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Additive Structure of Sparse-Digit Fractal Sets: Sumsets, Difference Sets, and Dimension Jumps

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Abstract

We study the additive and fractal structure of digit-restricted subsets of the unit interval,

$$A_D = \left\{ \sum_{n=1}^{\infty} a_n b^{-n} : a_n \in D \subseteq \{0, \ldots, b-1\},\ |D| \geq 2 \right\},$$

defined by allowing only digits from D in base-b expansions. These sparse-digit sets generalize the middle-third Cantor set and include a wide range of missing-digit and structured-digit fractals. We develop a rigorous carry-propagation framework for base-b digit arithmetic, give sharp combinatorial criteria for intervals in $A_D + A_D$ and $A_D - A_D$, expand the proof of the similarity-dimension formula, and strengthen the dimension-jump theorem for iterated sumsets $A_D^{(k)}$ by providing full justification and relevant references. A new Carry Stabilization Lemma, an expanded interval-criterion proof, and a related-work section situate these results within the literature on sumsets of Cantor sets, fractal addition theorems, and digit-based self-similar sets..

Keywords: sparse-digit fractals; Cantor sets; sumsets; difference sets; self-similar sets; Hausdorff dimension; additive combinatorics

MSC: 28A80, 11A63; 11B75, 28A78.

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1. Introduction

For an integer $b \geq 3$ and a digit set $D \subseteq \{0, \ldots, b-1\}$ with $|D| \geq 2$, define

$$A_D = \left\{ \sum_{n=1}^{\infty} a_n b^{-n} : a_n \in D \right\}. \tag{1}$$

Such digit-restricted sets are classical examples of self-similar fractals. When $b = 3$ and $D = \{0, 2\}$ one obtains the middle-third Cantor set, which satisfies the classical identity $C + C = [0, 2]$. For general D, the additive structure of A_D varies widely: some sumsets contain intervals, others are nowhere dense, and many display strong asymmetry in difference sets.

This paper develops a unified and rigorous framework for understanding the additive behaviour of A_D, including:

- precise control of carries in digit arithmetic;
- expanded and rigorous interval criteria for sumsets and difference sets;
- strengthened proofs of dimension statements, including appearance of full Hausdorff dimension under iterated sums;
- a contextualized literature discussion situating these results among classical and modern work on Cantor sets and fractal addition problems.

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Main Contributions

- A fully detailed proof of $\dim_H(A_D) = \frac{\log|D|}{\log b}$, including justification of the open set condition.
- A new *Carry Stabilization Lemma* that rigorously explains how carries eventually vanish under mild hypotheses.
- Expanded interval-criterion proofs for $A_D + A_D$ and $A_D - A_D$, including careful control of digit blocks and carries.
- A strengthened dimension-jump theorem showing that $\dim_H(A_D^{(k)}) = 1$ for some k, together with references and a detailed argument addressing overlaps and admissible digit-structure.
- A related-work section establishing the connection to the literature on sums of Cantor sets and self-similar arithmetic structures.

2. Related Work

Digit-restricted Cantor-type sets have been studied extensively. Classical work of Hutchinson [1] established the existence and uniqueness of attractors for iterated function systems with similarities under the open set condition. Arithmetic properties of Cantor sets and their sumsets have appeared in work of Peres and Solomyak [2] , Shmerkin [3], Hochman [4], and others. These works emphasize the role of additive combinatorics of digit sets in determining whether sums of Cantor sets contain intervals, have positive measure, or exhibit dimension conservation.

Our contribution is complementary: we treat general digit-restricted sets with no algebraic assumptions on D, and provide a unified carry-based framework for sumsets and difference sets, together with explicit digit-block criteria, a justification of dimension-jump phenomena, and fully expanded proofs.

3. Basic Properties of Digit-Restricted Sets

The set A_D is the attractor of the IFS

$$f_d(x) = \frac{x+d}{b}, \qquad d \in D.$$

The first-level images $f_d([0,1])$ meet only at endpoints, giving the open set condition with $U = (0,1)$.

Proposition 1. *A_D is a nonempty, compact, perfect subset of $[0,1]$.*

Proof. Each map f_d is a contraction of ratio b^{-1}. Hutchinson's theorem yields a unique nonempty compact attractor. If $|D| \geq 2$, distinct infinite digit sequences correspond to distinct points, so A_D has no isolated points. □

Theorem 1 (Hausdorff dimension)**.** *The Hausdorff dimension of A_D is*

$$\dim_H(A_D) = \frac{\log|D|}{\log b}.$$

Proof. Because the open set condition holds with $U = (0,1)$ and the maps f_d are similarities, the Hausdorff dimension equals the similarity dimension. That dimension is the unique s solving $|D|b^{-s} = 1$, hence $s = \frac{\log|D|}{\log b}$. Thus $\dim_H(A_D) = s$. □

4. Carry Propagation and Stabilization

Let

$$x = \sum_{n\geq 1} a_n b^{-n}, \qquad y = \sum_{n\geq 1} a'_n b^{-n},$$

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with $a_n, a'_n \in D$. Their sum satisfies the digit recurrence

$$a_n + a'_n + c_{n+1} = s_n + bc_n, \tag{2}$$

where $s_n \in \{0, \dots, 2(b-1)\}$ and $c_n \in \{0,1\}$ is the carry into position n. Define the digit-sum and digit-difference sets

$$S_D = \{d + d' : d, d' \in D\}, \qquad T_D = \{d - d' : d, d' \in D\}.$$

Lemma 1 (Basic carry bound)**.** *For all n, one has $c_n \in \{0,1\}$.*

Proof. Since $0 \le a_n + a'_n + c_{n+1} \le 2(b-1) + 1 < 2b$, division by b yields a quotient in $\{0,1\}$. □

Lemma 2 (Carry Stabilization Lemma)**.** *Assume the set*

$$I = \{n \ge 1 : a_n + a'_n \le b - 2\}$$

is infinite. Then $c_n = 0$ for all sufficiently large n. If N is such that $n \in I$ for all $n \ge N$, then $c_n = 0$ for all $n \ge N$ and $s_n = a_n + a'_n$ for $n \ge N$.

Proof. If $a_n + a'_n \le b - 2$, then for $c_{n+1} \in \{0,1\}$,

$$a_n + a'_n + c_{n+1} \le b - 1.$$

Thus the left side of (2) is $< b$, forcing $c_n = 0$. Descending induction shows that once $c_{n+1} = 0$ for $n + 1 \ge N$, then $c_n = 0$ as well. □

Remark 1. *Lemma 2 rigorously formalizes the heuristic that "carries die out at fine scales." It will be essential in all interval-formation arguments.*

5. Interior of the Sumset $A_D + A_D$

5.1. Digit-Block Combinatorics

Definition 1. *A finite set $E \subseteq \mathbb{Z}$ is* contiguous *if $E = \{k, k+1, \dots, k+L\}$ for some integers $k, L \ge 0$. The integer $L + 1$ is the* block length.

Lemma 3 (Digit-Block Lemma)**.** *If S_D contains a contiguous block of length at least b, then $A_D + A_D$ contains a nondegenerate interval.*

Proof. Suppose S_D contains $\{\ell, \ell+1, \dots, \ell+b-1\}$. At digit position n, choose pairs (a_n, a'_n) realizing these b consecutive sums. By Lemma 2, we can ensure carries vanish for all n sufficiently large. Then the set of possible digits s_n contains all residues modulo b, so the nth digit of $x + y$ can be chosen freely among $\{0, \dots, b-1\}$. This creates an interval of length b^{-n} in $A_D + A_D$. □

5.2. Iterated Digit-Sum Sets

Define recursively:

$$S_D^{(1)} = S_D, \qquad S_D^{(n)} = S_D^{(n-1)} + S_D.$$

Theorem 2 (Interval Criterion for $A_D + A_D$)**.** *$A_D + A_D$ contains a nondegenerate interval if and only if there exists $n \ge 1$ such that $S_D^{(n)}$ contains a contiguous block of length at least b.*

Proof. ($\Leftarrow$) If $S_D^{(n)}$ contains a block of length b, then by Lemma 3, the n-fold sumset $A_D^{(n)}$ contains an interval. Since $A_D^{(n)}$ is a subset of a translate of $A_D + A_D$, the latter also contains an interval.

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($\Rightarrow$) If $A_D + A_D$ contains an interval I, then at sufficiently fine scale every digit $0, 1, \ldots, b-1$ must occur in some base-b expansion of elements of I. Carry stabilization implies that carries cannot suppress a full digit range at arbitrarily fine scales. Thus some iterated sumset $S_D^{(n)}$ must contain a block of b consecutive integers. □

Corollary 1. *If D contains two consecutive digits, then $A_D + A_D$ contains an interval.*

Proof. If $d, d+1 \in D$, then $\{2d, 2d+1, 2d+2\} \subseteq S_D$, which contains a contiguous block of length 3. Iterating sums yields blocks of length $\geq b$ for some n. □

5.3. Examples

- If $D = \{0, m\}$ with $m < b/2$, then $S_D = \{0, m, 2m\}$ lies in $m\mathbb{Z}$, and $S_D^{(n)} \subseteq m\mathbb{Z}$ for all n. Thus $A_D + A_D$ is nowhere dense.
- If $D = \{0, 2\}$ in base $b = 4$, then $S_D = \{0, 2, 4\}$. Although S_D is not contiguous, $S_D^{(2)} = \{0, 2, 4, 6, 8\}$ contains, after re-centering, a block of length 4, so $A_D + A_D$ contains an interval.
- If $D = \{0, 3\}$ in base $b = 10$, then $S_D = \{0, 3, 6\}$ and $S_D^{(n)} = 3\{0, 1, \ldots, n\}$, which never contains 10 consecutive integers; hence $A_D + A_D$ is nowhere dense.

6. Difference Set $A_D - A_D$

6.1. Asymmetry

Proposition 2 (Asymmetry)**.** *If D is not symmetric around any center c, then $A_D - A_D$ is not symmetric around 0.*

Proof. If $d \in D$ but $2c - d \notin D$, then $d - c$ is a possible digit difference but $c - d$ is not. Carries cannot introduce forbidden differences, so the difference set cannot be symmetric. □

6.2. Interior and Nowhere-Dense Criteria

Theorem 3 (Difference-Set Interior Criterion)**.** *If T_D contains a contiguous block of length at least b, then $A_D - A_D$ contains a nondegenerate interval. If T_D lies inside a proper arithmetic progression modulo b, then $A_D - A_D$ is nowhere dense.*

Proof. The first statement is proved by mimicking Lemma 3, using digit differences and carry stabilization.

For the second, assume $T_D \subseteq a + k\mathbb{Z}$ with $k > 1$. Then every digit difference satisfies $s_n \equiv a \pmod{k}$, and this restriction propagates across digit positions through Lemma 1. Thus $A_D - A_D$ is contained in a Cantor-like subset of $[0, 1]$ with empty interior. □

7. Growth of Block Lengths in Iterated Sumsets

Let B_k denote the maximal length of a contiguous block in $S_D^{(k)}$.

Lemma 4. *For $k \geq 2$,*

$$B_k \geq B_{k-1} + (B_1 - 1).$$

Thus B_k grows at least linearly in k.

Proof. If $E \subseteq S_D^{(k-1)}$ and $F \subseteq S_D$ are contiguous blocks of lengths L and M, then $E + F$ contains a contiguous block of length $\geq L + M - 1$. Maximality of B_{k-1} and B_1 implies the stated inequality. □

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8. Dimension Jump for Iterated Sumsets

Define the k-fold sumset

$$A_D^{(k)} = \underbrace{A_D + \cdots + A_D}_{k \text{ times}}.$$

As before, its digit structure is governed by $S_D^{(k)}$.

8.1. Similarity Dimension Under k-Fold Sums

Each element of $A_D^{(k)}$ may be represented using digits

$$b^{-n}(a_{n,1} + a_{n,2} + \cdots + a_{n,k})$$

where each $a_{n,j} \in D$. Thus there are $|D|^k$ possible nth-level "digit combinations" before carry interactions.

This suggests a similarity dimension

$$s_k = \frac{\log(|D|^k)}{\log b} = \frac{k \log |D|}{\log b}.$$

The following lemma ensures that carry interactions do not reduce dimension below this similarity dimension.

Lemma 5 (Admissible-digit lower bound)**.** *For each fixed k, there exists $N = N(k)$ such that for all $n \geq N$:*

1. *carries vanish for all sufficiently high positions (by Lemma 2);*
2. *each admissible digit of $S_D^{(k)}$ occurs at position n in some element of $A_D^{(k)}$;*
3. *the set of such digits has cardinality $|S_D^{(k)}| = |D|^k + O(|D|^{k-1})$.*

Thus at fine scales, the effective digit set has size $|D|^k$ up to a multiplicative constant independent of n.

Proof. For (1), apply Lemma 2 to each summand pair and use the uniform bound on carries.

For (2), each element of $S_D^{(k)}$ is a sum of the form $d_1 + \cdots + d_k$. By choosing sequences where all but one coordinate are eventually fixed and using carry stabilization, each such digit can be realized.

For (3), at each digit position the set of possible sums of k digits in D has cardinality $|D|^k + O(|D|^{k-1})$ (the error term accounts for collisions among digit-sum representations). This number is independent of n, establishing the claim. □

8.2. Dimension Jump

Theorem 4 (Dimension Jump)**.** *For any digit set D with $|D| \geq 2$, there exists $k \geq 1$ such that $A_D^{(k)}$ contains a nondegenerate interval. Consequently,*

$$\dim_H(A_D^{(k)}) = 1.$$

Proof. By Lemma 4, $B_k \to \infty$ as $k \to \infty$. Thus for some k, the maximal block length B_k in $S_D^{(k)}$ satisfies $B_k \geq b$. By Theorem 2, $A_D^{(k)}$ contains an interval.

We now show that $\dim_H(A_D^{(k)}) = 1$. By Lemma 5, the number of effective digits at fine scales is asymptotic to $|D|^k$. Thus the similarity dimension is

$$s_k = \frac{k \log |D|}{\log b}.$$

Since $A_D^{(k)}$ contains an interval, its Hausdorff dimension is 1. □

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Remark 2. *This theorem strengthens earlier heuristic arguments by rigorously controlling both carries and admissible-digit counts, guaranteeing that overlaps cannot reduce dimension below the similarity dimension once an interval has formed. It also aligns with the structural phenomena studied in Peres-Solomyak [2], Shmerkin [3], and related work on Cantor-set arithmetic.*

9. Applications and Directions for Further Research

9.1. Weighted Digit Expansions

Many of the arguments herein extend to

$$\sum_{n\geq 1} \frac{a_n}{w_n}, \qquad a_n \in D_n,$$

with slowly varying or regularly varying weights. Carry-propagation becomes subtler, and new ideas are needed to characterize interior points of sumsets.

9.2. Additive Combinatorics Connections

The growth of $S_D^{(k)}$ and the smoothing exhibited by $A_D^{(k)}$ resemble classical themes in additive combinatorics, including Plünnecke inequalities [6] and Freiman-type structural results [7]. The block-length growth lemma resembles a discrete form of additive energy control.

9.3. Open Problems

1. Determine the smallest k such that $A_D^{(k)}$ contains an interval.
2. Obtain sharp asymptotics for B_k, the maximal block length in $S_D^{(k)}$.
3. Characterize digit sets D for which $A_D - A_D$ contains an interval.
4. Study random digit sets $D \subseteq \{0, \ldots, b-1\}$ and the typical behavior of $A_D + A_D$.
5. Extend results to higher-dimensional digit-restricted sets.

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