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Does Time Smoothen Space? Implications for Space-Time Representation

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Abstract: The continuous nature of space and time is a fundamental tenet of many scientific endeavours. That digital representation imposes granularity is well recognized but whether it is possible to address space completely remains unanswered. Part 1 argues that Hales' proof of Keppler's conjecture on the packing of hard spheres suggests the answer to be 'no', providing examples of why this matters in GIS generally and spatio-temporal GIS in particular. Part 2 seeks to resolve the dichotomy between continuous and granular space, showing how a continuous space may be emergent over a random graph. However, its projection into 3D/4D imposes granularity. Perhaps surprisingly, representing space and time as locally conjugate may be key to addressing a 'smooth' spatial continuum. This insight leads to the suggestion of Face Centered Cubic Packing as a space-time topology but also raises further questions for spatio-temporal representation.

Keywords: Discreteness, Topology, Space-Time Representation, Epistemology, Face Centered Cubic Packing.

1 Part 1 : Spatio-temporal granularity and sphere packing

Introduction to Part 1

The continuous nature of space and time is a fundamental tenet of many scientific endeavours as the background against which change is observed and so functional relationships understood. But it is a tenet which has long been challenged. Zeno's 'Race Course' paradox [82] suggests even Achilles could not catch up with a tortoise, or indeed reach the finishing, line for there will always be a 'half way' to be first passed ('ad inifintum'). It would seem no journey can be completed unless there is some minimum size of step i.e. a grain:

"Granularity is closely related, but not identical to imprecision. Granularity refers to the existence of clumps or grains in information, in the sense that individual elements in the grain cannot be distinguished or discerned apart." Duckham et al. [26]

Granularity in geodata may seem to pertain to scales far above paradoxes arising from infinitesimals. Yet, even the spatio-temporal precision of GPS is rarely matched by data resolution, e.g. a ship's location might be measured to a precision of 10m and timestamped to a second, but only every 6 minutes [112]. Similar sampling frequency issues are found in e.g. ecology and social sciences [22][131] and have practical relevance to GIS since spatial analyses often operate close to the data-grain or at multiple granularities [63].

Section 1.1 provides a brief background on some core ideas in mereology and how these underpin common principles in GIS. Section 1.2 provides several examples of common analytical operations where granularity can impact decisions, providing context for how Keppler's Conjecture on sphere packing may be relevant to spatial analyses (Section 1.3). Section 1.4 extends the issue to temporal aspects with a brief review of space-time representation methods. Section 1.5 provides some interim conclusions.

1.1 Background

Euclidean space is the prototypical geometry which seems closest to our perception of space as a continuum. Roy and Stell [96] argue that continuous space is *"convenient from a theoretical viewpoint, but has deficiencies when applied to real data"* since "*Data collected on geographical regions is by its very nature discrete—there will either be error bounds or an imposed granularity"* [96]. A continuum is foundational to calculus, yet Penrose argues for formalisms of calculus which do not require *"the existence of more points in the universe than there are rational numbers"* [87].

1.1.1 Parts and Wholes (and Holes)

If Euclid's Elements is the quintessential early text to reference ideas of continuous space, Camposampiero [11] cites from Christian Wolf (1679–1754) that Logistica (by Barlaam of Seminara 1290-1350) was the inspiration for Wolfs early work on mereology (the study of parts and wholes) within his work Ontologia [122]. Of particular relevance being Wolf's observation that: "*An actual part is a part that is contained in its own boundaries. A possible part, on the other hand, is a part whose boundaries can be arbitrarily assigned*" (Wolf [122], see Camposampiero [11]) i.e. the distinction between ontologies of naturally defined objects and Modifiable Areal Units. In a "Monist" [100] proposition, a spatial continuum 'is' and so can be arbitrarily divided by some unitary measure to any precision. Wolf argued for the opposite direction, *"treating relations between numbers as mereological relations"* [11] i.e. scale results from the relations between units, rather than units being merely divisions on a continuum. In a continuum topology between numbers is merely implicit while in Wolf's view topology is explicit and defining. Galton [128] clarifies the distinction for spatial (fields/objects) and temporal (fluents/events) contexts.

Wolf recognized the contemporary influence of ideas from Leibniz in his Ontologia [11]. In addition to proposing "infinitessimals" Leibniz also defined both the idea of the 'empty set' and of topological 'packing' both of which will be central to this paper. Euler also cited Leibniz, in his work on the sequencing of numbers as influential to what is today called graph theory, Euler of course being widely credited as originating the core ideas of topology [62]. The empty set is what is later here referred to as 'interstitial space' when discussing sphere packing, and topological invariance the necessary similarity between random graphs [41] applied in Part 2 as a way to find fundamental smoothness within a granular model of spatio-temporal relations.

Early works in GIS [17; 29; 60] cite "Elementary Concepts of Topology" by Alexandroff [2] as a key reference to set topology, and Shortridge and Goodchild cite Klain and Rota [67] and Solomon [105], as work in computational geometry on which they build [101]. While avoiding the term Monist, Shortridge and Goodchild state that "*Mapping is a divide-and-conquer activity.*

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Establishing the region to be mapped, identifying the spatial resolution at which phenomena will be measured, and effectively storing and retrieving information all involve partitioning space." [101]. Thus maps are "sets" of objects which are topologically distinct. This gives rise to the concept of "intersect" based topology since these sets change when elements of a set (e.g. its interior or boundary) intersect. From such basic building blocks one could then algorithmically analyse topological relations. The challenge was to find the minimum definition of such a set [32].

How to deal with holes was first addressed by Egenhofer et al. [32] based on a four intersect model but it only dealt with holes interior to the boundaries of regions. The 9-intersect model added relations between the complements of objects [33]. The complement of an object is that which surrounds it. That might be well defined or vaguely bounded [18] but must exist else to what would a boundary of an object partition its interior from? In addition to being a defacto standard topological model in most GI Systems, the 9-intersect model marked an important completion of representation by including the empty set.

The nine intersect model was implemented primarily on vector object space. Winter and Frank [121] applied it to raster data, creating an implicit lattice of cell boundaries (lattices having a topological complement [117]) arguing this could allow common operations regardless of data format [121] which could remove some sources of uncertainty [53]. Arguably the most important distinction between raster and vector formats is that raster granularity imposes a hard, non-isometric, limit on geometric precision. Potentially this limit has functional implications, such as for river connectivity in flow modelling [74].

The effect of data grain is less obvious in vector format. It typically comprises three sources of positional uncertainty; measurement precision, coordinate precision and computational precision [51]. However, metadata generally refers to a single 'mean error' [51] for measurements *on* the coordinate grid, while sampling over time produces uncertainty *between* grids [24][77][83] including from GPS epochs[1]. Determining inter-year change presents what Shortridge and Goodchild identify as a "Laplace extension of the Buffon problem" [101]. Furthermore, computers manage coordinates as arrays [124] indexed by finite numeric types [92; 37], yet operations may intersect the grain diagonally [124] exceeding available precision.

Another important consideration in array representation is whether a vector of change is "diagonalisable" i.e. can be rotated such that any eigenvector is exactly representable [103] and thus the function is frame independent [114].

"When we state that observables are not pure numbers but operators, and the observed values of these observables are the eigenvalues of these operators, that alone is sufficient to ensure that two operators which do not commute must be related by an uncertainty principle." [103]

Coordinates are effectively operators of displacement from an origin, with eigenvectors between. Usually analytical operations on spatial data are invertible [49] but not necessarily

[1] See http://www.nbmg.unr.edu/staff/pdfs/Blewitt_Encyclopedia_of_Geodesy.html (Accessed 15-07-2021)

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commutative. There are cases where variables are complementary, such as height, slope and aspect [108] or depth and roughness [91]. For a partition with zero length, concepts such as slope or roughness are meaningless in discrete space, so to measure them one must impose a finite grain. With spatial data there is generally the option to increase precision as necessary but when granular precision matters the principle applies[2].

1.1.2 Granularity and uncertainty in common GIS operations

1.1.2.1 Intersect and Overlay

Shortridge and Goodchild [101] estimated the probability of geometric intersections between objects and tilings. In particular, the probability that a circle of given radius will intersect with gridded points. A well known problem in rasterization is that small objects may "disappear" if no such intersect occurs. Figures 7a and 8 from that paper show how this probability varies with the resolution-object size ratio:

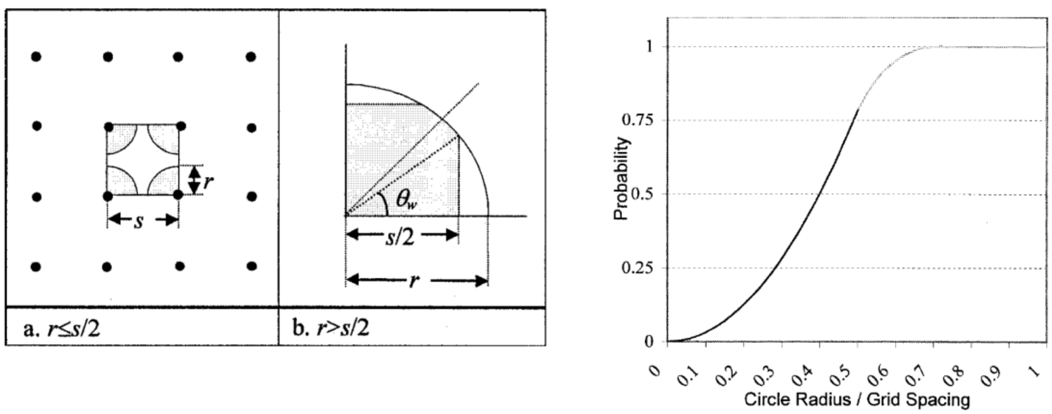

Figure 1 : Figure 7a (left) and Figure 8 (right) from Shortridge and Goodchild [101] (Reproduced by permission, licence 5104250425992)

Shortridge and Goodchild [101] show that the probability of intersection follows an S curve, with increasing slope up until about 0.5, thereafter the curve flattens out (in faint tone to indicate there is also overlap between the circles). The authors point out that "*the probability that a polygon is missed during conversion to raster is clearly a function of raster cell size and polygon shape*

[2] The Heisenberg Uncertainty Principle is often misconstrued as a property of, or an observation effect on, fundamental particles, but is in fact a result of the non-diagonalisable geometry of space time (see Siddharthan [103]). Heisenberg's genius was to recognise the relationship between this principle and the spatial discernibility of particles as Fourier wavelets (the smaller the spatial envelope, the less certain are derivatives across it such as momentum). So it is interesting to note that early work in GIS on partitioning cast the problem of appropriate statistical units in terms of discernibility of Fourier waves [20, 14; see 114], and that more recently the principle has also been shown for spectral decomposition of graphs [1].

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and area." [101]. It leaves open the question as to the appropriate raster representation when the circles overlap?

1.1.2.2 Visibility Analysis

Line of site intersection occurs in a projected space with occlusion affected by view orientation and angular precision [98], thus it cannot be validated in field or via test figures [37]. Angular precision matters in both raster and vector methods [19; 118]. Consequently, information about uncertainty in a viewshed must be generated following Hidden Surface Removal which is challenging for analytical purposes [37][39][98]. While typical ranges of uncertainty in viewshed can be ascertained for a terrain model of typical topographies [69] the range of possible viewsheds is in principle unbounded since a minute gap in a surface can provide extensive visibility. Thus, despite improvements in speed [13], modelling uncertainty in visual analysis requires consideration of perceptual issues such as autocorrelation and scene coherence [79, 99]. The problem is then to distinguish between Wolfs [122] "actual" and "possible" parts. Most visual landscape classifications have modifiable partitions but there are also higher order topologies (e.g. Euler Zones, see [97, 99]) forming (locally) viewpoint invariant discreet partitions on the more detailed aspect graph [90], a process called graph granulation [109]. Visibility is thus one area where geographical scale effects, which are hard to statistically bound, can arise from granular effects at extremely small grain sizes.

1.1.2.3 Graph Partition and Granulation

"*Any spatial partition has a corresponding dual graph*" [109]. The common 'dissolve' operation simplifies polygon data by combining adjacent partitions with common attributes. Implicitly the nodes of these partitions are combined to a single node in the dual graph. Subgraphs of a network partition it into a coarse grained representation of the full graph [109]. How coarse and detailed graphs relate across scales is important for problems such as efficient route finding [113] particularly when the location of the traveller is imprecise [25].

1.1.2.4 Path Analysis

Graph granularity also impacts path analysis in discreet space. For example topology is implicit in the partial ordering of a square raster but Triangular Irregular Networks (TIN) usually represent topology explicitly (e.g. [16, 44]) with some data structures offering more complete representations (e.g. a Quad-Edge Delaunay TIN [54] [44]) making more directions of travel available to routing and flow algorithms. Finer orientations can be achieved by using a larger step radius [64] and multiple scales ([75] figure 1) but Goodchild [45] shows movement is only asymptotically Euclidean with smaller grain.

1.1.3 From Tilings to Lattices

Spatial overlay, intersect, visibility, flow and network operations are all vulnerable to even miniscule uncertainties due to the binary nature of the decisions to be made at a partition.

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Overlay and intersect have the advantage that while errors may compound and propagate during analysis [115] the decision remains local to the objects concerned. In analyses such as visibility, least cost path and flow accumulation the error will be "transmitted" elsewhere in a manner which is hard to trace. So amplification of error, by projection or accumulation, is problematic not only with positional uncertainty (due to conceptual vagueness [38] or observational imprecision [80]) but also with topological incompleteness[3] due to granularity.

Both 9-intersect and Delaunay TIN build conceptually from points by extension to lines and areas [16]. Yet "*no one has ever perceived a point, or will ever do so*" (Simons [104] quoted in Stell [110]) in which case is it a meaningful concept for constructing a measure such as position? The Region Connection Calculus (RCC) [110] is an interesting alternative model of tiling space because it both draws directly on mereology [18; 73; 93; 96] and can be derived from lattice algebra [110]. Lattices are an example of partial ordering [45], which shall prove essential in Part 2. Lattices can also represent vagueness which is "closely related to granularity" [109] since space within discrete grains at one precision may be divided by the grain at another.

1.1.3.1 Measurement Precision and Topological Vagueness

Returning to the question left from the example of Figure 1 above, what happens when the circles overlap? The figure shows the probability for a circular polygon to intersect a vertex in a square sampling lattice. It also shows a lower bound for when the circle might intersect more than one grid point. If the grid is instead considered as a lattice of addresses, a circle can be defined around each point giving a complement equivalent in radius to the measurement precision, while the lattice spacing defines coordinate resolution. The experiment can then be reversed. Does increasing the coordinate resolution improve the probability of locating a point on a lattice of circles asymptotically - as Shortridge and Goodchild [101] showed for locating a circle on a lattice of points? If as coordinate resolution exceeds measurement precision complements overlap the address would be vague, if the circles "squash" they are no longer isometric. To be unique and isometric they must be considered 'hard'. So as resolution tends towards infinitely high, and measurement precision toward an infinitely small point, what happens to the odds that a random point will fall onto a unique address?

1.2 Keppler Conjecture

The Keppler Conjecture states that it is impossible to pack hard spheres in 3D such that the total volume enclosed by the spheres exceeds more than c74% of the total volume of their collective convex hull, more formally:

"The packing density δ(Λ) of any sphere packing Λ in R3 does not exceed π √18 ≈ 0.74048." [65]

Hales [56] demonstrated this conjecture is probably correct. So the degree to which precision affects representational completeness may be contingent on scene coherence [79; 69] or spatial

[3] 'Uncertainty' one is unsure as to degree, 'incompleteness' one does not know at all.

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autocorrelation [48] of the phenomena, but space itself appears to not be fully addressable - at least not by a static lattice. However, few if any coordinate systems can be considered truly static.

1.3 Granularity in Space and Time

1.3.1 Space Time Representation

Geographical sciences have long been aware of the intimate connection between space and time [89]. However, this has remained a largely Newtonian conceptualization of a 3D space *with* time. Methods proposed for space-time representation include modelling spatio-temporal data as points and tracks [63], sequential stacks and semantic events [5] or as fields and surfaces, tessellated such as kinetic Voronoi diagrams [76] or discrete prisms [55], the most commonly applied of which is the space-time cube [68]. Peuquet [88] presents a thorough theoretical overview of the issues in spatio-temporal mapping, and some early modelling approaches, which remains remarkably current. Ohori et al. [81] provide a more recent review of spatio-temporal models in GIS. They identify several additional approaches to those mentioned (composite methods and object oriented models plus some conceptual and semantic models) but conclude that most consider time to be a distinct dimension onto which snapshots or events are to be mapped. Excepting the fuzzy space-time coordinates of Brimicombe [9] and 4D partitioning work of Erwig and Schneider [34] these approaches to modelling space-time are better described as 3D+1, since the time dimension is not topologically integrated[4]. Ohori et al. [81] go on to present a 4D model of space time based on topological vector objects, which they state could in principle also support vector-field data structures, and argue that 'true' 4D requires time be topologically integrated with space.

The above methods all start from a 'monist' assumption of an extant space over which partitioning events unfold. There is also a long standing approach whereby space-time is 'constructed' from discrete elements, beginning well before the famous discussions of Newton, Leibnitz and Clarke on the relationship between space, time and continuity. This approach will be taken in Part 2 but what properties need to be considered? As the aim is only a consideration of space-time as an information formalism, not its physical existence (see [57] for the distinction) or our human perceptions, it seems reasonable to focus narrowly on the peculiar properties of time [130] identified by Peuquet [88] as of relevance to GIS:

a. Time is directional (the arrow of time).
b. Chaotic behavior may amplify small uncertainties through time.
c. Relative space and time are subjective and neither exist independently.
d. Both space and time are generally conceived as continuous, yet for purposes of objective measurement they are conventionally broken into discrete units.

[4] At least not in 4D, prisms have been implemented with time as the third dimension either as part of the object or by extrusion

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e. Temporal data is often incomplete. Science has traditionally viewed incompleteness as something to be overcome rather than to be openly acknowledged or willingly incorporated into models as a basic characteristic.

(summarized from Peuquet [88])

It may be argued that topological integration of time is unduly onerous given the different quality and magnitude in grain of temporal and spatial coordinates. However, the properties listed above are intimately linked with granularity. Spatial uncertainty can become incompleteness for a temporal question e.g. is an object observed twice at the same grain stationary? As GIS spreads well beyond geographic science, the use cases for spatio-temporal mapping (e.g. space-time datamining and real-time maps) are expanding rapidly [48][5]. Cases may arise where the interplay between spatial precision and time start to emerge from the theoretical shadows [7; 27].

1.5 Conclusion to Part 1

An interesting implication of Hales' proof [56] is that absolute risk of a point not finding an isometrically precise address is constant with resolution. For Buffon's Needle type problems [123] there will always be the same probability that the needle point falls in unaddressed space, which is determined not by precision but grain packing. Spatial relation ontologies are a long standing "Grand Challenge" in GIS [47]. A notion of scale-constant incompleteness would carry implications for how the relationship between Tobler's First Law and Heterogeneity is understood [3], particularly for fractal phenomena [46], by placing an upper bound on the proportion of space that can be sampled independently.

Space-Time poses additional complexities, in particular non-commutativity, 'butterfly effects', set discontinuity and inter-period incompleteness, which together may produce arbitrarily large effects from granular imprecision. Some of these issues are not unique to spatio-temporal data but, as will be seen in Part 2, how the time dimension is represented can exacerbate or help resolve them. In particular, developments such as real time spatial services, egocentric mapping, immersive views and other uses of a 'Digital Earth' [126] put the distinction between relative position and absolute position into focus at large enough scales for spatio-temporal granularity effects to be significant.

[5] A succinct overview of the relationship between 'big data' and sphere packing is available from Cohn, H., Arnold Ross Lecture, 2014, https://www.ams.org/publicoutreach/students/mathgame/cohn-slides-2014-web.pdf

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2 Part 2 : Smooth space-time from granular elements

Introduction to part 2

Part 1 explained why sphere packing limits suggest that a continuum cannot be completely addressed. Part 2 seeks to resolve that paradox by creating a smooth[6] space from discreet topological elements (section 2.1) in order to examine the relationship between space and time mechanistically and elucidate principles for its representation[7]. Based on these principles, Face Centered Cubic Packing is then proposed as a practical approach to model, if not fully resolve, the issue of incompleteness in spatio-temporal representation (Section 2.2).

2.1 Time as relative dimension in space

Premises

The work of mathematician Roger Penrose introduces a number of important concepts and principles which are central to the later arguments pursued here :

1. A necessary constraint is *"to get rid of the continuum and build up physical* [.. read spatial..] *theory from discreteness." [86]*
2. One should *"concentrate only on things which, in fact, are discrete in existing theory and try and use them as primary concepts—then to build up other things using these discrete primary concepts as the basic building blocks. Continuous concepts could emerge in a limit, when we take more and more complicated systems."[87].*
3. *"The most obvious physical concept that one has to start with...and which is connected with the structure of space-time in a very intimate way, is in angular momentum." [86]*

However, despite the importance of Penrose's thinking to this work and some apparent parallels with problems in physics, like Penrose: *"the picture I want to give here is just a model... I certainly don't want to suggest that the universe 'is' this picture or anything like that."* [86].

2.1.1.1 Building dimensions

Consider two or more observations which are measurably distinct, i.e. a 'natural' topology of real numbers [116] (after Wolff [122]). To establish an order of change a

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[6] The term 'smooth' is used, rather than continuous to distinguish between points as lie differentials on a function and nodes approximating that function in the latent space of their graph e.g. as per. 'smooth infinitesimal analysis' https://plato.stanford.edu/entries/continuity/ .

[7] Readers unfamiliar with the approach and notation in section 2.1 may find Supplemental 1 helpful.

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third is needed (a triad[8] [40]) which may connect as a graph [66]; a random graph [41] ensures no implicit background.

2.1.2 Building a probabilistically continuous dimension from a graph

The "threshold of connectivity" of a random graph is the number of edges needed for there to be a strong probability all nodes in the graph are connected [41] e.g. for a binomial random graph G(n,p) at *p(n) = log n/n* [59]. The number of spanning trees of a connected graph is an invariant N^{n-2} as given by Cayley's formula [15] but every connected graph contains *at least one* spanning tree [23, Theorem 4.12]. It is thus *always* possible for such a connected graph to randomly form an ordered set [59] (e.g. a tuple, line or tree etc.) where every node has a unique matrix of graph distance to all other nodes. If such a graph describes diffusion of information throughout a network then the maximum 'time' (in number of moves) it could take for information to diffuse throughout the network is when the spanning tree is an Euler path. Simple paths between other nodes will therefore require some fraction of this maximum. The probability of a topological chain (a tuple) of length no greater than *i* to form a path between any two particular nodes is given by **equation 1**:

$$P_i = \frac{2}{n} + \sum_{v=2}^{i} \frac{1}{n_v}$$

P = probability of a path between two points being achieved in i random steps where v = v^{th} node added to the path; n = total nodes. i.e. the initial chance of a direct connection (P = 2/n), plus the sequential sum of the chances that each further node added (v) to the chain will be the destination, each such chance being 1/n.

In equation 1, each *i*th additional node on the path increases the chance of a complete path by 1/n since nodes already in the path are still available. In this way information moves over/through the graph by a node randomly swapping its position in the graph to create the path.

2.1.3 Topological rotation

Consider a simple graph where one node connects all the others directly, thus others have a graph distance to each other of 2. Given a constant metric per edge, these points can describe a circular arc in 2D, while one point sits at the center i.e. a wheel graph [85][9]. Abstracting this further (a circulant [1]) the concept of rotation in a topological

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[8] Not to be confused with Peuquet's triad of locations, times and objects [89].

[9] for a graph of 2D or higher a chord-less cycle provides an alternative homeomorphic to a sphere.

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sense is known as an 'orbit' of the node under 'group action' a subset of which 'fix' the node by returning the system to its original state [127]. So even without literal "spinning", the concept of total angular momentum[10] is relevant and the term will be used for convenience to denote the ability of a node to swap position. The total momentum *M* is thus the set of all possible orbits (here after one 'revolution' of the system[11]). The maximum relative 'speed' information may attain over such a graph is the shortest path forming an arc of an orbit. How 'fast' that arc might be transited, relative to some external clock, is irrelevant but cannot be infinitely fast since if the start is returned to instantly no change can be observed (contradicting the natural topology premise [116]). With the property of angular momentum paths may not be equally likely**:**

$$P_{ij} = \frac{2m_i m_j}{M} + \sum_{v=2}^{a} \frac{m_v m_a}{M}$$

P = probability of a path; i = start node; j = end node; v = next node added; a = the node completing the path to the destination, m = a node with momentum; M = total angular momentum of the system.

In Equation 2 the probability of a direct path (*P*) forming is for two nodes *i* and *j* connecting directly given their fraction of the total momentum of the system M. For a path of three or more nodes one must then add the sequential sum of the probability (in proportion to *M*) that each subsequent node added to the tuple will complete the path directly to m_j.

2.1.4 Conservation of momentum

The analogy may be extended to conservation of angular momentum by instead considering *m* as the amount of information flowing *through* nodes[12] to evenly distribute *M*, forming a signal across the graph [1]. A higher net momentum between nodes means more frequent ('briefer') interactions but also a higher probability of new connections thus locally faster transmission rate (i.e. clustering [70]). The relationship between each cluster and the full set is given by **equation 3**:

[10] As also suggested by Penrose "*we must use total angular momentum ..rather than the angular momentum in some preassigned direction*" [87].
[11] No agreed mathematical term was found for this, suggestions welcome.
[12] The idea of a fundamental unit being a process rather than an object goes back at least to Whitehead [120]. However there is no metaphysical implication intended here.

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$$\exists \mathcal{M}\ m_i^j \sim d \in P,\ M \sim D,\ n \subseteq N \quad \forall i, j \in n \in \mathcal{M}$$

i.*e. There exists a set of points M, such that for any sub graph n and any points i, j on n the angular momentum m_{ij} follows distribution d defined by probabilities P (from equation 2), and total angular momentum M follows distribution D, and each sub graph n belongs to a set of points N.*

The probability of distribution $m_j^i \sim d$ (here after *f*) existing on *D* will then be the Cauchy product (or the discrete convolution) of *f* and *M~D* (the probability distribution of M here after g) in **equation 4:**

$$(f \star g)[n_m] = \sum_{m=0}^{M} f[m]g[n_m - m]$$

*i.e. the probability of a distribution of momentum within subgraph f being formed across the parent set g on n discrete elements in common of momentum m respectively, is the discrete convolution f *g.*

In the simplest case *f* is just an Euler path over g formed at random as per equation 2. As *m* accumulates into *f* so too does the probability of a path existing between *i* and *j*. Implicit here is that membership of a subgraph is defined by connection, i.e. *f* is a consistently connected graph while *g* is not. The more the momentum concentrates into some subgraphs over others, the more the overall graph becomes structured and the passage of information over the N subgraphs takes on a structured number of steps i.e. some branches become probabilistically more distant relative to others.

2.1.5 A lattice of communication

The individual nodes in the graph have no spatial relationship (they are not embedded in space at all), but they constitute a network over which information travels in discreet steps. But the *relative* communication time between one node and all others is in principle continuous; any number of nodes may be transited en route, and these may have any relative angular momentum, so any fraction in communication time is possible. The distribution of momentum values possible is discrete (being divided over nodes in the graph) but the metric of communication time, being based on arbitrary ratios of frequencies, may include complex and irrational numbers [78, p.14] so the emergent dimension of "Lattice Communication Time" (LCT) is genuinely continuous. Each path between two nodes on the graph can thus be mapped as a vector between points in this continuous, latent, hyperspace[13].

[13] The spatial representation of diffusion of signals on a graph, which partitions it into regions, is known as Graph Spectral Decomposition, in which "the Laplacian encodes a notion of smoothness on a graph" [1]. For a good

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It is clear from Equation 2 that time *t* for path *ij* is a probabilistic fraction of *M* (the LCT domain). Similarly, the relative 'volume' of *f* to *g* is bounded by the proportion of *M* in *f* if both are connected. But *g* is not necessarily a connected graph. Consider a case where every node in *g* has equal momentum and connects to another node, or not, at random. Over enough iteration, the mean graph distance between all nodes would be equal. The only geometric projection for this is if all nodes are at the same (fuzzy) point. Branching into connected subgraphs represents *divergence* from such a low order state e.g. a branch may be a path to a leaf node which, though multi-step, is still more probable than the leaf node connecting directly to any other node in G **(Equation 5)**:

$$P_{ij} > P_{iv} \qquad \forall \mathbf{t}_{ij} = \frac{m_{ij}}{M} \in \mathcal{F}_{\mathbf{ij}} \in \mathbb{G} \qquad \forall v \in \mathbb{G} : v \notin \mathcal{F}_{\mathbf{ij}}$$

F is a vector space of the LCT of a subgraph on G containing a path between nodes ij with momentum m. v is any node in G not in F. $\mathbf{t}_{ij}$ *is the LCT vector for path ij of length equal to the proportion of the domain M constituted by* m_{ij}*. For all cases where this is so, the Probability P of the path ij is such that ij has a greater probability than iv.*

Equation 5 defines the condition at which a path P_{ij} becomes more probable than alternative paths from *i* to *v*. Figure 2 illustrates this but taking a node *i* which projects to the same point in LCT space as the rest of *G* and a path from this node to *j*, which is reached via some structured route therefrom. This divergence "stretches" G from its projected point-like form into a vector. Other branches are added to illustrate that if structured paths exist from *j* to other points, these could also project to vectors which, in this simple case, are resolvable in 2D (Figure 2).

introduction see, Knyanzev, B, Spectral Graph Convolution Explained and Implemented Step By Step. https://towardsdatascience.com/spectral-graph-convolution-explained-and-implemented-step-by-step-2e495b57f801 (Accessed 4/6/2021).

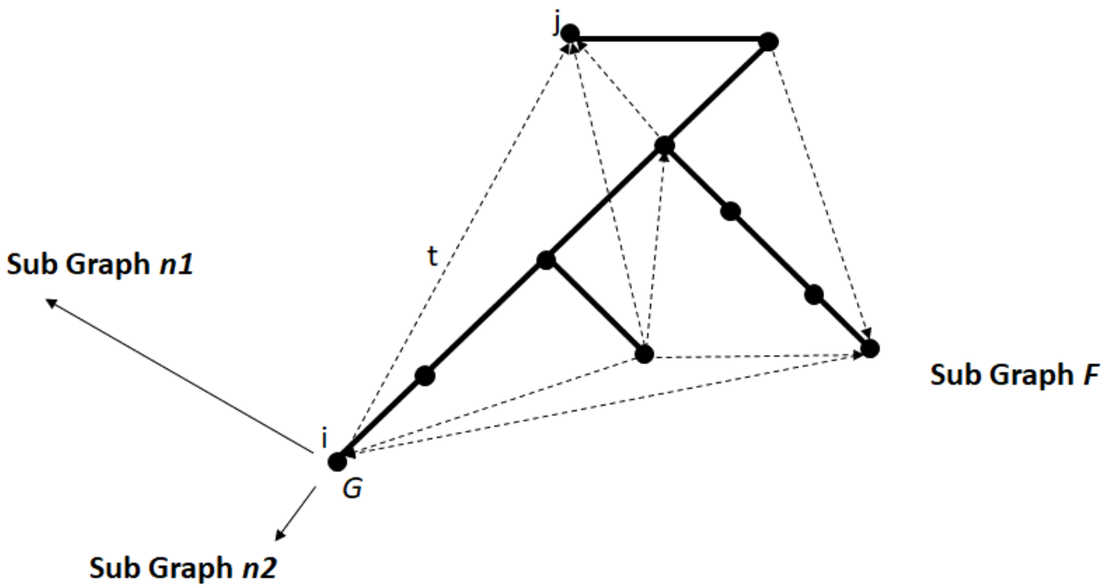

Figure 2 : Projecting the graph G into 2D. All nodes with equal probability of connecting project to point G. Nodes with higher probability of connecting to each other than to nodes in point G project to sub graphs with geometric position based on relative path length in units of fractions of total momentum of the system M.

Although relative speed is analogous to distance, G is a hyper-space with complex topology and *relative positions*, not a spatial volume such as $\mathbb{R}^n$, with *absolute locations*; but it has the potential to be similar. Consider the outcome that a single packet of information happens to take a route from one node in the graph to every other node through a single chain with no loops. G would be a 1D ordinal series of nodes with equal LCT time difference i.e. a scalar vector. If each node has differing angular momentum then sections of this vector would take different lengths of "time" to cross (per Equation 2). Either way a continuous spatial dimension [61] can emerge in the latent space [70] of a random graph. The LCT *field* is scalar, even if it may contain acute variations[14]. An emergent space with *independent* dimensions is more particular. Gawlik et al. [43] define such a field thus:

> *"Definition 2.1. (Discrete Zero-Forms.) A discrete zero-form (scalar field) is a column vector F ∈ $\mathbb{R}^n$. The components of such a vector F are regarded as the cell averages of a continuous scalar field f ∈ F(M), i.e. Fi = R Ci f dx / vol(Ci). The space of discrete zero-forms is denoted Ω 0 d (M)." [43].*

[14] If the points concerned were infinitesimal Lie differentials, such dilation would be a continuous diffeomorphism of the linear case. Gawlik et al [43] provides a table of correspondences between properties of continuous and discrete diffeomorphism groupings on continuous and discrete scalar fields.

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With complex topology the LCT will soon become irresolvable in one dimension so the next challenge is to add further dimensions while maintaining an isometric precision for the rational interval.

2.1.6 Building a probabilistically continuous volume

2.1.6.1 Dimensions

Additional dimensions result from the graph entering a state such that a new Linearly Independent Vector through the LCT is required. Carrell [12] defines an LIV thus:

> "*a set of vectors is linearly independent if no one of them can be expressed as a linear combination of the others*" [12, p. 116]

If a cycle exists with three or more nodes then each direction of rotation over that cycle represents different orderings of linear combinations of two independent vectors in any LCT metric space such that the sums are the same i.e. it needs two dimensions to represent it geometrically. Carrell (*ibid*) further states that:

> *"F_n can't contain more than n independent vectors. Our definition of dimension will in fact amount to saying that the dimension of an F-vector space V is the maximal number of independent vectors. This definition gives the right answer for the dimension of a line (one), a plane (two) and more generally F_n (n)." [12, p.121]*

To resolve the LCT into projected space, a new dimension must be added whenever a path cannot be completed by some combination of the existing independent vectors. There remains the question of the precision of that space in all *dependent* vectors through it **(Equation 6)**:

$$\mathbb{P} \equiv \sum_{V \in \mathbf{T}_{pn} \in \mathcal{F}} (V \oplus V)$$

*i.e. A manifold P is the direct sum of V independent vectors (each comprising a tuple **T** of n nodes with a probability p of existing (given by equation 2) over a discrete scalar field $\mathscr{F}$.*

2.1.6.2 Precision in projected space

Although discrete elements of each dimension are composed only of topological elements with non-oriented angular momentum, they now also have a probabilistic ordering in LCT. Each must occupy a discrete position in the LCT to some rational interval. If achieving this requires the LCT to be resolved into more than one dimension, then so too must that interval. Thus, the problem is not simply one of ordering nodes in one dimension, but of packing them in two or more dimensions and thus also packing their circular, spherical or hyper-spherical error margin.

Let $\mathbb{P}$ from Equation 6 be one instantiation of the metric space of the LCT and $\mathbf{T}_p$ a tuple on that space. Each element *e* of that tuple $\mathbf{T}_p$ has a spherical margin of error of radius *r* around it **(Equation 7)**:

$$e = \mathbf{B} \equiv \{x, y \in \mathbb{P} | d(x, y) \leq r\} \quad \forall e \in \mathbb{P}$$

i.e. each element e of the metric space $\mathbb{P}$ *is B, which has the dimensions dx, dy (being less than or equal to radius r).*

As per Keppler's conjecture (see part 1) no packing of **B** can fill space but this can be resolved by recognising that there is indeed the possibility for every *relative* position in LCT, but not every vector can be embedded to a common absolute precision in $\mathbb{R}^{\mathbf{n}}$ simultaneously. Thus, if the graph topology to resolve the LCT requires $\mathbb{R}^{\mathbf{n}}$ in a continuous space, realising this requires at least $\mathbb{R}^{\mathbf{n+1}}$. Another way of viewing this conclusion is to say that the lattice in $\mathbb{R}^{n}$ represents the most complete (probable) mapping between LCT and $\mathbb{R}^{n}$ achievable.

The most efficient 3D packing structure is hexagonal [28][15], such that each higher dimension places nodes at the center of the dual for the mesh in the dimension below it, which is also the center for the interstitial space. Expressed formally **(Equation 8)**:

$$(B_1^n \cap B_2^n \neq \emptyset) \rightarrow B_2 = V \equiv B_1^\star \in \mathbb{P}^{n+1}$$

i.e. *If the finite complement B of two elements intersect in n dimensional space, then* B_2 *must be an element of the* $B_{1,2}$ *vector dual V in the next dimension in* $\mathbb{P}$.

Some of the points in the projected map may have vague topology in 3D space alone, but it is explicitly recognized that only one of the possible nearest conditions to this may be defined at any one time. Any subset of the higher dimensional LCT may be 'foregrounded' [8] in the projected partition, so no vector is ever impossible, but others must be 'traced over' [8].

2.1.7 Stability of the LCT

Vectors in LCT are not created by the passing of a packet of momentum between two nodes, but by the ordered passing of that packet across many nodes. Relative LCT distance and position is the result of *net* relative connection speed across *all* nodes in

[15]There is some argument that finer granularity does not necessarily deliver higher information density [7] and that five dimensions may offer the most efficient generalization from a ball to a cubic space [58] Other options being 8 and 24 dimensions e.g. https://arxiv.org/abs/1603.06518 (Accessed12-07-2021). The question as to "necessary" dimensionality of physical space is long standing e.g. [119].

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the graph. Most probable is thus mapped as most direct, i.e. as per random walk theory [106, 22] (Figure 3):

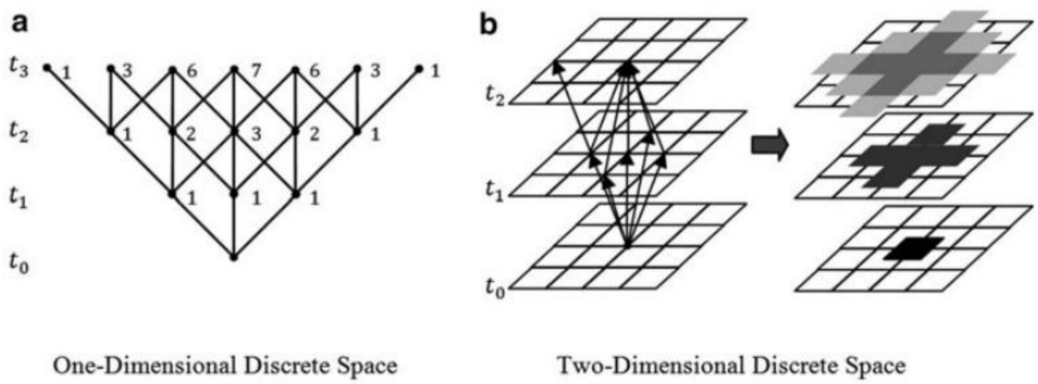

Fig. 12.5 Visit probability using random walk theory (Based on Winter and Yin 2010a)

(Figure 3: Figure 12.5 from Song and Miller [106] Visit probability using random walk theory, reproduced by permission license number 5104220298977)

Straight vectors in LCT represent the most probable path given the distribution of angular momentum, so following the principle of least action [125] Feyman's [35, 36] path integral formula can be applied:

Let **T** be a tuple with a Gaussian probability of describing a shortest path over the field $\mathcal{F}$ (from Equation 6) in n dimensions. Let V be a linear vector in $\mathcal{F}$ having the same start x_a and end point x_b as **T (Equation 9):**

$$\mathbf{P}_t(\complement(\mathbf{V}_x \cap \mathbf{T}_x) \neq \emptyset) = \mathbf{P}\left(\left(\left(\int_{\mathbf{x}(0)=\mathbf{x}}^{\mathbf{x}(\tau)=\mathbf{y}} \mathcal{D}\mathbf{x} \exp\left(-\int_0^t \frac{x^2}{2}\mathrm{d}t\right)\right) - \mathbf{V}\right) \neq \mathbf{0}\right) \leq (1/2)$$

i.e. *The probability* ***P*** *at an instance t, that the Complement* ∁ *to the intersect between the elements of the straight line V and the elements of the Tuple T is not empty (and thus the path taken was not linear) is equal to the probability that the path integral dt of the vector of a free particle moving from* x_a *to* x_b *over* **τ** *convolutions, minus the straight line vector V = [*x_a x_b*] does not equal zero given a potential set of paths* **Dxy**. ***P*** *must have a value equal to or less than 0.5 since for a Gaussian distribution a more direct (linear) route is more probable than an indirect route.*

The implication of Equation 9 is that a straight line be the most probable path over the LCT. Nodes which otherwise are further apart in $\mathbb{R}^3$ might occasionally find short cuts but, overall, long distances only resolve from the most probable paths (Figure 4).

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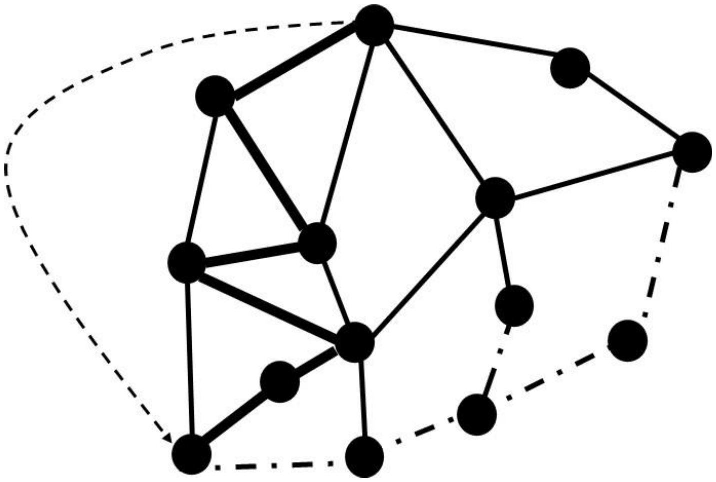

Figure 4 A graph projected into 2D space. Thicker and more solid lines represent higher momentum (shorter) paths in LCT showing how network centrality mitigates against links over large space-time distances.

A node is only likely to resolve into $\mathbb{R}^3$ when it provides an important link between many points in the LCT, rather than a 'tunnel' between just two points. It therefore seems that an extensive LCT, once formed, may exhibit incremental (rather than arbitrary) change to its large scale topology[16]. If only considering an instant the projected LCT is timeless and granular. Smoothness results from iterations reordering the packing topology. So to be strictly accurate it is "iteration" rather than time which smooths space. Thus 'over time' discrete space (d) becomes smoother (i.e. $3d^t \simeq 4D$). But this "topological time" is conjugate with the three spatial dimensions, it is not an independent "arrow" of time[17].

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[16] A reasonable analogy would be a pool of metal ball bearings magnetized to various strengths. Once the more magnetized have clustered, replacement of one bearing changes the field a little locally but not much globally.
[17] Rovelli shows the traditional sense of time as an independent dimension directing 'progression' could be emergent from partial ordering effects *at each spatial scale* [95]. Those looking for background on Rovelli's theory of emergent space time in the physical universe are recommended to first read the book "What is time? What is space?" [94] as a brief introduction.

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2.1.8 Section Summary

The core proposition of this section is that from a random graph, a latent field may emerge which is a spatial continuum in the sense of addressing any set of *relative* spatial relations to isometric precision but sphere packing imposes a grain when projected into *absolute* Euclidean location. This grain may be "smoothened" by iteration, resulting in the emergence of a fuzzy space-time continuum. Figure 5 summarizes these steps:

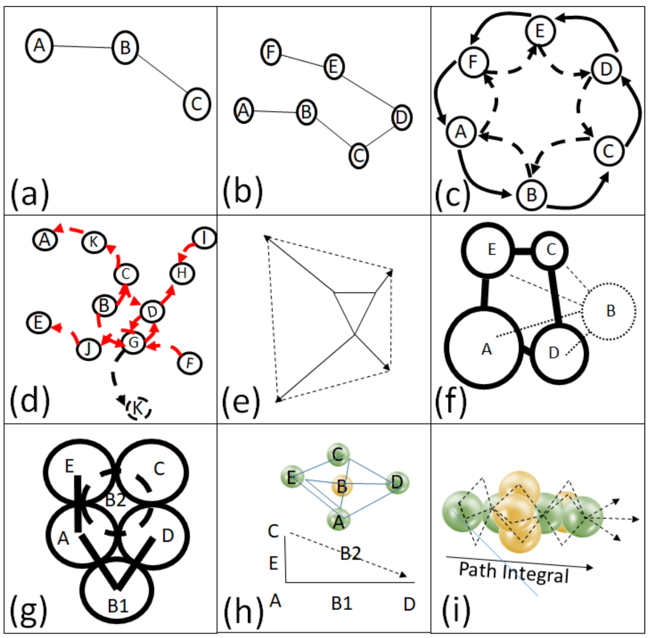

Figure 5 Steps for emergence of fuzzy space-time from a discreet graph : (a) a triad of three nodes; (b) a simple path chaining together triads; (c) an orbit (topological "clock"); (d) clustering of the graph; (e) latent vectors over the graph (f) uncertainty in relative position of nodes, represented geometrically as circular complement to the position (but not in projected space); (g) packing of isometric uncertainty complements in projected space (B1 and B2 represent solutions to paths ABD and EADCBAEBD respectively); (h) geometric projections of the graph (i) motion as a path integral through a hexagonal lattice, yellow sphere's represent possible positions for midway points between green spheres (i.e. the central node in each triad).

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2.2 Discussion

2.2.1 LCT as a spatio-temporal data structure?

The emergent LCT is smooth, its underlying graph is implicitly addressable and fuzzy coordinates are not new [9]. However, a fuzzily-emergent R^n has some obvious flaws as a spatial data-structure, not least that it is dynamic and a projection in R^n only remains smooth so long as it contains little data. There is no escaping entirely Keppler's conjecture on sphere packing. However, the journey to this conclusion is more important than the destination.

2.2.2 Lessons from the road

The process of attempting to build, rather than define, a Euclidean like space from discrete elements reveals mechanistically implications of the points **(a..e)** by Peuquet [88] in section 1.3:

- Space-time models often assume continuity but are measured in discrete units **(d,e)**
  Ideally, units are discreet but representation be still 'smoothly' space filling.
- Directional, non-commutative and incomplete **(a, c, e)** suggests a graph model.
  Ideally, Euclidean projections can represent these characteristics geometrically.
- Chaotic effects **(b)** may magnify granular uncertainties over time.
  Ideally, represent uncertainty isometrically.
- At the grain, time is locally conjugate with each spatial dimension **(c)**.
  Ideally, represent this conjugation explicitly.
- Time orders cause and effect **(a)** but this is scale dependent[18].
  Ideally, represent local uncertainty while respecting causality globally.

This thought experiment has also highlighted the logical paradox at the root of Penrose's discomfort with a 'pointless' continuum. Coordinate systems are crystalline approximations of space, of which the Cartesian lattice is only one, relatively incomplete, option. This raises two key questions:

i. Which lattice structure is most information dense and isomorphic in R^n?
   Implications being for e.g. cell class assignment [102].

ii. How can spatio-temporal conjugation be represented in a data-structure?
   Implications being for e.g. Buffam's Needles type problems [101] in 4D.

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[18] Galton et al. [131] observe that *"granularity effects may often confound an attempt to derive strict causation"*.

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2.2.3 FCCP for Space-Time representation?

The central issue to both question (i) and (ii) is how to best approximate the rational interval *r* and what to do with the interstitial space-time.

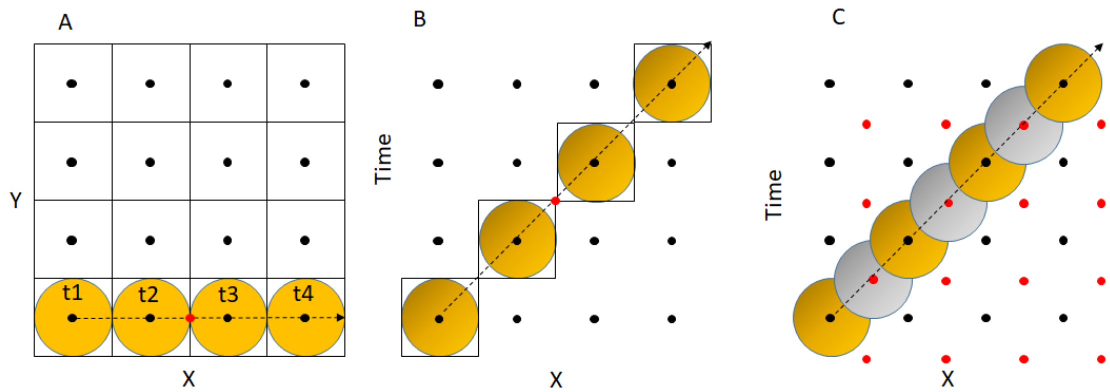

Figure 6 Representations of a trajectory in space time. 3A implicit time (e.g. animation). 3B one vertical side of a space-time cube. 3C FCCP, black dot in yellow circle = time step, red dot in grey circle = inter-time step.

If time is simply recorded as a non-geometric time stamp then uncertainty in the time at which the trajectory crosses from one unit of space to the next is not represented (red point in Figure 6A). In a space time prism (Figure 6B) rational units are maintained for change in space or time but not both. Geometrically this is no different to 2D raster tiling being non-isomorphic. However, in space-time this is arguably more fundamental because movement can *only* happen in both spatial and temporal dimensions *simultaneously* i.e. it will *always* be over non-axial vectors in space time (it is not diagonalizable).

Hexagonal data-structures [10] are more information dense than Cartesian coordinates but also less convenient to represent in a computer. Voronoi-Delaunay triangulation could also represent a hexagonal data structure [72] but computation for 4D would be hard [81]. A simpler option, easily applied to fields, is to use a Cartesian array but offset the origin for alternate layers (a Face Centered Cubic Packing, FCCP) and define the offset layers as representing spatio-temporal uncertainty between the non-offset layers and vice versa (Figure 6C). In effect, this interweaves two offset space-time cubes each of which provide topology for observations that are uncertain on the grain of the other, conceptually challenging but computationally trivial.

Langran suggests that all maps have a temporal reference, even if only an implicit 'now' [71]. The increasingly real time nature of data sources highlights that maps usually cover a period of collection, which has traditionally been allocated a publication date. Generalization of a period to a single time stamp is equivalent to an orthographic projection because all points in time project to the same plane, rather as height does to a 2D map. Ohori et al. [81] show how time can be represented by perspective projection also, which illustrates the important relationship between spatial and temporal scale; When considering analyses such as spatio-temporal

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clustering is 2 meters 2 seconds away closer than 1 meter 3 seconds? How equivalent are uncertainties in space and time for point-in-prism analyses [55] or object based view-shed analysis [98] where inter-visibility is dependent on temporal factors? If detecting motion in discrete space [42], should one assume a linear, constant, motion within each grain or consider a straight line to be the lower bound [50] on a probability function in position [7, 22]? All are examples where the spatial-to-temporal scale relationship might amplify uncertainty[19].

Peuquet suggests (in 2001 but the point remains valid) that the slow progress in addressing temporal aspects in GIS is partly because *"historically there has been a general emphasis on the short-term and implementation-oriented solution"* [88]. FCCP is suggested here simply to encourage discussion not asserted as *the* solution and whether a simple array or a 4D mesh would be an appropriate implementation depends on application. There are prior epistemic questions to be addressed such as; 'can incompleteness be represented topologically?'; 'can local conjugation of spatial and temporal uncertainty be represented geometrically?'; and 'what is the best scaling factor between space and time?'. Behind these questions are implications for how to model spatio-temporal smoothness.

3 Conclusions

3.1 On the representation of time

In calling for greater attention to development of a theory of space-time representation, Peuquet sets out four broad themes of which a few are highlighted as topics where this contributes:

Development of a theory of space-time representation
- *Particularly the interrelations between continuous and discrete space-time.*

Development of strategies for handling inexact and complex objects and relationships:
- *Particularly how do we represent fuzzy and inexact concepts in a computing context?*

Development of means to deal with missing data
- *Particularly what interpolation procedures would be appropriate to fill-in missing data?*

Development of a means to deal with multiple times and alternative histories
- *Particularly multiple times, as separate yet related pieces of information.*

Summarised from Peuquet [88]

Each of these issues has, of course, wider implications but examining them at a granular level ensures higher level theory is not built on 'Soritean sand':

Computable representation of space-time: Spatio-temporal modelling is fundamentally more exacting because space-time is non-commutative. Iterations and extrapolations can amplify

[19] Indeed increasing space-time resolution beyond observation period could introduce a form of coastline paradox since neighbouring points could be topologically further away in space-time than in space alone because the observation unit becomes too long to arrive at them in one step, (similar to how unit length impacts distance measurement https://en.wikipedia.org/wiki/Coastline_paradox..)

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issues through time. Fortunately, unlike particles [7], geographic phenomenon tend to have strongly positively auto-correlated error [51] so a suitable dual tessellation could bound this error by giving the 'incomplete part' its own topology.

How does the selection of an optimal grain size reflect the fact that there may be a non-linear uncertainty between a measurement made too early or too late, e.g. as to position of a shadow at 12 noon, which itself varies over seasons, scales and locations? In such a case, perhaps the flexible surface fitting properties of mesh would be merited while a lattice suffices when time scales are stable. Consider how spatio-temporal conjugation impacts the examples from Section 1.2.1. **Intersect** becomes intercept, *at the grain* precision in measuring speed must offset precision in estimating location. **Visibility Analysis** becomes not just a matter of where there is a line of sight but when and for how long (the effect of movement on attention is well known [6; 111]). **Graph granulation** is commonly applied to space-time applications, but doing so with vague location [25] means spatial and temporal uncertainty are not independent. Goodchild's [45] early work on **Path Analysis** must surely come into play but whether 4D lattice paths tend to the Euclidean length in the same asymptotical manner is an open question.

Incompleteness: That measurements only sample 'snapshots' overtime and space is, as Peuquet [88] points out, just a temporal extension of an accepted issue in spatial theory. But Hales proof [56] poses a deeper conundrum because it undermines the principle that absolute vs discrete space is a matter of differentiation; asymptotic approximation of Euclidean geometry [45] is pragmatically accepted but precision independent incompleteness needs further exploration.

Spatio-Temporal Scale(s): Although Peuquet agrees "it usually does not make sense to measure time in meters or feet." [88] the point has been moot since 1960 when the meter was defined in wave lengths of light. Perhaps the distinction between meters and the time by which they are measured, over-complicates space-time for 4D models? One would not, after all, model a terrain's height on a different mathematical base to its horizontal distance. On the other hand, seconds are not fundamental scientific units either, one might ask "seconds per what?" (Smart, see Harrison [57]). The term "related" in Tobler's First Law implies some common point of causation or interaction[20]. Galton points out that "At geographical scales.. change is mostly much slower than at the scale of individual humans" [128]. Could this be a more general, unified, interpretation of Tobler's first law; Near things are more *temporally* related than distant things? For example erosion of sand on a beach by a single rain storm and formation of a river delta by millennia of rainstorms shows how time scale becomes spatially embedded across spatial scales within a reference frame as to the speed of the phenomeno; time operates locally to the phenomena[21]. So identifying the correct reference frame [114] could be key to finding a

[20] The context in which Tobler first 'invoked' his first law being that of prediction based on history. [129]

[21] The concept of a "time cone" or "light cone" is well known from physics [95] where it is used to describe how multiple events have coalesced to create one specific condition at the present moment. One might then term this consequent spatial pattern a 'time shadow' from layers of past moments projecting onto our present (credit Robert Mann for the term). In this view, probabilistic, spatio-temporal change could be modelled as 'time-sheds' via multiple views linked by the 'shadows' cast from different viewpoints [98] with time as the 'z' axis of perspective projection [81].

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ratio by which to normalise spatial and temporal scale. For fractal phenomena, the fractal dimension seems like an obvious candidate for investigation. Spatial scientists have the advantage of not needing to determine 'plank units' which function for all cases but a sound theoretical basis for determining square spatio-temporal units suitable to a given dataset or phenomenon would be useful, as would theory for how, below these rational units, space and time become probabilistically conjugate.

3.2 On granularity and the emergence of spatial smoothness

Section 2.1 built on just two precepts. First, that 'values' are only meaningful if they identify difference relative to each other as part of an ordered set. Second, that the value measured is the ability to change position in that set. It is hoped that this satisfies Penrose's three principles set out at the start of part two. That the system should minimize difference in angular momentum could be argued to be introducing an axiom, but this is only adopting the widely used 'principle of least action' [125]. The paradox of a continuous volume, defined by relative position yet which cannot be filled by discrete positions may have a partial solution as presented here but this paper can only hope to add a little further weight of logical argument to the idea's long history, mathematical proof (if possible) is of huge complexity. Proof of the emergence of an LCT might be possible through simulation using random-pointer-networks, but whether this can in practice be calculated in reasonable time without setting pre-conditions such as would invalidate the experiment is an open question. Perhaps such a dynamic space would be interesting for modelling processes that are embedded in - not located on - space time, but its principle relevance to this paper is to clarify why a continuous space must be the product of a process and not a static condition. Progression through time happens in our minds when we observe [95]. Information space-time is not absolute but an analytical product.

This shift in perspective is also valuable if it encourages thinking beyond the Cartesian array to the idea of coordinates as data structures which crystallize an information space. It is interesting that this need not lead to abandonment of simple data structures such as arrays but there seems much to explore. Perhaps a-periodic tiling and quasi-crystals present options for avoiding periodic sample bias or better approximating Euclidean paths?

The most intriguing outcome from this work is that time may act as a 'smoothing' function on discrete space. Discussion within GIS on discreteness versus continuity generally rests epistemologically on Euclidean geometry and set theory. Yet, to draw the thread from Wolf's "possible parts" ([122],[11]) Bittner and Smith [8] observe set theory creates object discontinuity under membership change. FCCP offers a simple update to the space-time cube that 'sutures together' time slices via interstitial topology, reducing object discontinuity.

As one anonymous reviewer pointed out "it is often claimed that space and time cannot be separated in the context of geographical information.. but a proof is lacking". This paper provides a step toward that proof but there is work to be done. If as argued here, time can statistically modulate granular spatial uncertainty and it is generally agreed that "the relationship between the discrete space of computation and idealized continuous space is.. important for GIS" [96] perhaps the arrival in mainstream usage of real-time, immersive, spatio-temporal mapping (e.g. [107]) is an opportune moment to reexamine some key tenets as to their suitability for space-time representation?

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Ethics Statement

The author confirms there are no conflicts of interest in relation to this work.

Supplementary Materials

Supplementary 1, 'Time as relative dimension in space'.

Data Availability

No new data were created or analysed in this study. Data sharing is not applicable to this article.

3.2.1 References

[1] AGASKAR, A., LU, Y., A Spectral Graph Uncertainty Principle, *IEEE Transactions on Information Theory*, vol. 59, no. 7, pp. 4338-4356, July 2013, doi: 10.1109/TIT.2013.2252233.

[2] ARHANGEL'SKII, A., DRANISHNIKOV, A., 1997, P.S. Alexandroff and Topology: an introductory note , *Topology and its Applications,* 80, l-6 https://core.ac.uk/download/pdf/82183281.pdf

[3] ANSELIN, L., 1989, What Is Special about Spatial Data? Alternative Perspectives on Spatial Data Analysis. Santa Barbara, CA, National Center for Geographic Information and Analysis Technical Paper No 89-4

[4] COHN, A., BENNETT, B., GOODAY, J., GOTTS, N., 1997, Qualitative Spatial Representation and Reasoning with the Region Connection Calculus, *Geoinformatica*, 1, 1{44

[5] BEARD, K., Modelling Change in Space and Time: An Event Based Approach, In *Dynamic and Mobile GIS: Investigating Changes in Space and Time, Billen,R., Joao, E., Forrest, D., Eds. CRC Press, 2006, Ch 4, pp. 55-74 .*

[6] BISHOP, I.D., MILLER. D., 2007. Visual Assessment of Off-shore Wind Turbines: The Influence of Distance, Contrast, Movement, and Social Variables, *Renewable Energy* 32~5!:814–831

[7] BITTNER, T., 2019, Is There a Quantum Geography? In: Tambassi T. (eds) *The Philosophy of GIS*, Springer Geography, pp. 209-239

[8] BITTNER, T., SMITH, B., 2003, A theory of granular partitions. In: Duckham M, Goodchild, M., Worboys, M, 2003, *Foundations of geographic information science*. Taylor &

Francis, London, pp 117–151
https://philpapers.org/archive/BITATO-2.pdf

[9] BRIMICOMBE, A., 1998, A fuzzy co-ordinate system for locational uncertainty in space and time, In Carver, S., *Innovations in GIS 5*, Taylor & Francis, 1998, pp. 143-152

[10] BRIMKOV, V., BARNEVA, R., 2005, Analytical Honeycomb Geometry for Raster and Volume Graphics, *The Computer Journal*, 48 , 2, (2005) , 180-199.

[11] CAMPOSAMPIERO, M., (2019) Mereology and mathematics: Christian Wolff's foundational programme, *British Journal for the History of Philosophy*, 27:6, 1151-1172, DOI: 10.1080/09608788.2019.1568228

[12] CARRELL, J., 2003, Basic Concepts of Linear Algebra, http://www.math.ubc.ca/~jychen/m223/Book.pdf (Accessed 08-07-2021)

[13] CARVER, S., WASHTELL, J., 2012, Real-time visibility analysis and rapid viewshed calculation using a voxel-based modelling approach. *In Proceedings of the Twentieth Annual GIS Research UK Conference,* Lancaster, United Kingdom.

[14] CASETTI, E., 1966, "Analysis of Spatial Association by Trigonometric Polynomials", *Canadian Geographer*, 10:199-204

[15] CAYLEY, A., 2016, A theorem on trees, *Quart. J. Pure Appl. Math. 23, (1889), 376–378*

[16] GOLD,C., 2016, Tessellations in GIS: Part I—putting it all together, Geo-spatial Information Science, 19:1, 9-25, DOI: 10.1080/10095020.2016.1146440

[17] CLEMENTINI, E., DI FELICE, P., 1995, A comparison of methods for representing topological relationships, *Information Sciences - Applications*, Volume 3, Issue 3,1995, Pages 149-178, ISSN 1069-0115, https://doi.org/10.1016/1069-0115(94)00033-X

[18] COHN, A. G. AND GOTTS, N. M. 1994, A theory of spatial regions with indeterminate boundaries, in C. Eschenbach, C. Habel and B. Smith (eds), *Topological Foundations of Cognitive Science*. Hamburg: Graduiertenkolleg Kognitionswissenschaft.

[19] Coll, N., Fort, M., Madern, N., Sellarès, J., (2007) Multi-visibility maps of triangulated terrains, *International Journal of Geographical Information Science*, 21:10, 1115-1134, DOI: 10.1080/13658810701300097

[20] CURRY, L., 1966, "A Note on Spatial Association", *Professional Geographer*, 18:97-99.

[21] DARTNELL, L., 2020, Origins: How the Earth Shaped Human History, Penguine Random House, UK. ISBN 9781784705435.

[22] DEEPAK, E., KUJPERS, B., 2020, Visit Probability in Space-Time Prisms Based on Binomial Random Walk, *International Journal of Geo-Information*, *9*(9), 555; **https://doi.org/10.3390/ijgi9090555**

[23] DHARWADKER, A., SHARIEFUDDIN, P., 2011, Graph Theory, CreateSpace IPP, Indian Institute of Mathematics ISBN 1466254998

[24] DODD, M., 2011, Where are my quadrats? Positional accuracy in fieldwork, *Methods in Ecology and Evolution*, 2, 6, 567-584. https://doi.org/10.1111/j.2041-210X.2011.00118.x

[25] DUCKHAM, M., KULIK, L., WORBOYS, M., 2003, Imprecise Navigation, *Geoinformatica*, 7, 2, 79-94. http://duckham.org/matt/wp-content/uploads/2020/11/Imprecise-navigation.pdf

[26] DUCKHAM,M., MASON, K., STELL, J., WORBOYS, M., 2001 A formal approach to imperfection in geographic information, *Computers Environment and Urban Systems*, 25(1):89-103. DOI: 10.1016/S0198-9715(00)00040-5.

[27] DURAND, T., 2020, The Need for A Deep Space GPS, on line : https://www.spacelegalissues.com/the-need-for-a-deep-space-gps/ (Accessed 07-07-2021).

[28] EDELSBRUNNER, H., HAM, M., KURLIN, V., 2015, Relaxed Disk Packing, *CCCG 2015*,

Kingston, Ontario, August 10–12, 2015. https://arxiv.org/abs/1505.03402

[29] EGENHOFER M. J. FRANOZA, R.D., 1990, Point Set Topological Spatial Relations, *International Journal of Geographical Information Systems*, 5(2), 161-174.

[30] EGENHOFER,M., 1993, A model for detailed binary topological relationships, Geomatica 47(3)

[31] EGENHOFER, M. J, MARK, D., 1995, Naive Geography. In: Frank A.U., Kuhn W. (eds) Spatial Information Theory A Theoretical Basis for GIS. *COSIT 1995. Lecture Notes in Computer Science*, vol 988. Springer, Berlin, Heidelberg. https://doi.org/10.1007/3-540-60392-1_1

[32] EGENHOFER, M., CLEMENTINI., DI FELICE, P., 1994, Topological relations between regions with holes, *International Journal of Geographical Information Systems*, 8:2, 129-142, DOI: 10.1080/02693799408901990

[33] EGENHOFER, M., SHARMA, J., MARK, D., 2011, A critical comparison of the 4-intersection and 9-intersection models for spatial relations: formal analysis, In: R. McMaster and M. Armstrong (eds), *Autocarto* 11

[34] ERWIG, M., SCHNEIDER, M., 1999, The Honeycomb Model of Spatio-Temporal Partitions, *International Workshop on Spatio-Temporal Database Management*, Lecture Notes in Computer Science 1678, Berlin: Springer, 39–59.

[35] FEYMAN, R.F., 1983, *QED The strange theory of light*, Princeton University Press, 1983, https://www.scribd.com/document/358478820/QED-the-Strange-Theory-of-Light-and-Matter-PDF (Accessed 08-07-2021).

[36] FEYNMAN, R.P., 1965, HIBBS, A.R., *Quantum Mechanics and Path Integrals*, New York: McGraw-Hill., 1965, ISBN 0-07-020650-3.

[37] FISHER, P., 1993, Algorithm and implementation uncertainty in viewshed analysis, *International Journal of Geographical Information Systems*, 7, 331–347.

[38] FISHER, P., 2000, Sorites paradox and vague geographies, *Fuzzy Sets and Systems*, Volume 113, Issue 1, 1 July 2000, Pages 7-18

[39] FISHER, P.F., 1992, First experiments in viewshed uncertainty: simulating the fuzzy viewshed, *Photogrammetric Engineering and Remote Sensing*, 58, 345-352.

[40] FRANK, O., 1998, Triad Count Statistics, *Discrete Mathematics*, 72, pp. 141-149.

[41] FRIEZE, A., KAROŃSKI, M., 2015, Introduction to Random Graphs, Cambridge University Press ISBN 9781107118508

[42] GALTON, A., 2003, A generalized topological view of motion in discrete space, *Theoretical Computer Science*, 305, 111 – 134

[43] GAWLIK, E., MULLEN, P., PAVLOV, D., MARSDEN, J., DESBRUN, M., 2011, Geometric, Variational Discretization of Continuum Theories, *Physica D: Nonlinear Phenomena*, 240, 21, pp. 1724-1760.

[44] GOLD,, C., NANTEL,J., YANG, W., 1996, Outside-in: an alternative approach to forest map digitizing, *International Journal of Geographical Information Systems*, 10:3, 291-310, DOI: 10.1080/02693799608902080

[45] GOODCHILD M., 1977, An evaluation of lattice solutions to the problem of corridor location, *Environment and Planning A*. 9(7):727–38

[46] GOODCHILD, M., 1980, Fractals and the accuracy of geographical measures. *Mathematical Geology* 12: 85–98. [40]

[47] GOODCHILD, M., 2008, Geographic information science: the grand challenges. In J.P. Wilson and A.S. Fotheringham, editors, *The Handbook of Geographic Information Science*.

Malden, MA: Blackwell, pp. 596–608. [438]
[48] GOODCHILD, M., 2020, How well do we really know the world? Uncertainty in GIScience, *Journal Of Spatial Information Science*, 20 (2020), pp. 97–102
[49] GOODCHILD, M., MARK, D., 1988, The invertibility of distance matrices. In W. Coffey, editor, *Geographical Systems and Systems of Geography: Essays in Honour of William Warntz*. Department of Geography, University of Western Ontario, 141–151. [97]
[50] GOODCHILD, M., 2011, Scale in GIS : An Overview, *Geomorphology*, 130, pp. 5-9.
[51] GOODCHILD.M., 2002, Measurement-based GIS. In W. Shi, P.F. Fisher, and M.F. Goodchild, eds. *Spatial Data Quality*, CRC Press. ISBN 9780429219610.
[52] GOTTS, N., GOODAY, J., COHN, A., 1996, A connection based approach to common-sense topological description and reasoning, *The Monist*, Vol. 79 (1996), pp. 51-75
[53] GOTTSEGEN, J.; MONTELLO, D.; GOODCHILD, M.F. A comprehensive model of uncertainty in spatial data. In *Spatial Accuracy Assessment Land Information Uncertainty in Natural Resources*; Ann Arbor Press: Chelssea, MI, USA, 1999; pp. 175–182. [**Google Scholar**]
[54] GUIBAS, L., STOLFI, J., 1983, Primitives for the Manipulation of General Subdivisions and the Computation of Voronoi Diagrams. *ACM Transactions on Graphics*, TOG. 4. 221-234. 10.1145/282918.282923
[55] HÄGERSTRAND, T., 1982, Diorama, path and project. *Tijdschrift voor Economische en Sociale eografie*, 73, pp. 323-339.
[56] HALES, T., 2005, A proof of the Kepler conjecture, *Annals of Mathematics*, 162 (2005), pp. 1065–1185.
[57] HARRISON, C., 1972, On the Structure of Space Time, *Synthese, Space, Time and Geometry*, 24, 1/2, pp. 180-194. http://www.jstor.com/stable/20114831
[58] HAYES, B., 2011, An Adventure in the Nth Dimension, *American Scientist*, 99, pp. 442-446. http://www.peterbeerli.com/classes/images/2/20/AmSci2011Hayes2.pdf.
[59] HEFETZ, D., KRIVELEVICH, M., SZABO, T., 2012, Sharp Threshold for the appearance of certain spanning trees in random graphs. *Random Structures & Algorithms*, 41, 4, pp. 391-573. https://doi.org/10.1002/rsa.20472
[60] HERRING J.R., 1991, The Mathematical Modeling of Spatial and Non-Spatial Information in Geographic Information Systems. In: Mark D.M., Frank A.U. (eds) *Cognitive and Linguistic Aspects of Geographic Space*. NATO ASI Series (Series D: Behavioural and Social Sciences), vol 63. Springer, Dordrecht. https://doi.org/10.1007/978-94-011-2606-9_18
[61] HOFF, D., RAFTERY, A., HANDCOCK, M., 2002, Latent Space Approaches to Social Network Analysis, *Journal of the American Statistical Association*, December, 97, 460, DOI 10.1198/016214502388618906.
[62] HOPKINS, B., WILSON, R., 2004, The Truth about Königberg, *The college mathematics journal*,35, 3,p.207 https://www.maa.org/sites/default/files/pdf/upload_library/22/Polya/hopkins.pdf
[63] HORNSBY, K. AND EGENHOFER, M.J., 2002, Modeling Moving Objects over Multiple Granularities, *Annals of Mathematics and Artificial Intelligence*, 36(1-2), pp. 177-194.
[64] HUBER DL, CHURCH RL., 1985, Transmission Corridor Location Modeling. *Journal of Transportation Engineering-Asce*, 111(2):114–30
[65] JOSWIG, M., 2012, From Kepler to Hales and Back to Hilbert, *Documenta Math*, 62, p.439, http://www.kurims.kyoto-u.ac.jp/EMIS/journals/DMJDMV/vol-ismp/62_joswig-michael.pdf
[66] KINSEY, C., 1991, *The Topology of Surfaces*, Springer, New York.

http://www.maths.ed.ac.uk/~aar/papers/kinsey.pdf

[67] KLAIN, D., ROTA, G., 1997, *Introduction to Geometric Probability* (Cambridge: Cambridge University Press).

[68] KRAAK, MENNO-JAN, 2008, Geovisualization and time: new opportunities for the space-time cube". Dodge, M,, McDerby, M, Turner, M., (eds.). *Geographic visualization: concepts, tools and applications*, Chichester, England; Hoboken, NJ: Wiley, pp. 293–306. doi:10.1002/9780470987643.ch15.

[69] NACKAERTS, K., GOVERS, G., VAN ORSHOVEN, J.,1999, Accuracy assessment of probabilistic visibilities, *International Journal of Geographical Information Science*, 13:7, 709-721, DOI: 10.1080/136588199241076

[70] KRIVITSKY, P., HANDCOCK, M., RAFTERY, A., HOFF, P., 2009, Representing degree distributions, clustering, and homophily in social networks with latent cluster random effects models, *Social Networks*, 31, pp. 204–213.

[71] LANGRAN, G., 1991, Producing answers to spatial questions. *In Auto-Carto 10: Technical Papers*, ACSM-ASPRS Annual Convention (1991). https://cartogis.org/docs/proceedings/archive/auto-carto-10/pdf/producing-answers-to-spatial-questions.pdf (Accessed 08-07-2021)

[72] LEDOUX, H., 2006, Modelling three-dimensional fields in geoscience with the Voronoi diagram and its dual. PhD thesis, School of Computing, University of Glamorgan. https://ethos.bl.uk/OrderDetails.do?uin=uk.bl.ethos.435474

[73] LEŚNIEWSKI, S., 1983, On the foundation of mathematics, *Topoi*, Vol. 2 (1983), pp. 7-52

[74] LINDSAY, J. B., 2016, The practice of DEM stream burning revisited. *Earth Surf. Process. Landforms*, 41: 658– 668. doi: 10.1002/esp.3888.

[75] MEDRANO, F., 2021, Effects of raster terrain representation on GIS shortest path analysis, *PLOS ONE* 16(4): e0250106. https://doi.org/10.1371/journal.pone.0250106

[76] MIOC, M., ANTON, F., GOLD, C., MOULIN, B., 2009, On Kinetic Line Voronoi Operations and Finite Fields. Sixth annual international symposium on voronoi diagrams in science and engineering, pp. 65-70.

[77] MONMONIER, M., 2015, Geocoding, in *The History of Cartography, Volume 6* Cartography in the Twentieth Century, Edited by Mark Monmonier https://press.uchicago.edu/books/hoc/hoc_v6/hoc_volume6_g.pdf (Accessed 08-06-2021)

[78] MUSES, C., 1958, Foreword, In Rothstien, J., Ed., *Communication, Organization and Science,* Falcon's Wing Press, Indiana Hills, CO, Falcon's Wing Press, Indiana Hills, CO.

[79] ODE, A., HAGERHALL, C.M., SANG, N., 2010, Analysing visual landscape complexity: Theory and application, *Landscape Research*, 35 (1), pp. 111-131

[80] OGBURN,D., 2006, Assessing the level of visibility of cultural objects in past landscapes. *Journal of Archaeological Science*, 33. 405-413. 10.1016/j.jas.2005.08.005

[81] OHORI, K., LEDOUX, H., STOTER, J., 2017, Modeling and manipulating spacetime objects in a true 4D model, *Journal of Spatial Information Science*, 14, pp. 61–93

[82] ORILLIA, F., 2014 , This moment and the next moment, in Fano, V., Orilia, F. & Macchia, G. (2014). *Space and time : a priori and a posteriori studies* . Berlin, De Gruyter.

[83] OS, 2015, A guide to coordinate systems in Great Britain , An introduction to mapping coordinate systems and the use of GPS datasets with Ordnance Survey mapping, D00659 v2.3 Mar 2015 © Crown copyright https://www.bnhs.co.uk/2019/technology/grabagridref/OSGB.pdf (Accessed 08-06-2021)

[84] PAPADIMITRIOU F., 2020, Modelling and Visualization of Landscape Complexity

with Braid Topology. In: Edler D., Jenal C., Kühne O. (eds) *Modern Approaches to the Visualization of Landscapes*. RaumFragen: Stadt – Region – Landschaft. Springer VS, Wiesbaden. https://doi.org/10.1007/978-3-658-30956-5_5
[85] PEMMARAJU, S. SKIENA, S., 2003, Cycles, Stars, and Wheels, In *Computational Discrete Mathematics: Combinatorics and Graph Theory in Mathematica*. Cambridge, England: Cambridge University Press, pp. 248-249.
[86] PENROSE, R., 1971, *Angular Momentum: an approach to Combinatorial Space-Time Quantum Theory and Beyond*, edited by Ted Bastin, Cambridge University Press, pp. 151–180.
[87] PENROSE, R., 1972, On the Nature of Quantum Geometry in: Magic Without Magic, edited by J. Klauder, Freeman, San Francisco, 1972, pp. 333-354.
[88] PEUQUET D., 2001, Making Space for Time: Issues in Space-Time Data Representation, *GeoInformatica*, 5:1, pp. 11-32.
[89] PEUQUET D., 1994, It's about Time: A Conceptual Framework for the Representation of Temporal Dynamics in Geographic Information Systems, *Annals of the Association of American Geographers*, 84, 3, pp. 441-461
[90] PLANTINGA, H., DYER, C.R., 1990, Visibility, occlusion, and the aspect graph. *Int J Comput Vision* **5,** 137–160. https://doi.org/10.1007/BF00054919
[91] POATE, T., MASSELINK, G., AUSTIN, M. J., DICKSON, M., MCCALL, R., 2018, The role of bed roughness in wave transformation across sloping rock shore platforms, *Journal of Geophysical Research: Earth Surface*, 123, 97–123. https://doi.org/10.1002/2017JF004277
[92] POLHILL, G., LUIS R. I., GOTTS, N., 2006, What every agent-based modeller should know about floating point arithmetic, *Environmental Modelling and Software*, 21(3):283-309
[93] ROEPER, P., 1997, Region-based topology, *J. Philos. Logic*, 26, pp. 251-309
[94] ROVELLI, C., 2004, What is time? What is space?, Di Renzo Editore (English Translation), Rome.
[95] ROVELLI, C., 2017, The Order of Time, Penguin Science, Random House, UK.
[96] ROY, A., STELL, J., 2002, A Qualitative Account of Discrete Space, *Geographic Information Science, Proceedings of Second International Conference*, GIScience, Boulder, CO, USA, September 25-28.
[97] SANG, N., 2011, Visual Topology in SDI: A Data Structure for Modelling Landscape Perception, PhD Thesis, Univ. Glamorgan, BLL01016591924.
[98] SANG, N., GOLD, C., MILLER, D., 2016, The topological viewshed: embedding topological pointers into digital terrain models to improve GIS capability for visual landscape analysis, *International Journal of Digital Earth*, 9, pp. 1185-1205.
[99] SANG, N., HÄGERHÄLL, C., ODE, Å., 2015, The Euler character: a new type of visual landscape metric? *Environment and Planning B: Planning and Design*, 42 (1), 110-132
[100 SCHAFFER, J., 2018, Monism, in Zalta, E., *Stanford Encyclopedia of Philosophy*, 2018 Edition. https://plato.stanford.edu/entries/monism/#Tili (Accessed 05-07-2021)
[101] SHORTRIDGE, A., AND GOODCHILD, M., 2002, Geometric probability and GIS: some applications for the statistics of intersections. *International Journal of Geographical Information Science*, 16, 3, pp. 227–243.
[102] SHORTRIDGE, A., Geometric variability of raster cell classignment, *International Journal of Geographical Information Science*, 18, 6 2004, pp. 539-558.
[103] SIDDHARTHAN, R., 1998, The Uncertainty Principle for Dummies, Accessed 08062021 https://www.imsc.res.in/~rsidd/papers/uncert.pdf

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 October 2022 doi:10.20944/preprints202210.0336.v1

[104] SIMONS, P., 1987, *Parts: A Study in Ontology*, Clarendon Press, Oxford, UK.

[105] SOLOMON, H., 1978, Geometric Probability. *CBMS-NSF Regional Conference Series in Applied Mathematics*, 28 (Philadelphia: Society for Industrial and Applied Mathematics)

[106] SONG, Y., MILLER, H., 2015, Beyond the Boundary: New Insights from Inside the Space Time Prism, in Mei-Po Kwan, Douglas Richardson, Donggen Wang, Chenghu Zhou Eds., *Space Time Integration in Geography and GI Science*, Springer.

[107] SSIN, S., WALSH, J., SMITH, R., CUNNINGHAM,A., THOMAS, B., 2019, GeoGate: Correlating Geo-Temporal Datasets Using an Augmented Reality Space-Time Cube and Tangible Interactions, *IEEE Conference on Virtual Reality and 3D User Interfaces (VR)*, Osaka, Japan. DOI: 10.1109/VR.2019.8797812

[108] STAGE, A., SALAS, C., 2007, Interactions of Elevation, Aspect, and Slope in Models of Forest Species Composition and Productivity, *Forest Science*, 53, 4. https://www.fs.fed.us/rm/pubs_other/rmrs_2007_stag e_a002.pdf

[109] STELL, J., 1999, Granulation for Graphs, Spatial Information Theory. Cognitive and Computational Foundations of Geographic Information Science. *International Conference COSIT'99*, volume 1661 of Lecture Notes in Computer Science

[110] STELL, J., 2000, Boolean connection algebras: A new approach to the Region-Connection Calculus, *Artificial Intelligence*, Volume 122, Issues 1–2, Pages 111-136, ISSN 0004-3702, https://doi.org/10.1016/S0004-3702(00)00045-X

[111] SULLIVAN, R., KIRCHLER, L., COTHREN, J., WINTERS, S, 2013, Offshore Wind Turbine Visibility and Visual Impact Threshold Distances, *Environmental Practice*, Vol 15, Issue 1, Cambridge Core. DOI: https://doi.org/10.1017/S1466046612000464

[112] SVANBERG, M., SANTÉN, V., HÖRTEBORN, A., HOLM, H., & FINNSGÅRD, C., 2019, AIS in maritime research, *Marine Policy*, 106(103520). doi:https://doi.org/10.1016/j.marpol.2019.103520

[113] TIMPF, S., KUHN W., 2003, Granularity Transformations in Wayfinding. In: Freksa C., Brauer W., Habel C., Wender K.F. (eds) *Spatial Cognition III*, Lecture Notes in Computer Science (Lecture Notes in Artificial Intelligence), vol 2685. Springer, Berlin, Heidelberg. https://doi.org/10.1007/3-540-45004-1_5

[114] TOBLER, W., 1989, Frame independent spatial analysis, In Michael F. Goodchild, Sucharita Gopal Eds., *The Accuracy Of Spatial Databases*, Imprint CRC Press, eBook ISBN9780429208317 https://www.taylorfrancis.com/chapters/edit/10.1201/b12612-22/frame-independent-spatial-analysis-waldo-tobler

[115] VERMEER, M., 2014, Statistical uncertainty and error propagation, https://users.aalto.fi/~mvermeer/uncertainty.pdf (Accessed 05-07-2021)

[116] WAALDIJK, F., 2012, Natural Topology, Brouwer Society, Nijmegen, Netherlands, arXiv:1210.6288v1 [math.LO]

[117] WATSON, S., 1994, The number of complements in the lattice of topologies on a fixed set, *Topology and its Applications*, Volume 55, Issue 2, Pages 101-125, ISSN 0166-8641, https://doi.org/10.1016/0166-8641(94)90112-0.

[118] WATT, A., 2000, *3D Computer Graphics*. Addison-Wesley, pp, 1-13. 570

[119] WEBB, S., 2015, "Why-does-space-have-three-dimensions?" https://www.newscientist.com/article/mg22730370-700-why-does-space-have-three-dimensions/ (Accessed 08-07-2021)

[120] WHITEHEAD, A., 1929, *Process and Reality: An Essay in Cosmology*. 1979 corrected

edition, edited by David Ray Griffin and Donald W. Sherburne, Free Press, ISBN 0-02-934570-7

[121] WINTER, S., FRANK, A., 2000, Topology in Raster and Vector Representation, *GeoInformatica*, 4, 1, 35-65 DOI: 10.1023/A:1009828425380

[122] WOLFF, C., 1730, *Philosophia prima, sive Ontologia, methodo scientifica pertractata*, Frankfurt: Renger.

[123] WOOD, G., ROBERTSON, J., 1998, Buffon got it straight, *Statistics & Probability Letters*, Volume 37, Issue 4, 1998, Pages 415-421, ISSN 0167-7152, https://doi.org/10.1016/S0167-7152(97)00145-4

[124] WORBOYS, M., DUCKHAM, M., 2004, *GIS: A Computing Perspective*, Second Edition, Routledge, UK. ISBN 9780415283755

[125] TSONIS, A., 2016, Randomness: a property of the mathematical and physical systems, Hydrological Sciences Journal, 61:9, 1591-1610, DOI: 10.1080/02626667.2014.992434

[126] GOODCHILD, M. F. 2008. "The Use Cases of Digital Earth." International Journal of Digital Earth 1 (1): 31–42. doi:10. 1080/17538940701782528

[127] CARTER, N. (2009). *Visual group theory* [Washington, D.C.] :: Mathematical Association of America,.

[128] GALTON, A. (2019). Space, Time and the Representation of Geographical Reality. In: Tambassi, T. (eds) The Philosophy of GIS. Springer Geography. Springer, Cham. https://doi.org/10.1007/978-3-030-16829-2_4

[129] TOBLER, W., A Computer Movie Simulating Urban Growth in the Detroit Region Economic Geography, Vol. 46, Supplement: Proceedings. International Geographical Union. Commission on Quantitative Methods (Jun., 1970), pp. 234-240 Published by: Clark University Stable URL: http://www.jstor.org/stable/143141 .

[130] GALTON, A., 2011, Time flies but space does not: Limits to the spatialisation of time, Journal of Pragmatics, 43, 3, 695-703, ISSN 0378-2166, https://doi.org/10.1016/j.pragma.2010.07.002.

[131] GALTON, A., DUCKHAM, M., BOTH, A., 2015, Extracting Causal Rules from Spatio-Temporal Data. In: Fabrikant, S., Raubal, M., Bertolotto, M., Davies, C., Freundschuh, S., Bell, S. (eds) Spatial Information Theory, COSIT 2015. Lecture Notes in Computer Science(), vol 9368. Springer, Cham. https://doi.org/10.1007/978-3-319-23374-1_2.