_n \left[\frac{\tau}{2}\sqrt{C_n^2+m^2} - \frac{m^2\tau\epsilon^2}{\sqrt{C_n^2+m^2}}\right] \\
&\to E_0\tau - \frac{L}{2\pi}\int_{-\infty}^{\infty} dC\left[\frac{m^2\tau\epsilon^2}{\sqrt{C^2+m^2}}\right] \\
&= E_0\tau - \frac{Lm^2\tau\epsilon^2}{\pi}\ln\left(\frac{\Lambda+\sqrt{\Lambda^2+m^2}}{m}\right)
\end{aligned}$$

8. Renormalisation

In order to obtain the energy

$$E = -\frac{d}{d\tau}\left[S_{cl} + S_{ct} - \frac{1}{2}\sum_i \nu_i\right], \tag{27}$$

we must introduce counter terms and cancel the divergences from the sum of stability angles. The quadratically divergent piece is exactly the vacuum energy of the theory so will vanish. The more problematic is the logarithmically divergent piece.

Due to the field strength renormalisation which normally comes into the two-point function, we can use the standard trick and replace $\phi = Z^{1/2}\phi_r$ in the Lagrangian and write the bare mass m_0 and g_0 instead of the physical mass and coupling constant. Then we can introduce the standard:

$$\begin{aligned}
\delta_Z &= Z - 1 \\
\delta_m &= Zm_0^2 - m^2 \\
\delta_g &= Z^2 m_0^2 g_0 - m^2 g.
\end{aligned} \tag{28}$$

Inserting this into the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m_0^2\phi^2 + \frac{1}{4}m_0^2 g_0\phi^4 \tag{29}$$

We do not need to worry about ϕ^6 term since when we insert the counter terms into the WKB energy equation, terms will have to be evaluated at the classical field solution. And since we are only solving up to order less than $O(\epsilon^4)$, we can leave the ϕ^6 term out.

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$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}Z\left(\partial_\mu\phi_r\right)^2 - \frac{1}{2}Zm_0^2\phi_r^2 + \frac{1}{4}Z^2m_0^2g_0\phi_r^4 \\
&= \frac{1}{2}\left(\partial_\mu\phi_r\right)^2 - \frac{1}{2}m^2\phi_r^2 + \frac{1}{4}m^2g\phi_r^4 + \\
&+ \frac{1}{2}\delta_Z\left(\partial_\mu\phi_r\right)^2 - \frac{1}{2}\delta_m\phi_r^2 + \frac{1}{4}\delta_g\phi_r^4
\end{aligned} \tag{30}
$$

We can proceed exactly as in Peskin chapter 10.2 where he finds counter terms for the standard ϕ^4 theory. We only need to make a suitable identification of parameters between our Lagrangian and the standard ϕ^4 Lagrangian.

The identification is:

$$
\begin{aligned}
\phi^4 &= \text{oscillon} \\
\lambda &= -6m^2g \\
\delta_\lambda &= -6\delta_g
\end{aligned} \tag{31}
$$

Then by analyzing $2 \to 2$ scattering, we can find the counter terms.

The renormalisation conditions are the usual ones as in Peskin pp. 325.

Now

$$
i\mathcal{M} = 6im^2g + \left(6im^2g\right)^2\left[iV(s) + iV(t) + iV(u)\right] + 6i\delta_g, \tag{32}
$$

where

$$
iV(p^2) = \frac{i}{2}\int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2 - m^2}\frac{i}{(k+p)^2 - m^2}. \tag{33}
$$

The renormalisation condition demands that the amplitude equals $6im^2g$ at $s = 4m^2$ and $t = u = 0$.

Therefore

$$
\delta_g = 6m^4g^2\left[V(4m^2) + 2V(0)\right] \tag{34}
$$

Computing this integral gives after writing $l = k + xp$, Wick rotating and writing $\Delta = m^2 - x(1 - x)p^2$:

$$
\begin{aligned}
V(p^2) &= \frac{i}{2}\int_0^1 dx \int \frac{d^2l}{(2\pi)^2}\frac{1}{\left[l^2 + x(1-x)p^2 - m^2\right]^2} \\
&= -\frac{1}{2}\int_0^1 dx \int \frac{d^2l_E}{(2\pi)^2}\frac{1}{\left[l_E^2 + \Delta\right]^2} \\
&= -\frac{1}{4\pi}\int_0^1 dx \int_0^\infty \frac{l_E\,dl_E}{\left[l_E^2 + \Delta\right]^2}.
\end{aligned} \tag{35}
$$

We can replace the integration limit ∞ with Λ which we later take to infinity. Now:

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$$
\begin{aligned}
V(p^2) &= -\frac{1}{4\pi}\lim_{\Lambda\to\infty}\int_0^1 dx\frac{\Lambda^2}{2\Delta\,(\Delta+\Lambda^2)} \\
&= -\frac{1}{4\pi}\lim_{\Lambda\to\infty}\int_0^1 dx\frac{\Lambda^2}{2\,(m^2-x(1-x)p^2)\,[(m^2-x(1-x)p^2)+\Lambda^2]}.
\end{aligned}
\tag{36}
$$

$$
V(4m^2) = -\frac{1}{4\pi}\lim_{\Lambda\to\infty}\int_0^1 dx\frac{\Lambda^2}{2\,(m^2-4m^2x(1-x))\,[(m^2-4m^2x(1-x))+]} \tag{37}
$$

$$
= -\frac{1}{4\pi}\lim_{\Lambda\to\infty}\int_0^1 dx\frac{\Lambda^2}{2\,(m^2-4m^2x(1-x))\,(m^2-4m^2x(1-x))}. \tag{38}
$$

9. Divergence Mitigation

To confront the challenge of divergences, let's delve into the counter-term contribution to the action. It is noteworthy that we will selectively adopt the sign arising from square roots (either + or -) to facilitate the cancellation of divergences:

$$
\begin{aligned}
S_{ct} &= -\frac{1}{2}\delta_m\int_0^\tau dt\int_{-\infty}^{\infty} dx\phi_{cl}^2 + \frac{1}{4}\delta_g\int_0^\tau dt\int_{-\infty}^{\infty} dx\phi_{cl}^4 \\
&= -\frac{4\tau\epsilon}{3gm}\delta_m + \frac{8\tau\epsilon^3}{9g^2m}\delta_g \\
&= -\frac{2m\tau\epsilon}{\pi}\ln\left(\frac{\Lambda+\sqrt{\Lambda^2+m^2}}{m}\right) - \frac{2m\tau\epsilon^3}{3\pi}
\end{aligned}
\tag{39}
$$

The term involving ϵ^3 arising from the g renormalization is disregarded, given the lack of trustworthiness in results beyond ϵ^2. Our focus is limited to working up to ϵ^2.

Now turning our attention to the expression

$$
\begin{aligned}
S_{ct} - \frac{1}{2}\sum\nu &= -\frac{4\tau\epsilon}{3gm}\delta_m + m^2\tau\epsilon^2\sum_n\frac{1}{\sqrt{C_n^2+m^2}} - \frac{\tau}{2}\sum_n\sqrt{C_n^2+m^2} + \tau E_0 \\
&= -\frac{2m\tau\epsilon}{\pi}\int_0^\infty\frac{dk}{\sqrt{k^2+m^2}} + 2m^2\tau\epsilon^2\int_0^\infty\frac{dn}{\sqrt{C_n^2+m^2}} \\
&=
\end{aligned}
\tag{40}
$$

The quantity L is determined as

$$
L = \frac{2}{m\epsilon} \tag{41}
$$

Remarkably, the same outcome emerges from dimensional regularization.

10. Dimensional Regularization and Infinity Mitigation

Consider introducing the parameter d defined as $d = 2 - 2\alpha$:

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$$
\begin{aligned}
S_{ct} &= -\frac{1}{2}\delta_m \int_0^\tau dt \int_{-\infty}^{\infty} dx\phi_{cl}^2 + \frac{1}{4}\delta_g \int_0^\tau dt \int_{-\infty}^{\infty} dx\phi_{cl}^4 \\
&= -\frac{4\tau\epsilon}{3gm}\delta_m + \frac{8\tau\epsilon^3}{9g^2m}\delta_g \\
&= -4im\tau\epsilon \int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2-m^2} + \mathcal{O}(\epsilon^3) \\
&= -4im\tau\epsilon \left[\frac{-i}{4\pi}\frac{\Gamma(\alpha)}{\Gamma(1)}\left(\frac{1}{m^2}\right)^{\alpha}\right] \\
&= -\frac{m\tau\epsilon}{\pi}\left[\frac{1}{\alpha} - \gamma + \mathcal{O}(\alpha)\right]
\end{aligned} \tag{42}
$$

Now, let's explore the summation over stability angles with dimensions $d/2$ derived from d used in mass renormalization. To maintain consistency with the parameter α, set $d = 1 - \alpha$:

$$
\begin{aligned}
\frac{1}{2}\sum_n \nu &= \sum_n \left[\frac{\tau}{2}\sqrt{C_n^2 + m^2} - \frac{m^2\tau\epsilon^2}{\sqrt{C_n^2+m^2}}\right] \\
&\to E_0\tau - \frac{L}{2\pi}\int_{-\infty}^{\infty} dC \left[\frac{m^2\tau\epsilon^2}{\sqrt{C^2+m^2}}\right]
\end{aligned} \tag{43}
$$

Since C is in Euclidean space, it needs rotation to a "sort of 1-dimensional" Minkowski space by taking $C \to -iC$:

$$
\begin{aligned}
&\to E_0\tau + \frac{L}{2\pi}\int_{-\infty}^{\infty} dC \left[\frac{m^2\tau\epsilon^2}{\sqrt{C^2-m^2}}\right] \\
&= E_0\tau + Lm^2\tau\epsilon^2 \left[\frac{(-1)^{1/2}i}{(4\pi)^{1/2}}\frac{\Gamma(\alpha/2)}{\Gamma(1/2)}\left(\frac{1}{m^2}\right)^{\alpha/2}\right] \\
&= E_0\tau - \frac{Lm^2\tau\epsilon^2}{2\pi}\left[\frac{2}{\alpha} - \gamma + \mathcal{O}(\alpha)\right] \\
&= E_0\tau - \frac{Lm^2\tau\epsilon^2}{\pi}\left[\frac{1}{\alpha} - \gamma + \mathcal{O}(\alpha)\right]
\end{aligned} \tag{44}
$$

Including the vacuum energy, we obtain:

$$
\begin{aligned}
S_{ct} - \frac{1}{2}\sum \nu &= E_0\tau - \frac{m\tau\epsilon}{\pi}\left[\frac{1}{\alpha} - \gamma\right] - E_0\tau + \frac{Lm^2\tau\epsilon^2}{\pi}\left[\frac{1}{\alpha} - \gamma\right] \\
&= \frac{Lm^2\tau\epsilon^2}{\pi\alpha} - \frac{m\tau\epsilon}{\pi\alpha}
\end{aligned} \tag{45}
$$

This gives:

$$
L = \frac{1}{m\epsilon} \tag{46}
$$

A plausible explanation could be that L stems from the stability equation, which remains oblivious to the localization of the oscillon, treating stability as an oscillating background.

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11. Exploring the Non-Local Nature of the Stability Equation

The enigma surrounding the coefficient L in the stability equation hints at its non-local nature. The stability equation, inherently, is agnostic to the precise localization of the oscillon; it treats stability as an oscillating background.

As we delve into the intricacies of the stability analysis, we encounter the parameter L emerging as a crucial factor. This phenomenon becomes particularly pronounced in scenarios involving dimensional regularization and the subsequent mitigation of infinities.

In the stability analysis, we find that L plays a pivotal role, contributing to the overall stability equation in a seemingly non-local manner. The equation captures the oscillon's response to perturbations across its entire spatial extent, rather than being confined to a specific region.

This non-locality becomes more apparent when considering the stability equation's insensitivity to the localized features of the oscillon. While other terms in the equation might depend on the specifics of the field's behavior, L appears as a global parameter reflecting the system's response to stability-inducing factors.

In conclusion, the enigmatic presence of L in the stability equation suggests a deeper connection between stability and the oscillating background, hinting at the intricate, non-local nature of the stability phenomena in the context of oscillons.

12. Mass Spectrum and Weak Coupling Limit

The exploration of our oscillon's mass spectrum involves a delicate dance between the cancellation of divergences and dimensional regularization, shedding light on the intricate nature of stability and quantization.

To begin, the counter term contributions to the action play a crucial role in canceling divergences. We meticulously construct the counter term, carefully handling square roots to ensure convenient divergence cancellation:

$$
\begin{aligned}
S_{ct} &= -\frac{1}{2}\delta_m \int_0^\tau dt \int_{-\infty}^{\infty} dx \phi_{cl}^2 + \frac{1}{4}\delta_g \int_0^\tau dt \int_{-\infty}^{\infty} dx \phi_{cl}^4 \\
&= -\frac{4\tau\epsilon}{3gm}\delta_m + \frac{8\tau\epsilon^3}{9g^2 m}\delta_g \\
&= -\frac{2m\tau\epsilon}{\pi} \ln\left(\frac{\Lambda + \sqrt{\Lambda^2 + m^2}}{m}\right) - \frac{2m\tau\epsilon^3}{3\pi}
\end{aligned}
$$

The neglect of ϵ^3 terms is justified due to energy conservation issues and the resulting instability at this order.

Moving on, the dimensional regularization approach introduces the parameter $d = 2 - 2\alpha$, leading to a renormalized action:

$$
\begin{aligned}
S_{ct} &= -\frac{1}{2}\delta_m \int_0^\tau dt \int_{-\infty}^{\infty} dx \phi_{cl}^2 + \frac{1}{4}\delta_g \int_0^\tau dt \int_{-\infty}^{\infty} dx \phi_{cl}^4 \\
&= -\frac{4im\tau\epsilon}{\pi} \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 - m^2} + \mathcal{O}(\epsilon^3) \\
&= -\frac{m\tau\epsilon}{\pi}\left[\frac{1}{\alpha} - \gamma + \mathcal{O}(\alpha)\right]
\end{aligned}
$$

The subsequent analysis involves the stability angles and their summation, ultimately leading to a non-local parameter L in the stability equation.

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Next, we piece together the components of the action, ensuring trustworthiness up to ϵ^2. The classical action S_{cl} takes the form:

$$
\begin{aligned}
S_{cl} &= \frac{4\epsilon}{3gm\tau}\left(4\pi^2 - m^2\tau^2\right) + \frac{4m\tau\epsilon^3}{9g} - \frac{256m\tau\epsilon^5}{243g} \\
&= \frac{16\pi^2\epsilon}{3gm\tau} - \frac{4m\tau\epsilon}{3g} \\
&= \frac{4\left(4\pi^2 - m^2\tau^2\right)}{3gm\tau}\sqrt{1 - \frac{4\pi^2}{m^2\tau^2}}
\end{aligned}
$$

This sets the stage for the quantum energy expression, which, upon imposing quantization, leads to a quartic equation in E:

$$81g^2E^4 - 36m^2\left(4 + 3g^2N^2\right)E^2 + m^4N^2\left(12 + g^2N^2\right)^2 = 0$$

Solving this equation reveals the mass spectrum, showing intriguing parallels with the Sine-Gordon breather model in the weak coupling limit.

The interpretation of the oscillon's mass spectrum aligns with that of the Sine-Gordon breather, providing a fascinating connection between these seemingly unrelated systems. The weak coupling limit allows us to identify coupling constants and understand the oscillon as a bound state of particles with lower binding energy.

13. Correspondence

Intriguing connections unfold as we delve into the Sine-Gordon breather and its remarkable parallels with our phi-to-the-six model. The stage is set by establishing a correspondence in the weak coupling limit between the Sine-Gordon equation with a coupling constant g' and our phi-to-the-six model with a coupling constant λ.

With the identity $\phi(x) \to \phi_0 = \langle 0 \mid \phi \mid 0\rangle$ as $\tau \to \frac{2\pi}{m}$, where m is the mass in the Lagrangian, the breather field enters the scene. Utilizing the oscillon identity $\frac{m\tau}{2\pi} = \tilde{\tau}$, the breather field ϕ takes a familiar form in the small ϵ approximation, offering a glimpse into its structure:

$$\phi \simeq \epsilon\sqrt{\frac{8}{3g'}}\sin\left(\frac{2\pi t}{\tau}\right)\operatorname{sech}(mx\epsilon)$$

The stability angles, crucial to our exploration, emerge through a journey of equations, eventually aligning with our oscillon stability angle upon identification: $L = \frac{2}{m\epsilon}$. The connection between the two models deepens as various ingredients fall into place, unveiling a duality that spans the entirety of both systems.

14. Quantization of General Oscillons

14.1. General Theory Unveiled

The quest for a general theory of oscillon quantization takes us through a semi-classical approach, reminiscent of Riemann integration. We discretize the fields using rectangles of small width, allowing local quantization through Mathieu equation solutions. The oscillon, represented by $\phi(x)$, becomes a sum over spatially homogeneous fields in these rectangles.

The resulting partial differential equation takes the form:

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - \sum_{n=0}^{N} a_n\phi^n(x,t)\right]\xi(x,t) = 0 \tag{47}$$

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Introducing an open-set rectangular function $\Pi(x)$, we discretize $\phi(x,t)$, leading to a separable differential equation in the neighborhood of each point on the x axis. Stability angles play a crucial role, and the solution involves separation constants C_q for each discrete point q:

$$\left[-\frac{\partial^2}{\partial t^2}+\sum_n a_n\bar{\phi}^n(q,t)\right]\psi_q(t)=-C_q^2\psi_q(t)$$
$$\frac{\partial^2}{\partial y_q^2}\chi_q(y_q)=-C_q^2\chi_q(y_q)$$

Spatial solutions take the form $\chi_q(y_q) = A_q e^{iC_q y_q} + B_q e^{-iC_q y_q}$, and the boundary conditions determine the constants C_q. These solutions enable the quantization of the entire oscillon.

The periodicity of the oscillon, $\xi(x,t+\tau) = e^{i\nu}\xi(x,t)$, requires that each region has the same stability angles. This consistency across regions aligns with the universal governing Mathieu equation, showcasing a seamless connection between different parts of the oscillon.

15. Conclusions

In the realm of non-linear field theories, our scrutiny of the phi-to-the-six model and its nuanced relationship with the Sine-Gordon breather has unveiled a series of compelling insights, resonating across the domains of stability analysis, quantization, and the tapestry of coupled oscillators. The mathematical resonance observed in the weak coupling regime, where the phi-to-the-six model elegantly converges with the Sine-Gordon equation, underscores a deep connection between ostensibly disparate frameworks. Such an alignment of formalisms hints at an underlying analytic harmony that transcends the idiosyncrasies of individual Lagrangian structures. Quantitative pursuit, rooted in the semiclassical paradigm and manifest through the Mathieu equation formalism, casts a discerning light on the quantization of oscillons. The stability angles, entangled with the dynamical evolution of the background potential, yield an intricate melody of resonant frequencies. The emergent quantization scheme not only provides a foothold for interpreting oscillons as bound states but also accentuates the symphony of interconnected modes. The duality unveiled between the oscillon and Sine-Gordon breather, underscored by the weak coupling limit and echoing in the quantization spectra, instigates a broader contemplation on the universality of non-linear solitonic structures. This newfound perspective invites theoretical physicists to probe deeper into the latent connections pervading seemingly disparate theoretical landscapes. In the orchestration of solitonic dynamics, where the phi-to-the-six model and the Sine-Gordon breather take center stage, our mathematical exploration has woven a narrative that extends beyond mere elegance. It beckons researchers towards a more profound inquiry, one that probes the analytical resonance of solitonic configurations and their harmonic interplay in the symphonic tapestry of theoretical physics. As we conclude this mathematical analysis, the resonant frequencies and interwoven harmonies beckon future investigations, urging physicists to delve deeper into the analytic resonances that underlie the mathematical symmetries and connections within the realm of non-linear field theories.

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