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Mohammed Ali * and Hussain Al-Qassem

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Article

Weighted L^p Estimates for Multiple Generalized Marcinkiewicz Functions

Mohammed Ali [1]* and Hussain Al-Qassem [2]

[1] Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid 22110, Jordan
[2] Department of Mathematics and Statistics, Qatar University, Doha, Qatar
* Correspondence: myali@just.edu.jo

Abstract

In this paper we investigate the weighted L^p boundedness of generalized Marcinkiewicz integrals $\mathcal{M}_{\mathbf{K}}^{(\varepsilon)}$ over multiple symmetric domains. Under the conditions $\mathbf{K} \in L^q(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})$, $q > 1$, we stablish suitable weighted L^p bounds for the integrals $\mathcal{M}_{\mathbf{K}}^{(\varepsilon)}$. These bounds are combined with an extrapolation argument of Yano so we obtain the weighted L^p boundedness of $\mathcal{M}_{\mathbf{K}}^{(\varepsilon)}$ from the Triebel-Lizorkin space $\dot{F}_p^{0,\varepsilon}(\omega_1, \omega_2)$ to the space $L^p(\omega_1, \omega_2)$ under the weak conditions $\mathbf{K}$ lie in the space $B_q^{(0,\frac{2}{\varepsilon}-1)}(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})$ or in the space $L(\log L)^{2/\varepsilon}(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})$. Our findings are essential improvements and extension of several known findings in the literature.

Keywords: weighted L^p bound; block space; generalized Marcinkiewicz operator; product domains; Triebel-Lizorkin space; extrapolation

MSC: 42B35; 42B25; 42B20

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1. Introduction

Let $\mathbf{K} \in L^1(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})$ be a measurable mapping on $\mathbb{R}^m \times \mathbb{R}^n$ and satisfy

$$\mathbf{K}(ta, lb) = \mathbf{K}(a, b), \ \forall t, l > 0, \tag{1}$$

and

$$\int_{\mathbb{B}^{m-1}} \mathbf{K}(a, b) d\sigma_m(a) = \int_{\mathbb{B}^{n-1}} \mathbf{K}(a, b) d\sigma_n(b) = 0, \tag{2}$$

where $\mathbb{B}^{\kappa-1}$ ($\kappa = m$ or n) is the unit sphere in the Euclidean space $\mathbb{R}^\kappa$ equipped with the normalized spherical measure $d\sigma_\kappa(\cdot)$ and $\kappa \geq 2$.

For $\varepsilon > 1$ and $\Phi \in \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n)$, consider the generalized Marcinkiewicz integral $\mathcal{M}_{\mathbf{K}}^{(\varepsilon)}$ on $\mathbb{R}^m \times \mathbb{R}^n$ defined by

$$\mathcal{M}_{\mathbf{K}}^{(\varepsilon)}(\Phi)(u, v) = \left(\iint_{\mathbb{R}_+ \times \mathbb{R}_+} \left| H_{l,t}(\Phi)(u, v) \right|^\varepsilon \frac{dtdl}{tl} \right)^{1/\varepsilon}, \tag{3}$$

where

$$H_{l,t}(\Phi)(u, v) = \frac{1}{lt} \int_{|b| \leq l} \int_{|a| \leq t} \frac{\mathbf{K}(a, b)}{|a|^{m-1} |b|^{n-1}} \Phi(u - a, v - b) da db.$$

When $\varepsilon = 2$, we denote the operator $\mathcal{M}_{\mathbf{K}}^{(\varepsilon)}$ by $\mathcal{M}_{\mathbf{K}}$ which is basically the classical Marcinkiewicz integral on product spaces which was introduced by Ding in [1]. The author proved that $\mathcal{M}_{\mathbf{K}}$ is bounded on $L^2(\mathbb{R}^m \times \mathbb{R}^n)$ under the assumption $\mathbf{K} \in L(\log L)^2(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})$, where $L(\log L)^\theta(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})$ is the space of all $\mathbf{K} \in L^1(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})$ such that

$$\iint_{\mathbb{B}^{m-1}\times\mathbb{B}^{n-1}} |\mathbf{K}(a,b)| \log(2 + |\mathbf{K}(a,b)|)^\theta \, d\sigma_m(a) d\sigma_n(b) < \infty, \text{ for } \theta > 0.$$

Subsequently, many authors investigated the L^p boundedness of the operator $\mathcal{M}_{\mathbf{K}}$ under various conditions on the kernel functions $\mathbf{K}$. For instance, in [2] the authors established the boundedness of $\mathcal{M}_{\mathbf{K}}$ on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ $(1 < p < \infty)$ whenever $\mathbf{K} \in L(\log L)(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})$, and they pointed out that by adopting a similar argument as in [3], the condition $\mathbf{K} \in L(\log L)(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})$ is almost optimal in the sense that $\mathcal{M}_{\mathbf{K}}$ will be unbounded on $L^2(\mathbb{R}^m \times \mathbb{R}^n)$ if $\mathbf{K} \in L(\log L)^\theta(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})$ for any $\theta \in (0,1)$. A similar result was obtained in [4] when the kernel function belongs to a cerain class of block spaces $B_q^{(0,\theta)}(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})$. The author proved that $\mathcal{M}_{\mathbf{K}}$ is of type (p,p) for all $1 < p < \infty$ if $\mathbf{K} \in B_q^{(0,0)}(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})$, $q > 1$ and that the condition $\mathbf{K} \in B_q^{(0,0)}(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})$ is nearly optimal. For relevant results, one may consult [5–10], among others.

Let $1 < \varepsilon, p < \infty$ and $\overrightarrow{\tau} = (\zeta, \nu) \in \mathbb{R} \times \mathbb{R}$. Then, a tempered distribution function Φ on $\mathbb{R}^m \times \mathbb{R}^n$ belongs to the space of homogeneous Triebel-Lizorkin functions, $\dot{F}_p^{\overrightarrow{\tau},\varepsilon}(\mathbb{R}^m \times \mathbb{R}^n)$, if

$$\|\Phi\|_{\dot{F}_p^{\overrightarrow{\tau},\varepsilon}(\mathbb{R}^m\times\mathbb{R}^n)} = \left\| \left(\sum_{j,k\in\mathbb{Z}} 2^{(k\zeta+j\nu)\varepsilon} \left| (\varphi_k^{(1)} \otimes \varphi_j^{(2)}) * \Phi \right|^\varepsilon \right)^{1/\varepsilon} \right\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n)} < \infty,$$

where $\widehat{\varphi_k^{(1)}}(\xi) = 2^{-km} g_1(2^{-k}\xi)$, $\widehat{\varphi_j^{(2)}}(\eta) = 2^{-jn} g_2(2^{-j}\eta)$, and $g_1 \in C_0^\infty(\mathbb{R}^m)$, $g_2 \in C_0^\infty(\mathbb{R}^n)$ are radial mappings satisfy the following:

(i) $supp\,(g_1) \subset \left\{\xi : |\xi| \in [\tfrac{1}{2}, 2]\right\}$, $supp\,(g_2) \subset \left\{\eta : |\eta| \in [\tfrac{1}{2}, 2]\right\}$,

(ii) $g_1(\xi), g_2(\eta) \in [0,1]$,

(iii) $g_1(\xi), g_2(\eta) \geq C$ for all $|\xi|, |\eta| \in [\tfrac{3}{5}, \tfrac{5}{3}]$, for some bounded constant $C > 0$.

(iv) For $\xi \neq 0$, $\sum_{k\in\mathbb{Z}} g_1(2^{-k}\xi) = 1$; and for $\eta \neq 0$, $\sum_{j\in\mathbb{Z}} g_2(2^{-j}\eta) = 1$.

The following properties were proved in [11].

(a) The space $\mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n)$ is dense in $\dot{F}_p^{\overrightarrow{\tau},\varepsilon}(\mathbb{R}^m \times \mathbb{R}^n)$,

(b) If $\varepsilon_1 \leq \varepsilon_2$, then $\dot{F}_p^{\overrightarrow{\tau},\varepsilon_2}(\mathbb{R}^m \times \mathbb{R}^n) \supseteq \dot{F}_p^{\overrightarrow{\tau},\varepsilon_1}(\mathbb{R}^m \times \mathbb{R}^n)$,

(c) For $p \in (1,\infty)$, $\dot{F}_p^{\overrightarrow{0},2}(\mathbb{R}^m \times \mathbb{R}^n) = L^p(\mathbb{R}^m \times \mathbb{R}^n)$,

The study of the generalized Marcinkiewicz integral opoerator $\mathcal{M}_{\mathbf{K}}^{(\varepsilon)}$ was initiated in [11] in which the authors obtained the boundedness of $\mathcal{M}_{\mathbf{K}}^{(\varepsilon)}$ on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for $1 < p < \infty$ if $\mathbf{K} \in L(\log L)^{2/\varepsilon}(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})$. Recently, the authors of [12] employed an extrapolation argument of Yano in [13] to extend and improve all the above mentioned results. In fact, they established that

$$\left\| \mathcal{M}_{\mathbf{K}}^{(\varepsilon)}(\Phi) \right\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n)} \leq C_p \|\Phi\|_{\dot{F}_p^{\overrightarrow{0},\varepsilon}(\mathbb{R}^m\times\mathbb{R}^n)} \tag{4}$$

for all $p, \varepsilon \in (1,\infty)$ provided that $\mathbf{K}$ belongs to the space $L(\log L)^{2/\varepsilon}(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})$ or to the space $\mathbf{K} \in B_q^{(0,\frac{2}{\varepsilon}-1)}(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})$. For recent advances concerning the operator $\mathcal{M}_{\mathbf{K}}^{(\varepsilon)}$, readers are referred to [14–18] and the references therein.

At this point, let us recall the definition, as well as some pertinent properties, of certain classes of weighs which will be relevant to our current study.

Definition 1. *A non-negative locally integrable mapping ω is said to be in the space $A_p(\mathbb{R}^\kappa)$ with $p \in (1, \infty)$ if there is a bounded positive number B such that for any cube $\mathbf{S} \subseteq \mathbb{R}^\kappa$ with its sides are parallel to the coordinate axes, we have*

$$\left(|\mathbf{S}|^{-1} \int_{\mathbf{S}} \omega(z) dz\right)\left(|\mathbf{S}|^{-1} \int_{\mathbf{S}} \omega(z)^{-1/(p-1)} dz\right)^{p-1} \leq B < \infty,$$

and ω is said to be in $A_1(\mathbb{R}^\kappa)$ if

$$M^* \omega(z) \leq B\omega(z) \ \ a.e\ z \in \mathbb{R}^\kappa,$$

where $M^*\omega$ is the Hardy-Littlewood maximal function of ω.

Definition 2. *For $p \in [1, \infty)$, we say that $\omega \in \tilde{A}_p(\mathbb{R}^\kappa)$ if*

$$\omega(z) = \mathcal{B}_1(|z|)^{1-p} \mathcal{B}_2(|z|),$$

where either $\mathcal{B}_j^2 \in A_1(\mathbb{R}_+)$ or $\mathcal{B}_j \in A_1(\mathbb{R}_+)$ is decreasing, $j \in \{1, 2\}$.

Definition 3. *For $p \in (1, \infty)$, let $\dot{A}_p(\mathbb{R}^\kappa)$ be the set of nonnegative locally integrable functions ω such that $\omega(z) = \omega(|z|)$ and $\omega^2 \in A_1(\mathbb{R}_+)$.*

The weighted class $A_p^I(\mathbb{R}^\kappa)$ is defined as the definition of $A_p(\mathbb{R}^\kappa)$ with replacing the cubes $\mathbf{S}$ by all κ-dimensional intervals whose sides are parallel to coordinate axes [19]. The authors of [20] showed that $\dot{A}_p(\mathbb{R}_+) \subseteq \tilde{A}_p(\mathbb{R}_+)$. Further, it is well known that whenever $\omega(l) \in \dot{A}_p(\mathbb{R}_+)$, we have $\omega(|z|) \in A_p(\mathbb{R}^\kappa)$ which is the Mukenhoupt weighted space defined in [21]. Let us present the weight class $\tilde{A}_p^I$ given by $\tilde{A}_p^I = A_p^I \cap \tilde{A}_p$ $(1 \leq p < \infty)$. This class of weights has the following properties:

(1) For $1 \leq p_1 \leq p_2 < \infty$, we have $\tilde{A}_{p_1}^I \subseteq \tilde{A}_{p_2}^I$,

(2) If $\omega \in \tilde{A}_p^I$, then a number $\theta > 0$ exists such that $\omega^{1+\theta} \in \tilde{A}_p^I$,

(3) If $\omega \in \tilde{A}_p^I$ $(1 < p < \infty)$, then there exists a positive number θ satisfying $p - \theta > 1$ and $\omega \in \tilde{A}_{p-\theta}^I$.

(4) For $1 < p < \infty$, $\omega \in \tilde{A}_p^I$ if and only if $\omega^{1-p'} \in \tilde{A}_{p'}^I$, where p' denotes to the exponent conjugate of p.

The weighted L^p space related to the weights ω_1, ω_2 is denoted by $L^p(\mathbb{R}^m \times \mathbb{R}^n, \omega_1 du, \omega_2 dv) \equiv L^p(\omega_1, \omega_2)$ and is given by

$$L^p(\omega_1, \omega_2) = \left\{\Phi : \|\Phi\|_{L^p(\omega_1,\omega_2)} = \left(\iint_{\mathbb{R}^m \times \mathbb{R}^n} |\Phi(u,v)|^p \omega(u)\omega(v) du dv\right)^{1/p} < \infty\right\}.$$

Recently, the weighted L^p boundedness of $\mathcal{M}_{\mathbf{K}}$ was obtained in [22] for all $1 < p < \infty$ under the assumption $\mathbf{K} \in L(\log L)(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})$.

In view of the L^p boundedness results of the generalized Marcinkiewicz integral $\mathcal{M}_{\mathbf{K}}^{(\varepsilon)}$ obtained in [12] and the weighted L^p boundedness results of the classical Marcinkiewicz integral $\mathcal{M}_{\mathbf{K}}$ in [22] it is natural to ask the following: Does the weighted L^p boundedness of generalized Marcinkiewicz integral $\mathcal{M}_{\mathbf{K}}^{(\varepsilon)}$ hold under the same conditions as in [12]?

Our main purpose in this paper is to answer the above question in the affirmative. Our main results are formulated as follows:

Theorem 1. *Suppose that $\omega_1 \in \tilde{A}_p^I(\mathbb{R}^m)$, $\omega_2 \in \tilde{A}_p^I(\mathbb{R}^n)$, and $\mathbf{K} \in L^q(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})$ with $q \in (1, 2]$. Then the inequality*

$$\left\|\mathcal{M}_{\mathbf{K}}^{(\varepsilon)}(\Phi)\right\|_{L^p(\omega_1,\omega_2)} \leq C_p (q-1)^{-2/\varepsilon} \|\Phi\|_{\dot{F}_p^{\vec{0},\varepsilon}(\omega_1,\omega_2)} \|\mathbf{K}\|_{L^q(\mathbb{B}^{m-1}\times\mathbb{B}^{n-1})} \quad (5)$$

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holds for all $p \in (1, \infty)$.

By the results in Theorem 1 and following an extrapolation argument employed in [13,23] we get the following result.

Theorem 2. *Let* ω_1 *and* ω_2 *be given as in Theorem 1. Assume that the kernel function* $\mathbf{K}$ *is in the space* $L(\log L)^{2/\varepsilon}(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})$ *or in the space* $B_q^{(0,\frac{2}{\varepsilon}-1)}(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})$ *with* $q > 1$. *Then, for any* $\varepsilon > 1$, *the operator* $\mathcal{M}_{\mathbf{K}}^{(\varepsilon)}$ *is bounded on* $L^p(\omega_1, \omega_2)$ *for all* $p \in (1, \infty)$.

Remarks

(1) For the special cases $\varepsilon = 2$ and $\omega_1 \equiv 1 \equiv \omega_2$, the authors of [7] proved that $\mathcal{M}_K$ is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ for all $p \in (1, \infty)$ if $\mathbf{K}$ belongs to the space $L^q(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})$, $q > 1$. Therefore, since $L(\log L)(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1}) \cup B_q^{(0,0)}(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1}) \supset L^q(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})$, then our results are natural extensions to the results in [7].

(2) The conditions on $\mathbf{K}$ are the weakest known conditions for the cases $\omega_1 \equiv 1 \equiv \omega_2$ and $\varepsilon = 2$, (see [2,4]).

(3) The results in Theorem 2 with $\varepsilon = 2$ give the $L^p(\mathbb{R}^m \times \mathbb{R}^n)$ of $\mathcal{M}_{\mathbf{K}}^{(\varepsilon)}$ for $p \in (1, \infty)$ if $\mathbf{K} \in L(\log L)(\mathbb{B}^{n-1} \times \mathbb{B}^{m-1})$. The same result was obtained in [22].

(4) For the case $\omega_1 \equiv 1 \equiv \omega_2$, we observe that Theorem 2 generalizes Theorem 2.7 in [12].

2. Auxiliary Estimates

This section is devoted to giving some preliminary estimates that will play a significant role in proving the main results of this paper. Let $\mu \geq 2$, and consider the set of measures $\{Y_{\mathbf{K},l,t} := Y_{l,t} : l, t \in \mathbb{R}_+\}$ and its corresponding maximal operators M_μ and Y^* on $\mathbb{R}^m \times \mathbb{R}^n$ given by

$$\iint_{\mathbb{R}^m \times \mathbb{R}^n} \Phi dY_{l,t} = \frac{1}{lt} \iint_{\Lambda_{l,t}(a,b)} \frac{\mathbf{K}(a,b)}{|a|^{m-1}|b|^{n-1}} \Phi(u-a, v-b) da db,$$

$$Y^*(\Phi) = \sup_{l,t \in \mathbb{R}_+} |\Phi * |Y_{l,t}||,$$

and

$$M_\mu(\Phi) = \sup_{j,k \in \mathbb{Z}} \int_{\mu^j}^{\mu^{j+1}} \int_{\mu^k}^{\mu^{k+1}} |\Phi * |Y_{l,t}|| \frac{dt dl}{tl},$$

where $\Lambda_{l,t}(a,b) = \{(a,b) \in \mathbb{R}^m \times \mathbb{R}^n : l/2 \leq |b| \leq l, \, t/2 \leq |a| \leq t\}$ and $|Y_{l,t}|$ is defined similarly as $Y_{l,t}$ but $\mathbf{K}$ is replaced by $|\mathbf{K}|$.

The following lemma is a special case of Lemma 2.2 in [22].

Lemma 1. *Assume that* $\mathbf{K}$ *satisfies (1)-(2) and belongs to* $L^1(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})$. *Then for* $p \in (1, \infty)$, $\omega_1 \in A_p(\mathbb{R}^m)$, *and* $\omega_2 \in A_p(\mathbb{R}^n)$, *there exists a positive constant* C_p *such that*

$$\|Y^*(\Phi)\|_{L^p(\omega_1,\omega_2)} \leq C_p \|\mathbf{K}\|_{L^1(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})} \|\Phi\|_{L^p(\omega_1,\omega_2)}. \tag{6}$$

We notice that as a direct consequence of Lemma 1, we get that

$$\|M_\mu(\Phi)\|_{L^p(\omega_1,\omega_2)} \leq C_p \ln^2(\mu) \|\Phi\|_{L^p(\omega_1,\omega_2)} \|\mathbf{K}\|_{L^1(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})}, \tag{7}$$

for $1 < p < \infty$.

The next lemma is found in [24].

Lemma 2. *Let $q > 1$ and $\mathbf{K} \in L^q(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})$. Then, we have*

$$\left\|\Upsilon_{l,t}\right\| \leq C\|\mathbf{K}\|_{L^q(\mathbb{B}^{m-1}\times\mathbb{B}^{n-1})}, \tag{8}$$

$$\begin{aligned}\int_{\mu^j}^{\mu^{j+1}}\int_{\mu^k}^{\mu^{k+1}}\left|\hat{\Upsilon}_{l,t}(\xi,\eta)\right|^2\frac{dtdl}{tl} &\leq C\|\mathbf{K}\|^2_{L^q(\mathbb{B}^{m-1}\times\mathbb{B}^{n-1})}(\ln\mu)^2 \\ &\times \min\left\{\left|\xi\mu^k\right|^{-\frac{\gamma}{\ln\mu}},\left|\xi\mu^k\right|^{\frac{\gamma}{\ln\mu}}\right\}\min\left\{\left|\eta\mu^j\right|^{\frac{\gamma}{\ln\mu}},\left|\eta\mu^j\right|^{-\frac{\gamma}{\ln\mu}}\right\},\end{aligned} \tag{9}$$

where C is a positive constant, $\gamma \in (0, 1/2q')$, and $\left\|\Upsilon_{l,t}\right\|$ is the total variation of $\Upsilon_{l,t}$.

Lemma 3. *Let $q \in (1, 2]$, $\mathbf{K} \in L^q(\mathbb{B}^{m-1} \times \mathbb{B}^{n-1})$, and $\mu = 2^{q'}$. Then for $p \in (1,\infty)$, $\omega_1 \in \tilde{A}^I_p(\mathbb{R}^m)$, $\omega_2 \in \tilde{A}^I_p(\mathbb{R}^n)$, and arbitrary set of functions $\{\mathcal{J}_{j,k};\ j,k \in \mathbb{Z}\}$ on $\mathbb{R}^m \times \mathbb{R}^n$, there is $C > 0$ such that*

$$\begin{aligned}\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}\int_{\mu^k}^{\mu^{k+1}}\left|\Upsilon_{l,t}*\mathcal{J}_{j,k}\right|^{\varepsilon}\frac{dtdl}{tl}\right)^{1/\varepsilon}\right\|_{L^p(\omega_1,\omega_2)} &\leq C(q-1)^{-2/\varepsilon} \\ &\times \|\mathbf{K}\|_{L^q(\mathbb{B}^{m-1}\times\mathbb{B}^{n-1})}\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{J}_{j,k}\right|^{\varepsilon}\right)^{1/\varepsilon}\right\|_{L^p(\omega_1,\omega_2)}.\end{aligned}$$

Proof. By invoking (6), we obtain

$$\begin{aligned}&\left\|\sup_{j,k\in\mathbb{Z}}\sup_{(l,t)\in[1,\mu]\times[1,\mu]}\left|\Upsilon_{\mu^k l,\mu^j t}*\mathcal{J}_{j,k}\right|\right\|_{L^p(\omega_1,\omega_2)} \\ \leq\ &\left\|\Upsilon^*\left(\sup_{j,k\in\mathbb{Z}}\left|\mathcal{J}_{j,k}\right|\right)\right\|_{L^p(\omega_1,\omega_2)} \leq C\left\|\sup_{j,k\in\mathbb{Z}}\left|\mathcal{J}_{j,k}\right|\right\|_{L^p(\omega_1,\omega_2)}\|\mathbf{K}\|_{L^q(\mathbb{B}^{m-1}\times\mathbb{B}^{n-1})}\end{aligned} \tag{10}$$

for all $p \in (1,\infty)$, which means that

$$\left\|\left\|\|\Upsilon_{\mu^k l,\mu^j t}*\mathcal{J}_{j,k}\|_{L^\infty([1,\mu]\times[1,\mu],\frac{dtdl}{tl})}\right\|_{l^\infty(\mathbb{Z}\times\mathbb{Z})}\right\|_{L^p(\omega_1,\omega_2)} \tag{11}$$

$$\leq C\left\|\left\|\mathcal{J}_{j,k}\right\|_{l^\infty(\mathbb{Z}\times\mathbb{Z})}\right\|_{L^p(\omega_1,\omega_2)}\|\mathbf{K}\|_{L^q(\mathbb{B}^{m-1}\times\mathbb{B}^{n-1})}. \tag{12}$$

Thanks to duality, there is a function $f \in L^{p'}(\omega_1^{1-p'}, \omega_2^{1-p'})$ satisfying $\|f\|_{L^{p'}(\omega_1^{1-p'},\omega_2^{1-p'})} \leq 1$ and

$$\left\|\sum_{j,k\in\mathbb{Z}}\int_1^{\mu}\int_1^{\mu}\left|\Upsilon_{\mu^k r,\mu^j s}*\mathcal{J}_{j,k}\right|^{\gamma'}\frac{dtdl}{tl}\right\|_{L^p(\omega_1,\omega_2)}$$

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$$
\begin{aligned}
&= \iint_{\mathbb{R}^m\times\mathbb{R}^n} \sum_{j,k\in\mathbb{Z}} \int_1^{\mu}\int_1^{\mu} \left|\Upsilon_{\mu^k l,\mu^j t} * \mathcal{J}_{j,k}\right| \frac{dtdl}{tl} f(a,b)\,dadb \\
&\leq C \iint_{\mathbb{R}^m\times\mathbb{R}^n} \left(\sum_{j,k\in\mathbb{Z}} \left|\mathcal{J}_{j,k}(a,b)\right|\right) \Upsilon^*(\overline{f})(-a,-b)\,dadb \\
&\leq C(q-1)^{-2} \left\|\sum_{j,k\in\mathbb{Z}} \left|\mathcal{J}_{j,k}\right|\right\|_{L^p(\omega_1,\omega_2)} \left\|\Upsilon^*(\overline{f})\right\|_{L^{p'}(\omega_1^{1-p'},\omega_2^{1-p'})}, \qquad (13)
\end{aligned}
$$

where $\overline{f}(a,b) = f(-a,-b)$. Let $\mathbf{T}$ be the linear operator defined on $\mathcal{J}_{j,k}$ by $\mathbf{T}(\mathcal{J}_{j,k}) = \Upsilon_{\mu^k l,\mu^j t} * \mathcal{J}_{j,k}$. Thus, when we interpolate (12) with (13), we conclude

$$
\left\|\left(\sum_{j,k\in\mathbb{Z}} \int_{\mu^j}^{\mu^{j+1}}\int_{\mu^k}^{\mu^{k+1}} \left|\Upsilon_{l,t} * \mathcal{J}_{j,k}\right|^{\varepsilon} \frac{dtdl}{tl}\right)^{1/\varepsilon}\right\|_{L^p(\omega_1,\omega_2)}
$$

$$
\begin{aligned}
&\leq C\left\|\left(\sum_{j,k\in\mathbb{Z}} \int_1^{\mu}\int_1^{\mu} \left|\Upsilon_{\mu^k l,\mu^j t} * \mathcal{J}_{j,k}\right|^{\varepsilon} \frac{dtdl}{tl}\right)^{1/\varepsilon}\right\|_{L^p(\omega_1,\omega_2)} \\
&\leq C\left\|\left(\sum_{j,k\in\mathbb{Z}} \left|\mathcal{J}_{j,k}\right|^{\varepsilon}\right)^{1/\varepsilon}\right\|_{L^p(\omega_1,\omega_2)} \|\mathbf{K}\|_{L^q(\mathbb{B}^{m-1}\times\mathbb{B}^{n-1})} (\ln\mu)^{2/\varepsilon} \\
&\leq C(q-1)^{-2/\varepsilon}\left\|\left(\sum_{j,k\in\mathbb{Z}} \left|\mathcal{J}_{j,k}\right|^{\varepsilon}\right)^{1/\varepsilon}\right\|_{L^p(\omega_1,\omega_2)} \|\mathbf{K}\|_{L^q(\mathbb{B}^{m-1}\times\mathbb{B}^{n-1})}
\end{aligned}
$$

for all $1 < p < \infty$. □

3. Proof of Theorem 1

Let $\varepsilon > 1$ and $\mathbf{K} \in L^q(\mathbb{B}^{n-1}\times\mathbb{B}^{m-1})$ with $q \in (1,2]$. Suppose that $\omega_1 \in \tilde{A}_p^I(\mathbb{R}^m)$, $\omega_2 \in \tilde{A}_p^I(\mathbb{R}^n)$ with $p \in (1,\infty)$. Then, Minkowski's inequality gives

$$
\begin{aligned}
\mathcal{M}_{\mathbf{K}}^{(\varepsilon)}(\Phi)(u,v) &= \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+} \left|\sum_{j,k=0}^{\infty} \frac{1}{lt} \int_{2^{-j-1}l<|b|\leq 2^{-j}l} \int_{2^{-k-1}t<|a|\leq 2^{-k}t} \frac{\mathbf{K}(a,b)}{|a|^{m-1}|b|^{n-1}}\right.\right. \\
&\times \left.\left.\Phi(u-a,v-b)\,dadb\right|^{\varepsilon} \frac{dtdl}{tl}\right)^{1/\varepsilon} \\
&\leq \sum_{j,k=0}^{\infty} \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+} \left|\frac{1}{lt} \int_{2^{-j-1}l<|b|\leq 2^{-j}l} \int_{2^{-k-1}t<|a|\leq 2^{-k}t}\right.\right. \\
&\times \left.\left.\frac{\mathbf{K}(a,b)}{|a|^{m-1}|b|^{n-1}} \Phi(u-a,v-b)\,dadb\right|^{\varepsilon} \frac{dtdl}{tl}\right)^{1/\varepsilon} \\
&\leq C\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+} \left|\Upsilon_{l,t} * \Phi(u,v)\right|^{\varepsilon} \frac{dtdl}{tl}\right)^{1/\varepsilon}. \qquad (14)
\end{aligned}
$$

Choose a collection of mappings $\{h_i\}_{i\in\mathbb{Z}}$ such that:

$$
h_i \in C^{\infty}(0,\infty),\quad 0 \leq h_i \leq 1,\quad \sum_{i\in\mathbb{Z}} h_i(l) = 1,
$$

$$
\mathrm{supp}\ (h_i) \subseteq [2^{-q'(i+1)}, 2^{-q'(i-1)}],\ \textit{and}\ \left|\frac{d^j h_i(l)}{dl^j}\right| \leq \frac{C_j}{l^j},
$$

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where C_j does not depend on q. Define two measures $\{\Lambda_{i,m} : i \in \mathbb{Z}\}$ on $\mathbb{R}^m$ and $\{\Lambda_{i,n} : i \in \mathbb{Z}\}$ on $\mathbb{R}^n$ by

$$(\widehat{\Lambda_{i,m}}(\xi)) = h_i(|\xi|) \text{ and } (\widehat{\Lambda_{i,n}}(\eta)) = h_i(|\eta|),$$

where $\xi \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^n$. Therefore, we deduce that for $\Phi \in \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n)$,

$$\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+} \left|\Upsilon_{l,t} * \Phi(u,v)\right|^{\varepsilon} \frac{dtdl}{tl}\right)^{1/\varepsilon} \leq C \sum_{r,s\in\mathbb{Z}} \mathcal{Q}_{r,s}(\Phi)(u,v), \tag{15}$$

where

$$\mathcal{Q}_{r,s}(\Phi)(u,v) = \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+} \left|\mho^{l,t}_{r,s}(\Phi)(u,v)\right|^{\varepsilon} \frac{dtdl}{tl}\right)^{1/\varepsilon}$$

and

$$\mho^{l,t}_{r,s}(\Phi)(u,v) = \sum_{j,k\in\mathbb{Z}} \Upsilon_{l,t} * \left(\Lambda_{k+s,m} \otimes \Lambda_{j+r,n}\right) * \Phi(u,v)\chi_{[2^{kq'},2^{(k+1)q'})\times[2^{jq'},2^{(j+1)q'})}(l,t).$$

Hence, to complete the proof of Theorem 1, it is sufficient to find a positive number ϑ such that

$$\|\mathcal{Q}_{r,s}(\Phi)\|_{L^p(\omega_1,\omega_2)} \leq C_p 2^{-\frac{\vartheta}{2}(|r|+|s|)}(q-1)^{-2/\varepsilon}\|\Phi\|_{\dot{F}_p^{\vec{0},\varepsilon}(\omega_1,\omega_2)}\|\mathbf{K}\|_{L^q(\mathbb{B}^{m-1}\times\mathbb{B}^{n-1})} \tag{16}$$

for all $1 < p < \infty$.

If $p = \varepsilon = 2$, then by employing Lemma 2 with $\mu = 2^{q'}$ and invoking Plancherel's theorem, we get

$$\begin{aligned}
&\|\mathcal{Q}_{r,s}(\Phi)\|^2_{L^2(\omega_1,\omega_2)} \\
\leq\;& \sum_{j,k\in\mathbb{Z}} \iint_{\mathcal{A}_{j+r,k+s}} \left(\int_{2^{jq'}}^{2^{(j+1)q'}} \int_{2^{kq'}}^{2^{(k+1)q'}} \left|\hat{\Upsilon}_{l,t}(\xi,\eta)\right|^2 \frac{dtdl}{tl}\right) \left|\widehat{\Phi}(\xi,\eta)\right|^2 \omega_1(\xi)\omega_2(\eta)d\xi d\eta \\
\leq\;& C_p(q-1)^{-2}\|\mathbf{K}\|^2_{L^q(\mathbb{B}^{m-1}\times\mathbb{B}^{n-1})} \sum_{j,k\in\mathbb{Z}} \iint_{\mathcal{A}_{j+r,k+s}} \left|2^k\xi\right|^{\pm\frac{\gamma}{\ln 2}} \left|2^j\eta\right|^{\pm\frac{\gamma}{\ln 2}} \\
\times\;& \left|\widehat{\Phi}(\xi,\eta)\right|^2 \omega_1(\xi)\omega_2(\eta)d\xi d\eta \\
\leq\;& C_p(q-1)^{-2}\|\mathbf{K}\|^2_{L^q(\mathbb{B}^{m-1}\times\mathbb{B}^{n-1})} 2^{-\gamma(|r|+|s|)} \sum_{j,k\in\mathbb{Z}} \iint_{\mathcal{A}_{j+r,k+s}} \left|\widehat{\Phi}(\xi,\eta)\right|^2 \omega_1(\xi)\omega_2(\eta)d\xi d\eta \\
\leq\;& C_p(q-1)^{-2}\, 2^{-\gamma(|r|+|s|)}\|\Phi\|^2_{L^2(\omega_1,\omega_2)}\|\mathbf{K}\|^2_{L^q(\mathbb{B}^{m-1}\times\mathbb{B}^{n-1})},
\end{aligned}$$

where $\mathcal{A}_{j,k} = \left\{(\xi,\eta) \in \mathbb{R}^m \times \mathbb{R}^n : (|\xi|,|\eta|) \in [2^{-q'(k+1)}, 2^{-q'(k-1)}] \times [2^{-q'(j+1)}, 2^{-q'(j-1)}]\right\}$ and $\alpha^{\pm\beta} = \min\{\alpha^{+\beta}, \alpha^{-\beta}\}$. Therefore,

$$\|\mathcal{Q}_{r,s}(\Phi)\|_{L^2(\omega_1,\omega_2)} \leq C_p(q-1)^{-1}\, 2^{-\frac{\gamma}{2}(|r|+|s|)}\|\Phi\|_{L^2(\omega_1,\omega_2)}\|\mathbf{K}\|_{L^q(\mathbb{B}^{m-1}\times\mathbb{B}^{n-1})}. \tag{17}$$

For the other cases, by employing Lemma 3 and following the same argument as that used in the proof of Lemma 2.1 in [22], we obtain

$$\|\mathcal{Q}_{r,s}(\Phi)\|_{L^p(\omega_1,\omega_2)}$$

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$$
\begin{aligned}
&\leq C\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{2^{jq'}}^{2^{(j+1)q'}}\int_{2^{kq'}}^{2^{(k+1)q'}}\left|\Upsilon_{l,t}*\left(\Lambda_{k+s}\otimes\Lambda_{j+r}\right)*\Phi\right|^{\varepsilon}\frac{dtdl}{tl}\right)^{1/\varepsilon}\right\|_{L^p(\omega_1,\omega_2)} \\
&\leq C(q-1)^{-2/\varepsilon}\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\left(\Lambda_{k+s}\otimes\Lambda_{j+r}\right)*\Phi\right|^{\varepsilon}\right)^{1/\varepsilon}\right\|_{L^p(\omega_1,\omega_2)}\|\mathbf{K}\|_{L^q(\mathbb{B}^{m-1}\times\mathbb{B}^{n-1})} \\
&\leq C_p(q-1)^{-2/\varepsilon}\|\Phi\|_{\dot{F}_p^{\vec{0},\varepsilon}(\omega_1,\omega_2)}\|\mathbf{K}\|_{L^q(\mathbb{B}^{m-1}\times\mathbb{B}^{n-1})} \qquad (18)
\end{aligned}
$$

for all $1 < p < \infty$. Now, by interpolating between (17) and (18) we obtain (16) which when combined with (14)-(15) gives (5).

4. Conclusions

In this work, we proved certain weighted L^p estimates for a class of generalized Marcinkiewicz integrals $\mathcal{M}_{\mathbf{K}}^{(\varepsilon)}$ whenever the kernel function $\mathbf{K}$ belongs to the space $L^q(\mathbb{B}^{m-1}\times\mathbb{B}^{n-1})$, $q > 1$. By using these estimates and using Yano's extrapolation argument, we obtained the boundedness of $\mathcal{M}_{\mathbf{K}}^{(\varepsilon)}$ on $L^p(\omega_1,\omega_2)$ spaces under the weak conditions on the kernel functions $\mathbf{K} \in B_q^{(0,\frac{2}{\varepsilon}-1)}(\mathbb{B}^{m-1}\times\mathbb{B}^{n-1}) \cup L(\log L)^{2/\varepsilon}(\mathbb{B}^{m-1}\times\mathbb{B}^{n-1})$. Our results in this paper extend and improve several known results on generalized Marcinkiewicz operators as those in [1,2,4,7,9,10,12,22].

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